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A novel asymptotic formulation for partial slip half-plane frictional contact problems

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ABSTRACT

A method of solution and the necessary calibrations are given to permit the steady-state extent of slip to be found in contacts properly described within a half-plane formulation using only two parameters: the contact law and the first-order descriptions of tractions arising at the contact edges. The approach takes the assumption of full stick and corrects for the slip regions using an array of glide dislocations. This is a very versatile approach and is particularly appropriate when studying fretting fatigue, as it permits the region in which cracks nucleate to be defined very simply, and in a form which is transportable from contact to contact, including laboratory tests. The approach has the additional benefit of giving a relatively straightforward expression for the density of dislocations, from which the slip displacement and shear traction within the stick region may readily be calculated. An example implementation is provided in the case of a Hertzian contact in the absence of changes in bulk tension, for which we demonstrate the veracity of the predictions by comparing to previous asymptotic approaches that build upon the traction solution under the assumption of full sliding, as well as the known exact solution.

1. Introduction

Fretting fatigue and crack nucleation arising more generally may be quantified only through experiment. An ambition we have is to describe the behaviour of contact edges simply and rigorously in a small number of parameters so that the results of a laboratory experiment may be carried over with confidence to different prototypes. Here, we restrict ourselves to the study of convex (or incomplete) contacts, the majority of which may be modelled well within the basis of a half-plane formulation [1]. We are particularly interested in problems for which a steady set of loads is present alongside an oscillatory set, since this commonly occurs in mechanical components such as the dovetail joint of a gas turbine blade [2,3]. In particular, we want to find the size of the reversing slip zone at the contact edges in the steady state. In general, the loads applied to the problem will be the normal force, P; possibly a moment, M, tending to rock the contact (with the notable exception in the case of a cylindrical (Hertzian) contact where a moment could not be resisted and would cause the contact simply to roll); a shear force, Q; and differential tensions lying in the surface of the bodies of magnitude σ. The general configuration is shown in Fig. 1.

We have already solved a number of frictional partial slip contact problems of this general kind using one of two approaches. The first approach – the 'corrective stick' method, say – was introduced in [4,5], where we take the rigid-body sliding solution as our starting point, and use a superposed corrective term in the intended stick region, a method which is a generalisation of the Jäger–Ciavarella principle [6,7]. As discussed in [8–10], the second approach – the 'corrective slip' method – considers the reverse problem in which we take the full stick solution and use an array of glide dislocations along the interface to introduce regions of slip in the appropriate regions of load space.

There have been two motivations for looking at asymptotic representations. One is, as stated earlier, a desire to match experiment with complex prototype at the contact edges. The second is due to limitations in the two methods for solving partial slip problems discussed above. Although the corrective stick formulation can be applied to a problem of any complexity up to a simultaneous variation of (P, Q, σ, M), that class of solution applies only when, at all points in the loading cycle, the contact edge slip zones are of the same sign, so is only able to describe problems for which changes in σ are moderate [5]. On the other hand, the corrective slip solution is limited by our inability to determine the change in locked-in shear traction in the presence of a moment (except near the contact edges [11]) and so applies to problems up to

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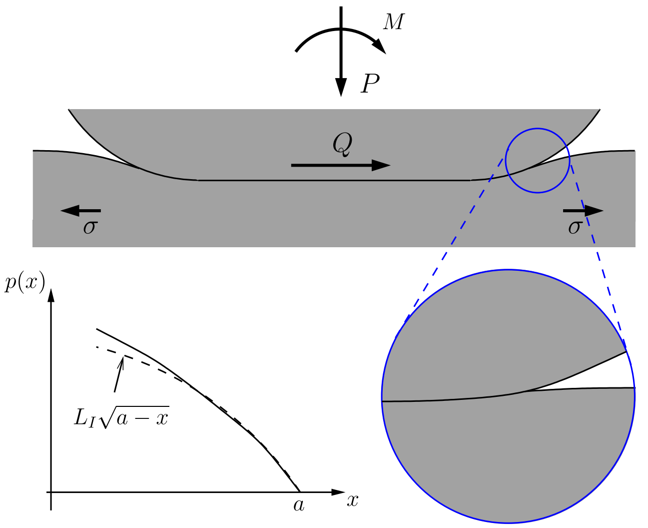

Fig. 1. The general contact problem in which a large, almost-flat indenter is pressed into an elastically-similar half-space. The applied loads are a normal load, P, a moment, M, a shear force Q and remote bulk tensions σ. As shown in the inset, we are interested in the asymptotic solution in the vicinity of one of the contact edges — here the right-hand edge is considered without loss of generality. In this region, the contact pressure is assumed to be well-approximated by a first-order asymptote, which is square-root bounded in character.

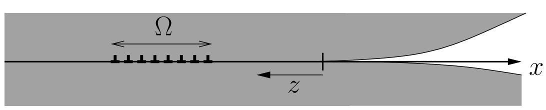

Fig. 2. An array of glide dislocations inserted along a subset Ω of the bond between two elastically-similar bodies. The coordinate $z > 0$ measures distance from the right-hand contact edge. The approach will centre around inserting such dislocations in the slip zones of the contact, in order to correct for the sliding traction there.

a simultaneous variation in (P, Q, σ) (notably including cases in which the slip zones are of opposite sign).

A partial remedy to these difficulties is to use an asymptotic form looking at one edge only (cf. the inset to Fig. 1). We have done this using the corrective stick approach, again using the sliding traction near-edge distribution and the generalised Jäger–Ciavarella principle to find the extent of the slip zone [12], but we have never attempted a partial slip solution using asymptotes where glide dislocations are used to introduce slip to a full stick problem, i.e. a corrective slip approach. Here we remedy this omission, and note that it permits further deductions to be made about local properties of the partial slip problem.

2. Asymptotic formulation in terms of dislocations

The starting point for the calculation of the slip zone size is to consider two elastically-similar half planes having a plane strain modulus E^* and bonded together over the half-line $z > 0$. Here z represents distance from the near-edge of the contact. Suppose that an array of glide dislocations is inserted in a subset Ω of the interface, as shown in Fig. 2. If the dislocation density is given by $B_x(z)$, the dislocations induce a shear traction along the interface, $q_d(z)$, given by [8]

$$q_d(z) = \frac{E^*}{2\pi\sqrt{z}} \fint_{\Omega} \frac{\sqrt{r}B_x(r)\,dr}{r-z}, \tag{1}$$

where the dash on the integral indicates that the integral should be interpreted in a principal value sense for $z \in \Omega$.

We now turn to the contact problem displayed in Fig. 1 and, for the purposes of the analysis, we shall assume that there is no applied

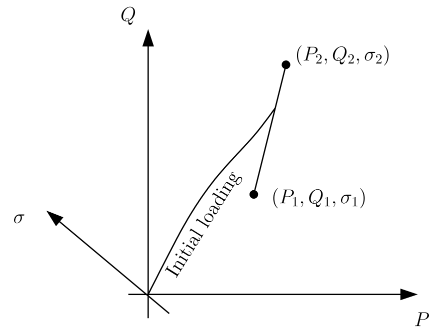

Fig. 3. The steady-state load path under consideration in the present study. In particular, we assume that the three loads, P, Q and σ, vary in proportion after the initial loading (i.e. that we move along a straight line in load space).

moment; that is, $M = 0$. We take (x, y) to be global Cartesian axes centred in the contact. We assume that the steady-state loading moves along a straight line between two points 1 and 2 a finite distance apart in (generally) a (P, Q, σ) load-space, i.e. so that the relation

$$\frac{\Delta P}{P-P_1} = \frac{\Delta Q}{Q-Q_1} = \frac{\Delta\sigma}{\sigma-\sigma_1} \tag{2}$$

holds, where

$$\Delta P = P_2 - P_1, \quad \Delta Q = Q_2 - Q_1, \quad \Delta\sigma = \sigma_2 - \sigma_1 \tag{3}$$

are the load changes and a subscript i indicates values at load state i. This is illustrated in Fig. 3. The contact half-width, a, is related to the normal load, P, by the contact law $a(P)$, which we take to be an increasing function. Moreover, we assume without loss of generality that $P_2 > P_1$ (i.e. $\Delta P > 0$) so that $a_2 = a(P_2) > a(P_1) = a_1$. We note that the simpler case in which $\Delta P = 0$ has previously been considered in detail, see [13] pp. 136–138.

The region of interest is in the vicinity of one of the contact edges, since it is in these regions that slip (and eventually fatigue) first occurs [14]. Here, we consider the right-hand contact edge without loss of generality. In this region, we assume that a first-order representation in an eigenfunction expansion of contact edge stress state applies (see [15] for a detailed consideration of the validity of this representation for different indenters), so that the contact pressure at end i of the loading trajectory is given by

$$p_i(x; P_i) = L_{I,i}\sqrt{a_i - x}, \tag{4}$$

where the coefficient $L_{I,i}$ will be discussed in more detail shortly in Section 2.1. Initially, the coefficient of friction is taken to be sufficient to inhibit all slip. Then, providing only that we exclude equality between the normal loads at each end of the cycle, we see that close to the contact edges the shear traction is simply a scaled version of the contact pressure [16], and so may write

$$q_i(x; P_i) = L_{II,i}\sqrt{a_i - x} \tag{5}$$

where, again, we will discuss the coefficient $L_{II,i}$ in Section 2.1.

Therefore, under the assumption of full stick, the change in shear tractions, $\Delta q(x)$ going from point 1 in the load cycle to point 2, is given by

$$\Delta q(x) = L_{II,2}\sqrt{a_2 - x} - L_{II,1}H(a_1 - x)\sqrt{a_1 - x} \quad \text{for} \quad x < a_2, \tag{6}$$

where $H(x)$ is the Heaviside function. So, relaxing the assumption of full stick, if the actual shear traction distribution at point 1 of the load cycle is $q_1(x)$, that arising at point 2 is given by

$$q_2(x) = q_1(x) + L_{II,2}\sqrt{a_2 - x} - L_{II,1}H(a_1 - x)\sqrt{a_1 - x} + \frac{E^*}{2\pi\sqrt{a_2 - x}} \fint_{a_2}^{b} \frac{\sqrt{a_2 - t}\,B_x^{1\to 2}(a_2 - t)\,dt}{t - x}, \tag{7}$$

for $x < a_2$, where we have inserted an array of dislocations in the slip zone, (b, a_2) and added the resulting traction by setting $z = a_2 - x$, $r = a_2 - t$ in (1). Moreover, note that we have added a superscript to the dislocation density to denote the direction of loading. In the slip zone, we must have

$$q_2(x) = \mu p_2(x) = \mu L_{I,2}\sqrt{a_2 - x} \quad \text{for} \quad b < x < a_2, \tag{8}$$

where μ is the coefficient of friction.

The contact is now unloaded from state 2 and returned to state 1, so that the change in shear traction, under full stick conditions, is given by

$$\Delta q(x) = -(L_{II,2}\sqrt{a_2 - x} - L_{II,1}H(a_1 - x)\sqrt{a_1 - x}) \quad \text{for} \quad x < a_2, \tag{9}$$

and hence, in the presence of the dislocation array, the actual shear traction distribution is

$$q_1(x) = q_2(x) - (L_{II,2}\sqrt{a_2 - x} - L_{II,1}H(a_1 - x)\sqrt{a_1 - x}) + \frac{E^*}{2\pi\sqrt{a_2 - x}} \fint_{a_2}^{b} \frac{\sqrt{a_2 - t}\, B_x^{2\to 1}(a_2 - t)\, dt}{t - x}, \tag{10}$$

for $x < a_2$, where a new superscript is added to the dislocation density to indicate the direction of the load path. We again write down a statement establishing the size of the slip zone at this point, i.e.

$$q_1(x) = -\mu p_1(x) = -\mu L_{I,1} H(a_1 - x)\sqrt{a_1 - x} \quad \text{for} \quad b < x < a_2. \tag{11}$$

It should be noted that, in Eqs. (7), (8), (10) and (11), the limits of integration and the interval over which the slip conditions are imposed are the same, since, in steady state, the stick-slip boundary at the extremes of loading (i.e. points 1 and 2 in Fig. 3) must be the same, as part of the requirement that material must be conserved. The second aspect of ensuring that material is preserved is to ensure that all the dislocations inserted during loading are annihilated during unloading, so that $B_x^{1\to 2}(x) = -B_x^{2\to 1}(x)$ for $b \leqslant x \leqslant a_2$.

These observations enable us to form a single equation by combining (7), (8), (10) and (11), which is

$$(\mu L_{I,1} + L_{II,1})H(a_1 - x)\sqrt{a_1 - x} + (\mu L_{I,2} - L_{II,2})\sqrt{a_2 - x} = \frac{E^*}{2\pi\sqrt{a_2 - x}} \fint_{a_2}^{b} \frac{\sqrt{a_2 - t}\, B_x^{1\to 2}(a_2 - t)\, dt}{t - x} \quad \text{for} \quad b < x < a_2. \tag{12}$$

This is a Cauchy singular integral equation along an open contour and may be inverted using standard techniques, see, for example, [17]. Since both the asymptotic form of the shear traction assumed in (5) is bounded in the slip zone and we require the final shear traction to also be bounded there, we seek a solution to the singular integral Eq. (12) that is bounded at both ends $x = a_2$ and $x = b$. The resulting dislocation density is given by

$$B_x^{1\to 2}(a_2 - x) = \frac{2\sqrt{x - b}}{\pi E^*}\left[(\mu L_{I,1} + L_{II,1}) \fint_b^{a_1} \sqrt{\frac{a_1 - t}{t - b}}\frac{dt}{t - x} + (\mu L_{I,2} - L_{II,2}) \fint_b^{a_2} \sqrt{\frac{a_2 - t}{t - b}}\frac{dt}{t - x}\right]. \tag{13}$$

Now, using the fact that

$$\int_\beta^1 \sqrt{\frac{1 - t}{t - \beta}}\frac{dt}{t - x} = \begin{cases} -\pi & \text{for} \quad \beta < x < 1, \\ -\pi\left(1 - \sqrt{\dfrac{x - 1}{x - \beta}}\right) & \text{for} \quad x > 1, \end{cases} \tag{14}$$

for $\beta < 1$, we may evaluate (13) explicitly, finding

$$B_x^{1\to 2}(a_2 - x) = -\frac{2\sqrt{x - b}}{E^*}\left(\mu(L_{I,1} + L_{I,2}) + L_{II,1} - L_{II,2}\right) + \frac{2(\mu L_{I,1} + L_{II,1})}{E^*} H(x - a_1)\sqrt{x - a_1} \tag{15}$$

for $b < x < a_2$. Additionally, we also require the consistency condition

$$0 = (\mu L_{I,1} + L_{II,1}) \int_b^{a_1} \sqrt{\frac{a_1 - t}{t - b}}\, dt + (\mu L_{I,2} - L_{II,2}) \int_b^{a_2} \sqrt{\frac{a_2 - t}{t - b}}\, dt \tag{16}$$

to hold. Solving this allows us to find the size of the slip zone, b, in terms of the contact half-widths a_i and the coefficients $L_{I,i}$, $L_{II,i}$ at load states $i = 1$ and $i = 2$, viz.:

$$b = \frac{(\mu L_{I,2} - L_{II,2})\, a_2 + (\mu L_{I,1} + L_{II,1})\, a_1}{(\mu L_{I,2} - L_{II,2}) + (\mu L_{I,1} + L_{II,1})}. \tag{17}$$

2.1. Calibrations for contact-edge multipliers

It remains to give the coefficients $L_{I,i}$ and $L_{II,i}$ in terms of the inputs to the problem. Recall that the system moves between load states (P_1, Q_1, σ_1) and (P_2, Q_2, σ_2) along a straight line so that there exists constants λ and η such that

$$\lambda = \frac{\Delta Q}{\Delta P} = \frac{Q - Q_1}{P - P_1}, \quad \eta = \frac{\Delta\sigma}{\Delta P} = \frac{\sigma - \sigma_1}{P - P_1}. \tag{18}$$

Now, the multipliers on the normal solution defining the contact pressure, $L_{I,i}$, as defined in (4) may be related to the incremental contact law $a(P)$ by,

$$L_I = \frac{1}{\pi}\sqrt{\frac{2}{a}}\frac{dP}{da}, \tag{19}$$

see, for example, [13] p. 129. Similarly, under the assumptions of full stick, we have that

$$L_{II} = \frac{1}{\pi}\sqrt{\frac{2}{a}}\frac{dQ}{da} + \frac{1}{4}\sqrt{2a}\frac{d\sigma}{da}, \tag{20}$$

(this is readily seen by considering the Mossakovskii–Barber solution in, for example, [16]). Hence, by the assumption of proportional loading, we must have

$$L_{II} = \frac{1}{\pi}\sqrt{\frac{2}{a}}\left(\lambda + \frac{\pi\eta a}{4}\right)\frac{dP}{da} = \left(\lambda + \frac{\pi\eta a}{4}\right)L_I. \tag{21}$$

Substituting (19) and (21) into the expression for the asymptotic size of the slip zone (17), we find that

$$b = \frac{\left(1 - \dfrac{\lambda}{\mu} - \dfrac{\pi a_2}{4}\dfrac{\eta}{\mu}\right)L_{I,2}a_2 + \left(1 + \dfrac{\lambda}{\mu} + \dfrac{\pi a_1}{4}\dfrac{\eta}{\mu}\right)L_{I,1}a_1}{\left(1 - \dfrac{\lambda}{\mu} - \dfrac{\pi a_2}{4}\dfrac{\eta}{\mu}\right)L_{I,2} + \left(1 + \dfrac{\lambda}{\mu} + \dfrac{\pi a_1}{4}\dfrac{\eta}{\mu}\right)L_{I,1}}. \tag{22}$$

where $L_{I,i}$ may be calculated from (19).

2.2. Necessary conditions for the asymptotic theory to be valid

The procedure of Section 2 relies on the applicability of the asymptotes (4) and (5), which are equivalent to assuming that the change in contact size from load state 1 to load state 2 is much smaller than the contact half-width at (without loss of generality) load state 1 and, moreover, that the size of the slip zone is much smaller the contact half-width in load state 1.

For the first of these, a straightforward application of Taylor's theorem states that there exists a $\xi \in (P_1, P_2)$ such that

$$a_2 = a(P_2) = a(P_1 + \Delta P) = a_1 + \Delta P a'(\xi), \tag{23}$$

where a prime indicates differentiation with respect to argument. Hence, provided that

$$\frac{\Delta P a'(P)}{a_1} \ll 1 \tag{24}$$

for all $P_1 < P < P_2$, we are guaranteed to have that $(a_2 - a_1)/a_1 \ll 1$, as required.

The second condition on the size of the slip zone is met provided that

$$\left|\frac{b - a_1}{a_1}\right| = \frac{a_2 - a_1}{a_1}\left|\left(1 + \frac{\left(1 + \dfrac{\lambda}{\mu} + \dfrac{\pi a_1}{4}\dfrac{\eta}{\mu}\right)L_{I,1}}{\left(1 - \dfrac{\lambda}{\mu} - \dfrac{\pi a_2}{4}\dfrac{\eta}{\mu}\right)L_{I,2}}\right)^{-1}\right| \ll 1. \tag{25}$$

Notably, since we already require that (24) holds, we are allowed significant scope in the size of the term in brackets in (25).

While cumbersome, the condition (25) is exact. A simpler expression may be found if we make the assumption that changes in the multiplier L_I are small, that is, following a similar argument to the contact law above,

$$\frac{\Delta P L_I'(P)}{L_{I,1}} \ll 1 \tag{26}$$

for all $P_1 < P < P_2$. Upon combining this with (24) and (25), we find that a necessary condition for the slip zone to be significantly smaller than the size of the contact is given by

$$\frac{\Delta P a'(P)}{a_1}\left|1 - \frac{\lambda}{\mu} - \frac{\pi\eta a_1}{4\mu}\right| \ll 1 \tag{27}$$

for all $P_1 < P < P_2$. Moreover, in this case, the size of the slip zone is then given by

$$\frac{b}{a_1} \approx 1 - \frac{\Delta P a'(P_1)}{2a_1}\left(\frac{\lambda}{\mu} + \frac{\pi a_1}{4}\frac{\eta}{\mu} - 1\right), \tag{28}$$

It is worth noting that (28) is independent of the geometry, aside from through the contact law $a(P)$, but we stress that we require (24),(26) and (27) to hold for this approximation to be valid.

3. Example application to a Hertzian problem

In some problems, it is envisaged that the procedure being developed will be applied to geometries where the calibration for the contact edge multipliers will have been found numerically, and where the contact law is also not known in closed form. In such cases, we will need to extract an approximation of $L_{I,i}$ in (22) or of $a'(P_1)$ in (28) from the numerical implementation.

However, there are several geometries for which it is possible to find explicit solutions (see [18–21] for some examples) and we will illustrate the veracity of our asymptotic approach for one such example, namely the contact of a cylinder, also known as a Hertzian contact. The contact law for a Hertzian problem is given by

$$a^2 = \frac{4PR}{\pi E^*}, \tag{29}$$

see, for example, [1]. We will further simplify the loading to one moving between load states specified in (P, Q) space, so that $\eta = 0$. Thus, we consider the size of the slip zone for a case where the normal load differs between the ends of the loading path, (P_2, Q_2) and (P_1, Q_1),with $P_2 > P_1$ so that the change of shear tractions under conditions of full stick is bounded, and we will do this in three ways:

(a) from the full solution for the contact geometry [2];
(b) using an asymptotic approach based on superposition of the full sliding solution — i.e. a corrective stick approach [12];
(c) using the dislocations-based method developed in Section 2 — i.e. a corrective slip approach.

In each case, we shall make the assumption that the change in size of the contact region and the size of the slip zone are much smaller than the contact half-width, as discussed in Section 2.2. For a Hertzian geometry, these conditions simplify somewhat, as we shall now show.

First, it is straightforward to see from (29) that $a'(P) = a/2P$, so that the condition for the change in contact size to be much smaller than the magnitude of the contact, (24), reduces to

$$\frac{\Delta P}{P_1} \ll 1 \tag{30}$$

for a Hertzian problem, that is, the change in load is much smaller than the magnitude of the load (importantly, we are *not* assuming that ΔP is small).

Second, by combining (19) and (29), the calibration for the contact pressure multiplier applied to a Hertzian geometry, is given by

$$L_I = \frac{E^*}{R}\sqrt{\frac{a}{2}} = \frac{2P}{\pi}\sqrt{\frac{2}{a^3}}, \tag{31}$$

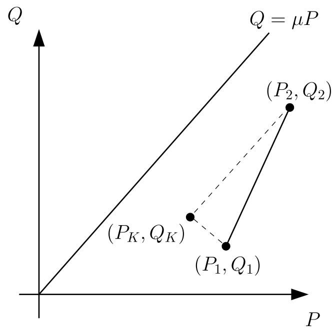

Fig. 4. The steady-state load path under consideration for the Hertzian example. The size of the permanent stick zone is given by $a(P_K)$, where (P_K, Q_K) is the intersection point between lines parallel to $\mp\mu P$ drawn through the load states 1 and 2, respectively.

so that, after some straightforward calculation, we see that $L_I'(P) = L_I/4P$. Thus, since we already require $\Delta P/P_1 \ll 1$, the condition (26) is automatically satisfied for the Hertzian problem. Hence, the conditions for the asymptotic solution to be valid in the Hertzian regime are simply

$$\frac{\Delta P}{P_1} \ll 1, \quad \frac{\Delta P}{P_1}\left|1 - \frac{\lambda}{\mu}\right| \ll 1. \tag{32}$$

In each of the following subsections, we shall therefore assume these conditions are satisfied.

3.1. (a) Solution derived from consideration of full contact

In [2] it is shown that, for a general steady-state loading 'loop' drawn in (P, Q)-space that is just enclosed by two tangential lines with gradients $\pm\mu$ and whose intersection, in turn, defines a point with abscissa P_K, a Hertzian contact has a stick zone of half width $b = a(P_K)$. So, this point is defined by

$$P_K = \frac{P_1 + P_2}{2} - \frac{Q_2 - Q_1}{2\mu}. \tag{33}$$

We depict the load path and the point (P_K, Q_K) in Fig. 4. Now, substituting (33) into (29), we see that

$$b^2 = \frac{2R}{\pi E^*}\left(P_1 + P_2 - \frac{Q_2 - Q_1}{\mu}\right), \tag{34}$$

so that, using the approximations (32),we deduce

$$b^2 = a_1^2\left(1 + \frac{\Delta P}{2\mu P_1}(\mu - \lambda)\right). \tag{35}$$

Hence, as $\Delta P/P_1 \to 0$, we find that

$$\frac{b}{a_1} \approx 1 - \left(\frac{\lambda}{\mu} - 1\right)\frac{1}{4}\frac{\Delta P}{P_1}. \tag{36}$$

3.2. (b) Asymptotic solution based on corrective stick

The asymptotic method based on full sliding to be used here is described in [12]. In that paper, the limit is taken of making the change in contact size become infinitesimal, so that the change in shear traction becomes the difference between two slightly different bounded solutions, and is hence singular. So, in addition to the contact pressure multipliers $L_{I,i}$ defined previously in (4), we need also to record the implied singular change in shear tractions, and it is straightforward, in this context, to introduce the effect of differential bulk tension, too, for completeness. The singular contact-edge shear traction distribution is given by

$$q(x) = \frac{K_{II}}{\sqrt{a - x}} \quad \text{as} \quad a - x \to 0, \tag{37}$$

where the calibration K_{II} is given by [13] p. 135,

$$K_{II} = \frac{Q}{\pi\sqrt{2a}} + \frac{\sigma}{4}\sqrt{\frac{a}{2}}. \tag{38}$$

and hence

$$\frac{\Delta K_{II}}{\Delta P} = \frac{\lambda}{\pi\sqrt{2a}} + \frac{\eta}{4}\sqrt{\frac{a}{2}}. \tag{39}$$

In [12], the asymptotic prediction for the size of the steady-state slip zone at point 1 of the load cycle is found to be

$$b = a_1 - d_1, \quad \text{where} \quad d_1 = \frac{\Delta K_{II}}{\mu L_I} - \frac{\Delta a}{2}. \tag{40}$$

Hence, utilising (31) and (39), we see that

$$b = a_1 - \frac{\Delta P R}{\pi\mu E^* a_1}\left(\lambda + \frac{\pi\eta a_1}{4}\right) + \frac{a_2 - a_1}{2}. \tag{41}$$

Thus, recalling the assumption that $\Delta P/P_1 \ll 1$ and the expansion for a_2 given by (23), we can expand the contact law (29) to show that

$$a_2 = a(P_1 + \Delta P) \approx a_1\left(1 + \frac{2R\Delta P}{\pi E^* a_1^2}\right). \tag{42}$$

Substituting this into (41), we find

$$b \approx \frac{a_1}{2}\left(2 + \frac{2R\Delta P}{\pi E^* a_1^2}\right) - \frac{\Delta P R}{\pi a_1 \mu E^*}\left(\lambda + \frac{\pi\eta a_1}{4}\right) \tag{43}$$

so that, recalling (29),

$$\frac{b}{a_1} \approx 1 - \left(\frac{\lambda}{\mu} + \frac{\eta}{\mu}\frac{\pi a_1}{4} - 1\right)\frac{\Delta P}{4P_1}, \tag{44}$$

which agrees with the result derived from the full solution (36) when we set $\eta = 0$.

3.3. (c) Asymptotic solution based on corrective slip

Following the above discussion that, for the Hertzian example, the condition (26) is automatically satisfied when (24) holds, we may write down the size of the slip zone by using (28) with $\eta = 0$; we again retrieve

$$\frac{b}{a_1} \approx 1 - \left(\frac{\lambda}{\mu} - 1\right)\frac{\Delta P}{4P_1}, \tag{45}$$

consistent with both (36) and (44).

3.4. Shear tractions

It is clear that we correctly retrieve the first term in a small-$\Delta P/P_1$ expansion of the exact solution for the size of the slip zone using each of the asymptotic methods: the corrective slip method presented in the current paper and the corrective stick method developed by [12]. To demonstrate an advantage of the current method, we conclude by comparing the asymptotic solution for the shear tractions to the known exact solution.

In order to do this, it is easiest to consider the difference between the shear tractions at each end of the load cycle, namely

$$q_{\text{diff}}(x) = q_2(x) - q_1(x). \tag{46}$$

By utilising the difference, we may neglect any tractions developed in the initial loading path (cf. Fig. 3). By a careful application of the Jäger-Ciavarella principle, the exact solution gives

$$q_{\text{diff}}(x) = \begin{cases} \mu(p(x,P_2) + p(x,P_1) - 2p(x,P_k)) & \text{for} \quad 0 < x < b \\ \mu(p(x,P_2) + p(x,P_1)) & \text{for} \quad b < x < a_1, \\ \mu p(x,P_2) & \text{for} \quad a_1 < x < a_2 \end{cases} \tag{47}$$

where P_K is given by (33) and the Hertzian contact pressure is given by

$$p(x,P) = \frac{2P}{\pi a^2}\sqrt{a^2 - x^2}. \tag{48}$$

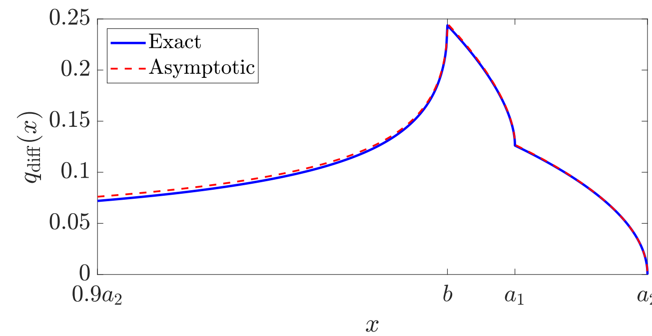

Fig. 5. The exact solution (47) (solid blue) and asymptotic approximation (49) (dashed red) solutions for $q_{\text{diff}}(x) = q_2(x) - q_1(x)$ for a Hertzian contact with illustrative values of $R = E^* = 1$, $\mu = 0.1$, $\lambda = 0.2$, $P_1 = 100$, $\Delta P = 5$.

Here we have exploited symmetry to consider $x > 0$ only.

By (7), the asymptotic solution is given by

$$q_{\text{diff}}(x) = L_{II,2}\sqrt{a_2 - x} - L_{II,1}H(a_1 - x)\sqrt{a_1 - x} + \frac{E^*}{2\pi\sqrt{a_2 - x}} \fint_{a_2}^{b} \frac{\sqrt{a_2 - t}\,B_x^{1\to 2}(a_2 - t)\,dt}{t - x}, \tag{49}$$

where the dislocation density is found from (15), (21), (29), (31) and (35).

We plot the exact solution (47) alongside the asymptotic approximation (49) in Fig. 5, where we have chosen some representative values of the parameters for illustrative purposes: $R = E^* = 1$, $\mu = 0.1$, $\lambda = 0.2$, $P_1 = 100$, $\Delta P = 5$. Notably, $\Delta P/P_1 = \Delta P/P_1|1 - \lambda/\mu| = 0.05 \ll 1$ so that the necessary conditions for the asymptotic solution to be valid, (32), hold. It is clear that the asymptotic solution does an excellent job of capturing the local shear tractions: the error in the approximation is only ≈ 0.004 at 10% of the distance in from the contact edge at $x = a_2$. In fact, the error for this example is only ≈ 0.01 at the centre of the contact, $x = 0$, although we stress that the values taken for this example are illustrative rather than physical. Nevertheless, we see very encouraging evidence that the asymptotic model presented herein provides an excellent description of the shear traction distribution in the vicinity of the contact edge.

4. Conclusion

A new solution for solving partial slip problems within a half-plane formulation using asymptotic representations of the near-edge solution, has been found under the assumption that the changes in the loads are directly proportional. In contrast to an earlier asymptotic method (cf. [12]), the approach starts with the assumption that the change of state occurs under full stick, and slip is introduced by distributing glide dislocations in the slip zones. The approach may be applied whether the shear tractions originate from the application of a shear force, or the generation of differential tension.

The method provides a straightforward approximation for the size of the slip zone

$$b = \frac{\left(1 - \frac{\lambda}{\mu} - \frac{\pi a_2}{4}\frac{\eta}{\mu}\right)L_{I,2}a_2 + \left(1 + \frac{\lambda}{\mu} + \frac{\pi a_1}{4}\frac{\eta}{\mu}\right)L_{I,1}a_1}{\left(1 - \frac{\lambda}{\mu} - \frac{\pi a_2}{4}\frac{\eta}{\mu}\right)L_{I,2} + \left(1 + \frac{\lambda}{\mu} + \frac{\pi a_1}{4}\frac{\eta}{\mu}\right)L_{I,1}}. \tag{50}$$

where μ is the coefficient of friction; a_i is the size of the contact at load state i; $\lambda = \Delta P/\Delta Q$ and $\eta = \Delta\sigma/\Delta P$ where ΔP, ΔQ and $\Delta\sigma$ are the changes in normal load, shear force and bulk tension respectively; and $L_{I,i}$ is the coefficient of the square-root term in the local contact pressure expansion at load state i. This expression further simplifies when the change in L_I is small compared to its absolute value, viz.:

$$\frac{b}{a_1} \approx 1 - \frac{\Delta P a'(P_1)}{2a_1}\left(\frac{\lambda}{\mu} + \frac{\pi a_1}{4}\frac{\eta}{\mu} - 1\right). \tag{51}$$

Eqs. (50) and (51) may be readily used in laboratory simulations of contact problems. In particular, we demonstrated explicitly the veracity of our predictions in the simple example of a Hertzian contact with no changes in bulk tension, for which we were able to show that the slip zone size was the same as that found from the full-sliding asymptotic approach [12], as well as the known exact solution [2].

For more general problems, given the ease with which the dislocation density, and hence the tractions themselves, may be found (cf. Eq. (15)), it may be that the present methodology has advantages over the sliding traction approach. We demonstrated the accuracy of our asymptotic method for predicting the shear traction distributions for a Hertzian problem: there is very strong agreement with the exact solution, with errors of less than 0.01% over the contact for an illustrative set of material and loading parameters. This provides a strong indicator of both the veracity and usefulness of the approach in future analyses and experimental investigations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] J.R. Barber, Elasticity, Springer, 2002.
[2] J.R. Barber, M. Davies, D.A. Hills, Frictional elastic contact with periodic loading, Int. J. Solids Struct. 48 (13) (2011) 2041–2047.
[3] F. Abbasi, G.H. Majzoobi, J. Mendiguren, A review of the effects of cyclic contact loading on fretting fatigue behavior, Adv. Mech. Eng. 12 (9) (2020) 1–17.
[4] H. Andresen, D.A. Hills, J.R. Barber, J. Vázquez, Frictional half-plane contact problems subject to alternating normal and shear loads and tension in the steady state, Int. J. Solids Struct. 168 (2019) 166–171.
[5] H. Andresen, D.A. Hills, J.R. Barber, J. Vázquez, Steady state cyclic behaviour of a half-plane contact in partial slip subject to varying normal load, moment, shear load, and moderate differential bulk tension, Int. J. Solids Struct. 182 (2020) 156–161.
[6] J. Jäger, A new principle in contact mechanics, ASME J. Tribol. 120 (1998) 677–684.
[7] M. Ciavarella, The generalized Cattaneo partial slip plane contact problem. I Theory, Int. J. Soilds Struct. 35 (18) (1998) 2349–2362.
[8] M.R. Moore, R. Ramesh, D.A. Hills, J.R. Barber, Half-plane partial slip contact problems with a constant normal load subject to a shear force and differential bulk tension, J. Mech. Phys. Solids 118 (2018) 245–253.
[9] H. Andresen, D.A. Hills, M.R. Moore, The steady state partial slip problem for half plane contacts subject to a constant normal load using glide dislocations, Eur. J. Mech. A Solids 79 (2020) 103868.
[10] D.A. Hills, M.R. Moore, Partial slip problems in contact mechanics, in: D. Dini, T. Liskiewicz (Eds.), Fretting Wear and Fretting Fatigue, Elsevier, 2022, in press.
[11] H. Andresen, D.A. Hills, M.R. Moore, Representation of incomplete contact problems by half-planes, Eur. J. Mech. A / Solids 85 (2021) 104138.
[12] H. Andresen, R.M.N. Fleury, M.R. Moore, D.A. Hills, Explicit and asymptotic solutions for frictional incomplete half-plane contacts subject to general oscillatory loading in the steady-state, J. Mech. Phys. Solids 146 (2021) 104214.
[13] D.A. Hills, H.N. Andresen, Mechanics of Fretting and Fretting Fatigue, Springer, 2021.
[14] O. Vingsbo, S. Söderberg, On fretting maps, Wear 126 (2) (1988) 131–147.
[15] M.R. Moore, D.A. Hills, Analysing the accuracy of asymptotic approximations in incomplete contact problems, Int. J. Solids Struct. (2022) 111557.
[16] D.A. Hills, M. Davies, J.R. Barber, An incremental formulation for half-plane contact problems subject to varying normal load, shear, and tension, J. Strain Anal. Eng. Des. 46 (6) (2011) 436–443.
[17] F.D. Gakhov, Boundary Value Problems, Pergamon, 1966.
[18] G. Schubert, Zur Frage der Druckverteilung unter elastisch gelagerten Tragwerken, Ing. Arch. 13 (3) (1942) 132–147.
[19] H. Andresen, D.A. Hills, J. Vázquez, Closed-form solutions for tilted three-part piecewise-quadratic half-plane contacts, Int. J. Mech. Sci. 150 (2019) 127–134.
[20] H.D. Conway, The pressure distribution between two elastic bodies in contact, Z. Angew. Math. Phys. 7 (5) (1956) 460–465.
[21] J. Jäger, New analytical and numerical results for two-dimensional contact profiles, Int. J. Solids Struct. 39 (4) (2002) 959–972.