CATEGORIFICATION AND THE QUANTUM GRASSMANNIAN

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Abstract. In [13] we gave an (additive) categorification of Grassmannian cluster algebras, using the category $\mathrm{CM}(A)$ of Cohen-Macaulay modules for a certain Gorenstein order A. In this paper, using a cluster tilting object in the same category $\mathrm{CM}(A)$, we construct a compatible pair (B, L), which is the data needed to define a quantum cluster algebra. We show that when (B, L) is defined from a cluster tilting object with rank 1 summands, this quantum cluster algebra is (generically) isomorphic to the corresponding quantum Grassmannian.

1. Introduction

Cluster algebras were created by Fomin and Zelevinsky [5]. They are subalgebras of a freely-generated (or purely transcendental) field, generated by cluster variables obtained by mutations from an initial seed. Fomin and Zelevinksy also showed in [5] that the homogeneous coordinate rings $\mathbb{C}[\mathrm{Gr}_{2,n}]$ of the simplest Grassmannians are cluster algebras (of finite type A_{n-3}). Scott [19] extended this result to all Grassmannian coordinate rings $\mathbb{C}[\mathrm{Gr}_{m,n}]$, although most are no longer of finite cluster type.

In [13], we gave an additive categorification of this cluster structure using the category $\mathrm{CM}(A)$ of Cohen-Macaulay modules for a certain algebra A, or equivalently, of equivariant Cohen-Macaulay modules for the plane curve singularity $x^m = y^{n-m}$. More precisely, in this categorification, clusters correspond to reachable cluster tilting objects and cluster variables to reachable indecomposable rigid objects in this category. This work built on work of Geiss, Leclerc and Schröer [7] on categorification of the cluster structure on the affine coordinate rings of the open cells in Grassmannians and, much more generally, of unipotent subgroups of semisimple Lie groups.

Quantum cluster algebras were introduced by Berenstein and Zelevinsky [2] as quantum analogues of the cluster algebras of Fomin and Zelevinsky. They are subalgebras of a 'quasi-free' skew field $\mathcal{F}$, generated by cluster variables that are obtained by mutations from an initial quantum seed. (By "quasi-free" here we mean freely-generated by a finite set of quasi-commuting variables.) Geiss, Leclerc and Schröer partially generalised their results in [7] to the quantum setting in [8], including the case of open cells in Grassmannians. By the delicate process of homogenising this result, Grabowski and Launois [11] proved that there is a

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quantum cluster algebra structure on the full Grassmannian quantum coordinate ring $\mathbb{C}_q[\mathrm{Gr}_{m,n}]$, or at least on a small modification of it (see Theorem 1.3 below for more detail).

A crucial aspect of quantising a cluster algebra is that classical clusters become quasi-free generating sets for $\mathcal{F}$. The quasi-commutation rule between these generators becomes an additional piece of data in the quantum seed, namely a skew-symmetric matrix L that must be 'compatible' with the exchange matrix B. Furthermore there is a corresponding mutation rule for L.

Our first main result in this paper is to show that this additional data in the quantum seed is a natural invariant of the corresponding cluster tilting object in the category $\mathrm{CM}(A)$. Our approach is similar to [8], but requires a new idea, because the Hom-spaces in $\mathrm{CM}(A)$ are infinite dimensional. More precisely, for any $M, N \in \mathrm{CM}(A)$, we consider the embedding

$$\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_Z(e_v M, e_v N),$$

where Z is the centre of A and e_v is a primitive idempotent in A corresponding to a vertex v of the quiver presenting A. Note that all choices of v are equivalent under a cyclic symmetry of A, so this choice really just breaks that symmetry. The cokernel $K_v(M,N)$ of this embedding is in fact finite dimensional and the dimension

$$\kappa(M,N) = \dim K_v(M,N)$$

is the new invariant that we use to categorify the quantum seed. More precisely, to a cluster tilting object $T = \oplus_i T_i$, we associate the standard exchange matrix $B(T)$ corresponding to the Gabriel quiver of $\mathrm{End}(T)$ and a matrix $L(T) = (\lambda_{ij})$ with

$$\lambda_{ij} = \kappa(T_j, T_i) - \kappa(T_i, T_j).$$

Using μ_k to denote mutation of quantum cluster seeds as in [2] and mutation of cluster tilting objects as [3], we prove:

Theorem 1.1 (Theorem 6.3)**.** *The two matrices B and L associated to T are compatible. Furthermore, mutation of cluster tilting objects is consistent with mutation of seed data, i.e. the pair associated to the mutated object $\mu_k(T)$ is the mutated pair $\mu_k(B, L)$.*

This means that our original category $\mathrm{CM}(A)$ knows the quasi-commutation rules for quantum clusters, by direct computation from cluster tilting objects. Note that we do not need to pass to some related category of graded objects with finite dimensional Hom-spaces, as one might have expected.

Recall from [13] that $\mathrm{CM}(A)$ contains a family of (rigid) rank one modules M_I, for each m-subset I of $\{1, \ldots, n\}$, whose cluster characters are the classical minors D_I in $\mathbb{C}[\mathrm{Gr}_{m,n}]$. Furthermore, $\mathrm{Ext}^1(M_I, M_J) = 0$ precisely when the labels I, J are 'non-crossing'. This is precisely the condition for the quantum minors Δ_I, Δ_J

in $\mathbb{C}_q[\mathrm{Gr}_{m,n}]$ to quasi-commute, as proved by Leclerc-Zelevinsky [16], who also determined the quasi-commutation rule

$$\Delta_I \Delta_J = q^{c(I,J)} \Delta_J \Delta_I.$$

Our second main result is to show that, as one might hope,

Theorem 1.2 (Theorem 6.5). *When Δ_I and Δ_J are quasi-commuting quantum minors,*

$$c(I,J) = \kappa(M_J, M_I) - \kappa(M_I, M_J).$$

As a consequence of this, and some combinatorial results of [17], there is a quantum cluster algebra $C_q(\mathrm{Gr}_{m,n})$, containing cluster variables X_I for all m-subsets I of $\{1,\ldots,n\}$, whose quasi-commutation rules match those of the quantum minors Δ_I. By showing further that the appropriate quantum exchange relations are precisely quantum Plücker relations (Proposition 8.3) we can deduce that $C_q(\mathrm{Gr}_{m,n})$ contains the quantum coordinate ring $\mathbb{C}_q[\mathrm{Gr}_{m,n}]$ in a way that identifies X_I with Δ_I, for all I.

One would like to believe that these two algebras coincide in general, but this is known only when the cluster structure has finite type [10]. What can be proved is that they conicide 'generically', in following sense.

Theorem 1.3 (Theorem 8.4). *There is an isomorphism*

$$\mathbb{C}_q[\mathrm{Gr}_{m,n}] \otimes_{\mathbb{C}[q^{\pm 1/2}]} \mathbb{C}(q^{1/2}) \simeq C_q(\mathrm{Gr}_{m,n}) \otimes_{\mathbb{C}[q^{\pm 1/2}]} \mathbb{C}(q^{1/2}).$$

This result is effectively what was already proved in [11], but we reprove it here in a more direct way, by applying the method of [8] in the homogeneous situation.

As a final point worth noting, in the case of rank one modules M_I, M_J, our new invariant gives a categorical interpretation of a known combinatorial invariant of the corresponding partitions λ_I and λ_J, namely

$$\kappa(M_I, M_J) = \mathrm{MaxDiag}(\lambda_J \setminus \lambda_I),$$

where the right-hand side is as defined in [18] and used there to compute lattice points of Newton-Okounkov bodies. See Remark 5.6 for details and a conjectural generalisation for more general modules.

The structure of the paper is as follows. In Sections 2, 3 and 4, we recall relevant background about quantum cluster algebras, quantum Grassmannians and the Grassmannian cluster category $\mathrm{CM}(A)$. In Section 5, we introduce the new invariant $\kappa(M,N)$ and discuss various interpretations of it. In Section 6, we use $\kappa(M,N)$ to construct quantum seed data from a cluster tiliting object in $\mathrm{CM}(A)$. We also show that this data correctly reproduces the quasi-commutation rules for quantum minors. In Section 7, we recall how the homogeneous coordinate ring of the Grassmannian becomes a cluster algebra and, in Section 8, we explain how far the arguments carry over to the quantum Grassmannian.

2. Quantum cluster algebras

We recall those aspects of the definition of a quantum cluster algebra which are relevant for this paper. For a complete discussion, we refer to the original work of Berenstein and Zelevinsky [2]. Let $B = (b_{ij})$ be an $n \times m$ integer matrix, with $n \geqslant m$, and let $L = (\lambda_{ij})$ be a skew-symmetric $n \times n$ integer matrix.

Definition 2.1. We say that B and L are *compatible* if

$$\sum_{j=1}^{n} b_{jk}\lambda_{j\ell} = \delta_{k\ell} d_k \tag{2.1}$$

for some positive integers d_k. That is, the matrix $B^{\mathrm{t}}L$ consists of two blocks, an $m \times m$ diagonal matrix D with positive diagonal entries $d_1, \dots, d_m$ and an $m \times (n-m)$ zero matrix.

The submatrix of B consisting of the first m rows is called the *principal part* of B. Note that, if a pair (B, L) is compatible, then B has full rank m and its principal part is skew-symmetrizable [2, Prop. 3.3].

Given a compatible pair (B, L), choose k in the 'mutable' range $1 \leqslant k \leqslant m$ and define $n \times n$ and $m \times m$ matrices $E = (e_{ij})$ and $F = (f_{ij})$ by

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k, \\ -1 & \text{if } i = j = k, \\ \max\{0, b_{ik}\} & \text{if } i \neq j = k, \end{cases} \qquad f_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k, \\ -1 & \text{if } i = j = k, \\ \max\{0, -b_{kj}\} & \text{if } i = k \neq j. \end{cases}$$

Then the mutated matrices are defined to be

$$\mu_k(L) = E^{\mathrm{t}} L E, \qquad \mu_k(B) = E^{\mathrm{t}} B F. \tag{2.2}$$

The new pair $\mu_k(B, L)$ is compatible [2, Prop. 3.4] and μ_k is an involution [2, Prop. 3.6], that is $\mu_k\mu_k(B, L) = (B, L)$.

Let q be a formal variable, with a chosen square root $q^{1/2}$. For clarity, note that $\mathbb{C}[q^{\pm 1/2}]$ denotes the Laurent polynomial ring $\mathbb{C}[q^{1/2}, q^{-1/2}]$.

Definition 2.2. The *based quantum torus* $\mathcal{T}(L)$ is the $\mathbb{C}[q^{\pm 1/2}]$-algebra generated by formal variables $X_1, \dots, X_n$ and their inverses $X_1^{-1}, \dots, X_n^{-1}$, subject to the quasi-commutation relations

$$X_i X_j = q^{\lambda_{ij}} X_j X_i. \tag{2.3}$$

For $\mathbf{a} = (a_i) \in \mathbb{Z}^m$, let

$$X^{\mathbf{a}} = q^{\gamma(\mathbf{a})} X_1^{a_1} \dots X_n^{a_n}. \tag{2.4}$$

where $\gamma(\mathbf{a}) = \frac{1}{2}\sum_{i>j} a_i a_j \lambda_{ij}$. Then $\{X^{\mathbf{a}} | \mathbf{a} \in \mathbb{Z}^n\}$ is a $\mathbb{C}[q^{\pm 1/2}]$-basis of $\mathcal{T}(L)$ and for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, we have

$$X^{\mathbf{a}} X^{\mathbf{b}} = q^{\frac{1}{2}\sum_{i>j}(a_i b_j - b_i a_j)\lambda_{ij}} X^{\mathbf{a}+\mathbf{b}} = q^{\sum_{i>j}(a_i b_j - b_i a_j)\lambda_{ij}} X^{\mathbf{b}} X^{\mathbf{a}}.$$

Note that $\mathcal{T}(L)$ is an Ore domain [2, Prop. A.1], that is, it can be embedded in its skew field of fractions $\mathcal{F}(L)$.

The datum $\Phi = (\{X_1, \dots, X_n\}, B, L)$ forms what is called a *quantum seed* (see [2, Def. 4.5] for a complete definition) and the (indexed) set $\{X_1, \dots, X_n\}$ is called a *cluster*. The mutation of pairs (B, L) described above extends to an involutive operation on quantum seeds as follows.

For any k in the mutable range $1 \leqslant k \leqslant m$, define a new cluster variable

$$X_k^* = X^{\mathbf{a}'} + X^{\mathbf{a}''}, \tag{2.5}$$

where

$$a'_j = \begin{cases} -1 & \text{if } j = k, \\ \max\{0, b_{jk}\} & \text{if } j \neq k, \end{cases} \quad \text{and} \quad a''_j = \begin{cases} -1 & \text{if } j = k, \\ \max\{0, -b_{jk}\} & \text{if } j \neq k. \end{cases}$$

Remark 2.3. Note that when $\ell \neq k$, we can rewrite (2.1) as

$$\sum_{j: b_{jk} > 0} b_{jk} \lambda_{j\ell} = \sum_{j: b_{jk} < 0} (-b_{jk}) \lambda_{j\ell}. \tag{2.6}$$

This is precisely the condition that $X^{\mathbf{a}'}$ and $X^{\mathbf{a}''}$ have the same quasi-commutation rule with X_ℓ, and hence that X_k^* quasi-commutes with X_ℓ. Indeed

$$X_k^* X_\ell = q^{\lambda_{k\ell}^*} X_\ell X_k^*,$$

where $\lambda_{k\ell}^* + \lambda_{k\ell}$ is the common value in (2.6). Comparing this to (2.2), we see that $\mu_k(L)$ is precisely the matrix $L^* = (\lambda_{ij}^*)$.

Denote by $\mu_k\{X_1, \dots, X_n\}$ the set $\{X_1, \dots, X_k^*, \dots, X_n\}$, i.e. X_k is replaced by X_k^* in the cluster $\{X_1, \dots, X_n\}$. The datum $\mu_k(\Phi) = \big(\mu_k\{X_1, \dots, X_n\}, \mu_k(B, L)\big)$ is called the *mutation* of Φ in direction k, and is again a quantum seed [2, Prop. 4.7] and thus $\mu_k\{X_1, \dots, X_n\}$ is another cluster in $\mathcal{F}(L)$.

Note that the initial variables $X_{m+1}, \dots, X_n$ are never mutated and appear in every cluster; they are called *frozen (cluster) variables.*

Definition 2.4. The quantum cluster algebra $C_q(\Phi)$ is the $\mathbb{C}[q^{\pm 1/2}]$-subalgebra of the skew field $\mathcal{F}(L)$ generated by all the cluster variables appearing in all quantum seeds obtained from Φ by all possible sequences of mutations.

Note that, by the quantum Laurent phenomenon [2, §5], we actually have $C_q(\Phi) \subseteq \mathcal{T}(L)$. Since $\mathcal{T}(L)$ is a free $\mathbb{C}[q^{\pm 1/2}]$-module, $C_q(\Phi)$ is torsion free and therefore free, because $\mathbb{C}[q^{\pm 1/2}]$ is a PID.

Recently, Geiss, Leclerc and Schröer proved that, under suitable assumptions, the specialisation of $C_q(\Phi)$ at $q^{1/2} = 1$ is the corresponding classical cluster algebra $C(X, B)$, where X stands for the cluster $\{X_1, \dots, X_n\}$ and (X, B) is a classical seed. Note that this is not an obvious result and is not just the statement that setting $q^{1/2} = 1$ in the construction of $C_q(\Phi)$ gives $C(X, B)$.

Theorem 2.5. [9] *Suppose that $C(X,B)$ is a $\mathbb{Z}$-graded cluster algebra with finite dimensional homogeneous components and that $C(X,B)$ coincides with the corresponding upper cluster algebra $U(X,B)$. Then, for a quantum seed $\Phi = (X,B,L)$, we have*

$$C_q(\Phi) \otimes_{\mathbb{C}[q^{\pm 1/2}]} \mathbb{C} = C(X,B),$$

where $\mathbb{C}$ here stands for the $\mathbb{C}[q^{\pm 1/2}]$-module $\mathbb{C}[q^{\pm 1/2}]/(q^{1/2}-1)$.

Remark 2.6. Thus, under the assumptions of the theorem, $C_q(\Phi)$ is a flat deformation of $C(X,B)$. Furthermore, $C_q(\Phi)$ will also be graded and each graded piece is a free $\mathbb{C}[q^{\pm 1/2}]$-module of finite rank, equal to the dimension of the corresponding graded piece of $C(X,B)$.

3. The Quantum Grassmannian

In this section, we work over the deformation ground ring $\mathbb{C}[q,q^{-1}]$ for simplicity, but we will later (in §8) extend scalars to $\mathbb{C}[q^{\pm 1/2}]$ without changing notation, in order to make comparisons with cluster algebras.

Denote by $\mathbb{C}_q[M_{m\times n}]$ the *quantum matrix algebra*, which is a q-deformation of the coordinate ring of the affine variety $M_{m\times n}$ of $m\times n$ complex matrices. This is a graded $\mathbb{C}[q,q^{-1}]$-algebra generated by degree 1 variables x_{ij}, for $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$, subject to the following homogeneous relations (cf. [4]):

$$x_{ij}x_{st} = \begin{cases} q x_{st}x_{ij} & \text{if } i=s \text{ and } j<t; \\ q x_{st}x_{ij} & \text{if } i<s \text{ and } j=t; \\ x_{st}x_{ij} & \text{if } i<s \text{ and } j>t; \\ x_{st}x_{ij} + (q-q^{-1})x_{sj}x_{it} & \text{if } i<s \text{ and } j<t. \end{cases}$$

This algebra is free as a $\mathbb{C}[q,q^{-1}]$-module, with finite rank graded pieces. Indeed, it has a basis consisting of reverse-lexicographically ordered monomials and the above relations provide a 'straightening law'.

Any $I\subseteq\{1,\dots,n\}$ with $|I|=m$ determines an $m\times m$-quantum minor

$$\Delta_I = \sum_{\sigma\in S_n} (-q)^{l(\sigma)} x_{1\sigma(1)}\dots x_{n\sigma(n)},$$

where S_n is the symmetric group of $\{1,\dots,n\}$ and $l(\sigma)$ is the length of $\sigma\in S_n$. Setting $q=1$, these become the ordinary minors that generate the coordinate ring $\mathbb{C}[\mathrm{Gr}_{m,n}]$ of the Grassmannian $\mathrm{Gr}_{m,n}$ of m-dimensional quotients of $\mathbb{C}^n$.

Definition 3.1. [15] The coordinate ring $\mathbb{C}_q[\mathrm{Gr}_{m,n}]$ of the *quantum Grassmanniann* is the $\mathbb{C}[q,q^{-1}]$-subalgebra of $\mathbb{C}_q[M_{m\times n}]$ generated by all $m\times m$-quantum minors. Thus it has an induced (and scaled) grading in which each quantum minor has degree 1.

Kelly, Lenagan and Rigal [14, Thm 2.7] constructed a homogeneous basis for $\mathbb{C}_q[\mathrm{Gr}_{m,n}]$ which specialises to a standard basis of $\mathbb{C}[\mathrm{Gr}_{m,n}]$ on setting $q = 1$. As an immediate consequence we have the following.

Theorem 3.2. *The quantum Grassmannian $\mathbb{C}_q[\mathrm{Gr}_{m,n}]$ is a flat deformation of the classical coordinate ring $\mathbb{C}[\mathrm{Gr}_{m,n}]$, which is the specialisation at $q = 1$. Each graded piece is a free $\mathbb{C}[q, q^{-1}]$-module of finite rank, equal to the dimension of the corresponding graded piece of $\mathbb{C}[\mathrm{Gr}_{m,n}]$.*

Two quantum minors Δ_I and Δ_J are said to *quasi-commute* if

$$\Delta_I \Delta_J = q^c \Delta_J \Delta_I,$$

for some integer c. Leclerc and Zelevinsky [16] describe the quasi-commuting quantum minors and give a combinatorial formula for computing the exponent c, as follows.

Definition 3.3. Two m-element subsets I and J of $\{1, \dots, n\}$ are said to be *non-crossing* (or *weakly separated)* if one of the following two conditions holds:

(i) $J \setminus I$ can be written as a disjoint union $J' \cup J''$ so that $J' < (I \setminus J) < J''$;

(ii) $I \setminus J$ can be written as a disjoint union $I' \cup I''$ so that $I' < (J \setminus I) < I''$,

where $I < J$ means that $i < j$ for all $i \in I$ and $j \in J$. When I and J are non-crossing, we define

$$c(I, J) = \begin{cases} |J''| - |J'| & \text{in case (i)} \\ |I'| - |I''| & \text{in case (ii)} \end{cases}$$

When both conditions hold, these two formulae give the same answer.

Remark 3.4. Note that, despite the formulation in Definition 3.3, the non-crossing condition depends only on the cyclic order the index set $\{1, \dots, n\}$. On the other hand, the definition of $c(I, J)$ and indeed of the quantum Grassmannian algebra itself, depends crucially on the total order of the index set.

Theorem 3.5. [16] *Two quantum minors Δ_I and Δ_J quasi-commute if and only if I and J are non-crossing. In which case, the quasi-commutation rule is*

$$\Delta_I \Delta_J = q^{c(I,J)} \Delta_J \Delta_I.$$

4. The Grassmannian cluster category

In this section, we recall from [13] how the Grassmannian cluster algebra has a categorification given by the category $\mathrm{CM}(A)$ of Cohen-Macaulay modules for a suitable algebra A.

Consider the action of $G = \{\zeta \in \mathbb{C}^* : \zeta^n = 1\}$ on $\mathbb{C}^2$, via

$$(x, y) \mapsto (\zeta x, \zeta^{-1} y) \tag{4.1}$$

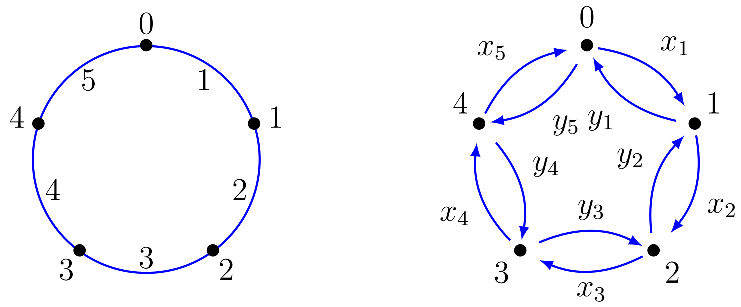

FIGURE 1. The circular graph C and double quiver $Q(C)$

so that G is identified with a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. Let

$$R = \mathbb{C}[[x, y]]/(x^k - y^{n-k}). \tag{4.2}$$

Then G acts R via (4.1), with invariant subring $Z = \mathbb{C}[[t]]$, where $t = xy$.

The category $\mathrm{mod}_G(R)$ of finitely generated G-equivariant R-modules is tautologically equivalent to the finitely generated module category $\mathrm{mod}(A)$ for the twisted group ring $A = R * G$.

Note that Z is precisely the centre of A, while R and A are free Z-modules of rank n and n^2, respectively. Furthermore, R is a Gorenstein (commutative) ring, while A is a non-commutative Z-order, which is also Gorenstein, in the sense of Buchweitz (cf. [13, Cor. 3.7]).

Exploiting the identification $\mathrm{mod}(A) = \mathrm{mod}_G(R)$, we will define

$$\mathrm{CM}(A) = \mathrm{CM}_G(R), \tag{4.3}$$

that is, the category of G-equivariant Cohen-Macaulay R-modules. This is a Frobenius category, in the sense of [12], as R is Gorenstein. As R is a finitely generated Z-module and Z is a PID, the Cohen-Macaulay R-modules are precisely those which are free over Z and thus $\mathrm{CM}(A)$ may also be described directly as the category of finitely generated A-modules which are free over Z.

Another way to describe the algebra A is as a quotient of the complete path algebra $\widehat{\mathbb{C}Q}$ of a quiver Q, which is the doubled quiver of a simple circular graph C. More precisely, let $C = (C_0, C_1)$ be the circular graph with vertex set $C_0 = \mathbb{Z}_n = G^\vee$ and edge set $C_1 = \{1, \ldots, n\}$, with edge i joining vertices $(i-1)$ and (i). The associated quiver $Q = Q(C)$ has vertex set $Q_0 = C_0$ and arrows set $Q_1 = \{x_a, y_a : a \in C_1\}$ with $x_a \colon (i-1) \to (i)$ and $y_a \colon (i) \to (i-1)$, as illustrated in Figure 1 in the case $n = 5$.

As is familiar from the McKay correspondence, $\mathbb{C}[[x, y]] * G$ is isomorphic to the complete preprojective algebra of type $\widetilde{A}_{n-1}$, that is, the quotient of $\widehat{\mathbb{C}Q}$ by (ideal generated by) the n relations $xy = yx$, one beginning at each vertex. If we quotient further by the n relations $x^k = y^{n-k}$, then we obtain $A = R * G$.

For any A-module M we can define its *rank*

$$\mathrm{rk}(M) = \mathrm{len}_{A\otimes_Z K}\big(M \otimes_Z K\big), \tag{4.4}$$

where K is the field of fractions of Z and we note that $A \otimes_Z K \cong M_n\big(K\big)$. For any M in $\mathrm{CM}(A)$ and every $j \in Q_0$, we have

$$\mathrm{rk}_Z(e_j M) = \mathrm{rk}(M).$$

In other words, every such M may be regarded as a representation of the quiver Q, with a free Z-module of rank $\mathrm{rk}(M)$ at each vertex and satisfying the relations $xy = t = yx$ and $x^k = y^{n-k}$.

In particular, it is possible [13, Prop 5.2] to classify the rank one modules in $\mathrm{CM}(A)$, as follows. For any m-subset $I \subseteq C_1$, define M_I in $\mathrm{CM}(A)$ as follows. For $j \in Q_0$, set $e_j M_I = Z$ and, for $a \in C_1$, set

$$x_a \colon Z \to Z \text{ to be (multiplication by) } 1, \text{ if } a \in I, \text{ or } t, \text{ if } a \notin I,$$
$$y_a \colon Z \to Z \text{ to be (multiplication by) } t, \text{ if } a \in I, \text{ or } 1, \text{ if } a \notin I.$$

Thus the rank one modules are in canonical one-one correspondence with the minors, or Plücker coordinates, D_I in $\mathbb{C}[\mathrm{Gr}_{m,n}]$ and indeed these are the cluster characters of the corresponding modules [13, §9].

A key point of the categorification is the following result [13, Prop 5.6]

Proposition 4.1. *Let I, J be m-subsets of C_1. Then $\mathrm{Ext}^1(M_I, M_J) = 0$ if and only if I and J are non-crossing.*

Note that the algebra A has a cyclic symmetry, induced by the graph automorphism that cycles the index sets C_0 and $C_1 = \{1, \ldots, n\}$. This explains why the Ext-vanishing condition is invariant under cycling the indices, just as the non-crossing condition is (cf. Remark 3.4).

It follows from Proposition 4.1 that, for any maximal non-crossing collection $\mathcal{S}$,

$$T_{\mathcal{S}} = \bigoplus_{J\in\mathcal{S}} M_J \tag{4.5}$$

is a cluster tilting object in $\mathrm{CM}(A)$ [13, Rem 5.7]. Note that in our context cluster tilting objects are the same as maximal rigid objects [3, Thm II.1.8] (see also [13, Rem 4.8]). One important additional property is the following.

Proposition 4.2. *For any basic maximal rigid module T in $\mathrm{CM}(A)$, the Gabriel quiver of the algebra $\mathrm{End}(T)$ has no loops or 2-cycles.*

Proof. Consider the projection functor $\pi\colon \mathrm{CM}(A) \to \mathrm{Sub}\, Q_k$, as in [13, Prop 4.3], whose kernel is generated by the projective P_n. Write $T = P_n \oplus T'$. The Gabriel quiver Q' of $\mathrm{End}(\pi T')$ in $\mathrm{Sub}\, Q_k$ has no loops or 2-cycles, by the arguments in [7, §8.1] and [6, Thm 2.2]. On the other hand, Q' is obtained from the Gabriel quiver Q of $\mathrm{End}(T)$ by deleting the vertex v corresponding to P_n and all incident arrows. Thus Q has no loops or 2-cycles at any vertices different from v, but, by the cyclic symmetry of A, the same argument applies when we replace P_n by any other projective-injective in $\mathrm{CM}(A)$ and so the required result follows. $\square$

5. The invariant $\kappa(M, N)$

To make the definition, we fix a vertex $v \in Q_0$. For $M \in \mathrm{CM}(A)$, we write M_v for the free Z-module $e_v M$ and $\mathrm{Hom}_v(M, N)$ for $\mathrm{Hom}_Z(M_v, N_v)$. The forgetful functor $\mathrm{mod}\, A \to \mathrm{mod}\, Z \colon M \mapsto M_v$ may also be written as

$$M_v = \mathrm{Hom}_A(Ae_v, M) = e_v A \otimes_A M$$

and hence has left and right adjoints $\mathrm{mod}\, Z \to \mathrm{mod}\, A$ given by $W \mapsto Ae_v \otimes_Z W$ and $W \mapsto \mathrm{Hom}_Z(e_v A, W)$, respectively. Thus, in particular, if we write, for any $M \in \mathrm{mod}\, A$,

$$P_v M = Ae_v \otimes_Z M_v, \qquad J_v M = \mathrm{Hom}_Z(e_v A, M_v),$$

then

$$\mathrm{Hom}_v(M, N) \cong \mathrm{Hom}_A(P_v M, N) \cong \mathrm{Hom}_A(M, J_v N). \tag{5.1}$$

Lemma 5.1. *When $M, N \in \mathrm{CM}(A)$, the natural maps $P_v M \to M$ and $N \to J_v N$ are injective with finite dimensional cokernels πM and ωN, respectively, i.e. we have short exact sequences*

$$0 \to P_v M \to M \to \pi M \to 0 \tag{5.2}$$

$$0 \to N \to J_v N \to \omega N \to 0 \tag{5.3}$$

Proof. In both cases, the restriction of the natural map to v is the identity map, so the kernel has rank 0 and hence, being in $\mathrm{CM}\, A$, it must be zero. The cokernel also has rank 0 and is finitely generated, so is finite dimensional. $\square$

Lemma 5.2. *For $M, N \in \mathrm{CM}(A)$, the map*

$$\phi_v \colon \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_v(M, N) \colon f \mapsto f_v$$

is injective, i.e. defining $K_v(M, N) = \mathrm{coker}\, \phi_v$, we have a short exact sequence

$$0 \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_v(M, N) \to K_v(M, N) \to 0. \tag{5.4}$$

Further, $K_v(M, N)$ is finite dimensional.

Proof. Under the isomorphism $\mathrm{Hom}_v(M, N) \cong \mathrm{Hom}_A(M, J_v N)$, the map ϕ_v is simply obtained by applying $\mathrm{Hom}_A(M, -)$ to the injective map $N \to J_v N$ and hence is injective. Furthermore, applying $\mathrm{Hom}_A(M, -)$ to (5.3), we can also see $K_v(M, N)$ as the kernel of the connecting homomorphism $\mathrm{Hom}_A(M, \omega N) \to \mathrm{Ext}^1(M, N)$ and hence it is finite dimensional, as claimed. $\square$

Note that $\mathrm{Ext}^1(M, J_v N) = 0$, for $M \in \mathrm{CM}\, A$, as $J_v N$ is injective in $\mathrm{CM}\, A$. Hence we also have a short exact sequence

$$0 \to K_v(M, N) \to \mathrm{Hom}_A(M, \omega N) \to \mathrm{Ext}^1_A(M, N) \to 0. \tag{5.5}$$

Remark 5.3. From above, we can also describe $K_v(M,N)$ as the image of the middle map in the four-term exact sequence

$$0 \to \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(M,J_vN) \to \mathrm{Hom}_A(M,\omega N) \to \mathrm{Ext}^1_A(M,N) \to 0 \tag{5.6}$$

obtained by applying $\mathrm{Hom}_A(M,-)$ to the short exact sequence (5.3). Hence $K_v(M,N)$ is also the image of the middle map in either of following two four-term exact sequences, which are isomorphic to (5.6).

$$0 \to \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(P_vM,N) \to \mathrm{Ext}^1_A(\pi M,N) \to \mathrm{Ext}^1_A(M,N) \to 0. \tag{5.7}$$

$$0 \to \mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(P_vM,J_vN) \to \mathrm{Hom}_A(\pi M,\omega N) \to \mathrm{Ext}^1_A(M,N) \to 0. \tag{5.8}$$

Note that (5.7) is obtained by appyling $\mathrm{Hom}_A(-,N)$ to (5.2). The isomorphisms between the inner terms in (5.6) and (5.8) and in (5.7) and (5.8) are the obvious natural maps and the commutation of the outer squares is straightforward. Thus the middle map in (5.8) may be taken to be the one through which the middle map in (5.6) factors and it remains to observe that the middle map in (5.7) also factors through the same map.

This observation amounts to the following. Given a map $g \in \mathrm{Hom}_A(M,J_vN)$, the fact that $\mathrm{Hom}_A(P_vM,\omega N)=0$ implies that there are unique $f \in \mathrm{Hom}_A(P_vM,N)$ and $h \in \mathrm{Hom}_A(\pi M,\omega N)$ that make up a map between the short exact sequences (5.2) and (5.3). The required observation is equivalent to the fact that the images in $\mathrm{Ext}^1_A(\pi M,N)$ of f and h under the appropriate coboundary maps coincide. This happens due to the general result that the map of short exact sequences induces an equivalence between the pushforward of (5.2) under f and the pullback of (5.3) under h.

Definition 5.4. We define $\kappa(M,N)=\dim K_v(M,N)$, where $K_v(M,N)$ is defined in Lemma 5.2.

Remark 5.5. Because of the cyclic symmetry, we could, without loss of generality, choose the special vertex to be $v=0\in\mathbb{Z}_n=C_0$. Put another way, it is precisely this choice that breaks the cyclic symmetry and gives the index set $C_1=\{1,\dots,n\}$ its total order, as required for the quantum Grassmannian.

Remark 5.6. From its description in (5.6), we can see that $K_v(T,N)$ is an $\mathrm{End}_A(T)$-module, for any $T\in\mathrm{CM}(A)$. In particular, for $T_{\mathcal{S}}$, as in (4.5), $K_v(T_{\mathcal{S}},N)$ is a representation of the quiver dual to a certain reduced plabic graph G with faces labelled by the maximal noncrossing set $\mathcal{S}$ (cf. [18, Def. 3.8], [1]). The dimension vector of this representation is given by

$$\mathbf{dim}\, K_v(T_{\mathcal{S}},N) = (\kappa(M_J,N) : J\in\mathcal{S}).$$

When $N=M_I$, observe that, in the notation of [18, Def 13.3],

$$\kappa(M_J,M_I)=\mathrm{MaxDiag}(\lambda_J\setminus\lambda_I) \tag{5.9}$$

where λ_I is the partition corresponding to the minor label I, under the standard correspondence (cf. [18, Def. 3.5]). This correspondence does depend the total ordering on $C_1 = \{1, \ldots, n\}$, i.e. on the choice of the vertex v. The essential relationship between the two sides of (5.9) is the observation that the (Ferrers) diagram of the partition λ_J is a (rotated) picture of πM_J.

Thus we can interpret [18, Cor 15.18] as saying that

$$\mathbf{dim}\, K_v(T_{\mathcal{S}}, M_I) = \mathrm{val}_G(D_I), \tag{5.10}$$

where val_G is defined in [18, Def 7.1] and D_I is the classical minor, or Plücker coordinate, with label I. It is then natural to conjecture that, for any $N \in \mathrm{CM}(A)$,

$$\mathbf{dim}\, K_v(T_{\mathcal{S}}, N) = \mathrm{val}_G(\Psi_N), \tag{5.11}$$

where Ψ_N is the cluster character of N.

Note that, the projective $P_v = M_{1..m}$ at the vertex v is always a summand of $T_{\mathcal{S}}$ and $K_v(P_v, N) = 0$ for any N. By convention, this trivial component is omitted from the definition of val_G.

6. Constructing seed data from a cluster tilting object

In this section, we construct two matrices B, in (6.1), and L, in (6.7), associated to a cluster tilting object T in $\mathrm{CM}(A)$. We show that they are compatible and mutate consistently when T mutates. The definition of B is standard, but that of L uses our new invariant κ from the previous section. We also show that, when $T = T_{\mathcal{S}}$ as in (4.5), the matrix L recovers the quasi-commutation rules for the corresponding quantum minors (see Theorem 3.5).

For any two modules M and N, denote by $\mathrm{Irr}(M, N)$ the space of irreducible maps from M to N. Note that the space of homomorphisms $\mathrm{Hom}(M, N)$ from M to N in $\mathrm{CM}(A)$ is a free $k[t]$-module.

Let $T = \oplus_{i=1}^{n} T_i \in \mathrm{CM}(A)$ be a cluster tilting object, where each T_i is indecomposable and the last $n - m$ summands are the indecomposable projective objects in $\mathrm{CM}(A)$. Let $B = B(T) = (b_{ij})$ be the $n \times m$-matrix defined by

$$b_{ij} = \dim \mathrm{Irr}(T_j, T_i) - \dim \mathrm{Irr}(T_i, T_j), \tag{6.1}$$

where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Note that by Proposition 4.2, at least one of $\dim \mathrm{Irr}(T_j, T_i)$ and $\dim \mathrm{Irr}(T_i, T_j)$ is zero. Hence, if $b_{ij} > 0$, then it is the number of arrows from j to i in the Gabriel quiver of $\mathrm{End}_A(T)$, while if $-b_{ij} > 0$, then it is the number of arrows from i to j.

Suppose that $1 \leqslant k \leqslant m$, so that T_k is a non-projective indecomposable summand of T. Then (cf. [6, §5], in particular Propositions 5.6 and 5.7) we have the following short exact 'exchange' sequences for T_k in $\mathrm{Add}(T)$:

$$0 \to T_k^* \to \bigoplus_{j : b_{jk} < 0} T_j^{-b_{jk}} \to T_k \to 0. \tag{6.2}$$

and

$$0 \to T_k \to \bigoplus_{j:b_{jk}>0} T_j^{b_{jk}} \to T_k^* \to 0 \tag{6.3}$$

Denote the two middle terms $\bigoplus_{j:b_{jk}<0} T_j^{-b_{jk}}$ and $\bigoplus_{j:b_{jk}>0} T_j^{b_{jk}}$ by B_k and B'_k, respectively. Note that the maps in these sequences are minimal left/right approximations, and hence are made up of the irreducible maps to and from T_k or T_k^*. The new cluster tiliting object $(T/T_k) \oplus T_k^*$ is called the *mutation* of T in direction k and denoted by $\mu_k(T)$.

Mutation of a cluster tilting object T is compatible with mutation of the matrix $B(T)$. For the convenience of the reader, we provide a proof, adapted from the proof in [3, Theorem II.1.6] to our setting with minor modifications.

Theorem 6.1. *Let T be a cluster tilting object and T_k be a non-projective indecomposable summand. Then*

$$B(\mu_k(T)) = \mu_k(B(T)).$$

Proof. The exchange sequences (6.3) and (6.2) provide an orientation-reversing correspondence between the arrows incident to vertex k in the Gabriel quivers of $\operatorname{End} T$ and $\operatorname{End} \mu_k(T)$. Thus

$$B(\mu_k(T))_{ik} = -b_{ik} = (\mu_k(B(T)))_{ik}.$$

So it remains to consider the entries $B(\mu_k(T))_{ij}$, where $i, j \neq k$. By definition (of B), we know that T_i and T_j are not both projective, so we may assume that T_i is not projective. By duality we only need to consider the case where there is at least one arrow from i to k and so no arrows from k to i.

$$i \longrightarrow k \cdots\cdots j.$$

There may or may not be any arrows between k and j.

Similar to the exchange sequences (6.3) and (6.2), we have

$$0 \to T_i \to \bigoplus_{\ell:b_{\ell i}>0} T_\ell^{b_{\ell i}} \to T_i^* \to 0 \tag{6.4}$$

and

$$0 \to T_i^* \to \bigoplus_{v:b_{\ell i}<0} T_\ell^{-b_{\ell i}} \to T_i \to 0. \tag{6.5}$$

Write $B'_i = \bigoplus_{\ell : b_{\ell i} > 0} T_\ell^{b_{\ell i}} = D'_i \oplus T_k^m$ with $m = b_{ki}$. We have the following commutative diagram, where all rows and all columns are short exact sequences,

$$\begin{tikzcd}
0 \arrow[r] & T_i \arrow[r] \arrow[d] & D'_i \oplus T_k^m \arrow[r] \arrow[d] & T_i^* \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & T_i \arrow[r, "a"] & D'_i \oplus (B'_k)^m \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & & (T_k^*)^m \arrow[r] & (T_k^*)^m. &
\end{tikzcd}$$

The fact that a is a composition of two left approximations in the commutative diagram implies that a is itself a left $T/(T_k \oplus T_i)$-approximation. Any homomorphsim $f \colon T_i \to T_k^*$ factors through the right approximation $r_k \colon B'_k \to T_k^*$ in (6.3)

$$\begin{tikzcd}
 & T_i \arrow[dl, dotted, "s"'] \arrow[d, "f"] \\
B'_k \arrow[r, "r_k"'] & T_k^*
\end{tikzcd}$$

As B'_k has no summand isomorphic to T_i or T_k^*, the map s factors through a and so f factors through a. This shows that a is a left $(T/(T_k \oplus T_i) \oplus T_k^*)$-approximation.

Constructing the push-out of (6.5) along the map $g : T^* \to X$ gives the commutative diagram of short exact sequences,

$$\begin{tikzcd}
0 \arrow[r] & T_i^* \arrow[r] \arrow[d, "g"'] & B_i \arrow[r] \arrow[d] & T_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r, "b"] \arrow[d] & T_i \arrow[r] & 0 \\
 & (T_k^*)^m \arrow[r] & (T_k^*)^m. & &
\end{tikzcd}$$

By Proposition 4.2, the Gabriel quiver of $\operatorname{End} T$ has no 2-cycles. Therefore B_i has no summand isomorphic to T_k and so $\operatorname{Ext}^1(T_k^*, B_i) = 0$. Thus $Y = B_i \oplus T_k^*$.

Further b is a right $(T/(T_k \oplus T_i) \oplus T_k^*)$-approximation, as b has a component that is the right approximation $B_i \to T_i$ and $\operatorname{Ext}^1(T_k^*, X) = 0$, which follows from

$$\operatorname{Ext}^1(T_k^*, T_k^*) = 0 = \operatorname{Ext}^1(T_k^*, T_i^*).$$

Denote the multiplicity of a module M in N by $\alpha_M N$. We have $(B(\mu_k(T)))_{ji}$ is the difference of the numbers of arrows from i to j and the number arrows from j

to i in the Gabriel quiver of $\operatorname{End}\mu_k(T)$ and so

$$\begin{aligned}(B(\mu_k(T)))_{ji} &= |\{\text{arrows } i \to j\}| - |\{\text{arrows } j \to i\}| \\ &= \alpha_{T_j}(D_i' \oplus (B_k')^m) - \alpha_{T_j}(B_i \oplus (T_k^*)^m) \\ &= \alpha_{T_j}(D_i') + m\alpha_{T_j}(B_k') - \alpha_{T_j}(B_i) \\ &= b_{ji} + mb_{jk} - b_{ij} \\ &= (\mu_k(B(T)))_{ji}.\end{aligned}$$

This completes the proof. □

Remark 6.2. The maps a and b, in the proof of Theorem 6.1, are not necessarily minimal approximations. However, by the nature of the two short exact sequences containing a and b, the middle terms Y and $D_i' \oplus (B_k')^m$ have the same number of extra summands that are isomorphic to T_j.

Recalling the definition of $\kappa(M, N)$ from Definition 5.4, we define

$$\lambda(M, N) = \kappa(N, M) - \kappa(M, N). \tag{6.6}$$

Given a cluster tilting object $T = \oplus_{i=1}^n T_i$, we define an $n \times n$-matrix $L = L(T) = (\lambda_{ij})$ associated to T, and the vertex v, by

$$\lambda_{ij} = \lambda(T_i, T_j). \tag{6.7}$$

It turns out that this is an appropriate generalisation of the construction of Geiss-Leclerc-Schroer [8, §10.2] to the current context, when $\operatorname{Hom}(M, N)$ is infinite dimensional, so their definition does not makes sense. Indeed, their proofs of [8, Props 10.1 & 10.2] can be adapted to give a similar result, as follows.

Theorem 6.3. *The two matrices B and L associated to T, as above, are compatible. Furthermore, mutation of cluster tilting objects is consistent with mutation of seed data, i.e. the pair associated to the mutated object $\mu_k(T)$ is the mutated pair $\mu_k(B, L)$.*

Proof. We consider the short exact sequences in (6.3) and, for brevity, write

$$T' = \bigoplus_{j: b_{jk} > 0} T_j^{b_{jk}} \quad \text{and} \quad T'' = \bigoplus_{j: b_{jk} < 0} T_j^{-b_{jk}}$$

for their middle terms. Applying $\operatorname{Hom}(T_\ell, -)$ and $\operatorname{Hom}_v(T_\ell, -)$ to (6.3), and using a degenerate case of the Snake Lemma, we obtain

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}(T_\ell, T_k) \arrow[r] \arrow[d] & \operatorname{Hom}(T_\ell, T') \arrow[r] \arrow[d] & \operatorname{Hom}(T_\ell, T_k^*) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{Hom}_v(T_\ell, T_k) \arrow[r] \arrow[d] & \operatorname{Hom}_v(T_\ell, T') \arrow[r] \arrow[d] & \operatorname{Hom}_v(T_\ell, T_k^*) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K_v(T_\ell, T_k) \arrow[r] & K_v(T_\ell, T') \arrow[r] & K_v(T_\ell, T_k^*) \arrow[r] & 0, \end{tikzcd}$$

where the vertical sequences are all short exact as in (5.4). Note that we have used the fact that $\mathrm{Ext}^1(T_\ell, T_k) = 0$ for all ℓ. Thus

$$\kappa(T_\ell, T') = \kappa(T_\ell, T_k^*) + \kappa(T_\ell, T_k). \tag{6.8}$$

Similarly, applying $\mathrm{Hom}(-, T_\ell)$ to (6.2), we obtain

$$\kappa(T'', T_\ell) = \kappa(T_k^*, T_\ell) + \kappa(T_k, T_\ell). \tag{6.9}$$

On the other hand, when $\ell \neq k$, we also have $\mathrm{Ext}^1(T_\ell, T_k^*) = 0 = \mathrm{Ext}^1(T_k^*, T_\ell)$ and can equally apply $\mathrm{Hom}(T_\ell, -)$ to (6.2) and $\mathrm{Hom}(-, T_\ell)$ to (6.3) to obtain

$$\kappa(T_\ell, T'') = \kappa(T_\ell, T_k^*) + \kappa(T_\ell, T_k), \tag{6.10}$$

$$\kappa(T', T_\ell) = \kappa(T_k^*, T_\ell) + \kappa(T_k, T_\ell). \tag{6.11}$$

Combining (6.8)–(6.11) gives $\kappa(T_\ell, T') = \kappa(T_\ell, T'')$ and $\kappa(T', T_\ell) = \kappa(T'', T_\ell)$, and hence $\lambda(T', T_\ell) = \lambda(T'', T_\ell)$, that is,

$$\sum_{j: b_{jk} > 0} b_{jk} \lambda(T_j, T_\ell) = \sum_{j: b_{jk} < 0} (-b_{jk}) \lambda(T_j, T_\ell).$$

In other words,

$$\sum_j b_{jk} \lambda_{j\ell} = 0,$$

which is the $\ell \neq k$ part of the condition that B and L are compatible.

In the case $\ell = k$, following [7, Prop 8.1] and [13, Cor 4.6], we have

$$\mathrm{Ext}^1(T_k, T_k^*) \cong \mathbb{C} \cong \mathrm{Ext}^1(T_k^*, T_k).$$

So when we apply $\mathrm{Hom}(-, T_\ell)$ and $\mathrm{Hom}_v(-, T_\ell)$ to (6.3), we obtain the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(T_k^*, T_k) \arrow[r] \arrow[d] & \mathrm{Hom}(T', T_k) \arrow[r] \arrow[d] & \mathrm{Hom}(T_k, T_k) \arrow[r] \arrow[d] & \mathbb{C} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}_v(T_k^*, T_k) \arrow[r] \arrow[d] & \mathrm{Hom}_v(T', T_k) \arrow[r] \arrow[d] & \mathrm{Hom}_v(T_k, T_k) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & \ker\psi \arrow[r] & K_v(T', T_k) \arrow[r, "\psi"] & K_v(T_k, T_k) & &
\end{tikzcd}$$

Applying the Snake Lemma to the middle two columns shows that ψ is surjective and that there is a short exact sequence

$$0 \to K_v(T_k^*, T_k) \to \ker\psi \to \mathbb{C} \to 0.$$

Hence

$$\kappa(T', T_k) = \kappa(T_k^*, T_k) + \kappa(T_k, T_k) + 1.$$

Similarly, applying $\mathrm{Hom}(T_\ell, -)$ to (6.2), we obtain

$$\kappa(T_k, T'') = \kappa(T_k, T_k^*) + \kappa(T_k, T_k) + 1.$$

Following the same steps as before, we now obtain $\lambda(T', T_\ell) + 1 = \lambda(T'', T_\ell) - 1$ and hence

$$\sum_j b_{jk}\lambda_{jk} = 2,$$

which is the $\ell = k$ part of the compatibility condition.

The equations (6.8)–(6.11) also show that

$$\begin{aligned}\lambda(T_k^*, T_\ell) + \lambda(T_k, T_\ell) = \lambda(T', T_\ell) &= \sum_{j:b_{jk}>0} b_{jk}\lambda(T_j, T_\ell) \\ = \lambda(T'', T_\ell) &= \sum_{j:b_{jk}<0} (-b_{jk})\lambda(T_j, T_\ell).\end{aligned}$$

Comparing this with Remark 2.3, we see that the mutated matrix $\mu_k(L)$ is defined by the function λ applied to $\mu_k(T)$. Together with Theorem 6.1, this completes the proof that mutation of cluster tilting objects is consistent with mutation of seed data. □

We now show that, when $T = T_{\mathcal{S}}$ as in (4.5), our matrix L computes the quasi-commutation rules for the corresponding quantum minors, as in Theorem 3.5.

Lemma 6.4. *Suppose that I and J are non-crossing and that $J \setminus I = J' \cup J''$ so that $J' < (I \setminus J) < J''$, i.e. case (i) of Definition 3.3. Then*

$$\kappa(M_I, M_J) = |J'| \quad \textit{and} \quad \kappa(M_J, M_I) = |J''|.$$

Proof. By Remark 5.4 in [13], $\mathrm{Hom}(M_I, M_J)$ is generated by t^α with $\alpha \in \mathbb{N}^{C_0}$ the minimal exponent vector satisfying

$$\alpha_{ha} - \alpha_{ta} = \begin{cases} 1 & \text{if } a \in J \setminus I, \\ -1 & \text{if } a \in I \setminus J, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, α decreases by 1 on each edge in J', then increases by 1 on each edge in $I \setminus J$, then decreases by 1 on each edge in J''. To be minimal α must be zero somewhere, which must then be on the vertices between J' and $I \setminus J$. Hence $\kappa(M_I, M_J) = \alpha_v = |J'|$, as required.

Similarly, $\mathrm{Hom}(M_J, M_I)$ is generated by t^β, where β increases by 1 on each edge in J', then decreases by 1 on each edge in $I \setminus J$, then increases by 1 on each edge in J''. Furthermore, β must be zero on the vertices between $I \setminus J$ and J'' and so $\kappa(M_J, M_I) = \beta_v = |J''|$. □

Theorem 6.5. *Let M_I, M_J be rank-1 modules with I and J non-crossing sets (or equivalently $\mathrm{Ext}^1(M_I, M_J) = 0$, cf. Proposition 4.1). Then*

$$c(I, J) = \lambda(M_I, M_J) = \kappa(M_J, M_I) - \kappa(M_I, M_J)$$

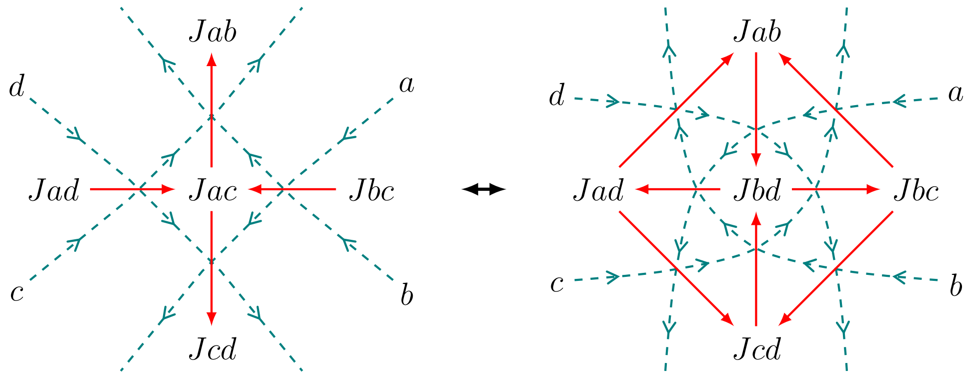

FIGURE 2. A geometric exchange (cf. [19, Fig. 7])

where $c(I,J)$ is defined in Definition 3.3 and is the exponent occurring in the quasi-commutation rule in Theorem 3.5:

$$\Delta_I \Delta_J = q^{c(I,J)} \Delta_J \Delta_I.$$

Proof. Swapping the roles of I and J if necessary, we may assume that case (i) of Definition 3.3 holds and then this is an immediate application of Lemma 6.4. □

7. THE GRASSMANNIAN CLUSTER ALGEBRA

To understand the quantum case, we start by recalling in some detail how the Grassmannian homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}_{m,n}]$ becomes a (graded) cluster algebra, We follow the original work of Scott [19], enhanced by the combinatorial fact, now proved by Oh-Postnikov-Speyer [17, Thm 1.6], that every maximal non-crossing set $\mathcal{S}$ (Def 3.3) corresponds (one-to-one) to a certain type of Postnikov alternating strand diagram, or a reduced plabic graph $G_\mathcal{S}$ or, dually, a quiver $Q_\mathcal{S}$ and thus an exchange matrix $B_\mathcal{S}$. The set $\mathcal{S}$ canonically labels the faces of $G_\mathcal{S}$, and hence the vertices of $Q_\mathcal{S}$ and the rows and columns of $B_\mathcal{S}$ (see [19, §5] or [1], [17] for details).

Then $X_\mathcal{S} = \{X_I : I \in \mathcal{S}\}$, together with the exchange matrix $B_\mathcal{S}$, is an initial seed for a cluster algebra $C(X_\mathcal{S}, B_\mathcal{S})$, which can be graded by giving all the X_I degree 1, since the quiver $Q_\mathcal{S}$ is balanced, in the sense of [13, Lemma 2.1].

Now, by [17, Thm 1.4], any two such maximal non-crossing sets can be related by a sequence of mutations or 'geometric exchanges'. More precisely, suppose that a, b, c, d are cyclically ordered indices and a maximal non-crossing set $\mathcal{S}$ contains the minor labels Jab, Jbc, Jcd, Jad and Jac, then there will be another maximal non-crossing set $\mathcal{S}'$ in which Jac is replaced by Jbd. Note that Jab is short-hand for $J \cup \{a, b\}$ and we are assuming that J is disjoint from $\{a, b, c, d\}$. The local change in the corresponding alternating strand diagram and quiver is illustrated in Figure 2. Thus, the quiver $Q_{\mathcal{S}'}$ is obtained from $Q_\mathcal{S}$ by quiver mutation.

As a consequence, we obtain a canonical isomorphism $C(X_{\mathcal{S}}, B_{\mathcal{S}}) \cong C(X_{\mathcal{S}'}, B_{\mathcal{S}'})$ by identifying the X_I for $I \in \mathcal{S} \setminus Jac = \mathcal{S}' \setminus Jbd$ and identifying X_{Jac} and X^*_{Jac} with X^*_{Jbd} and X_{Jbd}, respectively. Thus we have a single cluster algebra in which $X_{\mathcal{S}}$ and $X_{\mathcal{S}'}$ are two clusters related by mutation.

For any $\mathcal{S}$, there is a homogeneous map

$$\eta_{\mathcal{S}} \colon C(X_{\mathcal{S}}, B_{\mathcal{S}}) \subseteq \mathbb{C}[X_I^{\pm 1} : I \in \mathcal{S}] \to \mathbb{C}(\mathrm{Gr}_{m,n})$$

where $\mathbb{C}(\mathrm{Gr}_{m,n})$ is the field of fractions of $\mathbb{C}[\mathrm{Gr}_{m,n}]$, defined by sending each X_I to the corresponding (classical) minor D_I, for $I \in \mathcal{S}$. Since the local quiver for the mutation is as in Figure 2, the exchange relation precisely matches the short Plücker relations between minors

$$D_{Jac} D_{Jbd} = D_{Jab} D_{Jcd} + D_{Jad} D_{Jbc}$$

and so the maps $\eta_{\mathcal{S}}$ and $\eta_{\mathcal{S}'}$ conicide after making the canonical identification just described. Since every label is in some maximal non-crossing set and all such sets are linked by mutation, the image of $\eta_{\mathcal{S}}$ contains all the minors and so generates $\mathbb{C}(\mathrm{Gr}_{m,n})$ as a field. But the transcendence degree of $\mathbb{C}(\mathrm{Gr}_{m,n})$ is equal to $|\mathcal{S}|$, so the minors $\{D_J : J \in \mathcal{S}\}$ must be algebraically independent. Hence $\eta_{\mathcal{S}}$ is an embedding and identifies $C(X_{\mathcal{S}}, B_{\mathcal{S}})$ with a single cluster algebra $C(\mathrm{Gr}_{m,n}) \subset \mathbb{C}(\mathrm{Gr}_{m,n})$, independent of $\mathcal{S}$, for which all the minors D_J are cluster variables, so $\mathbb{C}[\mathrm{Gr}_{m,n}] \subset C(\mathrm{Gr}_{m,n})$.

It then takes a geometric argument (see [19, Thm 3]), applied to a special initial cluster $\{D_J : J \in \mathcal{S}\}$, to show that this inclusion is an equality and hence $\mathbb{C}[\mathrm{Gr}_{m,n}]$ is itself a cluster algebra.

8. The Grassmannian quantum cluster algebra

We can now try to follow the above argument in the quantum case and prove the analogous results as far as possible. We must start by bringing in the matrix $L_{\mathcal{S}}$ given by the Leclerc-Zelevinsky quasi-commutation rule of Theorem 3.5.

Proposition 8.1. *For every maximal non-crossing set $\mathcal{S}$, the pair $(B_{\mathcal{S}}, L_{\mathcal{S}})$ is compatible. Furthermore, any two such pairs are related by mutation.*

Proof. In the category $\mathrm{CM}(A)$, we have a cluster tiliting object $T_{\mathcal{S}}$, as in (4.5). By [1, Thm 10.3], we have $B_{\mathcal{S}} = B(T_{\mathcal{S}})$ and, by Theorem 6.5, we have $L_{\mathcal{S}} = L(T_{\mathcal{S}})$. Thus, by the first part of Theorem 6.3, $B_{\mathcal{S}}$ and $L_{\mathcal{S}}$ are compatible.

Given a geometric exchange from $\mathcal{S}$ to $\mathcal{S}'$, the two cluster tilting objects $T_{\mathcal{S}}$ and $T_{\mathcal{S}'}$ differ by only one indecomposable summand and so must be related by the mutation of cluster tiliting objects(cf. [6, Prop 4.5]). Thus [17, Thm 1.4] implies that any two such $T_{\mathcal{S}}$ are related by mutation of cluster tiliting objects, and hence the corresponding pairs $(B_{\mathcal{S}}, L_{\mathcal{S}})$ are also related by mutation of such pairs, by the second part of Theorem 6.3. □

As a consequence of this proposition, any maximal non-crossing set $\mathcal{S}$ determines a quantum seed $\Phi_{\mathcal{S}} = (X_{\mathcal{S}}, B_{\mathcal{S}}, L_{\mathcal{S}})$ and thus a quantum cluster algebra $C_q(\Phi_{\mathcal{S}}) \subseteq \mathcal{T}(L_{\mathcal{S}})$, as in Definition 2.4, which can be graded, for the same combinatorial reasons as the classical case, by giving the initial variables degree 1. As in the classical case, we have a canonical isomorphism $C_q(\Phi_{\mathcal{S}}) \cong C_q(\Phi_{\mathcal{S}'})$, when $\mathcal{S}$ and $\mathcal{S}'$ are related by a geometric exchange.

We now bring back the quantum Grassmannian $\mathbb{C}_q[\mathrm{Gr}_{m,n}]$, as in Definition 3.1 but with scalars extended from $\mathbb{C}[q, q^{-1}]$ to $\mathbb{C}[q^{\pm 1/2}]$. We denote its skew field of fractions by $\mathcal{F}(\mathrm{Gr}_{m,n})$. The first step in comparing $C_q(\Phi_{\mathcal{S}})$ to $\mathbb{C}_q[\mathrm{Gr}_{m,n}]$ is the following.

Lemma 8.2. *The canonical map $\eta_{\mathcal{S}} \colon \mathcal{T}(L_{\mathcal{S}}) \to \mathcal{F}(\mathrm{Gr}_{m,n}) \colon X_I \mapsto \Delta_I$ is injective.*

Proof. Note that this map is well-defined, because, by choice, $L_{\mathcal{S}}$ gives the quasi-commutation rules for the Δ_I with $I \in \mathcal{S}$. It suffices to prove that the restriction to the quasi-commuting polynomial subalgebra $\mathbb{C}[q^{\pm 1/2}][X_I : I \in \mathcal{S}]$ is injective. The image of this restriction lies in $\mathbb{C}_q[\mathrm{Gr}_{m,n}]$, which is free over $\mathbb{C}[q^{\pm 1/2}]$ with finite rank graded pieces. Hence the image is free and, since the specialisation to $q^{1/2} = 1$ is injective, the original map is injective. □

We claim that the (restricted) maps $\eta_{\mathcal{S}} \colon C_q(\Phi_{\mathcal{S}}) \to \mathcal{F}(\mathrm{Gr}_{m,n})$ are all compatible, as in the classical case, and hence identify each $C_q(\Phi_{\mathcal{S}})$ with a single quantum cluster algebra $C_q(\mathrm{Gr}_{m,n}) \subseteq \mathcal{F}(\mathrm{Gr}_{m,n})$, which contains clusters of quantum minors $\{\Delta_I : I \in \mathcal{S}\}$, for every maximal non-crossing set $\mathcal{S}$. This claim is true because the quantum exchange relation matches precisely with the corresponding short quantum Plücker relation:

Proposition 8.3. *Let $\eta_{\mathcal{S}} \colon C_q(\Phi_{\mathcal{S}}) \to \mathcal{F}(\mathrm{Gr}_{m,n})$ be the embedding associated to some maximal non-crossing set $\mathcal{S}$ and suppose that we can obtain $\mathcal{S}'$ by a geometric exchange centred at Jac. Then $\eta_{\mathcal{S}}(X^*_{Jac}) = \Delta_{Jbd}$. Thus, given the canonical isomorphism $C_q(\Phi_{\mathcal{S}}) \cong C_q(\Phi_{\mathcal{S}'})$, the maps $\eta_{\mathcal{S}'}$ and $\eta_{\mathcal{S}}$ coincide.*

Proof. The indices a, b, c, d are cyclically ordered and, without loss of generality, we can assume that $a < c$. Hence the actual ordering of the indices is either $a < b < c < d$ or $d < a < b < c$. In these two cases, the short quantum Plücker relation (cf. [14]) can be written, respectively,

$$\Delta_{Jbd} = q^{-1}\Delta_{Jac}^{-1}\Delta_{Jab}\Delta_{Jcd} + q\Delta_{Jac}^{-1}\Delta_{Jad}\Delta_{Jbc}, \tag{8.1}$$

$$\Delta_{Jbd} = q^{-1}\Delta_{Jad}\Delta_{Jbc}\Delta_{Jac}^{-1} + q\Delta_{Jcd}\Delta_{Jab}\Delta_{Jac}^{-1}. \tag{8.2}$$

We want to show that the right-hand sides are precisely what appears in the quantum exchange relation (2.5) for X^*_{Jac}, after identifying X_I with Δ_I for $I \in \mathcal{S}$. In other words, each right-hand side term is a quantum monomial as in (2.4):

$$X^{\mathbf{a}} = q^{\gamma(\mathbf{a})} X_1^{a_1} \dots X_n^{a_n},$$

where

$$\gamma(\mathbf{a}) = \tfrac{1}{2} \sum_{i>j} a_i a_j \lambda_{ij}$$

and $L = (\lambda_{ij})$ is the matrix of quasicommutation exponents. In this case, the entries of this matrix are $c(I_i, I_j)$, as in Definition 3.3. The precise calculation depends on the order of the indices a, b, c, d. In the case $a < b < c < d$, the first term exponent is

$$\gamma = \tfrac{1}{2}\big(c(Jcd, Jab) - c(Jab, Jac) - c(Jcd, Jac)\big) = \tfrac{1}{2}(-2 - 1 + 1) = -1$$

and the second term exponent is

$$\gamma = \tfrac{1}{2}\big(c(Jbc, Jad) - c(Jad, Jac) - c(Jbc, Jac)\big) = \tfrac{1}{2}(0 + 1 + 1) = 1,$$

as required. In the case $d < a < b < c$, the first term exponent is

$$\gamma = \tfrac{1}{2}\big(c(Jbc, Jad) - c(Jac, Jad) - c(Jac, Jbc)\big) = \tfrac{1}{2}(-2 + 1 - 1) = -1$$

and the second term exponent is

$$\gamma = \tfrac{1}{2}\big(c(Jab, Jcd) - c(Jac, Jcd) - c(Jac, Jab)\big) = \tfrac{1}{2}(0 + 1 + 1) = 1.$$

Each exponent agrees with that in (8.1) or (8.2), as required. □

Because $C_q(\mathrm{Gr}_{m,n})$ contains all quantum minors, we have

$$\mathbb{C}_q[\mathrm{Gr}_{m,n}] \subseteq C_q(\mathrm{Gr}_{m,n}) \tag{8.3}$$

and consequently the subfield of $\mathcal{F}(\mathrm{Gr}_{m,n})$ generated by $C_q(\mathrm{Gr}_{m,n})$ is the whole of $\mathcal{F}(\mathrm{Gr}_{m,n})$.

In the quantum case, there is no geometric argument to show that the inclusion (8.3) is an equality. Indeed, all we can prove is the following.

Theorem 8.4. *The inclusion* (8.3) *induces an isomorphism*

$$\mathbb{C}_q[\mathrm{Gr}_{m,n}] \otimes_{\mathbb{C}[q^{\pm 1/2}]} \mathbb{C}(q^{1/2}) \simeq C_q(\mathrm{Gr}_{m,n}) \otimes_{\mathbb{C}[q^{\pm 1/2}]} \mathbb{C}(q^{1/2}).$$

Proof. We have already noted that $\mathbb{C}_q[\mathrm{Gr}_{m,n}]$ is a flat deformation of $\mathbb{C}[\mathrm{Gr}_{m,n}]$ (Theorem 3.2). Furthermore, we know that $\mathbb{C}[\mathrm{Gr}_{m,n}]$ coincides with the classical cluster algebra $C(X_{\mathcal{S}}, B_{\mathcal{S}})$ and with its upper cluster algebra $U(X_{\mathcal{S}}, B_{\mathcal{S}})$ (see [19, Theorem 3] and its proof). Hence Theorem 2.5 and Remark 2.6 apply and we see that $C_q(\mathrm{Gr}_{m,n})$ is a flat deformation of $C(X_{\mathcal{S}}, B_{\mathcal{S}}) = \mathbb{C}[\mathrm{Gr}_{m,n}]$.

Thus the graded pieces of $\mathbb{C}_q[\mathrm{Gr}_{m,n}]$ and $C_q(\mathrm{Gr}_{m,n})$ have the same rank in equal degrees and so the inclusion becomes an isomorphism on tensoring with $\mathbb{C}(q^{1/2})$, as required. □

Now, the inclusion (8.3) is an equality in the finite type cases, namely $\mathrm{Gr}(2, n)$, for arbitrary n, and $\mathrm{Gr}(3, n)$, for $n = 6, 7, 8$. This follows, for $\mathrm{Gr}(2, n)$, because the cluster variables are all just (quantum) minors and, for these $\mathrm{Gr}(3, n)$, because the additional cluster variables have been computed in [10, §3.2] and are explicit polynomials in $\mathbb{C}_q[\mathrm{Gr}(3, n)]$. Note that in the $\mathrm{Gr}(3, n)$ cases, the degree 2 cluster variables do involve (odd) powers of $q^{1/2}$, showing that the extension of scalars to $\mathbb{C}[q^{\pm 1/2}]$ is already necessary here.

On the other hand, for the remaining infinite type cases, Theorem 8.4 is the strongest statement we know that could be informally rendered as "the quantum Grassmannian is a quantum cluster algebra". We do not prove the stronger theorem that $\mathbb{C}_q[\mathrm{Gr}_{m,n}] = C_q(\mathrm{Gr}_{m,n})$ and we do not know of any proof in the literature, but we have no reason to believe that it couldn't be true.

References

[1] K. Baur, A.D. King, R.J. Marsh, *Dimer models and cluster categories of Grassmannians*, Proc. L.M.S. **113** (2016) 213–260 (arXiv:1309.6524)

[2] A. Berenstein, A. Zelevinsky, *Quantum cluster algebras*, Adv. Math. **195** (2005) 405–455 (arXiv:math/0404446)

[3] A.B. Buan, O. Iyama, I. Reiten, J. Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, Compositio Math. **145** (2009) 1035–1079 (arXiv:math/0701557)

[4] L. Faddeev, N. Reshetikhin, L. Takhtadzhyan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. **1** (1990) 193–225

[5] S. Fomin, A. Zelevinsky, *Cluster algebras II: Finite type classification*, Invent. Math. **154** (2003) 63–121 (arXiv:math/0208229)

[6] C. Geiss, B. Leclerc, J. Schröer, *Rigid modules over preprojective algebras*, Invent. Math. **165** (2006) 589–632 (arXiv:math/0503324)

[7] C. Geiss, B. Leclerc, J. Schröer, *Partial flag varieties and preprojective algebras*, Ann. de l'Institut Fourier **58** (2008) 825–876 (arXiv:math/0609138)

[8] C. Geiss, B. Leclerc, J. Schröer, *Cluster structures on quantum coordinate rings*, Selecta Math. **19** (2013) 337–397 (arXiv:1104.0531)

[9] C. Geiss, B. Leclerc, J. Schröer, *Quantum cluster algebras and their specialization*, preprint arXiv:1807.09826

[10] J. Grabowski, S. Launois, *Quantum cluster algebra structures on quantum Grassmannians and their quantum Schubert cells: the finite-type cases*, Int. Math. Res. Not. **10** (2011) 2230–2262 (arXiv:0912.4397)

[11] J. Grabowski, S. Launois, *Graded quantum cluster algebras and an application to quantum Grassmannians*, Proc. L.M.S. **109** (2014) 697–732 (arXiv:1301.2133)

[12] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, L.M.S. Lecture Notes **119**, C.U.P. 1988.

[13] B.T. Jensen, A. King, X. Su, *A categorification of Grassmannian cluster algebras*, Proc. L.M.S. **113** (2016) 185–212 (arXiv:1309.7301)

[14] A.C. Kelly, T.H. Lenagan, L. Rigal, *Ring theoretic properties of quantum Grassmannians*, J. Algebra Appl. **03** (2004) 9–30 (arXiv:math/0208152)

[15] S. Launois, T. Lenagan, *Twisting the quantum grassmannian*, Proc. A.M.S. **139** (2011) 99–110 (arXiv:0910.0208)

[16] B. Leclerc, A. Zelevinsky, *Quasicommuting families of quantum Plücker coordinates*, in: Kirillov's Seminar on Representation Theory, A.M.S. Transl. Ser. 2 **181** (1998) 85–108

[17] S. Oh, A. Postnikov, D. Speyer, *Weak separation and plabic graphs*, Proc. L.M.S. **110** (2015) 721–754 (arXiv:1109.4434)

[18] K. Rietsch, L. Williams, *Newton-Okounkov bodies, cluster duality, and mirror symmetry for Grassmannians*, preprint arXiv:1712.00447

[19] J. Scott, *Grassmannians and cluster algebras*, Proc. L.M.S. **92** (2006) 345–380 (arXiv:math/0311148)

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