Spacecraft Attitude Control: A Consideration of Thrust Uncertainty

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To account for torque disturbances, and control trajectory error, a model of spacecraft attitude system is presented which replicates uncertainty in the class of continuous low-thrust systems. The generated uncertainty from each thruster is modeled as a Gaussian white noise process, multiplicative in control. An optimal stochastic control law is derived for precision pointing and three-axis stabilization. To derive the optimal control, a Hamilton-Jacobi-Bellman equation is formulated, and a power series-based method is employed to approximate the optimal control. The derived nonlinear control minimizes the objective function of the Lagrange problem in an infinite horizon setting. Stability and existence conditions of control are provided. The nonlinear stochastic optimal controller is compared to its deterministic counterpart for a 6U CubeSat model.

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I. Introduction

THE force and torque disturbances exerted on a spacecraft are often divided into two categories: those that are not caused by the spacecraft, and those that are results of the spacecraft's operation itself. The first group consists of the unwanted environmental torques and forces caused by phenomena such as solar radiation pressure, aerodynamic, and gravitational forces. In this paper, we focus on the second group. Specifically, we are interested in modeling and mitigating thrust-induced disturbances of continuous low-thrust spacecraft attitude maneuvers. Mission constraints such as propellant and power consumption, precision-pointing, and actuator lifetime are affected by unwanted effects of thrust fluctuations and hysteresis. Moreover, torque disturbances and excessive force cycling could jeopardize the structural integrity of the spacecraft if they exceed any loading constraints.

In modeling thrust-induced disturbances, we deviate from the rather traditional disturbance modeling and control practices. Instead, each thruster's disturbance is modeled as a Gaussian white noise process which is multiplicative in the commanded force. That is, each thruster's uncertainty is modeled as a multiplicative noise, in which it becomes the additive uncertain component of the generated total force. The consideration of existing noise in thrust enables us to embed the uncertainty information directly in the proposed control law. In this manner, we formulate a stochastic optimal controller that adjusts its behavior based on the best-known information on the severity of the disturbance. In designs where on/off thrusters are used, this stochastic controller can be implemented through use of pulse-width pulse-frequency (PWPF) modulation techniques. In the class of low-thrust propulsion engines, thrusters operate for a long range of time continuously; thus, the thrust fluctuations can be modeled as stochastic processes, as proposed here.

Several studies have previously addressed actuator uncertainty. In an influential work, McLane [1] derived the solution of the linear regulator problem for thrust-dependent noise in a physical system. Similarly, in the study of stochastic Hill's equations, Ostoja-Starzewski and Longuski [2] modeled the thrust as an additive random process. Gustafson [3] provided the numerical methods for the optimal feedback control of linear spacecraft system with thrusters. Jia and Zhao [4] investigated the attitude stabilization of a stochastic spacecraft system under additive disturbance. Other experimental studies, such as Nicolini et al. [5], have demonstrated the relation of increasing commanded thrust level to that of decreasing thrust accuracy.

In this work, spacecraft attitude dynamics are modeled as stochastic differential equations (SDE). For this model, we derive a stochastic control law that takes into consideration the multiplicative relation of increasing commanded thrust magnitude on the propagated thrust uncertainty. We assume perfect knowledge of initial conditions, and state variables at all times. A Hamilton-Jacobi-Bellman (HJB) equation is formulated, and its solution is approximated through a power series-based method [6]. Since the solution to the HJB equation involves the use of power series, the resulting optimal control is local in nature. That is, it may become suboptimal away from the origin. Though, in the operational domain of state-space, the derived control retains its approximate optimality, and its stability properties are desirable. To the best knowledge of the authors, there are three well-studied formulations of attitude dynamics. The first is the use of cascade structure as in Ref. [7] and [8]. The second method is the Hamiltonian formulation, introduced in Ref. [9], where through differentiating the kinematic differential equations, the dynamic and kinematic equations are reformulated as a second order differential equation system. The third approach is that of adjoining the kinematic and dynamic equations by constructing the state vector as an extension of the kinematic parameters, and body rotational rates. When working with singularity-free kinematic parameterizations, often the derivation of linear optimal control involves solving a state-dependent Riccati equation (SDRE). In fact, SDREs can arise even when using singular parameterizations, see Ref. [10]. In this paper, the combination of power series-based method [6], and a singular kinematic parameterization [11] gives an alternative to dealing with SDREs for nonlinear stochastic systems. Instead, we solve a form of algebraic Riccati equation (ARE) to obtain the linear control, where its existence (in a deterministic setting) relies on controllability properties of the system. Furthermore, stability and performance in a cascade structure depend on how fast the dynamics subsystem is. In addition, it must be shown that the cascaded structure is stable through a choice of a separate Lyapunov function. Using the third choice of state-space formulation mentioned above, we are able to approximate the HJB; hence the stability of the closed-loop nonlinear system follows in a neighborhood of the origin (see Theorem 2). The power series method employed in this paper allows the derivation and analysis of each control order separately, thus giving the control designer a choice in approximation.

The remaining of this paper is organized into five sections. The stochastic modeling of the spacecraft attitude system is carried out in section II. The proofs of optimality, existence, and stability, for both the linear and nonlinear control are given in section III. Section IV contains the computed control expressions up to the third order. Higher order control expressions can be found in Appendix A. The simulation results are tabulated and discussed in section V. Finally, section VI gives a brief conclusion discussing the presented research.

II. System Disturbance Modeling

The spacecraft attitude system is described by the Euler rigid body equations with addition of three or more differential equations describing the orientation of the spacecraft with respect to a reference frame. The dynamic equations are

$$I\dot{\omega} = S(\omega)I\omega + M$$

$$S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{1}$$

where, $I \in \mathbb{R}^{3\times3}$ is the principal moment of inertia matrix, $\omega \in \mathbb{R}^{3\times1}$ is the angular velocity vector about the body principal axes, and M is the total applied torque vector.

We will use the Tsiotras-Longuski parameterization [11] to describe orientation. The rotation matrix R describes the orientation of the body reference frame, with respect to the inertial reference frame. Matrix R is a result of two successive rotations: $R = R_2(w)R_1(z)$. To be precise, the parameterizing matrices are given by

$$R_1(z) = \begin{bmatrix} \cos(z) & \sin(z) & 0 \\ -\sin(z) & \cos(z) & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_2(w) = \frac{1}{1+w_1^2+w_2^2}\begin{bmatrix} 1+w_1^2-w_2^2 & 2w_1w_2 & -2w_2 \\ 2w_1w_2 & 1-w_1^2+w_2^2 & 2w_1 \\ 2w_2 & -2w_1 & 1-w_1^2-w_2^2 \end{bmatrix} \tag{2}$$

where $z \in \mathbb{R}$ is a rotation about the body z-axis, and $w = w_1 + iw_2 \in \mathbb{C}$ gives the coordinates of a point in the complex plane. Let the orientation of z-axis of the reference frame resulting from the rotation $R_1(z)$, be described by the direction cosines (a, b, c) in the body reference frame. Then, the mapping $w: S^2\backslash(0,0,-1) \to \mathbb{C}$, $w = \frac{b-ia}{1+c}$ is a stereographic projection describing the location of the rotated z-axis in the body reference frame. A more descriptive explanation would be that of a complex plane cutting through the unit-sphere at the equator. Then, connecting a line from the south pole of that sphere to the point (a, b, c) on the sphere, w is defined as the intersection point of this line with the plane (see the figures in Ref. [11]). Note that since $R_1(z), R_2(w) \in \mathrm{SO}(3)$, this parameterization, like every three-dimensional parameterization, is singular. The singularity occurs when w_1, $w_2 \to \infty$, i.e. pointing towards the south pole of the unit-sphere. The evolution of w, z parameters is given by the following differential equations [11]

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & w_2 \\ 0 & \frac{1}{2} & -w_1 \\ -w_2 & w_1 & 1 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(w_1^2 - w_2^2) & w_1w_2 & 0 \\ w_1w_2 & \frac{1}{2}(-w_1^2 + w_2^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{3}$$

The advantages of this parametrization are that the parameters describing orientation are equal to the rigid body's degrees of freedom. This reduces both nonlinearity and the dimension of the state-space. Moreover, the differential equations (3) contribute a linear component to the complete system (14), which is to be described below. As a result, the complete system (14) can be linearized about its defined state's origin. Specifically, in our application, we are interested in making an approximation to the optimal control, and hence, the structure of Eq. (3) allows a degree-by-degree approximation starting from the linear control. Additionally, as opposed to two possible singularities of Euler angles parametrization, the singularity is at a more desirable location.

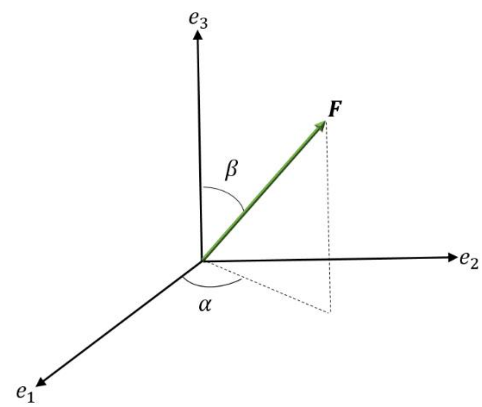

Fig. 1 Thruster force vector in spherical coordinates.

Consider a single thruster's force vector as shown in Fig. 1. Assume that $r = r_1 e_1 + r_2 e_2 + r_3 e_3$ is the vector from the center of gravity (center of the body frame) to the thruster of interest. Constant angles α and β are the thruster azimuth and elevation angles [12]. Then, the generated torque from a single thruster is calculated as

$$\tau = r \times F = b\acute{\boldsymbol{F}} = \begin{bmatrix} r_2 \cos(\beta) - r_3 \sin(\alpha)\sin(\beta) \\ r_3 \cos(\alpha)\sin(\beta) - r_1 \cos(\beta) \\ r_1 \sin(\alpha)\sin(\beta) - r_2 \cos(\alpha)\sin(\beta) \end{bmatrix} \acute{\boldsymbol{F}} \tag{4}$$

where, $\acute{\boldsymbol{F}}$ is the scalar magnitude of the force generated by the thruster, and the force vector F is

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\sin(\beta) \\ \cos(\beta) \end{bmatrix} \acute{\boldsymbol{F}} \tag{5}$$

Since thrusters are typically operated in pairs in attitude maneuvers [12], we assume that the spacecraft is equipped with multiple pairs of bi-directional thrusters, numbered by the index i. For further simplification, we assume that each thruster pair is mounted symmetrically as shown in Fig. 2, and hence, the vectors from the center of mass of the spacecraft to each thruster are of equal length. Let M_i be the torque τ generated by the i^{th} thruster pair. The forces due to thruster 1 and 2 of the i^{th} pair are denoted by F_{i_1} and F_{i_2} respectively.

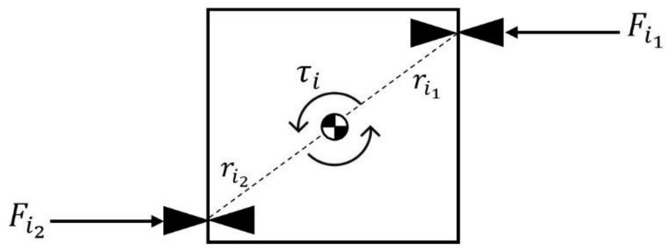

Fig. 2 Produced torque by a thruster pair.

For instance, for the lever arms $r_i = r_{i_1} = -r_{i_2}$, the generated torque by the i^{th} thruster pair is calculated as $\tau_i = r_{i_1} \times F_{i_1} + r_{i_2} \times F_{i_2} = r_{i_1} \times \left(\|F_{i_1}\| + \|F_{i_2}\|\right) \frac{F_{i_1}}{\|F_{i_1}\|}$. Let us denote expression $\left(\|F_{i_1}\| + \|F_{i_2}\|\right) \frac{F_{i_1}}{\|F_{i_1}\|}$ by F_i, that is an equivalent net force resulting in the generated torque by the i^{th} thruster pair. Then, $\tau_i = r_i \times F_i$ is the torque generated by the i^{th} pair, and the total generated torque τ is summation of torques generated by all the thruster pairs. For m thruster pairs, the torque vector is given by

$$\tau = \sum_{i=1}^{m} \tau_i = \sum_{i=1}^{m} b_i \acute{\boldsymbol{F}}_i \tag{6}$$

where $\acute{\boldsymbol{F}}_i = \|F_{i_1}\| + \|F_{i_2}\|$ is the scalar magnitude of the force generated by the i^{th} thruster pair, and b_i is given by Eq. (4). Expressing Eq. (6) in state-space notation, the total exerted torque, τ, is equivalent to

$$\tau = \sum_{i=1}^{m} b_i \acute{\boldsymbol{F}}_i = \sum_{i=1}^{m} b_i U_i(t) = bU(t) \tag{7}$$

where, $U \in \mathbb{R}^m$ is the control vector, and $b : \mathbb{R}^3 \to \mathbb{R}^m$ is a real valued 3-by-m matrix. The columns of b, namely b_i, give the orientation of each thruster pair in terms of angles α and β. In fact, b_i vectors are the axes about which the corresponding control torques $\|b_i\|U_i$ are applied [13]. We consider vectors b_i to be time invariant by assumption. The entries of vector U, describe the generated net force by each thruster pair. Substituting $bU(t)$ as the generated moment M in Eq. (1), the deterministic dynamic equations become

$$I\dot{\omega} = S(\omega)I\omega + bU(t) \tag{8}$$

In modeling thrust uncertainty, the main idea is to let generated uncertainty from the i^{th} thruster be modeled as a Gaussian white noise process $(\eta_t)_i$, where all the $(\eta_t)_i$ are independent. The uncertainty due to a thruster pair can then be represented as

$$((\eta_t)_1 + (\eta_t)_2) = \xi_t \tag{9}$$

where ξ_t is a Gaussian mean-zero white noise process. Then we have that

$$U_i(t) = u_i(t)(1 + (\xi_t)_i), \quad i = 1, \ \dots, m \tag{10}$$

and the control vector with multiplicative noise becomes

$$I^{-1}bU(t) = I^{-1}\sum_{i=1}^{m} b_i(u_i(t))(1 + (\xi_t)_i) \tag{11}$$

where $u \in \mathbb{R}^m$ is the nominal control vector. In general, ξ_t accounts for uncertainty in control input, such as thrust magnitude variations. As opposed to the additive noise model considered in Ref. [2], the multiplicative uncertainty structure provides a more accurate and realistic model where the magnitude of noise generated by the thruster pair is dependent on the magnitude of the control input itself. For instance, a small commanded nominal control u will result in $(\xi u) \approx 0$ for an arbitrary ξ. Furthermore, it is known that for a measurable function $\sigma(u(t))$,

$$\int \sigma(u(t))\xi_t \, dt, \quad \int \sigma(u(t)) \, dW_t \tag{12}$$

are statistically equivalent [14]. Hence, the differential equation (8) is statistically equivalent to

$$\omega_t = \omega_o + \int_{t_o}^{t} [I^{-1}S(\omega_s)I\omega_s + I^{-1}bu(s)] \, ds + \int_{t_o}^{t} \sigma\big(u(t)\big) \, dW_t \tag{13}$$

where W_t $t \geq 0$ is the m-dimensional standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\sigma(.)$ denotes the diffusion coefficient. Next, to adjoin the dynamic and kinematic equations (8) and (3), we define the state vector as $x = [\omega_1 \ \omega_2 \ \omega_3 \ w_1 \ w_2 \ z]^T$. Differentiating x, and letting $\tilde{I}_1 = \frac{I_2 - I_3}{I_1}$, $\tilde{I}_2 = \frac{I_3 - I_1}{I_2}$, and $\tilde{I}_3 = \frac{I_1 - I_2}{I_3}$, with I_i, $i = 1,2,3$ being the entries of the principal moment of inertia matrix, the complete system is described by the following SDE

$$dx = \big[Ax + f^{(2)}(x) + f^{(3)}(x) + Bu(t)\big]dt + \sigma(u(t))dW_t \tag{14}$$

where, $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$, $f^{(2)}(x) = \begin{bmatrix} \tilde{I}_1 x_2 x_3 \\ \tilde{I}_2 x_1 x_3 \\ \tilde{I}_3 x_1 x_2 \\ x_3 x_5 \\ -x_3 x_4 \\ -x_1 x_5 + x_2 x_4 \end{bmatrix}$, $f^{(3)}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2}(x_1 x_4^2 - x_1 x_5^2) + x_2 x_4 x_5 \\ \frac{1}{2}(-x_2 x_4^2 + x_2 x_5^2) + x_1 x_4 x_5 \\ 0 \end{bmatrix}$, and $B = \begin{bmatrix} I^{-1}b \\ 0_{3\times m} \end{bmatrix}$. The superscript in parenthesis gives the order of the terms in state. In the case of spacecraft thrusters with multiplicative noise, the diffusion coefficient is a function of control and is given by

$$\sigma(u) \stackrel{\text{def}}{=} \varepsilon B \begin{bmatrix} u_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & u_m \end{bmatrix} \tag{15}$$

where, $\varepsilon \geq 0$ is a real parameter scaling the thruster uncertainty effects. The diagonal form of the control matrix of Eq. (15) makes sure that each entry of the m-dimensional Wiener process is associated with its respective input $u_i(t)$, $i = 1, \dots, m$.

III. Optimal Attitude Control

Having derived a model of the spacecraft attitude system, we are now interested in finding a stochastic optimal control for the nonlinear constraint (14), which minimizes the expected cost function

$$\mathcal{J}(u) = \mathbb{E}_{x_o,t_o}\left[\int_{t_o}^{\infty} r(x,u)\, dt\right] \tag{16}$$

in an infinite horizon setting, given the initial time $t_o \geq 0$, and state $x_o \in \mathbb{R}^6$. At all times, the initial conditions are assumed to be known with probability 1. Let us define the running cost function $r: \mathbb{R}^6 \times \mathbb{R}^m \to \mathbb{R}$, as $r(x,u) = \frac{1}{2}u^T R u + \mathcal{r}(x)$ where $x \in \mathbb{R}^6$, $u \in \mathbb{R}^m$, $R \in \mathbb{R}^{m \times m}$, $R > 0$, $\mathcal{r}(x) = \sum_{i=2}^{m} \mathcal{r}^{(i)}(x)$ is a power series, and m is the order of the terms in x. For $m = 2$, the regulator has the form $\mathcal{r}(x) = \frac{1}{2}x^T Q x$, where $Q \in \mathbb{R}^{6\times 6}$, $Q \geq 0$. Let $\mathbb{P}$ denote the probability measure generated by trajectories starting at (x_o, t_o), driven by the Brownian motion W. Then $\mathbb{E}_{x_o,t_o}$ is an expected value with respect to the probability measure $\mathbb{P}$.

Consider the following HJB equation associated with the nonlinear SDE (14)

$$\min_u \{\mathcal{L}^u V(x) + r(x,u)\} = 0 \tag{17}$$

$$\mathcal{L}^u V(x) = \sum_{i=1}^{6} f_i(x,u) \frac{\partial V(x)}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6} a_{i,j}(u) \frac{\partial^2 V(x)}{\partial x_i \partial x_j} \tag{18}$$

Here, $a(u) \in \mathbb{R}^{6\times 6}$, $a_{i,j}(u) = (\sigma(u)\sigma(u)^T)_{i,j}$, σ is defined by (15), and $f(x,u) = Ax + f^{(2)}(x) + f^{(3)}(x) + Bu(t)$. The superscript u denotes the dependency of the infinitesimal generator (18) on control. The solution to the HJB (17), is the value function (minimum cost) $V(x): \mathbb{R}^{6\times 1} \to \mathbb{R}$. We are interested in finding approximations of the optimal value function, and consequently of the optimal control in the ring of formal power series over $\mathbb{R}$. The following theorem gives the optimality conditions of control.

Theorem 1 Suppose a form of $V(x)$ and $u(x)$ have been found. If $V(x)$ and $u(x)$ satisfy the conditions i-iii, then the control $u = u(x)$ is optimal and will minimize the functional (16) in infinite time.

i. The Lyapunov function, $V(x)$, satisfies the asymptotic stability conditions of Lyapunov's second method for stochastic dynamical systems (see Remark 2).

ii. Given the closed-loop system (14), $V(x)$ satisfies the equation $\mathcal{L}^u V(x) = -r(x, u(x))$, where $\mathcal{L}^u(.)$ is the infinitesimal generator of diffusion (18).

iii. The Hamiltonian $\mathcal{H}\big(x, \kappa, V(x)\big) = f(x,\kappa)^T \frac{\partial V(x)}{\partial x} + \frac{1}{2}\text{trace}\left(a(\kappa)\frac{\partial^2 V(x)}{\partial x^2}\right) + r(x,\kappa)$ is strictly convex in κ, and attains its minimum at $\kappa = u$.

Proof To show that the above's assertion holds, we follow the general steps of Theorem 1.1 in [6], for stochastic dynamics. Let $u(x)$ be the optimal control. Then from condition ii it follows that

$$V(x_o) = \mathbb{E}_{x_o,t_o}\left[\int_{t_o}^{t} r\big(x_s, u(x_s)\big)\, ds + V(x_t)\right] \tag{19}$$

Set $t_o = 0$ and let $t \to \infty$. By assumption, condition i of asymptotic stability applies so that for $t \to \infty$, $V(x_t) \to 0$. Thus, Eq. (19) becomes

$$V(x_o) = \mathbb{E}_{x_o,t_o}\left[\int_{t_o=0}^{\infty} r\big(x_s, u(x_s)\big)\, ds\right] \tag{20}$$

Next, assume that $u(x)$ is not optimal. That is, there exists some $u^*(x)$ such that

$$\mathbb{E}_{x_o,t_o}\left[\int_{t_o=0}^{\infty} r\big(x_s, u^*(x_s)\big)\, ds\right] < \ \mathbb{E}_{x_o,t_o}\left[\int_{t_o=0}^{\infty} r\big(x_s, u(x_s)\big)\, ds\right] \tag{21}$$

From condition iii, we have

$$\mathcal{H}\big(x, u^*, V(x)\big) > \mathcal{H}\big(x, u, V(x)\big) \tag{22}$$

Integrating (22) with respect to time, the inequality becomes

$$\int_{t_o}^{t} \mathcal{L}^{u^*} V(x_s) + \int_{t_o}^{t} r(x, u^*) ds - \left(\int_{t_o}^{t} \mathcal{L}^{u} V(x_s)\, ds + \int_{t_o}^{t} r(x, u) ds\right) > 0$$

$$\Rightarrow\ \mathbb{E}_{x_o,t_o}\left[\int_{t_o}^{t} \mathcal{L}^{u^*} V(x_s) + \int_{t_o}^{t} r(x, u^*) ds\right] - \mathbb{E}_{x_o,t_o}\left[\int_{t_o}^{t} \mathcal{L}^{u} V(x_s)\, ds + \int_{t_o}^{t} r(x, u) ds\right] > 0 \tag{23}$$

Applying Itô lemma [15] to $V(x)$, we obtain the expression

$$V(x_t) - V(x_o) = \mathbb{E}_{x_o,t_o}\left[\int_{t_o}^{t} \mathcal{L}^{u^*} V(x_s)\, ds + \int_{t_o}^{t} \left(\frac{\partial V(x_s)}{\partial x}\right)^T \sigma\big(u^*(x_s)\big)\, dW_s\right] \tag{24}$$

Substituting expression (24) in inequality (23) for both processes driven by u^* and u, we obtain

$$V\big(x_t^{u^*}\big) - V(x_o) + \mathbb{E}_{x_o,t_o}\left[\int_{t_o}^{t} r(x,u^*) ds\right] - \left(V(x_t^u) - V(x_0) + \mathbb{E}_{x_o,t_o}\left[\int_{t_o}^{t} r(x,u) ds\right]\right) > 0 \tag{25}$$

Similarly set $t_o = 0$ and let $t \to \infty$. By condition i, Eq. (25) becomes

$$\mathbb{E}_{x_o,t_o}\left[\int_{t_o=0}^{\infty} r(x,u^*) ds\right] > \mathbb{E}_{x_o,t_o}\left[\int_{t_o=0}^{\infty} r(x,u) ds\right] \tag{26}$$

which is a contradiction, proving that u is optimal if conditions i, ii, and iii are satisfied. □

In the succeeding sections, conditions of existence and stability of different orders of optimal control are discussed. First, for simplicity, we shall introduce some notation. Let $H: \mathbb{R}^6 \to \mathbb{R}^{6\times 6}$ be a diagonal second order differential function defined as

$$H(V) = \begin{bmatrix} \frac{\partial^2 V}{\partial x_1^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial^2 V}{\partial x_2^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial^2 V}{\partial x_3^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial^2 V}{\partial x_4^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial^2 V}{\partial x_5^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 V}{\partial x_6^2} \end{bmatrix} \tag{27}$$

then inspecting the noise term of HJB (17), assuming a diagonal matrix $I^{-1}b$, we have that

$$\frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6} a_{i,j}(u) \frac{\partial^2 V(x)}{\partial x_i \partial x_j} = \frac{1}{2} trace\left(\sigma(u)\sigma(u)^T \frac{\partial^2 V(x)}{\partial x^2}\right)$$

$$\Rightarrow \sum_{i=1}^{6}\sum_{j=1}^{6} a_{i,j}(u) \frac{\partial^2 V(x)}{\partial x_i \partial x_j} = \varepsilon^2 u^T B^T H[V(x)] B u \tag{28}$$

Conditions of Theorem 1 imply that $u(x)$ and $V(x)$ resulting from HJB (17) are optimal. It remains now to find approximations of such solutions in form of truncated series for a choice of $m \geq 2$. To do so, suppose $u(x)$ and $V(x)$

are the optimal solutions of (17). Substituting $u(x)$ and $V(x)$ back in (17), we obtain the Hamiltonian Eq. (29). Additionally, differentiating Eq. (29) with respect to u, the control equation (30) is obtained

$$\left(Ax + f^{(2)}(x) + f^{(3)}(x)\right)^T \frac{\partial V(x)}{\partial x} + (Bu)^T \frac{\partial V(x)}{\partial x} + \frac{1}{2}\varepsilon^2 u^T B^T H[V(x)]Bu + \frac{1}{2}u^T Ru + r(x) = 0 \tag{29}$$

$$B^T \frac{\partial V(x)}{\partial x} + Ru + \varepsilon^2 B^T H[V(x)]Bu = 0 \tag{30}$$

Note that Eq. (29) and (30) form a system of equations in which their solutions are the assumed optimal control and optimal value function. Following Al'brekht's method of approximation [6], we assume that $V(x)$ and $u(x)$ possess a power series form of

$$V(x) = V^{(2)}(x) + V^{(3)}(x) + V^{(4)}(x) + V^{(5)}(x) + V^{(6)}(x) + V^{(7)}(x) + \cdots + V^{(m)}(x) \tag{31}$$

$$u(x) = k^{(1)}(x) + k^{(2)}(x) + k^{(3)}(x) + k^{(4)}(x) + k^{(5)}(x) + k^{(6)} + \cdots + k^{(m-1)}(x) \tag{32}$$

where $m \geq 2$ is the order of the term in x and $k^{(1)}(x) = Kx$, with linear optimal gain $K \in M_{3\times 6}[\mathbb{R}]$.

Proposition 1 Given the dynamical system (14), linear control

$$k^{(1)}(x) = Kx = -\left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right)B\right)^{-1} B^T Px \tag{33}$$

asymptotically stabilizes the linear dynamics in probability, and is optimal with respect to the quadratic Hamiltonian, if there exists a positive definite Hermitian matrix $P \in M_6[\mathbb{R}]$ satisfying the quadratic expansion of (29). That is, if the following two conditions hold simultaneously:

i. The pair (A, B) satisfies the Kalman rank condition, and for $Q = C^T C$, the pair (C, A) is detectable.

ii. $\sup\left\{\left|\frac{\Pi_1(1_{6\times 6})}{2(\lambda_1+\lambda_2)}\right|, \left|\frac{\Pi_2(1_{6\times 6})}{2(\lambda_3+\lambda_4)}\right|, \left|\frac{\Pi_3(1_{6\times 6})}{2(\lambda_5+\lambda_6)}\right|, \left|\frac{2\lambda_1\lambda_2\Pi_1(1_{6\times 6})}{\lambda_1+\lambda_2}\right|, \left|\frac{2\lambda_3\lambda_4\Pi_2(1_{6\times 6})}{\lambda_3+\lambda_4}\right|, \left|\frac{\lambda_5\lambda_6\Pi_3(1_{6\times 6})}{2(\lambda_5+\lambda_6)}\right|\right\} < 1$, where λ_i, $i = 1, \ldots, 6$ are the eigenvalues of $A + BK$, $\Pi_j(1_{6\times 6}) = \frac{\varepsilon^2 B_j^4}{R_j^2 + \varepsilon^2 B_j^2 R_j}$, $j = 1,2,3$ and $B_j, R_j \in \mathbb{R}$ are the j_{th} nonzero entries of matrices B and diagonal R respectively.

Proof To argue existence of stabilizing linear control, we will use a series of existing results on existence and uniqueness of the Lyapunov function (31) in association with (17) for $m = 2$. First, consider the linear part of the attitude dynamics (14)

$$dx_t = \left[Ax_t + Bk^{(1)}(x)\right]dt + \sigma\left(k^{(1)}(x)\right)dW_t \tag{34}$$

Suppose there exists, in a neighborhood of the origin $\mathcal{X} \subseteq \mathbb{R}^6$, a twice differentiable positive definite function $V(x)$, such that $\lim_{x\to 0} V(x) = 0$. Then if $\mathcal{L}^u V(x) < 0$ in $\mathcal{X}$, the trajectories of the linear SDE (34), starting within $\mathcal{X}$, approach the trivial solution and the trivial solution is asymptotically stable in probability by Khasminskii's Corollary 5.1 [16]. Hence, if there exists a positive definite Hermitian matrix $P \in M_6[\mathbb{R}]$ such that the Lyapunov function $V^{(2)}(x) = \frac{1}{2}x^T P x$ decreases along the trajectories of (34), then the linear SDE (34) is asymptotically stable in probability. Certainly if P is positive definite, then $V^{(2)}(x)$ is positive definite. Under the condition of optimality of control for $m = 2$, negative definiteness of $\mathcal{L}^u V^{(2)}(x)$ is guaranteed if

$$\mathcal{L}^u V^{(2)}(x) = -r\left(x, k^{(1)}(x)\right) \tag{35}$$

for x in $\mathcal{X}$. The equality (35) is in fact the quadratic terms of the Hamiltonian (29). Applying the generator (18) on $V^{(2)}(x)$, substituting for the linear dynamics (34) and the quadratic running cost, we have

$$\left(Ax + Bk^{(1)}(x)\right)^T \frac{\partial V^2(x)}{\partial x} + \frac{1}{2}\varepsilon^2 k^{(1)}(x)^T B^T H[V^2(x)] B k^{(1)}(x) = -\frac{1}{2}x^T Q x - k^{(1)}(x)^T R k^{(1)}(x) \tag{36}$$

Simplifying Hamiltonian (36), the following algebraic Riccati equation (ARE) is obtained

$$x^T\left[Q + PA + A^T P + 2PBK + K^T\left(R + \varepsilon^2 B^T \tilde{P} B\right)K\right]x = 0 \tag{37}$$

The linear control $k^{(1)}(x) = -\left(R + \varepsilon^2 B^T \tilde{P} B\right)^{-1}(B)^T P x$ is then computed through differentiating Eq. (36) with respect to control. This is equivalent to solving for linear terms of Eq. (30). Substituting for K, we obtain

$$Q + PA + A^T P - PB(R + \varepsilon^2 B^T H[V^2(x)]B)^{-1}(PB)^T = 0 \tag{38}$$

For a linear deterministic system, Eq. (38) will have the form $Q + PA + A^T P - PBR^{-1}(PB)^T = 0$. For such a system, stabilizability of the pair (A, B) is a sufficient condition for existence of matrix P (see Theorem 3 in Ref. [17]). However, ARE (38) of the stochastic dynamics (34) requires additional consideration. Due to Wonham [18], Theorem 4.1, we have that the following form of the ARE

$$Q + PA + A^T P - PBR^{-1}(PB)^T + \Pi(P) = 0 \tag{39}$$

admits at least one positive semidefinite solution P, if for $Q = C^T C$, the pair (C, A) is detectable, and (A, B) is stabilizable, such that

$$\inf_K \left\| \int_0^\infty e^{(A+BK)^T t} \Pi(1_{6\times 6}) e^{(A+BK)t}\, dt \right\|_\infty < 1 \tag{40}$$

where $\Pi: M_6(\mathbb{R}) \to M_6(\mathbb{R})$ is a linear map from the space of symmetric matrices onto itself, and $\|\,.\,\|_\infty$ denotes the spectral norm. In addition, if (C, A) is observable, then P is positive definite and unique [18]. The difference between ARE (39) and a deterministic Riccati is the term $\Pi(P)$, and the bound (40) implies that $\Pi(P)$ is not too large.

Suppose K^* is the gain yielding the smallest spectral norm and let K^o be the optimal gain. If $K^* = K^o$, then bounding the spectral norm by K^o satisfies bound (40). Now suppose $K^* \neq K^o$. If the spectral norm due to K^o is bounded above by 1, then surely, spectral norm due to K^* is also bounded by 1. Here, we set $K = K^o$ and seek conditions where bound (40) holds for the optimal gain satisfying Eq. (37). Doing so of course yields a stronger condition enabling us to find closed-form expressions guaranteeing bound (40) to be satisfied.

Now consider the ARE (38). Applying the Woodbury identity on $(R + \varepsilon^2 B^T H[V^2(x)]B)^{-1}$, we rewrite Eq. (38) in the general form of the Eq. (39), where

$$\Pi(P) = \varepsilon^2 PB\left(R^{-1}B^T\left(H\left(V^{(2)}(P,x)\right)^{-1} + \varepsilon^2 BR^{-1}B^T\right)^{-1} BR^{-1}\right)(PB)^T \tag{41}$$

and hence, condition (40) applies as an existence condition of ARE (38). Note that the norm of bound (40) is the largest singular value of its argument. Define $\mathcal{T} \stackrel{\text{def}}{=} \int_0^\infty e^{(A+BK)^T t}\Pi(1_{6\times6})e^{(A+BK)t}\,dt$, then $\|\mathcal{T}\|_\infty = (\lambda_{\max}(\mathcal{T}^*\mathcal{T})\ )^{\frac{1}{2}}$, where $\lambda_{\max}$ denotes the largest eigenvalue, and $\mathcal{T}^*$ is the conjugate transpose of $\mathcal{T}$. The integral of (40) converges if all the eigenvalues of $A + BK$ are real and negative. Such a requirement can be satisfied given the controllability of the system. Computing the convergent integral, we obtain

$$\mathcal{T} = \begin{bmatrix} -\frac{\Pi_1(1_{6\times6})}{2(\lambda_1+\lambda_2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\Pi_2(1_{6\times6})}{2(\lambda_3+\lambda_4)} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\Pi_3(1_{6\times6})}{2(\lambda_5+\lambda_6)} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2\lambda_1\lambda_2\Pi_1(1_{6\times6})}{\lambda_1+\lambda_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2\lambda_3\lambda_4\Pi_2(1_{6\times6})}{\lambda_3+\lambda_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\lambda_5\lambda_6\Pi_3(1_{6\times6})}{2(\lambda_5+\lambda_6)} \end{bmatrix} \tag{42}$$

where $\lambda_1, \dots, \lambda_6$ are the eigenvalues of $A + BK$, and $\Pi_j(1_{6\times6})$, $j = 1,2,3$ are the nonzero entries of $\Pi(1_{6\times6})$. Since $\mathcal{T}$ has converged to a symmetric matrix, we have that $\|\mathcal{T}\|_\infty = |\lambda_{max}(\mathcal{T})|$. It then follows that bounding the absolute value of all the eigenvalues of $\mathcal{T}$ by 1 implies $\|\mathcal{T}\|_\infty < 1$.

To conclude, given that matrices A and B satisfy the Kalman rank condition, and that for $Q = C^T C$, matrices C and A are detectable such that $\|\mathcal{T}\|_\infty < 1$, then there exists a positive definite Hermitian matrix P satisfying the ARE (39). This in turn implies the existence of a quadratic Lyapunov function satisfying the equality (35), and therefore asymptotic stability of the dynamical system (34) in probability. ☐

Proposition 2 Let $L_1 \equiv \left((A + BK)x\right)^T \frac{\partial}{\partial x}$ and $L_2 \equiv \frac{1}{2}\varepsilon^2 (BKx)(BKx)^T H$ be linear operators, $L_1, L_2 : \mathbb{R}^6 \to \mathbb{R}$. If the minimum eigenvalue of L_1 acting on $V^{(m)}(x)$ is greater than the maximum eigenvalue of L_2 in magnitude, and the linear deterministic part of the dynamics (14) is asymptotically stable, then the nonlinear stochastic control for $m > 2$ exists in some $\mathcal{X}_m \subset \mathbb{R}^6$ containing the origin and is given by

$$k^{(m-1)}(x) = -\left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right)B\right)^{-1}\left[(B)^T \frac{\partial V^{(m)}(x)}{\partial x} + \sum_{i=2}^{m-1} \varepsilon^2\, B^T H\left(V^{(i+1)}(x)\right) B\, k^{(m-i)}(x)\right] \tag{43}$$

Proof To show the existence of higher order control, solvability of Lyapunov function (31) for $m > 2$ must be shown. To do so, we study the invertibility of the linear operator mapping $V^{(m)}(x)$ to a polynomial of the same order. As the series-based method [6] has allowed us, we inspect the linear operator acting on $V(x)$, order by order. Let us first derive expression of these linear operators for $m > 2$. To do so, we substitute series (31) and (32) into the Hamiltonian (29). This expansion is given by

$$\begin{aligned} &(Ax)^T \frac{\partial V^{(m)}(x)}{\partial x} + f^{(2)^T}(x) \frac{\partial V^{(m-1)}(x)}{\partial x} + f^{(3)^T}(x) \frac{\partial V^{(m-2)}(x)}{\partial x} + \sum_{j=2}^{m} \left(Bk^{(m-j+1)}\right)^T \frac{\partial V^{(j)}(x)}{\partial x} \\ &+ \frac{\left(2 - \delta_{\alpha'\beta'}\right)}{2}\left(k^{(\alpha')}(x)\right)^T R k^{(\beta')}(x) + \frac{\left(2 - \delta_{\alpha\beta}\right)}{2}\left(k^{(\alpha)}(x)\right)^T \left(\varepsilon^2 B^T H\left(V^{(\gamma)}(x)\right)B\right) k^{(\beta)}(x) \\ &\qquad = -r^{(m)}(x) \end{aligned} \tag{44}$$

for $\alpha + \gamma + \beta = m - 2$, and $\alpha' + \beta' = m$, where $\alpha, \gamma, \beta, \alpha', \beta' \in \mathbb{N}$, and δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$, when $i = j$ and $\delta_{ij} = 0$ otherwise). Similarly, we substitute expansions (31) and (32) in the optimal control equation (30). Grouping, rearranging, and solving for every control order $k^{(m-1)}(x)$ separately, the nonlinear control, as a function of $V^{(m)}(x)$ becomes as given by (43) $\forall\, m > 2$. Then to solve the system (29)-(30), we substitute the expressions of optimal control (43) back into Hamiltonians (44) to arrive at $m - 1$ equations with $m - 1$ unkowns, one unknown per equation. For $j = 3,4, \dots, m - 1$ the solution of equation $j - 1$ is an input to equation j. In particular, we substitute for

$\left(Bk^{(m-j+1)}\right)^T \frac{\partial V^{(j)}(x)}{\partial x}$, $j = 2,3,4, \dots, m-1$, in every m^{th} order Hamiltonian equation resulting from Eq. (44). To construct these terms, let us rewrite Eq. (43) as

$$(B)^T \frac{\partial V^{(m)}(x)}{\partial x} = -\left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right) B\right) k^{(m-1)}(x) - \sum_{i=2}^{m-1} \varepsilon^2 \, B^T H\left(V^{(i+1)}(x)\right) B \, k^{(m-i)}(x) \tag{45}$$

then multiplying by $\left(k^{(m-j+1)}\right)^T$, $j = 2,3,4, \dots, m-1$, we obtain the modified control expressions

$$\begin{aligned} \left(k^{(m-j+1)}\right)^T \frac{\partial V^{(j)}(x)}{\partial x} \\ &= -\left(k^{(m-j+1)}\right)^T \left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right) B\right) k^{(j-1)}(x) \\ &\quad - \left(k^{(m-j+1)}\right)^T \sum_{i=2}^{j-1} \varepsilon^2 \, B^T H\left(V^{(i+1)}(x)\right) B \, k^{(j-i)}(x) \end{aligned} \tag{46}$$

for $j = 2,3,4, \dots, m-1$. Substituting (46) in every m^{th} order Hamiltonian and carrying out cancellations, we arrive at $m-1$ equations of m^{th} order, which are the expansion of Eq. (29). This system of equations is summarized by

$$\left((A + BK)x\right)^T \frac{\partial V^{(m)}(x)}{\partial x} + \frac{1}{2}(Kx)^T \left(\varepsilon^2 B^T H\left(V^{(m)}(x)\right) B\right)(Kx) = \Psi^{(m)}(x) \tag{47}$$

where $\Psi^{(m)}(x)$ is the summation of all the polynomials of order m with a known form. Notice that $V^{(m)}(x)$ is the only unknown of this equation. To determine the existence of coefficients of $V^{(m)}(x)$ that would satisfy Eq. (47), let us define the linear operator L acting on an arbitrary function $\theta(x)$ as

$$L\,\theta(x) \equiv \left((A + BK)x\right)^T \frac{\partial \theta(x)}{\partial x} + \frac{1}{2}(Kx)^T\left(\varepsilon^2 B^T H\big(\theta(x)\big)B\right)(Kx) \tag{48}$$

where $\theta: \mathbb{R}^6 \to \mathbb{R}$ is a twice differentiable arbitrary polynomial function. We would like to determine if $\Psi^{(m)}(x)$ are in the image of the mapping $L: \theta(x) \to L\big(\theta(x)\big)$. To do so, we consider the conditions of nonresonance [19] of the homological equation (47). This is of course equivalent to invertibility of the linear operator L, i.e. being able to compute $V^{(m)}(x) = L^{-1}\Psi^{(m)}(x)$. We shall show this by considering the eigenvalues of the operators. However, observing that $L\,\theta(x) = (L_1 + L_2)\,\theta(x)$, we treat the invertibility of each linear map L_1 and L_2 separately, and will thereafter prove the invertibility of L. This is because given Eq. (14), eigenvalues of $A + BK$ and BK are not identical.

Let (v^i, λ^i), $i = 1, \dots, 6$ denote a left eigenvector of the matrix $A + BK \in \mathbb{R}^{6\times 6}$, and corresponding eigenvalue. A polynomial in x, $\theta^{(m)}(x)$ can be represented in the basis

$$\theta^{(m)}(x) = \langle \ell_1, x\rangle\langle \ell_2, x\rangle \dots \langle \ell_m, x\rangle \tag{49}$$

where $\ell_i \in \mathbb{R}^6$, $i = 1, \dots, m$ are arbitrary vectors, and $\langle .,. \rangle$ denotes a dot product operation [20]. As the base case, let $m = 3$. Since for system (14), $A + BK$ has a full set of linearly independent eigenvectors, any polynomial $\theta^{(3)}(x)$ in the basis of the eigenvectors of $A + BK$, is given by

$$\theta^{(3)}(x) = \sum_{i,j,k=1}^{6} c_{ijk}^{(3)} \theta_{ijk}^{(3)}(x) \tag{50}$$

where $\theta_{ijk}^{(3)}(x) = \langle \nu^i, x\rangle\langle \nu^j, x\rangle\langle \nu^k, x\rangle$, and $c_{ijk}^{(3)} \in \mathbb{R}$ is a constant for $i, j, k = 1, \dots, 6$. If $L_p\, \theta_{ijk}^{(3)}(x) = s_{ijk}^{(3)} \theta_{ijk}^{(3)}(x)$ for $p = 1{,}2$ and some s_{ijk}, then we conclude that s_{ijk} is the ijk^{th} eigenvalue of L_p.

Consider the first order additive portion of the operator, L_1, acting on a basis function of $\theta_{ijk}^{(3)}(x)$. We have that

$$L_1 \theta_{ijk}^{(3)}(x) = x^T (A + BK)^T\, \partial/\partial x [\langle \nu^i, x\rangle\langle \nu^j, x\rangle\langle \nu^k, x\rangle]$$

$$\Longrightarrow\; L_1 \theta_{ijk}^{(3)}(x) = x^T (A + BK)^T [(\nu^i)^T \langle \nu^j, x\rangle\langle \nu^k, x\rangle + (\nu^j)^T \langle \nu^i, x\rangle\langle \nu^k, x\rangle + (\nu^k)^T \langle \nu^i, x\rangle\langle \nu^j, x\rangle] \tag{51}$$

Using the relation $\nu^i (A + BK) = \lambda^i \nu^i$, for $i = 1, \dots, 6$, we make the following substitutions

$$L_1 \theta_{ijk}^{(3)}(x) = x^T\, [\lambda^i \nu^i \langle \nu^j, x\rangle\langle \nu^k, x\rangle + \lambda^j \nu^j \langle \nu^i, x\rangle\langle \nu^k, x\rangle + \lambda^k \nu^k \langle \nu^i, x\rangle\langle \nu^j, x\rangle]$$

$$\Longrightarrow\; L_1 \theta_{ijk}^{(3)}(x) = [\lambda^i \langle \nu^i, x\rangle\langle \nu^j, x\rangle\langle \nu^k, x\rangle + \lambda^j \langle \nu^i, x\rangle\langle \nu^j, x\rangle\langle \nu^k, x\rangle + \lambda^k \langle \nu^i, x\rangle\langle \nu^j, x\rangle\langle \nu^k, x\rangle]$$

$$\Longrightarrow\; L_1 \theta_{ijk}^{(3)}(x) = (\lambda^i + \lambda^j + \lambda^k) \theta_{ijk}^{(3)}(x) \tag{52}$$

Now let $(\tilde{\nu}^i, \tilde{\lambda}^i)$, $i = 1, \dots, 6$, denote a left eigenvector of the matrix $BK \in \mathbb{R}^{6\times 6}$, and corresponding eigenvalue. For system (14), BK also has a full set of eigenvectors. As the continuation of the base case, consider the second order portion of the operator (48), L_2, acting on a basis of $\theta_{ijk}^{(3)}(x)$. We have that

$$L_2 \theta_{ijk}^{(3)}(x) = \frac{\varepsilon^2}{2} x^T (BK)^T H[\langle \tilde{\nu}^i, x\rangle\langle \tilde{\nu}^j, x\rangle\langle \tilde{\nu}^k, x\rangle](BK)x$$

$$\Longrightarrow\; L_2 \theta_{ijk}^{(3)}(x) = \frac{\varepsilon^2}{2} x^T (BK)^T [(\tilde{\nu}^i \tilde{\nu}^j + \tilde{\nu}^j \tilde{\nu}^i)\langle \tilde{\nu}^k, x\rangle + (\tilde{\nu}^i \tilde{\nu}^k + \tilde{\nu}^k \tilde{\nu}^i)\langle \tilde{\nu}^j, x\rangle + (\tilde{\nu}^j \tilde{\nu}^k + \tilde{\nu}^k \tilde{\nu}^j)\langle \tilde{\nu}^i, x\rangle](BK)x$$

$$\Longrightarrow\; L_2 \theta_{ijk}^{(3)}(x) = \frac{\varepsilon^2}{2} x^T \left[2\left(\tilde{\nu}^i \tilde{\lambda}^i \tilde{\lambda}^j (\tilde{\nu}^j)^T\, \langle \tilde{\nu}^k, x\rangle + \tilde{\nu}^i \tilde{\lambda}^i \tilde{\lambda}^k (\tilde{\nu}^k)^T\, \langle \tilde{\nu}^j, x\rangle + \tilde{\nu}^j \tilde{\lambda}^j \left(\tilde{\lambda}^k\right)^T \tilde{\nu}^k\, \langle \tilde{\nu}^i, x\rangle\right)\right] x$$

$$\Longrightarrow\; L_2 \theta_{ijk}^{(3)}(x) = \varepsilon^2 \left(\tilde{\lambda}^i \tilde{\lambda}^j + \tilde{\lambda}^i \tilde{\lambda}^k + \tilde{\lambda}^j \tilde{\lambda}^k\right) \langle \tilde{\nu}^i, x\rangle\langle \tilde{\nu}^j, x\rangle\langle \tilde{\nu}^k, x\rangle \tag{53}$$

Hence, the linear operators L_1 and L_2 acting on $\theta_{ijk}^{(3)}(x)$, have eigenvalues $\lambda^i + \lambda^j + \lambda^k$, and $\varepsilon^2\left(\tilde{\lambda}^i \tilde{\lambda}^j + \tilde{\lambda}^i \tilde{\lambda}^k + \tilde{\lambda}^j \tilde{\lambda}^k\right)$ respectively for $i, j, k \in \{1, \dots, 6\}$ when $m = 3$.

Next, let us assume that for $m = k$, and $k \geq 3$, L_1 and L_2 acting on the basis $\theta_{n_1 \dots n_k}^{(k)}(x)$ are given by expressions

$$L_1\theta^{(k)}_{n_1\dots n_k}(x) = \left(\sum_{r=1}^{k}\lambda^{n_r}\right)\prod_{r=1}^{k}\langle\nu^{n_r}, x\rangle \tag{54}$$

$$L_2\theta^{(k)}_{n_1\dots n_k}(x) = \varepsilon^2\left(\sum_{i=1}^{k-1}\sum_{j>i}^{k}\tilde{\lambda}^{n_i}\tilde{\lambda}^{n_j}\right)\prod_{p=1}^{k}\langle\tilde{\nu}^{n_p}, x\rangle \tag{55}$$

respectively, for $n_1, \dots, n_m \in \{1, \dots, 6\}$. Specifically, $\theta^{(k)}_{n_1\dots n_k}(x) = \prod_{r=1}^{k}\langle\nu^{n_r}, x\rangle$ in basis of eigenvectors of $A + BK$, and $\theta^{(k)}_{n_1\dots n_k}(x) = \prod_{p=1}^{k}\langle\tilde{\nu}^{n_p}, x\rangle$ in basis of eigenvectors of BK. Then to compute the inductive step, let $m = k + 1$. Applying L_1 on the eigenfunction of order $k + 1$, we have

$$L_1\theta^{(k+1)}_{n_1\dots n_{k+1}}(x) = L_1\left(\langle\nu^{n_q}, x\rangle\prod_{\substack{r=1\\ r\neq q}}^{k+1}\langle\nu^{n_r}, x\rangle\right)$$

$$\Longrightarrow\ L_1\theta^{(k)}_{n_1\dots n_k}(x) = x^T(A + BK)^T\nu^{n_q}\prod_{\substack{r=1\\ r\neq q}}^{k+1}\langle\nu^{n_r}, x\rangle + \langle\nu^{n_q}, x\rangle\, L_1\prod_{\substack{r=1\\ r\neq q}}^{k+1}\langle\nu^{n_r}, x\rangle \tag{56}$$

Using the assumption (54) in Eq. (56), and realizing that $\big((A + BK)x\big)^T\nu^{n_q} = \lambda^{n_q}\langle\nu^{n_q}, x\rangle$, we obtain

$$L_1\theta^{(k)}_{n_1\dots n_k}(x) = \lambda^{n_q}\langle\nu^{n_q}, x\rangle\prod_{\substack{r=1\\ r\neq q}}^{k+1}\langle\nu^{n_r}, x\rangle + \langle\nu^{n_q}, x\rangle\left(\sum_{\substack{r=1\\ r\neq q}}^{k+1}\lambda^{n_r}\right)\prod_{\substack{r=1\\ r\neq q}}^{k+1}\langle\nu^{n_r}, x\rangle$$

$$\Longrightarrow\ L_1\theta^{(k)}_{n_1\dots n_k}(x) = \left(\sum_{r=1}^{k+1}\lambda^{n_r}\right)\prod_{r=1}^{k+1}\langle\nu^{n_r}, x\rangle \tag{57}$$

Continuing the inductive step, we now apply L_2 on the eigenfunction of order $k + 1$

$$L_2\theta^{(k)}_{n_1\dots n_k}(x) = L_2\left(\langle\tilde{\nu}^{n_{k+1}}, x\rangle\prod_{p=1}^{k}\langle\tilde{\nu}^{n_p}, x\rangle\right)$$

$$\Longrightarrow\ L_2\theta^{(k)}_{n_1\dots n_k}(x) = \varepsilon^2 x^T(BK)^T\tilde{\nu}^{n_{k+1}}\sum_{q=1}^{k}\left((\tilde{\nu}^{n_q})^T BKx\prod_{\substack{p=1\\ p\neq q}}^{k}\langle\tilde{\nu}^{n_p}, x\rangle\right) + \langle\tilde{\nu}^{n_{k+1}}, x\rangle\, L_2\prod_{p=1}^{k}\langle\tilde{\nu}^{n_p}, x\rangle \tag{58}$$

Applying the assumption (55) to Eq. (58), and again substituting using the eigenvector equation, we obtain

$$L_2\theta^{(k)}_{n_1\dots n_k}(x) = \varepsilon^2\tilde{\lambda}^{n_{k+1}}\langle\tilde{\nu}^{n_{k+1}}, x\rangle\left(\sum_{q=1}^{k}\tilde{\lambda}^{n_q}\langle\tilde{\nu}^{n_q}, x\rangle\prod_{\substack{p=1\\ p\neq q}}^{k}\langle\tilde{\nu}^{n_p}, x\rangle\right) + \varepsilon^2\left(\sum_{i=1}^{k-1}\sum_{j>i}^{k}\tilde{\lambda}^{n_i}\tilde{\lambda}^{n_j}\right)\langle\tilde{\nu}^{n_{k+1}}, x\rangle\prod_{p=1}^{k}\langle\tilde{\nu}^{n_p}, x\rangle$$

$$\Longrightarrow\; L_2\theta^{(k)}_{n_1\dots n_k}(x) = \varepsilon^2 \sum_{q=1}^{k}\left(\tilde{\lambda}^{n_{k+1}}\tilde{\lambda}^{n_q}\right)\prod_{p=1}^{k+1}\langle \tilde{v}^{n_p}, x\rangle + \varepsilon^2\left(\sum_{i=1}^{k-1}\sum_{j>i}^{k}\tilde{\lambda}^{n_i}\tilde{\lambda}^{n_j}\right)\prod_{p=1}^{k+1}\langle \tilde{v}^{n_p}, x\rangle$$

$$\Longrightarrow\; L_2\theta^{(k)}_{n_1\dots n_k}(x) = \varepsilon^2\left(\sum_{q=1}^{k}\left(\tilde{\lambda}^{n_{k+1}}\tilde{\lambda}^{n_q}\right) + \sum_{i=1}^{k-1}\sum_{j>i}^{k}\tilde{\lambda}^{n_i}\tilde{\lambda}^{n_j}\right)\prod_{p=1}^{k+1}\langle \tilde{v}^{n_p}, x\rangle$$

$$\Longrightarrow\; L_2\theta^{(k)}_{n_1\dots n_k}(x) = \varepsilon^2\left(\sum_{i=1}^{k}\sum_{j>i}^{k+1}\tilde{\lambda}^{n_i}\tilde{\lambda}^{n_j}\right)\prod_{p=1}^{k+1}\langle \tilde{v}^{n_p}, x\rangle \tag{59}$$

Hence, the eigenvalues of the linear operators L_1 and L_2 for $m > 2$ are given by the expressions

$$\sum_{r=1}^{m}\lambda^{n_r} \tag{60}$$

$$\varepsilon^2\sum_{i=1}^{m-1}\sum_{j>i}^{m}\tilde{\lambda}^{n_i}\tilde{\lambda}^{n_j} \tag{61}$$

respectively. Therefore, the linear mappings L_1 and L_2 are each invertible if eigenvalues (60) and (61) are nonresonant. It can be computed that the eigenvalues of $A + BK$ and BK are real and negative due to stability of the linear controller, hence the operators L_1 and L_2 are both invertible. It remains to show that $L = L_1 + L_2$ is invertible as a result. Notice that $(L_1 + L_2)^{-1} = L_1^{-1}(1 + L_2L_1^{-1})^{-1}$, where L_1 is already shown to be invertible. We have that $(1 + L_2L_1^{-1})^{-1}$ is invertible if $\|L_2L_1^{-1}\|_\infty < 1$, where $\|.\|_\infty$ denotes the spectral norm as before [21]. Since,

$$\|L_2L_1^{-1}\|_\infty \le \|L_2\|_\infty\, \|L_1^{-1}\|_\infty \le \|L_2\|_\infty\,(1 - \|1 - L_1\|_\infty)^{-1} \tag{62}$$

holds, showing that $\|L_2\|_\infty\,(1 - \|1 - L_1\|_\infty)^{-1} < 1$ will guarantee the invertibility of $L_1 + L_2$. Having already computed the spectral norms of L_1 and L_2

$$\|L_2\|_\infty\,(1 - \|1 - L_1\|_\infty)^{-1} < 1$$

$$\Longrightarrow\; \max\left|\varepsilon^2\sum_{i=1}^{m-1}\sum_{j>i}^{m}\tilde{\lambda}^{n_i}\tilde{\lambda}^{n_j}\right| < 1 - \max\left|\left(1 - \sum_{r=1}^{m}\lambda^{n_r}\right)\right|$$

$$\Longrightarrow\; \left|\widetilde{\Lambda}^{(m)}_{\max}\right| < \left|\Lambda^{(m)}_{\min}\right| \tag{63}$$

where $\Lambda^{(m)}_{\min}$ is the minimum eigenvalue of L_1 and $\widetilde{\Lambda}^{(m)}_{\max}$ is the maximum eigenvalue of L_2 in magnitude. Hence we have shown the invertibility of L, and as a result the existence of nonlinear control for $m > 2$ when the linear deterministic part of the closed-loop system is asymptotically stable, and the minimum eigenvalue of L_1 acting on $V^{(m)}(x)$ is greater than the maximum eigenvalue of L_2 in magnitude. □

Remark 1 The deterministic analogue of Proposition 2 was shown by Lyapunov in 1892 (see Theorem 1 in Ref. [22] chapter 2, pages 71-79, or Theorem 1 in Ref. [23] part 21, pages 57-58). Similar to SDE (14), in a deterministic setting, the existence of higher order Lyapunov function is guaranteed by stability of the linear dynamics. Specifically, if a differential equation of the form $\dot{x} = M_{i1}x_1 + M_{i2}x_2 + \cdots + M_{in}x_n$, $i = n$, for an arbitrary $n > 0$, has eigenvalues which do not have a relation of the form $c_1\lambda_1 + c_2\lambda_2 + \cdots + c_n\lambda_n = 0$, for a given positive integer $c = c_1 + c_2 + \cdots + c_n$, where c_i are non-negative constants, then there exists a polynomial $V(x)$ of order m, satisfying the equation $\sum_{i=1}^{n}(M_{i1}x_1 + M_{i2}x_2 + \cdots + M_{in}x_n)\frac{\partial V^{(m)}(x)}{\partial x_i} = \Psi^{(m)}(x)$, where $\Psi^{(m)}(x)$ is known polynomial sum of the same order. In fact, this is the same condition shown in Proposition 2 when the first order linear operator is applied to $V^{(m)}(x)$.

We may now assume that existence conditions of Propositions 1 and 2 are satisfied. Though, one may want to consider the conditions of stability for the complete system (14) with a nonlinear control. Studying the stability in this sense will provide the region of attraction of the $(m-1)^{\text{th}}$ order control, i.e. for which set in $\mathcal{X} \subset \mathbb{R}^6$, the $(m-1)^{th}$ order control asymptotically stabilizes the system. We state the following classical theorem without proof. The results are pertaining to a continuous time-variant SDE, but are general enough to apply to the case of SDE (14).

Theorem 2 [24] Consider a generalization of the SDE (14): $dx_t = f(t, x_t)dt + \sigma(t, x_t)dW_t,\ t \geq t_o$, where $f(t, x) \in \mathbb{R}^n$ is an arbitrary drift function, $\dim(x) = n$. Assume that this SDE satisfies the existence and uniqueness conditions [24], and has continues coefficients in t. Let us further define a time-dependent counterpart of the infinitesimal generator: $\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i(t, x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(t, x)\frac{\partial^2}{\partial x_i \partial x_j}$. Suppose there exists a positive definite function $V(t, x)$ defined on $[t_o, \infty) \times \mathcal{X}$, where $\mathcal{X} = \{x: |x| < h\}$, $h > 0$. Let such a function be twice differentiable in x, and differentiable in t. Then if

$$\mathcal{L}V(t, x) < -CV(t, x) \tag{64}$$

for some arbitrary constant $C > 0$, $\mathcal{L}V(t, x)$ is negative definite. As a result, the trivial solution of the SDE is asymptotically stable in probability.

Remark 2 In this paper, given the form of the SDE (14), we have assumed that the preliminary existence, and uniqueness conditions are satisfied. Moreover, we have shown the existence of a Lyapunov function of order m. It must also be pointed out that for $V(x)$ to be positive definite on $\mathcal{X} \subset \mathbb{R}^6$, $V(t,x) > 0$, $t^+ \to \infty$, $\forall x \in \mathcal{X} \backslash \{0\}$ is a sufficient condition. Furthermore, if the condition $\mathcal{L}V(t,x) < -CV(t,x)$ holds on the local set $\mathcal{X}$, then the system (14) is asymptotically stable in probability, where $\mathcal{X}$ is the region of attraction for a m^{th} order control. A refined version of Theorem 2, as stated in Ref. [25], specifies that if $C_1(|x|) \le V(x,t) \le C_2(|x|)$ and $\mathcal{L}V(x) \le -C_3(|x|)$, are satisfied for strictly increasing continuous functions C_1,C_2,C_3, and radially unbounded functions C_1,C_2, such that, $C_i(0) = 0$, and $i = 1,2,3$, then, the trivial solution of the SDE is asymptotically stable. For the case of the complete power series (31), since we have that $\forall x \neq 0$,

$$\mathcal{L}^u V(x) = -r(x) < 0 \tag{65}$$

$$V(x) > 0 \tag{66}$$

where $\mathcal{L}^u$ is given by Eq. (18), with argument u the solution to Eq. (30), the conditions of asymptotic stability are satisfied $\forall x \in \mathcal{X}$ where inequalities (65) and (66) hold.

IV. Computation of Control

To compute the nonlinear control, we use Eq. (43) along with Eq. (44). Here, we will demonstrate the computation of the optimal stochastic control up to cubic order (and up to sextic order in Appendix A). Every order of control will have unknowns that come from Hamiltonians with one order higher than that of the control. We begin by expanding and simplifying Eq. (30) for $m - 1 = 1, \dots ,6$. The Control expressions are

$$k^{(1)}(x) = Kx = -\left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right)B\right)^{-1}\left[(B)^T \frac{\partial V^{(2)}(x)}{\partial x}\right] \tag{67}$$

$$k^{(2)}(x) = -\left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right)B\right)^{-1}\left[(B)^T \frac{\partial V^{(3)}(x)}{\partial x} + \varepsilon^2 B^T H\left(V^{(3)}(x)\right)BKx\right] \tag{68}$$

$$\begin{aligned} k^{(3)}(x) = -\left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right)B\right)^{-1}&\left[(B)^T \frac{\partial V^{(4)}(x)}{\partial x} + \varepsilon^2 B^T H\left(V^{(3)}(x)\right)Bk^{(2)}(x)\right. \\ &\left. + \varepsilon^2 B^T H\left(V^{(4)}(x)\right)BKx\right] \end{aligned} \tag{69}$$

To find the unknown $V(x)$ terms, we solve for every $V^{(m)}(x)$ through system (29)-(30). Specifically, since we have arbitrarily solved up to sixth order control, we require the value function (31) to be known for $m = 7$. Hence, we expand Eq. (44) for $m = 2,3, \dots ,7$. The quadratic through quartic Hamiltonian expansions become

$$(Ax)^T \frac{\partial V^{(2)}(x)}{\partial x} + (BKx)^T \frac{\partial V^{(2)}(x)}{\partial x} + \frac{1}{2}(Kx)^T RKx + \frac{1}{2}\varepsilon^2 (Kx)^T B^T H\left(V^{(2)}(x)\right) BKx + \frac{1}{2}x^T Qx = 0 \tag{70}$$

$$\begin{aligned} &(Ax)^T \frac{\partial V^{(3)}(x)}{\partial x} + f^{(2)^T}(x) \frac{\partial V^{(2)}(x)}{\partial x} + (BKx)^T \frac{\partial V^{(3)}(x)}{\partial x} + \left(Bk^{(2)}(x)\right)^T \frac{\partial V^{(2)}(x)}{\partial x} \\ &+ \frac{1}{2}\varepsilon^2 (Kx)^T B^T H\left(V^{(3)}(x)\right) BKx + (Kx)^T \left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right) B\right) k^{(2)}(x) + r^{(3)}(x) = 0 \end{aligned} \tag{71}$$

$$\begin{aligned} &(Ax)^T \frac{\partial V^{(4)}(x)}{\partial x} + f^{(2)^T}(x) \frac{\partial V^{(3)}(x)}{\partial x} + f^{(3)^T}(x) \frac{\partial V^{(2)}(x)}{\partial x} + (BKx)^T \frac{\partial V^{(4)}(x)}{\partial x} \\ &+ \left(Bk^{(2)}(x)\right)^T \frac{\partial V^{(3)}(x)}{\partial x} + \left(Bk^{(3)}(x)\right)^T \frac{\partial V^{(2)}(x)}{\partial x} + \frac{1}{2}\varepsilon^2 (Kx)^T B^T H\left(V^{(4)}(x)\right) BKx \\ &+ \frac{1}{2}\left(k^{(2)}(x)\right)^T \left(R + \varepsilon^2 B^T H\left(V^2(x)\right)B\right) k^{(2)}(x) + \varepsilon^2 (Kx)^T B^T H\left(V^{(3)}(x)\right) Bk^{(2)}(x) \\ &+ (Kx)^T \left(R + \varepsilon^2 B^T H\left(V^2(x)\right)B\right) k^{(3)}(x) + r^{(4)}(x) = 0 \end{aligned} \tag{72}$$

Orders quintic through septic are listed in Appendix A. Note that in Eq. (70)-(72) and (79)-(81), the expansions are in terms of lower orders of known control forms. To solve for $V^{(m)}(x)$, we substitute the resulting terms of Eq. (46) into every m^{th} order Hamiltonian. Reordering, grouping, and simplifying expressions based on their order m, we obtain the quadratic through quartic simplified Hamiltonians as

$$x^T(PA + A^T P)x - x^T(PB)\left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right) B\right)^{-1} (PB)^T x = -x^T Qx \tag{73}$$

$$(Ax)^T \frac{\partial V^{(3)}(x)}{\partial x} + f^{(2)^T}(x)Px + x^T K^T B^T \frac{\partial V^{(3)}(x)}{\partial x} + \frac{1}{2}\varepsilon^2 (Kx)^T B^T H\left(V^{(3)}(x)\right) BKx + r^{(3)}(x) = 0 \tag{74}$$

$$\begin{aligned} &(Ax)^T \frac{\partial V^{(4)}(x)}{\partial x} + f^{(2)^T}(x) \frac{\partial V^{(3)}(x)}{\partial x} + f^{(3)^T}(x)Px + (BKx)^T \frac{\partial V^{(4)}(x)}{\partial x} \\ &+ \frac{1}{2}\varepsilon^2 (Kx)^T B^T H\left(V^{(4)}(x)\right) B(Kx) \\ &- \frac{1}{2}\left(k^{(2)}(x)\right)^T \left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right) B\right) k^{(2)}(x) + r^{(4)}(x) = 0 \end{aligned} \tag{75}$$

Remark 3 The Hamiltonian (73) is the Riccati equation (38) discussed in Proposition 1. There are established numerical methods (i.e. see Rami and Zhou [26]) that approximate the solution P of Eq. (38). For instance, the YALMIP optimization toolbox in MATLAB is a useful tool for implementation of Linear Matrix Inequality (LMI) method. Though, for the particular case of system (14), the Riccati equation is solvable without the use of these methods. To do so, we assume a tridiagonal form of the solution matrix P and control matrix Q. Then solution to Eq. (38) will reduce to the problem of root finding of nine equations.

V. Simulations

The specific model of interest is a 6U CubeSat (6 University-class spacecraft in a 2-by-3 configuration) with three thruster pairs. The standard dimensions of 6U CubeSat are $10 \times 20 \times 30$ centimeters, and the maximum mass is 6 kg. As a result, the entries of moment of inertia tensor in principal axes (in units of kilogram meter squared) are calculated as $I_1 = 0.05$, $I_2 = 0.065$, $I_3 = 0.025$. We further set $b_1^T = [1 \quad 0 \quad 0]$, $b_2^T = [0 \quad 1 \quad 0]$, $b_3^T = [0 \quad 0 \quad 1]$. The Monte Carlo experiment below compares the stochastic nonlinear controller with quadratic cost criteria to a LQR (linear quadratic regulator) controller using MATLAB's built-in function, and then to the deterministic nonlinear controller by setting $\varepsilon = 0$ in the equations of control. Following the numerical experiments of Ref. [3] which considered the thrust variations of 10%-20% inspired by experimental propulsion studies, all the controllers in this paper are tested with uncertainty having standard deviations of 1%, 10%, and 20% from the nominal thrust. We study the rest-to-rest maneuvers with nonlinear and stochastic controller using two different constant gain choices:

A) A conservative choice of gains where the strong condition ii of Proposition 1 is satisfied.

B) An aggressive control gain choice where the strong condition ii of Proposition 1 is not satisfied.

The gain set A is as follows

$$Q = \begin{bmatrix} 1.17 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 1.14 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 1.3 & 0 & 0 & 0.5 \\ 0.7 & 0 & 0 & 0.35 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.6 \end{bmatrix}, R = \begin{bmatrix} 223.8 & 0 & 0 \\ 0 & 135 & 0 \\ 0 & 0 & 489.8 \end{bmatrix}$$

yielding a Riccati solution of

$$P = \begin{bmatrix} 0.9591 & 0 & 0 & 0.4463 & 0 & 0 \\ 0 & 0.9699 & 0 & 0 & 0.4817 & 0 \\ 0 & 0 & 0.8257 & 0 & 0 & 0.4343 \\ 0.4463 & 0 & 0 & 0.1043 & 0 & 0 \\ 0 & 0.4817 & 0 & 0 & 0.0108 & 0 \\ 0 & 0 & 0.4343 & 0 & 0 & 0.6407 \end{bmatrix}$$

for $\varepsilon = 0.1$. Similarly, the gain set B constants are

$$Q = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

yielding a Riccati solution of

$$P = \begin{bmatrix} 0.0219 & 0 & 0 & 0.0217 & 0 & 0 \\ 0 & 0.0266 & 0 & 0 & 0.0262 & 0 \\ 0 & 0 & 0.0147 & 0 & 0 & 0.0145 \\ 0.0217 & 0 & 0 & 0.0215 & 0 & 0 \\ 0 & 0.0262 & 0 & 0 & 0.0261 & 0 \\ 0 & 0 & 0.0145 & 0 & 0 & 0.0144 \end{bmatrix}$$

for $\varepsilon = 0.1$. The Riccati solution for all noise variations for both sets A and B are given in Appendix B. Notice that effort has been made to keep the entries of the optimal solution matrix bounded above by 1. This is because the nonlinearity, i.e. order of the entries of P, grows as the order of the control equations increases, contributing to the radius of attraction shrinking. In extreme cases, noise may cause the state trajectories to exit the region of attraction contributing to loss of stability.

Table 1 Comparison of $\|\mathcal{T}\|_\infty$ of stochastic and deterministic controllers varying $\boldsymbol{\varepsilon}$

Spectral Norms Gain Set A	$\boldsymbol{\varepsilon = 0.01}$	$\boldsymbol{\varepsilon = 0.1}$	$\boldsymbol{\varepsilon = 0.2}$
Stochastic	0.0003	0.0272	0.0985
Deterministic	0.001	0.098	0.3579
Spectral Norms Gain Set B	$\boldsymbol{\varepsilon = 0.01}$	$\boldsymbol{\varepsilon = 0.1}$	$\boldsymbol{\varepsilon = 0.2}$
Stochastic	4884.37	7837.75	7663.75
Deterministic	4961.84	8012.92	8050.42

The singular values of gain set A are relatively closer to each other than the singular values computed for gain set B. It is also clear from Table. 1 that the spectral norm increases with the variance of noise, which is proportional to ε^2. It must be pointed out that tuning gains in a manner to bound both $\|\mathcal{T}\|_\infty$ and all the entries of P by 1 is rather difficult. However, when accomplished, it is observed that condition ii of Proposition 1 results in similar behaviors of stochastic and deterministic controllers as tabulated on the right portion of Tables 2-4. The simulations are carried out for 2000 particles, during a rest-to-rest maneuver starting from $x = [0\ 0\ 0\ 1\ 1\ 1]^T$ and stabilizing at $x = [0\ 0\ 0\ 0\ 0\ 0]^T$.

Table 2 Mean cost comparison of stochastic controller to LQR and deterministic nonlinear controllers for $\varepsilon = 0.01$ using gain set A

Control Order	Stochastic Control Cost	Stochastic State Cost	Total Cost	(%) Improvement compared to LQR	Deterministic Control Cost	Deterministic State Cost	Total Cost	(%) Improvement compared to Deterministic
LQR				0.0000	0.3221	0.5060	0.8280	
Linear	0.3220	0.5060	0.8280	0.0012	0.3221	0.5061	0.8282	0.0228
Quadratic	0.3575	0.3769	0.7344	11.3019	0.3576	0.3770	0.7346	0.0252
Cubic	0.2932	0.4341	0.7272	12.1711	0.2933	0.4341	0.7274	0.0253
Quartic	0.2686	0.4547	0.7233	12.6451	0.2683	0.4554	0.7237	0.0560
Quintic	0.2931	0.4301	0.7232	12.6566	0.2933	0.4302	0.7235	0.0370
Sextic	0.3117	0.4117	0.7235	12.6284	0.3118	0.4118	0.7236	0.0147

Table 3 Mean cost comparison of stochastic controller to LQR and deterministic nonlinear controllers for $\varepsilon = 0.1$ using gain set A

Control Order	Stochastic Control Cost	Stochastic State Cost	Total Cost	(%) Improvement compared to LQR	Deterministic Control Cost	Deterministic State Cost	Total Cost	(%) Improvement compared to Deterministic
LQR				0.0000	0.3255	0.5096	0.8351	
Linear	0.3212	0.5119	0.8331	0.2468	0.3260	0.5088	0.8349	0.2177
Quadratic	0.3576	0.3832	0.7407	11.3010	0.3621	0.3801	0.7423	0.2056
Cubic	0.2937	0.4407	0.7344	12.0604	0.2980	0.4374	0.7354	0.1292
Quartic	0.2680	0.4620	0.7300	12.5860	0.2719	0.4582	0.7302	0.0217
Quintic	0.2928	0.4361	0.7288	12.7298	0.2973	0.4331	0.7304	0.2127
Sextic	0.3108	0.4173	0.7281	12.8174	0.3149	0.4136	0.7285	0.0605

Table 4 Mean cost comparison of stochastic controller to LQR and deterministic nonlinear controllers for $\varepsilon = 0.2$ using gain set A

Control Order	Stochastic Control Cost	Stochastic State Cost	Total Cost	(%) Improvement compared to LQR	Deterministic Control Cost	Deterministic State Cost	Total Cost	(%) Improvement compared to Deterministic
LQR				0.0000	0.3388	0.5201	0.8589	
Linear	0.3210	0.5313	0.8523	0.7710	0.3383	0.5191	0.8574	0.5891
Quadratic	0.3574	0.4022	0.7596	11.5600	0.3759	0.3898	0.7658	0.7977
Cubic	0.2927	0.4596	0.7523	12.4180	0.3105	0.4461	0.7565	0.5645
Quartic	0.2668	0.4810	0.7478	12.9431	0.2824	0.4655	0.7479	0.0174
Quintic	0.2919	0.4565	0.7483	12.8792	0.3084	0.4402	0.7487	0.0472
Sextic	0.3111	0.4373	0.7484	12.8635	0.3299	0.4245	0.7543	0.7804

Comparing the stochastic controller to the LQR, every order of the nonlinear controller outperforms the conventional LQR controller by minimizing the total cost. Generally, as the order of the nonlinear control increases, the percent improvement in cost optimization compared to LQR controller increases as well. Few exceptions to this trend are present due to numerical error in computation of the control. On the other hand, comparison of stochastic and deterministic nonlinear controllers of all orders reveals significantly smaller improvements due to the choice of gain set A. Given a controller K, the spectral norm (40) quantifies the total energy/variance of noise due to application of K over time. If a controller satisfies condition ii of Proposition 1, then the variance of the controller becomes narrowly bounded, hence contributing to the small difference between stochastic and deterministic nonlinear controls satisfying the condition. However, as the difference between the spectral norms corresponding to stochastic and deterministic controls increases, so does the improvement due to the stochastic controller. The following simulation results tabulated in Tables 5-7 are using gain set B which violates condition ii of Proposition 1.

Table 5 Mean cost comparison of stochastic controller to LQR and deterministic nonlinear controllers for $\varepsilon = 0.01$ using gain set B

Control Order	Stochastic Control Cost	Stochastic State Cost	Total Cost	(%) Improvement compared to LQR	Deterministic Control Cost	Deterministic State Cost	Total Cost	(%) Improvement compared to Deterministic
LQR				0.0000	0.0220	0.0221	0.0441	
Linear	0.0218	0.0221	0.0439	0.3583	0.0219	0.0220	0.0440	0.1910
Quadratic	0.0220	0.0220	0.0439	0.2468	0.0221	0.0219	0.0440	0.1251
Cubic	0.0216	0.0222	0.0439	0.4289	0.0217	0.0222	0.0439	0.1415
Quartic	0.0217	0.0223	0.0439	0.3385	0.0218	0.0222	0.0441	0.3230
Quintic	0.0217	0.0220	0.0438	0.6176	0.0219	0.0220	0.0439	0.1625
Sextic	0.0217	0.0221	0.0438	0.5825	0.0218	0.0220	0.0438	−0.0057

Table 6 Mean cost comparison of stochastic controller to LQR and deterministic nonlinear controllers for $\varepsilon = 0.1$ using gain set B

Control Order	Stochastic Control Cost	Stochastic State Cost	Total Cost	(%) Improvement compared to LQR	Deterministic Control Cost	Deterministic State Cost	Total Cost	(%) Improvement compared to Deterministic
LQR				0.0000	0.0334	0.0333	0.0667	
Linear	0.0206	0.0417	0.0624	6.5498	0.0339	0.0338	0.0676	7.8148
Quadratic	0.0205	0.0407	0.0613	8.1817	0.0334	0.0331	0.0666	7.9644
Cubic	0.0201	0.0410	0.0611	8.3855	0.0344	0.0347	0.0691	11.5576
Quartic	0.0203	0.0405	0.0608	8.9225	0.0353	0.0355	0.0708	14.1990
Quintic	0.0201	0.0402	0.0603	9.5799	0.0346	0.0346	0.0692	12.7832
Sextic	0.0199	0.0401	0.0600	10.0456	0.0328	0.0327	0.0655	8.3409

Table 7 Mean cost comparison of stochastic controller to LQR and deterministic nonlinear controllers for $\varepsilon = 0.2$ using gain set B

Control Order	Stochastic Control Cost	Stochastic State Cost	Total Cost	(%) Improvement compared to LQR	Deterministic Control Cost	Deterministic State Cost	Total Cost	(%) Improvement compared to Deterministic
LQR				0.0000	0.2095	0.2065	0.4159	
Linear	0.0126	0.1165	0.1291	68.9626	4.8204	4.7460	9.5664	98.6505
Quadratic	0.0126	0.1149	0.1275	69.3395	0.1908	0.1879	0.3787	66.3203
Cubic	0.0120	0.1153	0.1273	69.3917	0.1677	0.1669	0.3346	61.9504
Quartic	0.0120	0.1156	0.1276	69.3310	1.9908	1.9607	3.9515	96.7717
Quintic	0.0124	0.1108	0.1232	70.3798	0.9506	0.9381	1.8887	93.4768
Sextic	0.0125	0.1101	0.1225	70.5386	0.1994	0.1971	0.3965	69.0942

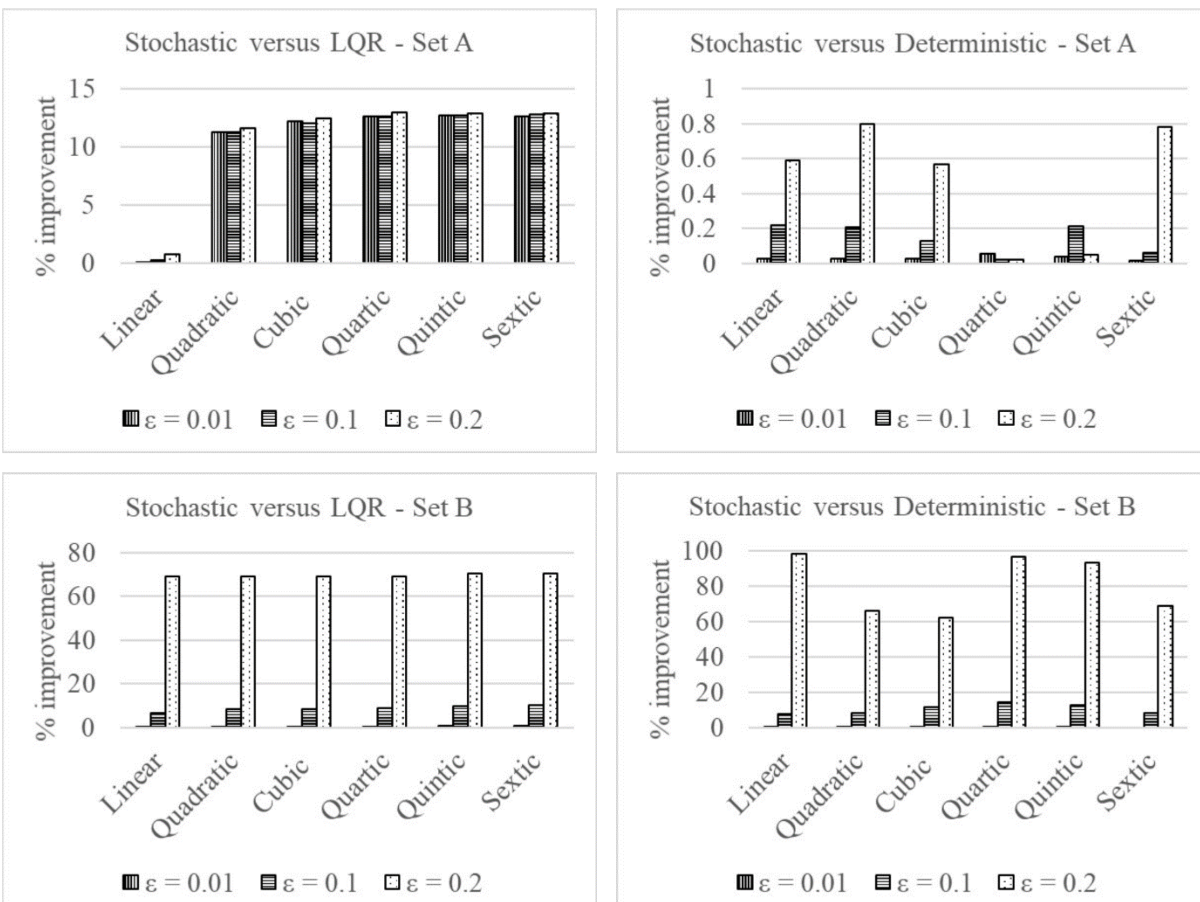

Fig. 3 Bar chart comparisons of LQR versus stochastic control (on the left), and stochastic versus deterministic control (on the right) with gain set A (on the top), and gain set B (on the bottom)

Simulations of gain set B also demonstrate how increasing ε leads to a greater difference between the total optimized cost in comparison of the stochastic and the LQR controllers in Tables 5-7. Furthermore, it is shown that nonlinear stochastic control optimizes the total cost better as its order is increased. Since the gains are more aggressive (i.e. leading to larger control constant values), and that the spectral norms tabulated in Table. 1 are larger, we see a greater difference between total cost optimized in comparison of the nonlinear stochastic and deterministic controllers. This in fact can be attributed to the multiplicative nature of control uncertainty, where larger thrust contributes to higher variations in thrust. This difference is then amplified as the noise standard deviation is increased, as shown in Tables 5-7. The main results of Tables 2-7 are summarized as bar charts in Fig. 3 above.

The performance comparison of a linear deterministic controller such as the LQR to nonlinear stochastic controller can also be seen in Fig. 4 and 5. Using gain set A, the cumulative distribution function (CDF) in Fig. 4 shows that the sextic nonlinear controller has a higher probability of achieving lower total cost compared to its LQR counterpart under a standard deviation of 20%. This trend can also be seen in Fig. 5 when the controllers are tuned using gain set B.

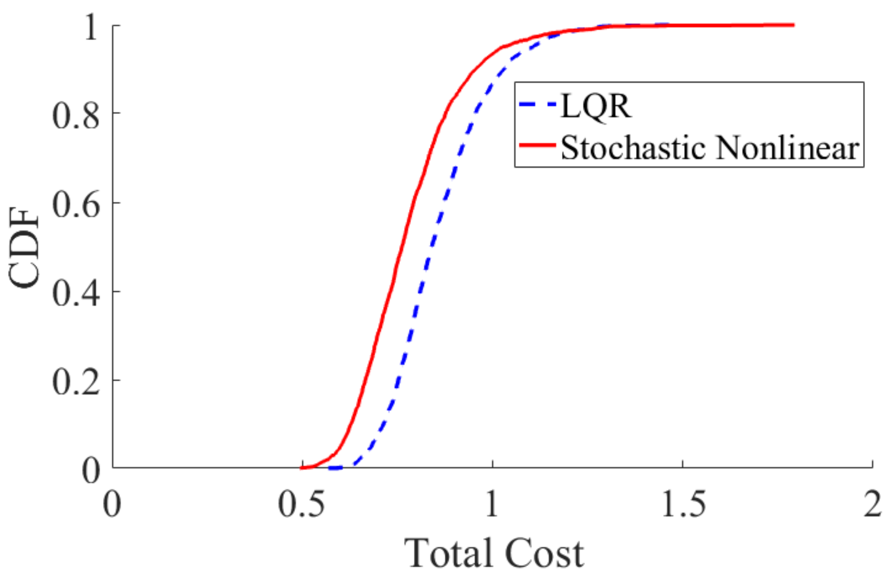

Fig. 4 Cumulative distribution function comparison of 2000 stabilization realizations of sextic stochastic and LQR controllers when $\boldsymbol{\varepsilon = 0.2}$ using gain set A.

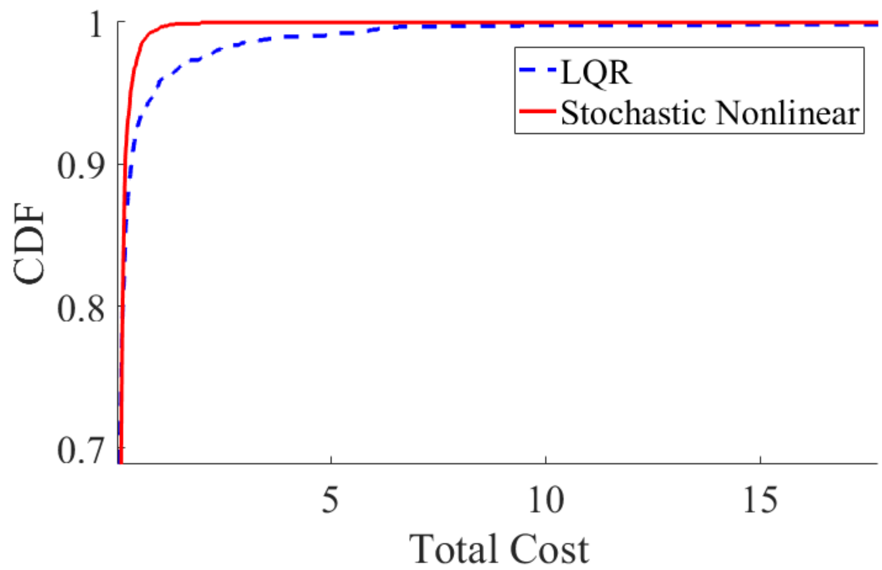

Fig. 5 Cumulative distribution function comparison of 2000 stabilization realizations of sextic stochastic and LQR controllers when $\varepsilon = 0.2$ using gain set B.

The following figures depict the stabilization trajectories of 2000 particles starting at $x = [0\ 0\ 0\ 1\ 1\ 1]^T$ using two different gains, where only the maximum and minimum trajectories are plotted. Figure 6 shows the stabilization realizations due to control gain set A, and Fig. 7 depicts the realizations due to control gain set B.

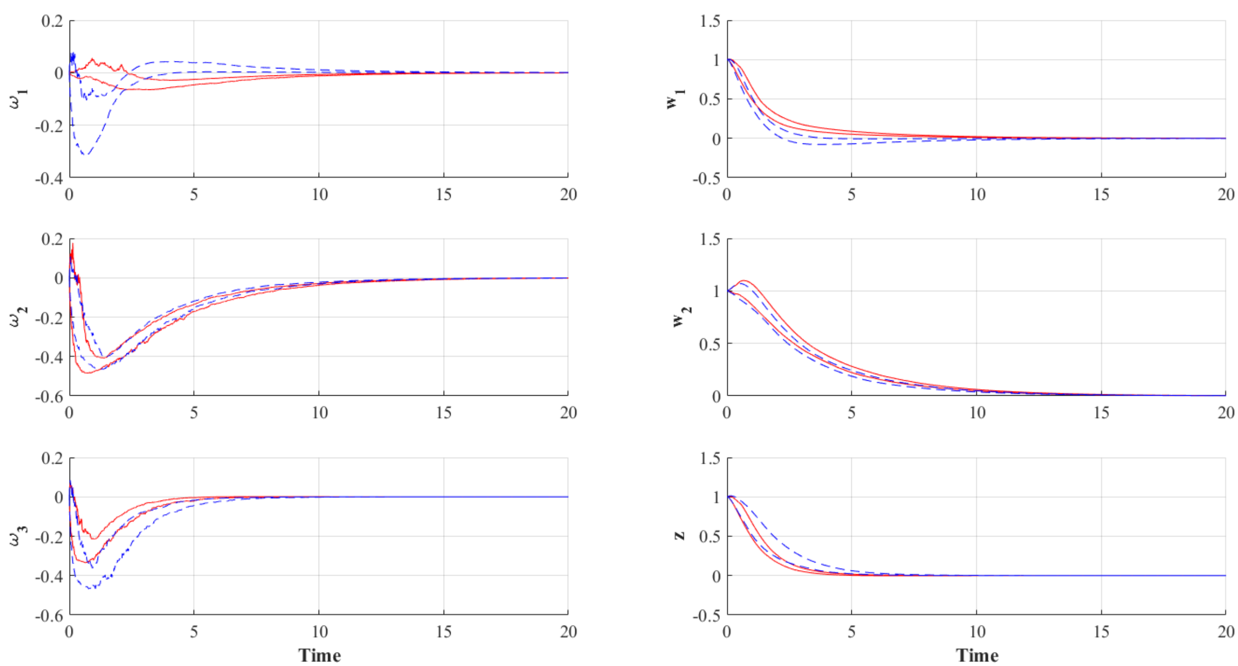

Fig. 6 Stabilization Trajectories of 100 realizations of sextic stochastic control (solid line) and LQR control (dashed line) for $\varepsilon = 0.2$ using gain set A.

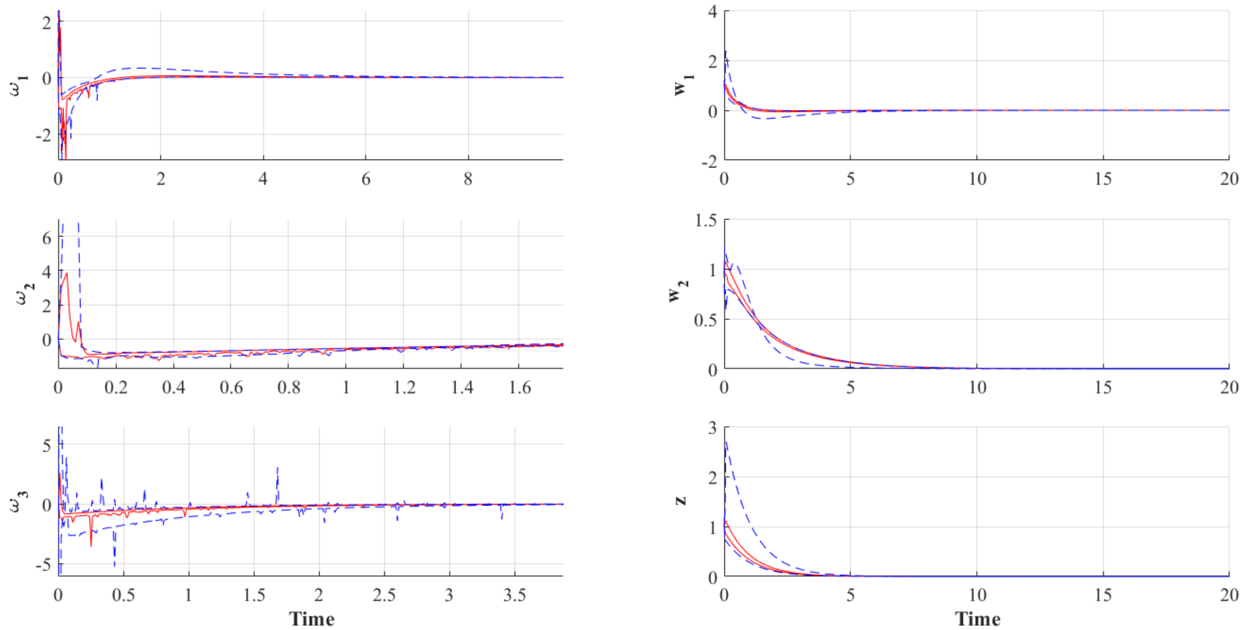

Fig. 7 Stabilization Trajectories of 100 realizations of sextic stochastic control (solid line) and LQR control (dashed line) for $\varepsilon = 0.2$ using gain set B.

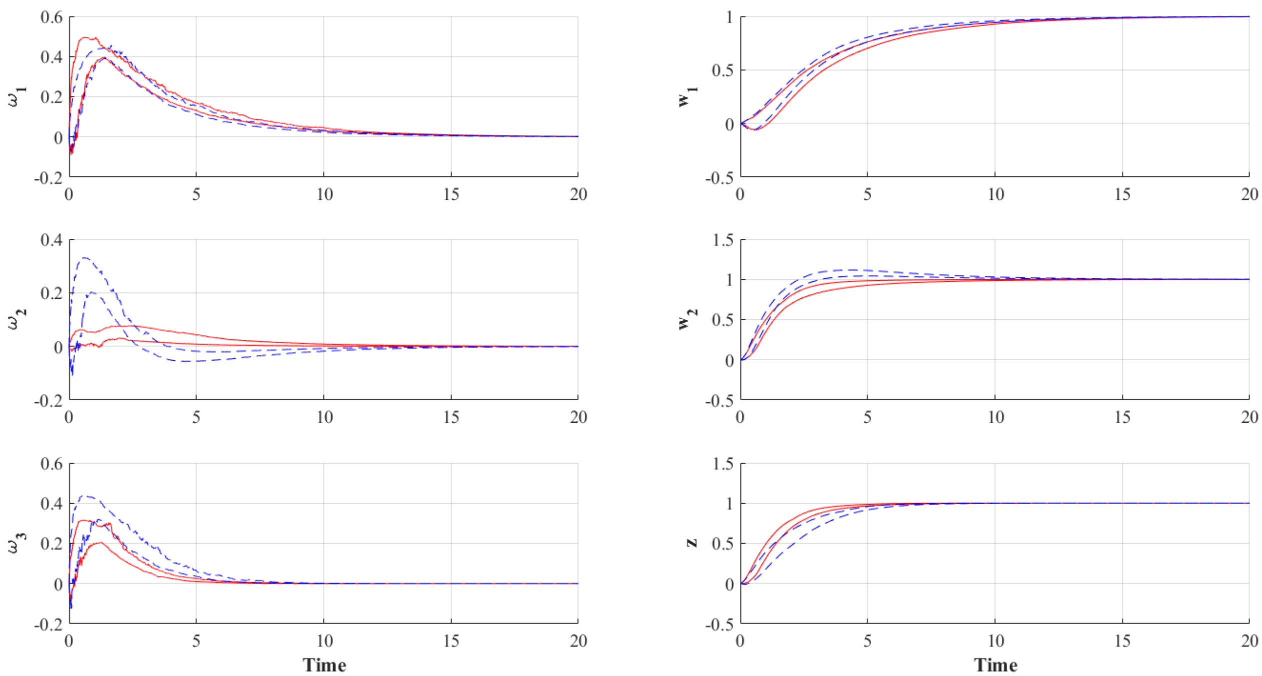

Fig. 8 Targeting control of 100 realizations of sextic stochastic control (solid line) and LQR control (dashed line) for $\varepsilon = 0.2$ using gain set A with reference point $x = [0\ 0\ 0\ 1\ 1\ 1]^T$.

Above, Fig. 8 demonstrates the targeting ability of the stochastic nonlinear control under standard deviation of 20% starting from the origin as the initial condition. Comparing the choice of gain sets A and B, Fig. 6 and 7 demonstrate that the aggressive gains of set B introduce higher variations among the realizations, i.e. the angular velocity trajectories of Fig. 6 are mostly bounded between ± 0.5, while Fig. 7 shows angular velocity of some particles overshooting above 5 using the gain set B. Although this may become a controller design choice at the end, the control design engineer may study the uncertainty induced by the controller of interest through such a multiplicative structure where the uncertainty induced is proportional to the inputted control energy. Another fact to point out here is that the gain set A satisfied a strong condition, meaning that there may exist other controllers with larger K constants which may satisfy bound (40) and demonstrate stability. Control gain set B is an instance of such controllers which exhibits stability properties while violating condition ii of Proposition 1. Furthermore, comparing the bandgap between the maximum and minimum trajectories of LQR and stochastic control, it is observed that the stochastic control is better at keeping a narrower gap. This amounts to having the probable trajectories of the system closer together when using the stochastic control, hence reducing uncertainty.

Earlier, we chose the Riccati solutions to be bounded above by one to ensure that the initial conditions are within the region of attraction. In the following Monte Carlo experiment, we demonstrate how this requirement could be mitigated through choosing the initial conditions which are within norm 1 of the origin. The demonstrated comparison in Table. 8 below is for initial condition of $x = [0\ 0\ 0\ 0.4\ 0.4\ 0.4]^T$ when $\varepsilon = 0.2$. The simulations are for 2000 particles and the gains chosen for the experiment are

$$Q = 1000 \times \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 900 & 0 & 0 \\ 0 & 900 & 0 \\ 0 & 0 & 900 \end{bmatrix}$$

which for $\varepsilon = 0.2$, yield $\|\mathcal{T}\|_\infty = 0.0561$, and a Riccati solution of

$$P = \begin{bmatrix} 75.0511 & 0 & 0 & 72.4710 & 0 & 0 \\ 0 & 89.5490 & 0 & 0 & 85.9329 & 0 \\ 0 & 0 & 55.4350 & 0 & 0 & 52.7248 \\ 72.4710 & 0 & 0 & 71.2035 & 0 & 0 \\ 0 & 85.9329 & 0 & 0 & 84.1620 & 0 \\ 0 & 0 & 52.7248 & 0 & 0 & 51.4036 \end{bmatrix}$$

Table 8 Mean cost comparison of stochastic controller to LQR controller for $\varepsilon = 0.2$ when entries of P are large in magnitude

Control Order	Control Cost	State Cost	Total Cost	(%) Improvement compared to LQR
LQR	24.9884	24.5791	49.5674	0.0000
Linear	9.6121	24.7539	34.3659	30.6683
Quadratic	9.3526	23.4565	32.8091	33.8092
Cubic	9.1873	23.3142	32.5015	34.4297
Quartic	9.0996	23.2447	32.3443	34.7469
Quintic	9.1491	23.3681	32.5172	34.3982
Sextic	9.2217	23.5727	32.7944	33.8389

Inspecting Table. 8 reveals that even when the Riccati solution has entries with magnitude larger than 1, choosing the initial condition within norm 1 of the origin will help to retain stability and optimality properties of the control. In fact, this is demonstration of a special case where condition ii of Proposition 1 is satisfied, but the entries of P are larger than 1. Although being limited to norm 1 of the origin may be restrictive in many applications, this problem may be alleviated by use of planning methods where the controller is set to achieve reference waypoints which are apart by norm 1 of their origin.

In this section we have shown how tuning strategies for nonlinear controllers may help reduce uncertainty due to control input. We have shown a case where stochastic nonlinear controller outperforms its deterministic counterpart, and another case where stochastic and deterministic controllers perform similarly, with smaller differences in the total cost optimized. The results of this section are useful in understanding how control-induced noise is amplified or reduced during a design process. Moreover, the numerical results presented here could aid in the design of optimal feedback controllers, which are fault tolerant. In many applications, estimating ε can help stochastic controllers attenuate uncertainty and noise regardless of the design gain choices.

VI. Conclusion

Thrust uncertainty causes state error and accumulated state error is detrimental to mission objectives. A stochastic control method has been presented that on average will reduce and regulate the diffusion of uncertainty and its effects in nonlinear systems. In the framework outlined in this paper, the choice of control's degree can be made based on factors such as need for accuracy, computational resources available, and the actuators themselves. Through this study, a control designer may better understand how different gain-tuning regimes could amplify or alleviate the thrust-induced uncertainty. In addition, the presented experiments may give insight into how different orders of nonlinearity in controllers improve the desired criteria. In general, the disturbance suppression properties of the presented stochastic controllers may increase the success chance of space missions. Specifically, in operations where control and state trajectory need to be precise, i.e. during docking operations, torque disturbances could be unwanted, or even hazardous. Moreover, in a hostile environment such as space, consumption minimization of scarce resources such as power and propellants is highly desirable. The discussed stochastic method considers the existing thrust disturbances and satisfies the optimal criteria on average.

Appendix A: Higher Order Control

This Appendix provides the higher order control along with their respective Hamiltonian equations. The following are the derived quartic through sextic control equations

$$\begin{aligned} k^{(4)}(x) = -\left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right) B\right)^{-1} \bigg[(B)^T \frac{\partial V^{(5)}(x)}{\partial x} + \varepsilon^2 B^T H\left(V^{(3)}(x)\right) B k^{(3)}(x) \\ + \varepsilon^2 B^T H\left(V^{(4)}(x)\right) B k^{(2)}(x) + \varepsilon^2 B^T H\left(V^{(5)}(x)\right) B K x\bigg] \end{aligned} \tag{76}$$

$$\begin{aligned} k^{(5)}(x) = -\left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right) B\right)^{-1} \bigg[(B)^T \frac{\partial V^{(6)}(x)}{\partial x} + \varepsilon^2 B^T H\left(V^{(3)}(x)\right) B k^{(4)}(x) \\ + \varepsilon^2 B^T H\left(V^{(4)}(x)\right) B k^{(3)}(x) + \varepsilon^2 B^T H\left(V^{(5)}(x)\right) B k^{(2)}(x) \\ + \varepsilon^2 B^T H\left(V^{(6)}(x)\right) B K x\bigg] \end{aligned} \tag{77}$$

$$\begin{aligned} k^{(6)}(x) = -\left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right) B\right)^{-1} \bigg[(B)^T \frac{\partial V^{(7)}(x)}{\partial x} + \varepsilon^2 B^T H\left(V^{(3)}(x)\right) B k^{(5)}(x) \\ + \varepsilon^2 B^T H\left(V^{(4)}(x)\right) B k^{(4)}(x) + \varepsilon^2 B^T H\left(V^{(5)}(x)\right) B k^{(3)}(x) \\ + \varepsilon^2 B^T H\left(V^{(6)}(x)\right) B k^{(2)}(x) + \varepsilon^2 B^T H\left(V^{(7)}(x)\right) B K x\bigg] \end{aligned} \tag{78}$$

The following equations are the quintic through the septic Hamiltonian equations

$$\begin{aligned} (Ax)^T \frac{\partial V^{(5)}(x)}{\partial x} + f^{(2)^T}(x) \frac{\partial V^{(4)}(x)}{\partial x} + f^{(3)^T}(x) \frac{\partial V^{(3)}(x)}{\partial x} + (BKx)^T \frac{\partial V^{(5)}(x)}{\partial x} \\ + \left(Bk^{(2)}(x)\right)^T \frac{\partial V^{(4)}(x)}{\partial x} + \left(Bk^{(3)}(x)\right)^T \frac{\partial V^{(3)}(x)}{\partial x} + \left(Bk^{(4)}(x)\right)^T \frac{\partial V^{(2)}(x)}{\partial x} \\ + \frac{1}{2}\varepsilon^2 (Kx)^T B^T H\left(V^{(5)}(x)\right) BKx + \frac{1}{2}\varepsilon^2 \left(k^{(2)}(x)\right)^T B^T H\left(V^{(3)}(x)\right) Bk^{(2)}(x) \\ + \varepsilon^2 (Kx)^T B^T H\left(V^{(4)}(x)\right) Bk^{(2)}(x) + \varepsilon^2 (Kx)^T B^T H\left(V^{(3)}(x)\right) Bk^{(3)}(x) \\ + (Kx)^T \left(R + \varepsilon^2 B^T H\left(V^2(x)\right) B\right) k^{(4)}(x) \\ + \left(k^{(2)}(x)\right)^T \left(R + \varepsilon^2 B^T H\left(V^2(x)\right) B\right) k^{(3)}(x) + r^{(5)}(x) = 0 \end{aligned} \tag{79}$$

$$
\begin{aligned}
&(Ax)^T \frac{\partial V^{(6)}(x)}{\partial x} + f^{(2)^T}(x) \frac{\partial V^{(5)}(x)}{\partial x} + f^{(3)^T}(x) \frac{\partial V^{(4)}(x)}{\partial x} + (BKx)^T \frac{\partial V^{(6)}(x)}{\partial x} \\
&\quad + \left(Bk^{(2)}(x)\right)^T \frac{\partial V^{(5)}(x)}{\partial x} + \left(Bk^{(3)}(x)\right)^T \frac{\partial V^{(4)}(x)}{\partial x} + \left(Bk^{(4)}(x)\right)^T \frac{\partial V^{(3)}(x)}{\partial x} \\
&\quad + \left(Bk^{(5)}(x)\right)^T \frac{\partial V^{(2)}(x)}{\partial x} + \frac{1}{2}(Kx)^T \left(\varepsilon^2 B^T H\left(V^{(6)}(x)\right) B\right) Kx \\
&\quad + \frac{1}{2}\left(k^{(2)}(x)\right)^T \left(\varepsilon^2 B^T H\left(V^{(4)}(x)\right) B\right) k^{(2)}(x) \\
&\quad + \frac{1}{2}\left(k^{(3)}(x)\right)^T \left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right) B\right) k^{(3)}(x) \\
&\quad + (Kx)^T \left(\varepsilon^2 B^T H\left(V^{(5)}(x)\right) B\right) k^{(2)}(x) + \varepsilon^2 (Kx)^T \left(B^T H\left(V^{(4)}(x)\right) B\right) k^{(3)}(x) \\
&\quad + (Kx)^T \left(\varepsilon^2 B^T H\left(V^{(3)}(x)\right) B\right) k^{(4)}(x) + (Kx)^T \left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right) B\right) k^{(5)}(x) \\
&\quad + \left(k^{(2)}(x)\right)^T \left(\varepsilon^2 B^T H\left(V^{(3)}(x)\right) B\right) k^{(3)}(x) \\
&\quad + \left(k^{(2)}(x)\right)^T \left(R + \varepsilon^2 B^T H\left(V^{(2)}(x)\right) B\right) k^{(4)}(x) + r^{(6)}(x) = 0
\end{aligned} \tag{80}
$$

$$
\begin{aligned}
&(Ax)^T \frac{\partial V^{(7)}(x)}{\partial x} + \left(f^{(2)}(x)\right)^T \frac{\partial V^{(6)}(x)}{\partial x} + \left(f^{(3)}(x)\right)^T \frac{\partial V^{(5)}(x)}{\partial x} + (BKx)^T \frac{\partial V^{(7)}(x)}{\partial x} \\
&\quad + \left(Bk^{(2)}(x)\right)^T \frac{\partial V^{(6)}(x)}{\partial x} + \left(Bk^{(3)}(x)\right)^T \frac{\partial V^{(5)}(x)}{\partial x} + \left(Bk^{(4)}(x)\right)^T \frac{\partial V^{(4)}(x)}{\partial x} \\
&\quad + \left(Bk^{(5)}(x)\right)^T \frac{\partial V^{(3)}(x)}{\partial x} + \left(Bk^{(6)}(x)\right)^T \frac{\partial V^{(2)}(x)}{\partial x} \\
&\quad + \frac{1}{2}(Kx)^T \left(\varepsilon^2 B^T H\left(V^{(7)}(x)\right) B\right) Kx + \frac{1}{2}\left(k^{(2)}(x)\right)^T \left(\varepsilon^2 B^T H\left(V^{(5)}(x)\right) B\right) k^{(2)}(x) \\
&\quad + \frac{1}{2}\left(k^{(3)}(x)\right)^T \left(\varepsilon^2 B^T H\left(V^{(3)}(x)\right) B\right) k^{(3)}(x) \\
&\quad + (Kx)^T \left(\varepsilon^2 B^T H\left(V^{(6)}(x)\right) B\right) k^{(2)}(x) + (Kx)^T \left(\varepsilon^2 B^T H\left(V^{(5)}(x)\right) B\right) k^{(3)}(x) \\
&\quad + (Kx)^T \left(\varepsilon^2 B^T H\left(V^{(4)}(x)\right) B\right) k^{(4)}(x) + (Kx)^T \left(\varepsilon^2 B^T H\left(V^{(3)}(x)\right) B\right) k^{(5)}(x) \\
&\quad + (Kx)^T\left(R + \varepsilon^2 B^T H\left(V^2(x)\right) B\right) k^{(6)}(x)
\end{aligned} \tag{81}
$$

$$+\left(k^{(2)}(x)\right)^T\left(\varepsilon^2B^TH\left(V^{(4)}(x)\right)B\right)k^{(3)}(x)+\left(k^{(2)}(x)\right)^T\left(\varepsilon^2B^TH\left(V^{(3)}(x)\right)B\right)k^{(4)}(x)$$

$$+\left(k^{(2)}(x)\right)^T\left(R+\varepsilon^2B^TH\left(V^2(x)\right)B\right)k^{(5)}(x)$$

$$+\left(k^{(3)}(x)\right)^T\left(R+\varepsilon^2B^TH\left(V^2(x)\right)B\right)k^{(4)}(x)+r^{(7)}(x)=0$$

Simplifying Eq. (79)-(81) through substituting Eq. (46), the quintic through the septic Hamiltonian equations become

$$\begin{aligned}&(Ax)^T\frac{\partial V^{(5)}(x)}{\partial x}+f^{(2)^T}(x)\frac{\partial V^{(4)}(x)}{\partial x}+f^{(3)}(x)\frac{\partial V^{(3)}(x)}{\partial x}+(BKx)^T\frac{\partial V^{(5)}(x)}{\partial x}\\&+\frac{1}{2}\varepsilon^2(Kx)^TB^TH\left(V^{(5)}(x)\right)B(Kx)-\frac{1}{2}\varepsilon^2\left(k^{(2)}(x)\right)^TB^TH\left(V^{(3)}(x)\right)Bk^{(2)}(x)\\&-\left(k^{(2)}(x)\right)^T\left(R+\varepsilon^2B^TH\left(V^{(2)}(x)\right)B\right)k^{(3)}(x)+r^{(5)}(x)=0\end{aligned}\tag{82}$$

$$\begin{aligned}&(Ax)^T\frac{\partial V^{(6)}(x)}{\partial x}+f^{(2)^T}(x)\frac{\partial V^{(5)}(x)}{\partial x}+f^{(3)^T}(x)\frac{\partial V^{(4)}(x)}{\partial x}+(BKx)^T\frac{\partial V^{(6)}(x)}{\partial x}\\&+\frac{1}{2}(Kx)^T\left(\varepsilon^2B^TH\left(V^{(6)}(x)\right)B\right)(Kx)\\&-\frac{1}{2}\left(k^{(2)}(x)\right)^T\left(\varepsilon^2B^TH\left(V^{(4)}(x)\right)B\right)k^{(2)}(x)\\&-\frac{1}{2}\left(k^{(3)}(x)\right)^T\left(R+\varepsilon^2B^TH\left(V^{(2)}(x)\right)B\right)k^{(3)}(x)\\&-\left(k^{(2)}(x)\right)^T\varepsilon^2B^TH\left(V^{(3)}(x)\right)Bk^{(3)}(x)\\&-\left(k^{(2)}(x)\right)^T\left(R+\varepsilon^2B^TH\left(V^{(2)}(x)\right)B\right)k^{(4)}(x)+r^{(6)}(x)=0\end{aligned}\tag{83}$$

$$\begin{aligned}&(Ax)^T\frac{\partial V^{(7)}(x)}{\partial x}+\left(f^{(2)}(x)\right)^T\frac{\partial V^{(6)}(x)}{\partial x}+\left(f^{(3)}(x)\right)^T\frac{\partial V^{(5)}(x)}{\partial x}+(BKx)^T\frac{\partial V^{(7)}(x)}{\partial x}\\&+\frac{1}{2}(Kx)^T\left(\varepsilon^2B^TH\left(V^{(7)}(x)\right)B\right)Kx-\frac{1}{2}\left(k^{(2)}(x)\right)^T\left(\varepsilon^2B^TH\left(V^{(5)}(x)\right)B\right)k^{(2)}(x)\\&-\frac{1}{2}\left(k^{(3)}(x)\right)^T\left(\varepsilon^2B^TH\left(V^{(3)}(x)\right)B\right)k^{(3)}(x)\\&-\left(k^{(3)}(x)\right)^T\left(\varepsilon^2B^TH\left(V^{(4)}(x)\right)B\right)k^{(2)}(x)\\&-\left(k^{(3)}(x)\right)^T\left(R+\varepsilon^2B^TH\left(V^{(2)}(x)\right)B\right)k^{(4)}(x)\end{aligned}\tag{84}$$

$$- \left(k^{(4)}(x)\right)^T \left(\varepsilon^2 B^T H \left(V^{(3)}(x)\right) B\right) k^{(2)}(x) - \left(k^{(5)}(x)\right)^T \left(R + \varepsilon^2 B^T H \left(V^{(2)}(x)\right) B\right) k^{(2)}(x) + r^{(7)}(x) = 0$$

Appendix B: Riccati Solution

Here we provide the optimal solutions of Riccati for both gain sets A and B, across different cases where $\varepsilon = 0.01$, $\varepsilon = 0.1$, and $\varepsilon = 0.2$. We start by providing the optimal solution for gain set A

$$P = \begin{bmatrix} 0.9500 & 0 & 0 & 0.4426 & 0 & 0 \\ 0 & 0.9606 & 0 & 0 & 0.4777 & 0 \\ 0 & 0 & 0.8128 & 0 & 0 & 0.4286 \\ 0.4426 & 0 & 0 & 0.1025 & 0 & 0 \\ 0 & 0.4777 & 0 & 0 & 0.0088 & 0 \\ 0 & 0 & 0.4286 & 0 & 0 & 0.6377 \end{bmatrix}, \text{ for } \varepsilon = 0.01$$

$$P = \begin{bmatrix} 0.9591 & 0 & 0 & 0.4463 & 0 & 0 \\ 0 & 0.9699 & 0 & 0 & 0.4817 & 0 \\ 0 & 0 & 0.8257 & 0 & 0 & 0.4343 \\ 0.4463 & 0 & 0 & 0.1043 & 0 & 0 \\ 0 & 0.4817 & 0 & 0 & 0.0108 & 0 \\ 0 & 0 & 0.4343 & 0 & 0 & 0.6407 \end{bmatrix}, \text{ for } \varepsilon = 0.1$$

$$P = \begin{bmatrix} 0.9875 & 0 & 0 & 0.4579 & 0 & 0 \\ 0 & 0.9987 & 0 & 0 & 0.4941 & 0 \\ 0 & 0 & 0.8667 & 0 & 0 & 0.4522 \\ 0.4579 & 0 & 0 & 0.1096 & 0 & 0 \\ 0 & 0.4941 & 0 & 0 & 0.0170 & 0 \\ 0 & 0 & 0.4521 & 0 & 0 & 0.6501 \end{bmatrix}, \text{ for } \varepsilon = 0.2$$

The Riccati solutions for gain set B are also as follows

$$P = \begin{bmatrix} 0.0160 & 0 & 0 & 0.0159 & 0 & 0 \\ 0 & 0.0208 & 0 & 0 & 0.0206 & 0 \\ 0 & 0 & 0.0080 & 0 & 0 & 0.0080 \\ 0.0159 & 0 & 0 & 0.0158 & 0 & 0 \\ 0 & 0.0206 & 0 & 0 & 0.0205 & 0 \\ 0 & 0 & 0.0080 & 0 & 0 & 0.0079 \end{bmatrix}, \text{ for } \varepsilon = 0.01$$

$$P = \begin{bmatrix} 0.0219 & 0 & 0 & 0.0217 & 0 & 0 \\ 0 & 0.0266 & 0 & 0 & 0.0262 & 0 \\ 0 & 0 & 0.0147 & 0 & 0 & 0.0145 \\ 0.0217 & 0 & 0 & 0.0215 & 0 & 0 \\ 0 & 0.0262 & 0 & 0 & 0.0261 & 0 \\ 0 & 0 & 0.0145 & 0 & 0 & 0.0144 \end{bmatrix}, \text{ for } \varepsilon = 0.1$$

$$P = \begin{bmatrix} 0.0473 & 0 & 0 & 0.0463 & 0 & 0 \\ 0 & 0.0507 & 0 & 0 & 0.0495 & 0 \\ 0 & 0 & 0.0450 & 0 & 0 & 0.0431 \\ 0.0463 & 0 & 0 & 0.0458 & 0 & 0 \\ 0 & 0.0495 & 0 & 0 & 0.0489 & 0 \\ 0 & 0 & 0.0431 & 0 & 0 & 0.0422 \end{bmatrix}, \text{for } \varepsilon = 0.2$$

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