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Optimal Distributed Control of an Extended Model of Tumor Growth with Logarithmic Potential

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Abstract
This paper is intended to tackle the control problem associated with an extended phase field system of Cahn–Hilliard type that is related to a tumor growth model. This system has been investigated in previous contributions from the viewpoint of well-posedness and asymptotic analyses. Here, we aim to extend the mathematical studies around this system by introducing a control variable and handling the corresponding control problem. We try to keep the potential as general as possible, focusing our investigation towards singular potentials, such as the logarithmic one. We establish the existence of optimal control, the Lipschitz continuity of the control-to-state mapping and even its Fréchet differentiability in suitable Banach spaces. Moreover, we derive the first-order necessary conditions that an optimal control has to satisfy.

Keywords Distributed optimal control · Tumor growth · Phase field model · Cahn–Hilliard equation · Optimal control · Necessary optimality conditions · Adjoint system

Mathematics Subject Classification 35K61 · 35Q92 · 49J20 · 49K20 · 92C50

1 Introduction

In this paper, we deal with a distributed optimal control problem for a system of partial differential equations whose physical context is that of tumor growth dynamics. Our aim is to devote this section to explain the general purpose of the work and we postpone all the technicalities for the forthcoming sections. In the next one, we will state precisely the problem and have the care to present in detail our notation and the mathematical framework in which set the problem. Here, let us only mention that with

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$\Omega \subset \mathbb{R}^3$ we denote the set where the evolution takes place and, for a given final time $T > 0$, we fix

$$Q := \Omega \times (0, T) \quad \text{and} \quad \Sigma := \Gamma \times (0, T).$$

The distributed control problem, referred as **(CP)**, consists of minimizing the so-called cost functional

$$\begin{aligned}\mathcal{J}(\varphi, \sigma, u) = {} & \frac{b_1}{2}\|\varphi - \varphi_Q\|^2_{L^2(Q)} + \frac{b_2}{2}\|\varphi(T) - \varphi_\Omega\|^2_{L^2(\Omega)} + \frac{b_3}{2}\|\sigma - \sigma_Q\|^2_{L^2(Q)} \\ & + \frac{b_4}{2}\|\sigma(T) - \sigma_\Omega\|^2_{L^2(\Omega)} + \frac{b_0}{2}\|u\|^2_{L^2(Q)},\end{aligned} \tag{1.1}$$

subject to the control constraints

$$u \in \mathcal{U}_{\text{ad}} := \{u \in \mathcal{L}^\infty(Q) : u_* \le u \le u^* \text{ a.e. in } Q\}, \tag{1.2}$$

and to the state system

$$\alpha\partial_t\mu + \partial_t\varphi - \Delta\mu = P(\varphi)(\sigma - \mu) \quad \text{in } Q \tag{1.3}$$

$$\mu = \beta\partial_t\varphi - \Delta\varphi + F'(\varphi) \quad \text{in } Q \tag{1.4}$$

$$\partial_t\sigma - \Delta\sigma = -P(\varphi)(\sigma - \mu) + u \quad \text{in } Q \tag{1.5}$$

$$\partial_n\mu = \partial_n\varphi = \partial_n\sigma = 0 \quad \text{on } \Sigma \tag{1.6}$$

$$\mu(0) = \mu_0,\ \varphi(0) = \varphi_0,\ \sigma(0) = \sigma_0 \quad \text{in } \Omega. \tag{1.7}$$

Let us give just some overall indications on the involved quantities of the above equations. The symbols b_0, b_1, b_2, b_3, b_4 represent nonnegative constants, not all zero, while $\varphi_Q, \varphi_\Omega, \sigma_Q, \sigma_\Omega, u_*$, and u^* denote given functions. As regards these latter, the first four model some targets, while the last two fix the box in which the control variable u can be chosen. Furthermore, F and P are nonlinearities, while (1.6) and (1.7) are the boundary conditions and the initial conditions, respectively.

During the last decades, lots of models based on continuum mixture theory have been derived. The above state system constitutes a variation on an approximation to a diffuse interface model for the dynamics of tumor growth proposed in [23] (see also [24,28]), in which the velocity contributions are neglected and the attention is focused on the behavior of the state variables that model the fractions of the tumor cells and the nutrient-rich extracellular water, respectively. Moreover, let us refer to [16–21], where transport mechanisms such as chemotaxis and active transport are also taken into account. Further investigations and mathematical models related to biology can be found e.g. in [13] and [15].

Let us spend some words about the interpretation of the system (1.3)–(1.7), and on the involved variables. The unknown φ is an order parameter which describes the tumor cell fraction and assumes values between -1 and $+1$. These two extremes represent the pure phases, say the tumor phase and the healthy cell phase, respectively. The second unknown μ has the interpretation, as usual for Cahn–Hilliard equation, of chemical potential and its relation with φ is precisely expressed by (1.4). The third unknown σ consists of the nutrient-rich extracellular water volume fraction and we assume that it takes values between 0 and 1 with the following property: the closer to one, the richer of water the extracellular fraction is, while the closer to zero, the poorer it is. As the nonlinearities are concerned, we have that F stands for a double-well potential, while P models a proliferation function which we assume to be nonnegative and dependent on the phase variable. To conclude the overview of the model, worth to point out the different role of α and β. When $\alpha = 0$, Eqs. (1.3)–(1.4) becomes of viscous Cahn–Hilliard type or it is pure Cahn–Hilliard equation depending on the fact that β is strictly positive or vanishes, respectively. On the other hand, the presence of α gives to (1.3) a parabolic structure with respect to the variable μ.

As for the interpretation of the **(CP)** problem, our goal consists of finding a "smarter" choice of $u \in \mathcal{U}_{\mathrm{ad}}$ such that, with its corresponding solution to (1.3)–(1.7), minimizes (1.1). Note that the control variable u appears in (1.5), the equation describing the nutrient evolution process. Thus, from the viewpoint of the model, it could represent a supply of a nutrient or a drug in chemotherapy. The cost functional we choose is a tracking-type one, namely we have fixed some a priori targets, say some a priori final configurations for the tumor cells and on the nutrient, and we try to find the control variable whose corresponding solutions approximate better this fixed configuration. Worth to insist on this fact: even if the better situation is the health of the patient, our efforts are neither in the direction of minimizing the variable φ, that has the meaning of leading to the healthier configuration nor minimizing the variable σ to reduce the tumor expansion. In fact, we only try to handle the whole evolution process, acting on the choice of the control variable, to force a final configuration that for some practical reason should be desirable. Obviously the ratios among the constants b_0, b_1, b_2, b_3, b_4 implicitly describe which targets hold the leading part in our application. To conclude the analysis, we focus our attention on the last term of (1.1). From an abstract viewpoint, it represents the cost we have to pay to implement u, thus in our framework it should be read as the rate of risks to afflict harm to the patient by following that strategy. Finally, observe that we do not consider the cost functional to be dependent on the chemical potential. Indeed, from an interpretation point of view, we mainly care to handle the phase dynamics, and it is not clear if including the variable μ in the analysis is interesting for applications (see the forthcoming Remark 4.1).

At this general stage, let us perform a little overview of the literature. The first systematic study on this system was carried out in [4,14], where well-posedness and long-time behavior of the solutions were investigated for a system very close to ours. Moreover, quite recently, the system has been investigated with particular interest on the asymptotic analysis as the constants α and β go to zero. To this concern, we address to [4,9], and [5], where the asymptotic analyses represent the core of the works. To our best knowledge, as the control theory is concerned, there are very few contributions to this kind of system. In this regard, we refer to [8], where a control problem for a system without relaxation terms is performed. Even though we take inspiration from this work, the functional framework and the potentials setting significantly differ from ours. Nevertheless, the control theory related to different phase-field models based on the Cahn–Hilliard equation presents more contributions. Among others, we mention [2,6,7,10–12]. Furthermore, since particular attention is devoted to singular potentials, we point out [3,22,26] and the vast list of references therein.

To conclude, let us sketch an outline of the work. The first section is devoted to fix our notation and state the established results. The second one contains all the proofs corresponding to the analysis of the state system, while the last one is completely devoted to the control problem. Namely, the last section faces the analysis of the existence of optimal control, the linearized problem, the investigation of the Fréchet differentiability of the control-to-state mapping and the adjoint problem. Moreover, it contains the necessary conditions that a control has to satisfy to be optimal.

2 General Assumptions and Results

In the following, we intend to fix the notation, state the problem in a precise form, and announce the main results.

The introduction should not have created any confusion since the employed notation is quite standard. We assume Ω to be a smooth, bounded and connected open set in $\mathbb{R}^3$, whose boundary is denoted by Γ. From the smoothness property, it is almost everywhere well defined the unit normal vector n of Γ and the symbol ∂_n represents the outward derivative in that direction. Moreover, for a fixed $T > 0$, which stands for the final time involved in the evolution process, we set

$$\begin{aligned} &Q_t := \Omega \times (0,t) \quad \text{and} \quad \Sigma_t := \Gamma \times (0,t) \quad \text{for every } t \in (0,T], \\ &Q := Q_T, \quad \text{and} \quad \Sigma := \Sigma_T. \end{aligned}$$

As the functional spaces are concerned, it turns out to be very convenient to introduce the following

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \left\{ v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \Gamma \right\},$$

and endow them with their standard norms indicated by $\| \cdot \|_\bullet$, where $\bullet$ stands for the referred space or is completely omitted if it is clear from the context which norm should be. In the same way, we write $\| \cdot \|_p$ for the usual norm in $L^p(\Omega)$. The above definitions yield that (V, H, V^*) forms a Hilbert triplet, that is, the following injections $V \subset H \equiv H^* \subset V^*$ are both continuous and dense. As a consequence, we also have that $\langle u, v \rangle = \int_\Omega uv$ for every $u \in H$ and $v \in V$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the dual V^* and V itself.

Now, we state the general assumptions on the problem.

H1 b_0, b_1, b_2, b_3, b_4 are nonnegative constants, but not all zero.
H2 $\varphi_Q, \sigma_Q \in L^2(Q), \varphi_\Omega, \sigma_\Omega \in H^1(\Omega), u_*, u^* \in L^\infty(Q)$ with
$u_* \leq u^*$ a.e. in Q.
H3 $\alpha, \beta > 0$.
H4 $\mu_0 \in H^1(\Omega) \cap L^\infty(\Omega), \varphi_0 \in H^2(\Omega), \sigma_0 \in H^1(\Omega)$.
H5 $P \in C^2(\mathbb{R})$ is nonnegative, bounded and Lipschitz continuous.
H6 $\widehat{B} : \mathbb{R} \to [0, \infty]$ is convex, proper and lower semicontinuous, with $\widehat{B}(0) = 0$.
H7 $\widehat{\pi} \in C^3(\mathbb{R})$ and $\pi := \widehat{\pi}'$ is Lipschitz continuous.

We define the potential $F : \mathbb{R} \to [0, \infty]$ and the graph $B \subseteq \mathbb{R} \times \mathbb{R}$ by

$$F := \widehat{B} + \widehat{\pi} \quad \text{and} \quad B := \partial \widehat{B}, \tag{2.1}$$

and note that B is a maximal monotone operator (see, e.g., [1, Ex. 2.3.4, p. 25]) with domain denoted by $D(B)$. Furthermore, we assume that B, when restricted to its domain $D(B)$, is a smooth function. Indeed, we require that

H8 $D(B) = (r_-, r_+)$, with $-\infty \le r_- < 0 < r_+ \le +\infty$, $B(0) = 0$, $F_{|D(B)} \in C^3(r_-, r_+)$, and $\lim_{r \to r_\pm} F'(r) = \pm\infty$.
H9 $r_- < \inf \varphi_0 \le \sup \varphi_0 < r_+$.
H10 $1/\beta\,(\mu_0 + \Delta\varphi_0 - B(\varphi_0) - \pi(\varphi_0)) \in L^2(\Omega)$.

It is worth to underline that from the above requirements, it follows that both $\widehat{B}(\varphi_0)$ and $B(\varphi_0)$ are both in $L^\infty(\Omega)$, thus a fortiori in $L^1(\Omega)$. In the literature, with a slight abuse of notation, F' usually denotes the sum of B, the subdifferential of $\widehat{B}$, and π, namely $F' = B + \pi$. Here, since F is regular, B exactly represents the derivative of $\widehat{B}$ in (r_-, r_+).

Notwithstanding (**H6**)–(**H10**), let us point out that there are significant classes of double-well potentials that fit the assumptions. Standard choices are the regular potential and the, physically more relevant, logarithmic one. Written as (2.1), they read as

$$F_{reg}(r) := \frac{1}{4}(r^2 - 1)^2 = \frac{1}{4}r^4 - \frac{1}{4}(2r^2 - 1) \quad \text{for } r \in \mathbb{R}, \tag{2.2}$$

$$F_{log}(r) := ((1 - r)\log(1 - r) + (1 + r)\log(1 + r)) - kr^2 \quad \text{for } |r| < 1, \tag{2.3}$$

where in the latter k is a constant large enough to kill convexity. Moreover, it is usually helpful to extend (2.3) by continuity imposing that it assumes the value $+\infty$ outside its actual domain. Note that both (2.2) and (2.3) do fit our framework with $D(B) = (-\infty, +\infty)$ and $D(B) = (-1, +1)$, respectively. Furthermore, if we take into account F_{reg}, due to its regularity, all the results we are going to prove still hold true even in a slightly weaker framework. However, since F_{reg} is introduced as an approximation of more general potentials, we try to focus our attention on the singular ones, such as F_{log}, which is more relevant for the applications. Before starting with the statements, we introduce another notation.

Let $\mathcal{U}_R$ be an open set in $L^2(Q)$ such that $\mathcal{U}_{\text{ad}} \subset \mathcal{U}_R$ and $\|u_2\| \le R$ for all $u \in \mathcal{U}_R$.

As it usually occurs in control problems, the requirements (**H2**)–(**H10**) are far from sharp in terms of the well-posedness and regularity result of (1.3)–(1.7) are concerned. Anyhow, they are all useful in order to deal with the corresponding control problem.

Let us proceed this section by listing the obtained results.

Theorem 2.1 (well-posedness and separation results) *Under the hypotheses* (**H2**)–(**H10**)*, and for every $u \in \mathcal{U}_R$, the following results hold true.*

(i) *The system* (1.3)–(1.7) *has a unique strong solution* (μ, φ, σ) *which satisfies*

$$\varphi \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \subset C^0([0,T];C^0(\bar{\Omega})) \tag{2.4}$$

$$\mu, \sigma \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \subset C^0([0,T];V) \tag{2.5}$$

$$\mu \in L^\infty(Q) \tag{2.6}$$

that is, there exists a constant $C_1 > 0$*, which depends on* R*,* α *and* β*, and on the data of the system, such that*

$$\begin{aligned}&\|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W)} + \|\mu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)\cap L^\infty(Q)}\\ &\quad+\|\sigma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le C_1.\end{aligned} \tag{2.7}$$

(ii) *There exists a compact subset* K *of* (r_-, r_+) *such that*

$$\varphi(x,t) \in K \quad \textit{for all } (x,t) \in Q; \tag{2.8}$$

in particular, there exists a constant $C_2 > 0$*, which depends on* R*,* α *and* β*,* K *and on the data of the system, such that*

$$\|\varphi\|_{C^0(\overline{Q})} + \max_{0\le i\le 3} \|F^{(i)}(\varphi)\|_{L^\infty(Q)} + \max_{0\le j\le 2} \|P^{(j)}(\varphi)\|_{L^\infty(Q)} \le C_2. \tag{2.9}$$

Theorem 2.2 (continuous dependence on the control) *Assume* (**H2**)–(**H10**)*. Then there exists a constant* $C_3 > 0$*, which depends only on* R*,* α *and* β*, and on the data of the system such that, if* $u_i \in \mathcal{U}_R$ *and* $(\mu_i, \varphi_i, \sigma_i)$ *are the corresponding solutions with the same initial value,* $i = 1, 2$*, it holds*

$$\begin{aligned}&\|\alpha(\mu_1 - \mu_2) + (\varphi_1 - \varphi_2) + (\sigma_1 - \sigma_2)\|_{L^\infty(0,T;V^*)} + \|\mu_1 - \mu_2\|_{L^2(0,T;H)}\\ &\quad+\|\varphi_1 - \varphi_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\sigma_1 - \sigma_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}\\ &\le C_3\|u_1 - u_2\|_{L^2(0,T;H)}.\end{aligned} \tag{2.10}$$

Let us remark that in the proof of the above result we do not account for point (ii) of Theorem 2.1. In fact, we will see that this first continuous dependence result is not sufficient to handle the **(CP)** (particularly to prove the Fréchet differentiability of the control-to-state mapping $\mathcal{S}$, cf. Sect. 4.3), then in the beneath lines there is an improvement that, this time, take strongly into account the second part of Theorem 2.1.

Theorem 2.3 *In the same framework of Theorem* 2.2*, there exists a constant* $C_4 > 0$*, possibly smaller than* C_3*, which depends only on* R*,* α *and* β*, and on the data of the system such that*

$$\begin{aligned}&\|\mu_1 - \mu_2\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi_1 - \varphi_2\|_{H^1(0,T;H)\cap L^\infty(0,T;V)}\\ &\quad\le C_4\|u_1 - u_2\|_{L^2(0,T;H)}.\end{aligned} \tag{2.11}$$

Since we have already provided the well-posedness of the state system in Theorem 2.1, we can introduce the so-called control-to-state mapping $\mathcal{S}$ that will cover a central role in the control theory. It consists of the map that assigns to every admissible control u the corresponding solution triple (μ, φ, σ), which components belong to the functional spaces pointed out by (2.4)–(2.6). Moreover, it allows us to present the so-called reduced cost functional as follows

$$\begin{aligned}&\tilde{\mathcal{J}} : \mathcal{U}_R \to \mathbb{R}, \quad \text{defined by} \quad \tilde{\mathcal{J}}(u) := \mathcal{J}(\mathcal{S}_{2,3}(u), u),\\ &\text{where } \mathcal{S}_{2,3}(u) \text{ represents the couple of the second and third components}\\ &\quad \text{of the solution triple } \mathcal{S}(u) = (\mu, \varphi, \sigma).\end{aligned} \tag{2.12}$$

In this view, Theorem 2.2 established the Lipschitz continuity of $\mathcal{S}$ in this natural functional framework.

At this point, we introduce a well-posedness result for the linearized system, which comes out naturally from the investigation of the control problem. First of all, let us present the mentioned problem. Fixed $\bar{u} \in \mathcal{U}_R$, we denote $(\bar{\mu}, \bar{\varphi}, \bar{\sigma}) = \mathcal{S}(\bar{u})$ the corresponding solution to (1.3)–(1.7). Then, for any $h \in L^2(Q)$, the linearized system reads as

$$\alpha \partial_t \eta + \partial_t \vartheta - \Delta \eta = P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu})\vartheta + P(\bar{\varphi})(\rho - \eta) \quad \text{in } Q \tag{2.13}$$

$$\eta = \beta \partial_t \vartheta - \Delta \vartheta + F''(\bar{\varphi})\vartheta \quad \text{in } Q \tag{2.14}$$

$$\partial_t \rho - \Delta \rho = -P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu})\vartheta - P(\bar{\varphi})(\rho - \eta) + h \quad \text{in } Q \tag{2.15}$$

$$\partial_n \rho = \partial_n \vartheta = \partial_n \eta = 0 \quad \text{on } \Sigma \tag{2.16}$$

$$\rho(0) = \vartheta(0) = \eta(0) = 0 \quad \text{in } \Omega. \tag{2.17}$$

Here the existence and uniqueness result follows.

Theorem 2.4 (well-posedness of the linearized system) *Under the assumptions* (**H2**)–(**H10**)*, and for every $h \in L^2(Q)$, the system* (2.13)–(2.17) *possesses a unique solution triple (η, ϑ, ρ) which satisfies*

$$\eta, \vartheta, \rho \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \subset C^0([0, T]; V); \tag{2.18}$$

that is, there exists a constant $C_5 > 0$, which depends on the data of the system, and possibly on α and β, such that

$$\begin{aligned}&\|\eta\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\vartheta\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)}\\ &\quad + \|\rho\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \le C_5.\end{aligned}$$

In the following, we prove that $\mathcal{S}$ is even Fréchet differentiable in suitable Banach spaces.

Theorem 2.5 (Fréchet differentiability of $\mathcal{S}$) *Assume* (**H2**)–(**H10**). *Then the control-to-state mapping $\mathcal{S}$ is Fréchet differentiable in $\mathcal{U}_R$ as a mapping from $L^2(Q)$ into the state space $\mathcal{Y}$, where*

$$\mathcal{Y} := \Big(H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \Big)^3. \tag{2.19}$$

Moreover, for any $\bar{u} \in \mathcal{U}_R$, the Fréchet derivative $D\mathcal{S}(\bar{u})$ is a linear and continuous operator from $L^2(Q)$ to $\mathcal{Y}$, and for every $h \in L^2(Q)$, $D\mathcal{S}(\bar{u})h = (\eta, \vartheta, \rho)$ where (η, ϑ, ρ) is the unique solution to the linearized system (2.13)–(2.17) associated with h.

Theorem 2.6 (Existence of optimal control) *Assume* (**H2**)–(**H10**). *Then the optimal control problem* **(CP)** *has at least a solution $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$.*

As the necessary optimality condition is concerned, we recall the reduced cost functional (2.12) and the fact that $\mathcal{U}_{\mathrm{ad}}$ is convex. Therefore, the optimal inequality we are looking for turns out to be

$$\langle D\tilde{\mathcal{J}}(\bar{u}), v - \bar{u} \rangle \geq 0 \quad \text{for every } v \in \mathcal{U}_{\mathrm{ad}}, \tag{2.20}$$

where $D\tilde{\mathcal{J}}(\bar{u})$ represents the differential of $\tilde{\mathcal{J}}$, at least in the Gâteaux sense. Accounting for Theorem 2.5 and the chain rule, (2.20) develops as follows.

Corollary 2.7 *Suppose that the assumptions* (**H1**)–(**H10**) *are fulfilled. Let $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ be an optimal control for* **(CP)** *with his corresponding optimal state $(\bar{\mu}, \bar{\varphi}, \bar{\sigma}) = \mathcal{S}(\bar{u})$. Then we have*

$$\begin{aligned} &b_1 \int_Q (\bar{\varphi} - \varphi_Q)\vartheta + b_2 \int_\Omega (\bar{\varphi}(T) - \varphi_\Omega)\vartheta(T) + b_3 \int_Q (\bar{\sigma} - \sigma_Q)\rho \\ &\quad + b_4 \int_\Omega (\bar{\sigma}(T) - \sigma_\Omega)\rho(T) + b_0 \int_Q \bar{u}(v - \bar{u}) \geq 0 \quad \forall v \in \mathcal{U}_{\mathrm{ad}}, \end{aligned} \tag{2.21}$$

where ϑ and ρ are the second and third components of the unique solution triple (η, ϑ, ρ) to the linearized system (2.13)–(2.17) associated with $h = v - \bar{u}$.

To eliminate the presence of the variables ϑ and ρ in the previous inequality, we introduce the so-called adjoint problem that consists of the following system of partial differential equations.

$$\beta\partial_t q - \partial_t p + \Delta q - F''(\bar{\varphi})q + P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu})(r - p) = b_1(\bar{\varphi} - \varphi_Q) \quad \text{in } Q \tag{2.22}$$

$$q - \alpha\partial_t p - \Delta p + P(\bar{\varphi})(p - r) = 0 \quad \text{in } Q \tag{2.23}$$

$$-\partial_t r - \Delta r + P(\bar{\varphi})(r - p) = b_3(\bar{\sigma} - \sigma_Q) \quad \text{in } Q \tag{2.24}$$

$$\partial_n q = \partial_n p = \partial_n r = 0 \quad \text{on } \Sigma \tag{2.25}$$

$$\begin{aligned} &p(T) - \beta q(T) = b_2(\bar{\varphi}(T) - \varphi_\Omega), \\ &\alpha p(T) = 0, \quad r(T) = b_4(\bar{\sigma}(T) - \sigma_\Omega) \quad \text{in } \Omega. \end{aligned} \tag{2.26}$$

Here the existence and uniqueness result for the adjoint problem is stated.

Theorem 2.8 (Well-posedness of the adjoint problem) *Under the assumptions* (**H1**)–(**H10**)*, the system* (2.22)–(2.26) *has a unique solution* (q, p, r) *that satisfies the following regularity requirements*

$$q, p, r \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \subset C^0([0, T]; V). \tag{2.27}$$

Finally, the well-posedness of the adjoint system allows us to improve Corollary 2.7 leading to a second necessary condition. Namely, we achieve the following result.

Theorem 2.9 (Necessary optimality condition) *Assume* (**H1**)–(**H10**). *Let* $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ *be an optimal control with his corresponding optimal state* $(\bar{\mu}, \bar{\varphi}, \bar{\sigma}) = \mathcal{S}(\bar{u})$ *and let* (p, q, r) *be the solution to the corresponding adjoint system. Then we have*

$$\int_Q (r + b_0\bar{u})(v - \bar{u}) \geq 0 \quad \forall v \in \mathcal{U}_{\mathrm{ad}}. \tag{2.28}$$

To conclude the section, let us introduce further notation and recall some well-known inequalities and general facts related to the Cahn–Hilliard equation. First of all, we remind the Young inequality

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \geq 0 \text{ and } \delta > 0. \tag{2.29}$$

Furthermore, for given $v \in V^*$ and $\underline{v} \in L^1(0, T; V^*)$, we introduce their generalized mean values $v^\Omega \in \mathbb{R}$ and $\underline{v}^\Omega \in L^1(0, T)$ by

$$v^\Omega := \frac{1}{|\Omega|} \langle v, 1 \rangle, \quad \text{and} \quad \underline{v}^\Omega(t) := \big(\underline{v}(t)\big)^\Omega \quad \text{for a.a. } t \in (0, T), \tag{2.30}$$

where (2.30) reduces to the usual mean values when it is applied to elements of H or $L^1(0, T; H)$. In addition, we often owe to the Poincaré inequality

$$\|v\|_V^2 \leq C_\Omega\big(\|\nabla v\|_H^2 + |v^\Omega|^2\big) \quad \text{for every } v \in V, \tag{2.31}$$

where we stressed the fact that C_Ω depends on Ω. Since it will be convenient to interpret some partial differential equations in the framework of the Hilbert triplet (V, H, V^*), we introduce the Riesz isomorphism associated with V. That is, we define the map $\mathcal{A} : V \to V^*$ as follows

$$\langle \mathcal{A}u, v \rangle = (u, v)_V = \int_\Omega (\nabla u \cdot \nabla v + uv) \quad \text{for every } u, v \in V. \tag{2.32}$$

Observe that when restricted to its domain W, $\mathcal{A}$ turns out to be the operator $-\Delta + I$ endowed with homogeneous Neumann boundary conditions, where I denotes the identity map of W. A little investigation on $\mathcal{A}$ leads to the following identities

$$\langle \mathcal{A}u, \mathcal{A}^{-1}v^* \rangle = \langle v^*, u \rangle \quad \text{for all } u \in V \text{ and } v^* \in V^*, \tag{2.33}$$

$$\langle u^*, \mathcal{A}^{-1} v^* \rangle = (u^*, v^*)_* \quad \text{for all } u^*, v^* \in V^*, \tag{2.34}$$

where $(\cdot, \cdot)_*$ stands for the inner product of V^*, whence also

$$2\langle \partial_t v^*(t), \mathcal{A}^{-1} v^*(t) \rangle = \frac{d}{dt} \|v^*(t)\|_*^2 \quad \text{for a.a. } t \in (0, T). \tag{2.35}$$

Remark 2.10 Let us explain a convention that we are going to use throughout the paper. Since we have to deal with a lot of estimates, we agree that the symbol c stands for any constants which depend only on the final time T, on Ω, the shape of the nonlinearities, on the norms of the involved functions, and possibly on α and β. For this reason, the meaning of c might change from line to line and even in the same chain of inequalities. Conversely, the capital letters are devoted to denote precise constants.

3 State System and Continuous Dependence Results

From this section on, we will focus our attention to prove the statements. This section is devoted to the investigation of the state system, namely we aim at checking Theorems 2.1, 2.2, and 2.3. Let us begin dealing with the first one.

Proof of Theorem 2.1 In [4, Thm. 2.2, p. 2426] it has been shown that the system (1.3)–(1.7) possesses a unique strong solution with the following regularity

$$\mu, \varphi, \sigma \in H^1(0, T; H) \cap L^2(0, T; W) \subset C^0([0, T]; V),$$

in the homogeneous case $u \equiv 0$. Since we admit that u can be chosen in $\mathcal{U}_R$, only straightforward modifications are needed in order to prove that, for every choice of u in $\mathcal{U}_R$, there exists a unique corresponding solution (μ, φ, σ) satisfying the same regularity mentioned above. Let us point out that conditions (**H2**)–(**H10**) perfectly fit the framework of [4]. In fact, the strong requirement (2.6) of [4] is only needed to handle the asymptotic behavior, whereas it can be substituted by the weak requirement (**H7**) as the investigation of the well-posedness and regularity of the system are concerned.

In the following, we proceed formally; as a matter of fact, we should introduce suitable approximation of the potential depending on a small parameter ε and then, after showing sufficient compactness property, let $\varepsilon \searrow 0$ as made in [4]. Anyhow, we will take care in referring to works in which this strategy is properly employed to justify all the passages that we present only in a formal level.

With the following estimates, we aim at improving the regularity of the unique solution to (1.3)–(1.7) in view of the forthcoming control investigation. Once obtained, it is a standard matter to conclude by compactness arguments that the solution triple satisfies (2.4)–(2.6). The rigorous treatment of the first three estimates, with little variations, can be found in [4, eqs. (4.4)–(4.12), pp. 2431–2432].

First estimate We add to both the sides of (1.4) the term φ, multiply (1.3) by μ, this new second equation by $-\partial_t \varphi$ and (1.5) by σ, then we add the resulting equations and integrate over Q_t and by parts. A little rearrangements of the terms produce

$$
\begin{aligned}
&\frac{\alpha}{2}\int_\Omega |\mu(t)|^2 + \int_{Q_t} |\nabla\mu|^2 + \beta\int_{Q_t} |\partial_t\varphi|^2 + \frac{1}{2}\int_\Omega |\varphi(t)|^2 + \frac{1}{2}\int_\Omega |\nabla\varphi(t)|^2 \\
&\quad + \int_\Omega \widehat{B}(\varphi(t)) + \frac{1}{2}\int_\Omega |\sigma(t)|^2 + \int_{Q_t} |\nabla\sigma|^2 + \int_{Q_t} P(\varphi)(\sigma-\mu)^2 \\
&= \frac{\alpha}{2}\int_\Omega |\mu_0|^2 + \frac{1}{2}\int_\Omega |\varphi_0|^2 + \frac{1}{2}\int_\Omega |\nabla\varphi_0|^2 + \int_\Omega \widehat{B}(\varphi_0) \\
&\quad + \frac{1}{2}\int_\Omega |\sigma_0|^2 + \int_{Q_t} u\sigma + \int_{Q_t} \varphi\,\partial_t\varphi - \int_{Q_t} \pi(\varphi)\partial_t\varphi,
\end{aligned}
$$

where we split F' as sum of B and π, and where the former, multiplied by $\partial_t\varphi$, consists of the derivative with respect to time of $\widehat{B}(\varphi(t))$. All the terms on the left-hand side are nonnegative since they are squares and P and $\widehat{B}$ are nonnegative by (**H5**) and (**H6**), respectively. The first five terms on the right-hand side are easily managed owing to (**H4**) and to the properties of $\widehat{B}$, while the others were denoted by I_1, I_2, I_3. Accounting for the Young inequality (2.29), we obtain

$$
|I_1| + |I_2| + |I_3| \le \frac{1}{2}\int_{Q_t} |u|^2 + \frac{1}{2}\int_{Q_t} |\sigma|^2 + 2\delta\int_{Q_t} |\partial_t\varphi|^2 + c_\delta\int_{Q_t} (|\varphi|^2+1).
$$

We choose $0<\delta<\beta/2$, and invoke the Gronwall lemma to conclude that

$$
\begin{aligned}
&\|\mu\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\widehat{B}(\varphi)\|_{L^\infty(0,T;L^1(\Omega))} \\
&\quad + \|\sigma\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c.
\end{aligned}
\tag{3.1}
$$

Second estimate Now we multiply (1.3) by $\partial_t\mu$ and (1.5) by $\partial_t\sigma$, add the resulting equations and integrate over Q_t. Using (3.1) and the boundedness of P, and arguing as above lead to

$$
\|\mu\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\sigma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c. \tag{3.2}
$$

Third estimate Equations (1.3) and (1.5) show a parabolic structure with respect to μ and σ, respectively. Moreover, it follows from the previous estimates that their forcing terms are both in $L^2(0,T;H)$. Therefore, since the initial data (1.7) are in V (cf. (**H4**)), parabolic regularity theory with Neumann homogeneous boundary conditions gives

$$
\|\mu\|_{L^2(0,T;W)} + \|\sigma\|_{L^2(0,T;W)} \le c. \tag{3.3}
$$

Fourth estimate As above we would like to obtain more regularity for the phase variable φ by comparing terms in (1.4). Since it is more delicate, worth to show the detailed procedure. In fact, we can rearrange (1.4) as follows

$$
-\Delta\varphi + B(\varphi) = f, \quad \text{where } f := \mu - \beta\,\partial_t\varphi - \pi(\varphi). \tag{3.4}
$$

The previous estimates entails that $f \in L^2(0,T;H)$. Now, we multiply the above inequality by $(-\Delta\varphi)$ and integrate over Ω. Actually, note that this choice is rigorously forbidden since too few regularity is known on the phase variable. Anyhow, this choice can be formally justified by introducing a suitable Faedo–Galerkin scheme. So, for a.a. $t \in (0,T)$, we get the following inequality

$$\int_\Omega |\Delta\varphi(t)|^2 + \int_\Omega B'(\varphi(t))\,|\nabla\varphi(t)|^2 \;\le -\int_\Omega f(t)\,\Delta\varphi(t) \le \frac{1}{2}\int_\Omega |\Delta\varphi(t)|^2 + \frac{1}{2}\int_\Omega |f(t)|^2,$$

owing to (2.29). Both the terms on the left-hand side are nonnegative since B' is so. Hence, we realize that

$$\|\Delta\varphi\|_{L^2(0,T;H)} \le c.$$

Moreover, by elliptic regularity theory, the boundary conditions (1.6), and by comparison in (3.4), we conclude that

$$\|\varphi\|_{L^2(0,T;W)} + \|B(\varphi)\|_{L^2(0,T;H)} \le c. \tag{3.5}$$

Fifth estimate We continue to proceed formally, in order to keep the proof as short and easy as possible. Here, for a precise and detailed treatment it will be necessary to introduce time steps and suitable translations, and show some estimates for this new functions. This procedure will become quite technical. Anyhow, for the interested reader, we refer to [4, Proof of Thm. 2.6 (iii), p. 2436], where the correct procedure is performed to establish a slightly different estimate.

So, we differentiate (1.4) with respect to the time variable, multiply it by $\partial_t\varphi$, and integrate over Q_t to get

$$\int_{Q_t} \partial_t\mu\,\partial_t\varphi = \beta\int_{Q_t} \partial_{tt}\varphi\,\partial_t\varphi - \int_{Q_t} (\Delta\partial_t\varphi)\,\partial_t\varphi + \int_{Q_t} (B'(\varphi)+\pi'(\varphi))\,|\partial_t\varphi|^2.$$

Using the integration by parts and the boundary conditions (1.6), we deduce that

$$\frac{\beta}{2}\int_\Omega |\partial_t\varphi(t)|^2 + \int_{Q_t} |\nabla\partial_t\varphi|^2 + \int_{Q_t} B'(\varphi)|\partial_t\varphi|^2 = \frac{\beta}{2}\int_\Omega |(\partial_t\varphi)(0)|^2 - \int_{Q_t} \pi'(\varphi)|\partial_t\varphi|^2 + \int_{Q_t} \partial_t\mu\,\partial_t\varphi,$$

where the terms on the left-hand side are all nonnegative. The first term of the right-hand side is under control, due to (1.4) and (**H10**). Moreover, the last two integrals can be estimate as follows

$$-\int_{Q_t} \pi'(\varphi)|\partial_t\varphi|^2 + \int_{Q_t} \partial_t\mu\,\partial_t\varphi \le c\int_{Q_t} |\partial_t\varphi|^2 + \frac{1}{2}\int_{Q_t} |\partial_t\mu|^2,$$

owing to the Lipschitz continuity of π' and (2.29). Thus, thanks to (3.1) and (3.2) we obtain

$$\|\varphi\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} \leq c. \tag{3.6}$$

Sixth estimate We take again into account equation (3.4). Due to the previous estimates, we can infer that f is more regular than we pointed out before. In fact, now we have that $f \in L^\infty(0, T; H)$. Therefore, the test by $-\Delta\varphi$ leads to the estimate $\|\Delta\varphi\|_{L^\infty(0,T;H)} \leq c$. Moreover, by the boundary conditions, the elliptic regularity and comparison in (3.4), we deduce that

$$\|\varphi\|_{L^\infty(0,T;W)} + \|B(\varphi)\|_{L^\infty(0,T;H)} \leq c, \tag{3.7}$$

which gives, by the Sobolev embeddings, also

$$\|\varphi\|_{L^\infty(Q)} \leq c. \tag{3.8}$$

Furthermore, an application of the well-known embedding results (see e.g., [27, Sect. 8, Cor. 4]) directly recovers the continuity of the solution variables. Namely, as the variables μ and σ are concerned, due to (3.1)–(3.3), we infer that they belong to $C^0([0, T]; V)$. Since the phase variable φ satisfies, in addition, the estimates (3.5) and (3.6)–(3.8), we deduce that φ is more regular and it belongs to $C^0([0, T]; C^0(\bar{\Omega}))$.

Now, we start to approach the separation result (ii). This property will be crucial in order to handle the potential and its higher order derivatives. Indeed, if (ii) holds true, it acts on functions which values are well detached from the boundary of the domain of B. In this way F and his higher order derivatives do not blow up and they turn out to be Lipschitz continuous and bounded functions.

First of all, we need to show the boundedness of the chemical potential in the whole of Q. In this direction, we would like to apply [25, Thm. 7.1, p. 181] to Eq. (1.3). The key point is the parabolic structure with respect to μ that (1.3) possesses. Indeed, by simply rearranging the terms, we get

$$\alpha\partial_t\mu - \Delta\mu = g, \quad \text{where } g := P(\varphi)(\sigma - \mu) - \partial_t\varphi.$$

Roughly speaking, the result formalizes the following idea: if the initial datum is bounded in Ω and the forcing term g satisfies a suitable summability regularity with respect to space and time, then it is natural to expect that the variable μ stay bounded in the whole of Q. Actually, from (**H4**) the property on the initial data is already satisfied. Moreover, (**H5**) and the previous estimates immediately yield that

$$\|g\|_{L^\infty(0,T;H)} \leq c.$$

This allows us to apply [25, Thm. 7.1, p. 181] and infer that there exists a positive constant c such that

$$\|\mu\|_{L^\infty(Q)} \leq c. \tag{3.9}$$

Note that (3.9) ends the proof of (i) and turns out to be fundamental in order to proceed with the second part of Theorem 2.1.

As before, let us emphasize that the estimate we are going to prove in the following is formal. Anyhow, it can be reproduced correctly by introducing a suitable approximation scheme, as made in [3, Proof of Thm. 2.6, pp. 992-994].

Seventh estimate We multiply (1.4) by $|B(\varphi)|^{p-1}\,\mathrm{sign}\varphi = |B(\varphi)|^{p-2}B(\varphi)$, for a fixed $p > 2$, and integrate over Q_t. Moreover, we set $f := \mu - \pi(\varphi)$ and observe that f belongs to $L^\infty(Q)$ due to (3.8), (3.9) and (**H7**). We infer, for every $t \in [0, T]$, that

$$\beta \int_\Omega \mathcal{B}_p(\varphi(t)) + (p-1)\int_{Q_t} B'(\varphi)|B(\varphi)|^{p-2}|\nabla\varphi|^2 + \int_{Q_t} |B(\varphi)|^p = \beta \int_\Omega \mathcal{B}_p(\varphi_0) + \int_{Q_t} f\,|B(\varphi)|^{p-1}\mathrm{sign}\varphi, \tag{3.10}$$

where $\mathcal{B}_p(r) := \int_0^r |B(s)|^{p-1}\mathrm{sign}s\,ds$. Furthermore, all the terms on the left-hand side are nonnegative. As the right-hand side is concerned, we manage the first term in the following way. From (**H9**) we know that $|B(\varphi_0)|$ is bounded by a positive constant M, hence, we infer that

$$\beta \int_\Omega \mathcal{B}_p(\varphi_0) \le \beta M^{p-1}\int_\Omega |\varphi_0| \le c^p.$$

Moreover, the last term can be estimated by

$$\int_{Q_t} f\,|B(\varphi)|^{p-1}\mathrm{sign}\varphi \le \frac{1}{p}\,c^p + \frac{1}{p'}\int_{Q_t} |B(\varphi)|^{(p-1)p'} \le c^p + \frac{1}{p'}\int_{Q_t} |B(\varphi)|^p,$$

owing to the general version of the Young inequality, where p' stands for the conjugate exponent of p. Using the above estimates, we can rearrange (3.10) to conclude that

$$\frac{1}{p}\int_{Q_t} |B(\varphi)|^p \le c^p,$$

that implies

$$\|B(\varphi)\|_{L^p(Q)} \le c,$$

where the constant c is independent of p. Since the above procedure can be iterated for every $p > 2$, we realize that $\|B(\varphi)\|_{L^\infty(Q)} \le c$ and from this we recover $\|F'(\varphi)\|_{L^\infty(Q)} \le c$. In view of (**H8**), this establishes that

$$r_- < \inf\varphi \le \sup\varphi < r_+ \quad \text{for a.a. } (x,t) \in Q,$$

as we claimed. □

At this point, we prove the continuous dependence results.

Proof of Theorem 2.2 For this proof we have largely taken inspiration from [4, Sect. 3, pp. 2429-2430]. First of all, we set

$$\mu := \mu_1 - \mu_2, \quad \varphi := \varphi_1 - \varphi_2, \quad \sigma := \sigma_1 - \sigma_2, \quad u := u_1 - u_2. \tag{3.11}$$

Writing (1.3)–(1.7) for $(\mu_i, \varphi_i, \sigma_i)$, $i = 1, 2$, and taking the difference, we obtain the following equations and conditions:

$$\alpha\partial_t\mu + \partial_t\varphi - \Delta\mu = R_1 - R_2 \quad \text{in } Q \tag{3.12}$$
$$\mu = \beta\partial_t\varphi - \Delta\varphi + F'(\varphi_1) - F'(\varphi_2) \quad \text{in } Q \tag{3.13}$$
$$\partial_t\sigma - \Delta\sigma = -(R_1 - R_2) + u \quad \text{in } Q \tag{3.14}$$
$$\partial_n\mu = \partial_n\varphi = \partial_n\sigma = 0 \quad \text{on } \Sigma \tag{3.15}$$
$$\mu(0) = \varphi(0) = \sigma(0) = 0 \quad \text{in } \Omega \tag{3.16}$$

where $R_i := P(\varphi_i)(\sigma_i - \mu_i)$, $i = 1, 2$. Now, we take the sum of (3.12) and (3.14), then add to both the members of this new equation $\mu + \sigma$. This gives

$$\partial_t(\alpha\mu + \varphi + \sigma) + \mathcal{A}(\mu + \sigma) = u + \mu + \sigma \quad \text{in } Q, \tag{3.17}$$

owing to (2.32). Keeping in mind (2.32)–(2.35), we multiply (3.17) by $\mathcal{A}^{-1}(\alpha\mu+\varphi+\sigma)$, (3.13) by $-\varphi$ and (3.14) by σ, add them, and integrate over Q_t. We deduce that

$$\begin{aligned}
&\frac{1}{2}\|(\alpha\mu + \varphi + \sigma)(t)\|_*^2 + \int_{Q_t} (\mu + \sigma)(\alpha\mu + \varphi + \sigma) - \int_{Q_t} \varphi\mu \\
&+ \frac{\beta}{2}\int_\Omega |\varphi(t)|^2 + \int_{Q_t} |\nabla\varphi|^2 + \int_{Q_t} (F'(\varphi_1) - F'(\varphi_2))\varphi + \frac{1}{2}\int_\Omega |\sigma(t)|^2 \\
&+ \int_{Q_t} |\nabla\sigma|^2 = \int_0^t \langle u + \mu + \sigma, \mathcal{A}^{-1}(\alpha\mu + \varphi + \sigma)\rangle \\
&- \int_{Q_t} (P(\varphi_1) - P(\varphi_2))\,(\sigma_1 - \mu_1)\sigma - \int_{Q_t} P(\varphi_2)\,(\sigma - \mu)\,\sigma + \int_{Q_t} u\sigma,
\end{aligned} \tag{3.18}$$

where the second and third terms of the right-hand side come from a simple rearrangement of $R_1 - R_2$. We develop the second term of the left-hand side as

$$\alpha\int_{Q_t} |\mu|^2 + \int_{Q_t} |\sigma|^2 + \int_{Q_t} \mu\varphi + (1+\alpha)\int_{Q_t} \mu\sigma + \int_{Q_t} \sigma\varphi,$$

and move the last three terms of the above sum to the right-hand side of (3.18). Observe that the term that involves the double-well potential should be decomposed as

$$\int_{Q_t} (F'(\varphi_1) - F'(\varphi_2))\varphi = \int_{Q_t} (B(\varphi_1) - B(\varphi_2))\varphi + \int_{Q_t} (\pi(\varphi_1) - \pi(\varphi_2))\varphi,$$

where the first term of the right-hand side is nonnegative by the monotonicity of B, while the second one can be moved to the right-hand side of (3.18) and easily managed, since π is Lipschitz continuous by (**H7**). If we rearrange (3.18) according to the above observations, we obtain

$$
\begin{aligned}
&\frac{1}{2}\|(\alpha\mu+\varphi+\sigma)(t)\|_*^2+\alpha\int_{Q_t}|\mu|^2+\int_{Q_t}|\sigma|^2+\frac{\beta}{2}\int_\Omega|\varphi(t)|^2\\
&\quad+\int_{Q_t}|\nabla\varphi|^2+\int_{Q_t}(B(\varphi_1)-B(\varphi_2))\varphi+\frac{1}{2}\int_\Omega|\sigma(t)|^2+\int_{Q_t}|\nabla\sigma|^2\\
&=\int_0^t\langle u+\mu+\sigma,\mathcal{A}^{-1}(\alpha\mu+\varphi+\sigma)\rangle-\int_{Q_t}(P(\varphi_1)-P(\varphi_2))\,(\sigma_1-\mu_1)\sigma\\
&\quad-\int_{Q_t}P(\varphi_2)\,(\sigma-\mu)\,\sigma-(1+\alpha)\int_{Q_t}\mu\sigma-\int_{Q_t}\sigma\varphi\\
&\quad+\int_{Q_t}u\sigma-\int_{Q_t}(\pi(\varphi_1)-\pi(\varphi_2))\varphi,
\end{aligned}
$$

where all the terms on the left-hand side are nonnegative. As the right-hand side is concerned, we denote $I_1,\dots,I_7$ the seven integrals, in that order. Using (2.29) and (2.34) we have

$$
\begin{aligned}
|I_1|&=\left|\int_0^t(u+\mu+\sigma,\alpha\mu+\varphi+\sigma)_*\right|\\
&\le\delta\int_0^t\|u+\mu+\sigma\|_*^2+c_\delta\int_0^t\|\alpha\mu+\varphi+\sigma\|_*^2,
\end{aligned}
$$

where the first term of the this inequality can be estimated by virtue of the embedding of V^* in H and the Young inequality as follows

$$
\delta\int_0^t\|u+\mu+\sigma\|_*^2\le c\delta\int_0^t\|u+\mu\ +\sigma\|_H^2\le\frac{\alpha}{4}\int_{Q_t}|\mu|^2+c\int_{Q_t}|u|^2+c\int_{Q_t}|\sigma|^2
$$

provided δ is sufficiently small. Moreover, combining the Hölder inequality and the Sobolev continuous embedding $V\subset L^q(\Omega)$, which holds for every $q\in[1,6]$, we realize that

$$
\begin{aligned}
|I_2|&\le c\int_{Q_t}|\varphi|(|\sigma_1|+|\mu_1|)|\sigma|\le c\int_0^t\|\varphi\|_2(\|\sigma_1\|_4+\|\mu_1\|_4)\|\sigma\|_4\\
&\le c\int_0^t\|\varphi\|_H(\|\sigma_1\|_V+\|\mu_1\|_V)\|\sigma\|_V\le\frac{1}{2}\int_{Q_t}\left(|\sigma|^2+|\nabla\sigma|^2\right)\\
&\quad+c\int_0^t(\|\sigma_1\|_V^2+\|\mu_1\|_V^2)\|\varphi\|_H^2\le\frac{1}{2}\int_{Q_t}\left(|\sigma|^2+|\nabla\sigma|^2\right)+c\int_0^t\|\varphi\|_H^2,
\end{aligned}
$$

where in the first line we use the Lipschitz continuity of P stated by (**H5**), in the second we apply the Young inequality, while in the latter we made use of estimate (2.7) for the solutions σ_1 and μ_1. Furthermore, in view of (2.29), we obtain

$$|I_3| \le c \int_{Q_t} |\sigma - \mu|\,|\sigma| \le \frac{\alpha}{4} \int_{Q_t} |\mu|^2 + c \int_{Q_t} |\sigma|^2,$$

and finally, from (2.29), (2.9), and (**H7**) we have that

$$|I_4| + |I_5| + |I_6| + |I_7| \le \frac{\alpha}{4} \int_{Q_t} |\mu|^2 + \frac{1}{2} \int_{Q_t} |u|^2 + c \int_{Q_t} |\sigma|^2 + c \int_{Q_t} |\varphi|^2.$$

Combining the above estimates, we have shown that for every $t \in [0, T]$ it holds that

$$\begin{aligned} &\frac{1}{2} \|(\alpha\mu + \varphi + \sigma)(t)\|_*^2 + \frac{\alpha}{4} \int_{Q_t} |\mu|^2 + \int_{Q_t} |\sigma|^2 + \frac{\beta}{2} \int_\Omega |\varphi(t)|^2 \\ &\quad + \int_{Q_t} |\nabla\varphi|^2 + \frac{1}{2} \int_\Omega |\sigma(t)|^2 + \frac{1}{2} \int_{Q_t} |\nabla\sigma|^2 \le c \int_{Q_t} |\sigma|^2 \\ &\quad + c \int_{Q_t} |\varphi|^2 + c \int_0^t \|(\alpha\mu + \varphi + \sigma)(s)\|_*^2 \, ds + c \int_{Q_t} |u|^2. \end{aligned}$$

Therefore, we invoke the Gronwall lemma and achieve

$$\begin{aligned} &\|\alpha\mu + \varphi + \sigma\|_{L^\infty(0,T;V^*)} + \|\mu\|_{L^2(0,T;H)} + \|\varphi\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \\ &\quad + \|\sigma\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c\|u\|_{L^2(0,T;H)}, \end{aligned}$$

where the variables are defined by (3.11). □

We conclude this section by proving a sharper estimate.

Proof of Theorem 2.3 Here, we account for (2.10) and make heavily use of the second part of Theorem 2.1. We consider again that the variables are defined by (3.11).

First estimate We consider again the system (3.12)–(3.14). We add to both sides of (3.13) the term $-\varphi$, test (3.12) by μ, and this new second equation by $-\partial_t\varphi$. Adding the equations and integrating over Q_t, we obtain

$$\begin{aligned} &\frac{\alpha}{2} \int_\Omega |\mu(t)|^2 + \int_{Q_t} |\nabla\mu|^2 + \beta \int_{Q_t} |\partial_t\varphi|^2 + \frac{1}{2} \int_\Omega |\varphi(t)|^2 + \frac{1}{2} \int_\Omega |\nabla\varphi(t)|^2 \\ &\quad \le \int_{Q_t} (R_1 - R_2)\mu - \int_{Q_t} (F'(\varphi_1) - F'(\varphi_2))\, \partial_t\varphi + \int_{Q_t} \varphi\, \partial_t\varphi. \end{aligned}$$

As before we call I_1, I_2, I_3 the three contributions on the right-hand side and proceed with a separate investigation. Due to (2.29) and Hölder's inequality, we have that

$$\begin{aligned} |I_2| + |I_3| &\le 2\delta \int_{Q_t} |\partial_t\varphi|^2 + c_\delta \int_{Q_t} |\varphi|^2 + c_\delta \int_{Q_t} |F'(\varphi_1) - F'(\varphi_2)|^2 \\ &\le 2\delta \int_{Q_t} |\partial_t\varphi|^2 + c_\delta \int_{Q_t} |\varphi|^2, \end{aligned}$$

where in the last estimate we invoke the fact that, by (ii) of Theorem 2.1, F' turns out to be Lipschitz continuous. Furthermore, by virtue of (**H5**), (2.10), and Hölder's inequality and Sobolev's embeddings, we conclude

$$\begin{aligned}
|I_1| &\le \int_{Q_t} P(\varphi_2)(\sigma - \mu)\mu + \int_{Q_t} (P(\varphi_1) - P(\varphi_2))(\sigma_1 - \mu_1)\mu \\
&\le c \int_{Q_t} |\sigma|^2 + c \int_{Q_t} |\mu|^2 + c \int_{Q_t} |\varphi|(|\sigma_1| + |\mu_1|)\,|\mu| \\
&\le c \int_{Q_t} |\sigma|^2 + c \int_{Q_t} |\mu|^2 + c \int_0^t \|\varphi\|_4(\|\sigma_1\|_4 + \|\mu_1\|_4)\|\mu\|_2 \\
&\le c \int_{Q_t} |\sigma|^2 + c \int_{Q_t} |\mu|^2 + c \int_0^t \|\varphi\|_V^2(\|\sigma_1\|_V^2 + \|\mu_1\|_V^2) \le c\|u\|_{L^2(0,T;H)}^2,
\end{aligned}$$

where the fact that σ_1 and μ_1 satisfy (2.7) turn out to be fundamental. On account of the previous estimates, we can choose $0 < \delta < \beta/2$, and apply the Gronwall lemma in order to conclude that

$$\|\mu\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\varphi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c\|u\|_{L^2(0,T;H)}.$$

□

4 The Control Problem

The current section represents the most challenging part of the work, since it contains the proof of the Fréchet differentiability of the control-to-state mapping $\mathcal{S}$, the investigation of both the linearized and the adjoint systems, and the necessary conditions that a control has to satisfy to be optimal.

4.1 Existence of Optimal Control

In the following, we are going to prove the existence of an optimal control. We remind that in general nothing can be said about the uniqueness. The strategy of the proof is quite standard and mainly lies on the semicontinuity property of the cost functional $\mathcal{J}$ and on standard weak compactness arguments.

Proof of Theorem 2.6 Let $\{u_n\}_n$ be a minimizing sequence for the control problem **(CP)** constituted of elements of $\mathcal{U}_{\mathrm{ad}}$ and for every $n \in \mathbb{N}$, let $(\mu_n, \varphi_n, \sigma_n)$ be the corresponding state. Therefore, the estimate (2.7) yields that there exist $\bar{u} \in \mathcal{U}_{\mathrm{ad}}$ and a triple $(\bar{\mu}, \bar{\varphi}, \bar{\sigma})$ such that, possibly for a subsequence which is not relabelled, it holds true the following

$$\begin{aligned}
&u_n \to \bar{u} \quad \text{weakly star in } L^\infty(Q), \\
&\mu_n \to \bar{\mu} \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W) \cap L^\infty(Q),
\end{aligned}$$

$$\varphi_n \to \bar{\varphi} \text{ weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W),$$
$$\sigma_n \to \bar{\sigma} \text{ weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W).$$

Furthermore, owing to standard compactness results (cf., e.g., [27, Sect. 8, Cor. 4]), we recover even some strong convergences. Indeed, we infer that

$$\varphi_n \to \bar{\varphi} \text{ strongly in } C^0([0,T];C^0(\bar{\Omega})).$$

This latter, paired with **(H5)**–**(H8)** and Theorem 2.1, allows us to manage the nonlinearities, since now

$$F'(\varphi_n) \to F'(\bar{\varphi}) \text{ and } P(\varphi_n) \to P(\bar{\varphi}),$$

with the same uniform convergence. Then, we can pass to the limit as n goes to infinity in the variational formulation of (1.3)–(1.7) written for $(\mu_n, \varphi_n, \sigma_n)$. Therefore, we realize that $\mathcal{S}(\bar{u}) = (\bar{\mu}, \bar{\varphi}, \bar{\sigma})$ and $\bar{u}$ itself are admissible solution for the **(CP)**. By the weak sequentially lower semicontinuity of $\mathcal{J}$ we finally realize that $\bar{u}$ is an optimal control that we were looking for. □

4.2 Towards Necessary Conditions: The Linearized Problem

Our first efforts are intended to establish the well-posedness of the linearized system (2.13)–(2.17), namely to prove Theorem 2.4.

Proof of Theorem 2.4 *Existence* The well-known spectral property of the operator $\mathcal{A}$ allow us to apply a Faedo-Galerkin scheme. We consider the family $\{w_j\}_j$ of eigenfunctions for the eigenvalue problem

$$-\Delta w_j + w_j = \lambda_j w_j \text{ in } \Omega, \quad \partial_n w_j = 0 \text{ on } \Gamma,$$

which constitutes a Galerkin basis in V. Moreover, let $\{w_j\}_j$ represent a complete orthonormal system in $(H, (\cdot,\cdot))$ which is also orthogonal in $(V, (\cdot,\cdot))$. For fixed n, we set $\mathcal{W}_n := span\,\{w_1, \dots, w_n\}$, and we expect that the solutions to the approximated problem possess the following structure

$$\eta_n(x,t) = \sum_{k=1}^n a_k^n(t) w_k(x), \quad \vartheta_n(x,t) = \sum_{k=1}^n b_k^n(t) w_k(x), \quad \rho_n(x,t) = \sum_{k=1}^n c_k^n(t) w_k(x),$$

for suitable unknown sequences a_k^n, b_k^n, c_k^n. Namely, we try to solve (2.13)–(2.17) in which the variables are replaced by the above expressions and we will refer to this problem as (P_n). Since (2.14) only depends on the variables a_i^n and b_i^n, $1 \le i \le n$, by comparison, we can express the unknowns a_i^n in terms of $\{b_1^n, \dots, b_n^n\}$. In this way, (P_n) can be reformulated as a Cauchy problem for a linear system of $2n$ first-order ODE in the $2n$ unknowns b_i^n, c_i^n, $1 \le i \le n$. By Cauchy-Lipschitz theorem, there exists a unique solution to this linear system satisfying $(b_1^n, \dots, b_n^n, c_1^n, \dots, c_n^n) \in (C^1(0,T))^{2n}$. This proves the existence and uniqueness of solution to (P_n), and it is straightforward to realize that $(\eta_n, \vartheta_n, \rho_n) \in C^1([0,T];\mathcal{W}_n)^3$.

At this point, we would like to obtain an existence result, for the solution to (2.13)–(2.17) itself. To do that, we look for some a priori estimates on the approximated solutions that involve constants that may depend on the data of the problem, but are independent of n, thus we will be able to pass to the limit as $n \nearrow +\infty$ to prove the existence of solutions. To prevent a heavy notation in the following estimates, we avoid writing every time the subscript n under the variables, while we will reintroduce the correct notation at the end of each estimate.

First estimate First of all, we add the term to both the members of (2.14) ϑ. Then we test (2.13) by η, this new second equation by $-\partial_t \vartheta$, and (2.15) by ρ, add the resulting equalities and integrate over Q_t and by parts to obtain

$$\begin{aligned}
&\frac{\alpha}{2}\int_\Omega |\eta(t)|^2 + \int_{Q_t} |\nabla\eta|^2 + \beta\int_{Q_t} |\partial_t\vartheta|^2 + \frac{1}{2}\int_\Omega |\vartheta(t)|^2 + \frac{1}{2}\int_\Omega |\nabla\vartheta(t)|^2 \\
&\quad +\frac{1}{2}\int_\Omega |\rho(t)|^2 + \int_{Q_t} |\nabla\rho|^2 + \int_{Q_t} P(\bar\varphi)(\rho-\eta)^2 = \int_{Q_t} h\rho \\
&\quad -\int_{Q_t} F''(\bar\varphi)\,\vartheta\,\partial_t\vartheta + \int_{Q_t} P'(\bar\varphi)(\bar\sigma-\bar\mu)\vartheta(\eta-\rho) + \int_{Q_t} \vartheta\,\partial_t\vartheta \\
&\le |I_1| + |I_2| + |I_3| + |I_4|,
\end{aligned}$$

where $I_1, \ldots, I_4$ represent, in that order, the integrals in the right-hand side. It is worth to note that all the terms of the left-hand side are nonnegative since they all contain squares and P attains nonnegative values by (**H5**). Clearly, by Young's inequality it turns out that

$$|I_1| \le \frac{1}{2}\int_{Q_t} (|h|^2 + |\rho|^2), \quad \text{and} \quad |I_2| + |I_4| \le 2\delta\int_{Q_t} |\partial_t\vartheta|^2 + c_\delta\int_{Q_t} |\vartheta|^2,$$

respectively. Moreover, by virtue of (2.7), (2.9), Hölder's inequality, and the Sobolev embeddings, we have that

$$\begin{aligned}
|I_3| &\le \int_{Q_t} (|\bar\sigma| + |\bar\mu|)\,|\vartheta|\,(|\eta| + |\rho|) \le c\int_0^t (\|\bar\sigma\|_6 + \|\bar\mu\|_6)\,\|\vartheta\|_3\,(\|\eta\|_2 + \|\rho\|_2) \\
&\le c\int_0^t (\|\bar\sigma\|_V + \|\bar\mu\|_V)\,\|\vartheta\|_V\,(\|\eta\|_H + \|\rho\|_H) \le c\int_0^t \left(\|\bar\sigma\|_V^2 + \|\bar\mu\|_V^2\right)\|\vartheta\|_V^2 \\
&\quad +c\int_{Q_t} |\eta|^2 + c\int_{Q_t} |\rho|^2 \le c\int_0^t \|\vartheta\|_V^2 + c\int_{Q_t} |\eta|^2 + c\int_{Q_t} |\rho|^2,
\end{aligned}$$

where in the second line we also apply (2.29). Thus, by fixing $0 < \delta < \beta/2$, the Gronwall lemma yields

$$\begin{aligned}
&\|\eta_n\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\vartheta_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \\
&\quad +\|\rho_n\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c\|h\|_{L^2(0,T;H)}.
\end{aligned} \tag{4.1}$$

Second estimate We test (2.14) by $\Delta\vartheta$, which is perfectly admissible. Indeed, in our approximating scheme $\Delta\vartheta$ actually stands for $\Delta\vartheta_n$, which belongs to $\mathcal{W}_n$. Using

(2.9), the previous estimate and the Young inequality, we deduce $\|\Delta\vartheta_n\|_{L^2(0,T;H)} \le c$. Therefore, by elliptic regularity we realize that

$$\|\vartheta_n\|_{L^2(0,T;W)} \le c. \tag{4.2}$$

Third estimate We multiply (2.13) by $\partial_t\eta$, (2.15) by $\partial_t\rho$, integrate over Q_t and by parts to get

$$\begin{aligned}
&\alpha\int_{Q_t}|\partial_t\eta|^2 + \frac{1}{2}\int_\Omega|\nabla\eta(t)|^2 + \int_{Q_t}|\partial_t\rho|^2 + \frac{1}{2}\int_\Omega|\nabla\rho(t)|^2 \\
&= -\int_{Q_t}\partial_t\vartheta\,\partial_t\eta + \int_{Q_t}P'(\bar\varphi)(\bar\sigma-\bar\mu)\vartheta\,\partial_t\eta + \int_{Q_t}P(\bar\varphi)(\rho-\eta)\,\partial_t\eta \\
&\quad -\int_{Q_t}P'(\bar\varphi)(\bar\sigma-\bar\mu)\vartheta\,\partial_t\rho - \int_{Q_t}P(\bar\varphi)(\rho-\eta)\,\partial_t\rho + \int_{Q_t}h\,\partial_t\rho,
\end{aligned}$$

where we denote the six terms of the right-hand side by $I_1,\dots,I_6$, in that order. As I_2 and I_4 are concerned, we invoke (2.7) and (2.9) to obtain

$$\begin{aligned}
|I_2|+|I_4| &\le c\int_{Q_t}(|\bar\sigma|+|\bar\mu|)|\vartheta||\partial_t\eta| + c\int_{Q_t}(|\bar\sigma|+|\bar\mu|)|\vartheta||\partial_t\rho| \\
&\le c\int_0^t(\|\bar\sigma\|_6+\|\bar\mu\|_6)\|\vartheta\|_3\|\partial_t\eta\|_2 + c\int_0^t(\|\bar\sigma\|_6+\|\bar\mu\|_6)\|\vartheta\|_3\|\partial_t\rho\|_2 \\
&\le \delta\int_{Q_t}(|\partial_t\eta|^2+|\partial_t\rho|^2) + c_\delta\int_0^t(\|\bar\sigma\|_V^2+\|\bar\mu\|_V^2)\|\vartheta\|_V^2,
\end{aligned}$$

where in the second line we apply first the Hölder inequality and then the Sobolev continuous embedding of $V\subset L^6(\Omega)$. Furthermore, the last estimate is obtained by the Young inequality combining the fact that $\bar\sigma$ and $\bar\mu$, as solutions, satisfy (2.7) and the above estimate (4.1). Accounting for the Young inequality, (2.7) and (2.9), we also conclude that

$$\begin{aligned}
|I_1|+|I_3|+|I_5|+|I_6| &\le 2\delta\int_{Q_t}(|\partial_t\eta|^2+|\partial_t\rho|^2) + c_\delta\int_{Q_t}|\partial_t\vartheta|^2 \\
&\quad + c_\delta\int_{Q_t}|P(\bar\varphi)(\rho-\eta)|^2 + c_\delta\int_{Q_t}|h|^2 \\
&\le 2\delta\int_{Q_t}(|\partial_t\eta|^2+|\partial_t\rho|^2) \\
&\quad + c_\delta\int_{Q_t}(|\partial_t\vartheta|^2+|\rho|^2+|\eta|^2+|h|^2).
\end{aligned}$$

Choosing $0<\delta<\min\{\alpha/3, 1/3\}$, we can apply the Gronwall lemma which gives

$$\|\eta_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\rho_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c. \tag{4.3}$$

Fourth estimate Now, we test (2.13) by $-\Delta\eta$, and (2.15) by $-\Delta\rho$, respectively. Summing the resulting equalities and integrating over Q_t, we obtain

$$\begin{aligned}\int_{Q_t}|\Delta\eta|^2+\int_{Q_t}|\Delta\rho|^2 = \alpha\int_{Q_t}\partial_t\eta\,\Delta\eta+\int_{Q_t}\partial_t\vartheta\,\Delta\eta-\int_{Q_t}P'(\bar\varphi)(\bar\sigma-\bar\mu)\vartheta\,\Delta\eta\\ -\int_{Q_t}P(\bar\varphi)(\rho-\eta)\Delta\eta+\int_{Q_t}\partial_t\rho\,\Delta\rho\\ +\int_{Q_t}P'(\bar\varphi)(\bar\sigma-\bar\mu)\vartheta\,\Delta\rho\\ +\int_{Q_t}P(\bar\varphi)(\rho-\eta)\Delta\rho-\int_{Q_t}h\,\Delta\rho,\end{aligned}$$

where we convey to denote the integrals on the right-hand side by $I_1,\dots,I_8$. Except $|I_3|$ and $|I_6|$, the other terms on the right-hand side that multiply $-\Delta\eta$ or $-\Delta\rho$ can be easily managed by means of the Young inequality since they are estimated with respect to the $L^2(0,T;H)$ norm. Moreover, owing to the Hölder and Young inequalities and (2.9), we infer that

$$\begin{aligned}|I_3|+|I_6| &\le \int_{Q_t}\Big|P'(\bar\varphi)(\bar\sigma-\bar\mu)\vartheta\,\Delta\eta\Big|+\int_{Q_t}\Big|P'(\bar\varphi)(\bar\sigma-\bar\mu)\vartheta\,\Delta\rho\Big|\\ &\le c\int_{Q_t}(|\bar\sigma|+|\bar\mu|)|\vartheta||\Delta\eta|+c\int_{Q_t}(|\bar\sigma|+|\bar\mu|)|\vartheta||\Delta\rho|\\ &\le c\int_0^t(\|\bar\sigma\|_6+\|\bar\mu\|_6)\|\vartheta\|_3\|\Delta\eta\|_2+c\int_0^t(\|\bar\sigma\|_6+\|\bar\mu\|_6)\|\vartheta\|_3\|\Delta\rho\|_2\\ &\le \delta\int_{Q_t}(|\Delta\eta|^2+|\Delta\rho|^2)+c_\delta\int_0^t(\|\bar\sigma\|_V^2+\|\bar\mu\|_V^2)\|\vartheta\|_V^2,\end{aligned}$$

where in the last two lines, we have used the Sobolev embeddings, the fact that $\bar\sigma$ and $\bar\mu$ solve (1.3)–(1.7) and the previous estimate. In conclusion, owing to (2.29) we can manage the other terms and obtain

$$\int_{Q_t}|\Delta\eta|^2+\int_{Q_t}|\Delta\rho|^2\le 4\delta\int_{Q_t}(|\Delta\eta|^2+|\Delta\rho|^2)+c_\delta.$$

Furthermore, we fix $0<\delta<1/4$ in order to find that

$$\|\eta_n\|_{L^2(0,T;W)}+\|\rho_n\|_{L^2(0,T;W)}\le c. \tag{4.4}$$

Conclusion of the proof Collecting all these informations, by standard compactness arguments, it follows that, up to a subsequence, suitably relabeled, $(\eta_n,\vartheta_n,\rho_n)$ converges weakly star to a limit (η,ϑ,ρ) that solves (2.13)–(2.17) and has the following regularity

$$\eta,\vartheta,\rho\in H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W).$$

Finally, the standard embedding results applied to each variable, imply that they all belong to $C^0([0,T];L^r(\Omega))$ for every $r<6$.

Uniqueness As the uniqueness is concerned we consider (2.13)–(2.17) written for the variables $(\eta_i,\vartheta_i,\rho_i)$, $i=1,2$, and subtract the equations. Then we denote $\eta:=\eta_1-\eta_2$, $\vartheta:=\vartheta_1-\vartheta_2$, $\rho:=\rho_1-\rho_2$ and observe that they solve (2.13)–(2.17) with $h\equiv 0$. Then it immediately follows that $\eta=\vartheta=\rho=0$. ◻

4.3 Fréchet Differentiability of the Control-to-state Mapping

In the following, we prove Theorem 2.5. Let us fix $\bar u\in\mathcal{U}_R$ and denote $(\bar\mu,\bar\varphi,\bar\sigma)=\mathcal{S}(\bar u)$ the corresponding solution to (1.3)–(1.7). Since we are going to work with small increments h and $\mathcal{U}_R$ is open, we assume h to be small enough in order that $\bar u+h$ belongs to $\mathcal{U}_R$ as well. For h fixed, we define

$$(\mu^h,\varphi^h,\sigma^h):=\mathcal{S}(\bar u+h),\ \zeta:=\mu^h-\bar\mu-\eta,\quad \psi:=\varphi^h-\bar\varphi-\vartheta,\quad \text{and}\quad \chi:=\sigma^h-\bar\sigma-\rho\,.$$

Therefore, we aim at providing a property such as

$$\mathcal{S}(\bar u+h)=\mathcal{S}(\bar u)+[D\mathcal{S}(\bar u)](h)+o(\|h\|_{L^2(0,T;H)})\quad\text{as}\quad \|h\|_{L^2(0,T;H)}\to 0.$$

In view of the investigation of the linearized system, by rearranging the terms, we realize that it suffices to prove that

$$\|(\zeta,\psi,\chi)\|_{\mathcal{Y}}\le c\|h\|^2_{L^2(0,T;H)}\quad\text{as}\quad \|h\|_{L^2(0,T;H)}\to 0, \tag{4.5}$$

where $\mathcal{Y}$ stands for the space to which belongs (ζ,ψ,χ). According to Theorem 2.1 and Theorem 2.4, we have that

$$\mathcal{Y}=\Big(H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)\Big)^3.$$

Proof of Theorem 2.5 Consider (1.3)–(1.7) associated to $\bar u+h$, and subtract (1.3)–(1.7) associated to $\bar u$ and (2.13)–(2.17). By combining them, we obtain that (ζ,ψ,χ) solves the following system

$$\alpha\partial_t\zeta+\partial_t\psi-\Delta\zeta=\Theta\quad\text{in } Q \tag{4.6}$$
$$\zeta=\beta\partial_t\psi-\Delta\psi+Z\quad\text{in } Q \tag{4.7}$$
$$\partial_t\chi-\Delta\chi=-\Theta\quad\text{in } Q \tag{4.8}$$
$$\partial_n\zeta=\partial_n\psi=\partial_n\chi=0\quad\text{on } \Sigma \tag{4.9}$$
$$\zeta(0)=\psi(0)=\chi(0)=0\quad\text{in } \Omega. \tag{4.10}$$

where Z and Θ are defined as follows

$$Z:=F'(\varphi^h)-F'(\bar\varphi)-F''(\bar\varphi)\,\vartheta,$$

$$\Theta := P(\varphi^h)(\sigma^h - \mu^h) - P(\bar\varphi)(\bar\sigma - \bar\mu) - P'(\bar\varphi)(\bar\sigma - \bar\mu)\,\vartheta - P(\bar\varphi)(\rho - \eta).$$

Taylor's theorem with integral remainder, and some easy calculations, allow us to write

$$\begin{aligned}
Z &= F''(\bar\varphi)\psi + R_1^h(\varphi^h - \bar\varphi)^2,\\
\Theta &= P(\bar\varphi)(\chi - \zeta) + \Big(P(\varphi^h) - P(\bar\varphi)\Big)\Big((\sigma^h - \bar\sigma) - (\mu^h - \bar\mu)\Big)\\
&\quad + P'(\bar\varphi)(\bar\sigma - \bar\mu)\psi + (\bar\sigma - \bar\mu)R_2^h(\varphi^h - \bar\varphi)^2,
\end{aligned}$$

where

$$R_1^h := \int_0^1 (1-z)\, F'''(\bar\varphi + z\,(\varphi^h - \bar\varphi))dz, \quad R_2^h := \int_0^1 (1-z)\, P''(\bar\varphi + z\,(\varphi^h - \bar\varphi))dz,$$

respectively. Before starting with the core of the proof, we introduce some preparatory estimates that will be useful later on.

Preliminary estimates First of all, thanks to (2.9) and (**H5**)–(**H8**), we have

$$\|R_1^h\|_{L^\infty(Q)} + \|R_2^h\|_{L^\infty(Q)} \le c. \tag{4.11}$$

By (2.11), the previous estimate and the Sobolev embeddings, we infer that for every $t \in [0, T]$, it holds

$$\begin{aligned}
&\int_0^t \left\| R_1^h(s)\Big(\varphi^h(s) - \bar\varphi(s)\Big)^2 \right\|_H^2 ds \le c \int_{Q_t} |\varphi^h - \bar\varphi|^4\\
&\quad \le c\int_0^t \|\varphi^h - \bar\varphi\|_4^4 \le c\|\varphi^h - \bar\varphi\|_{L^\infty(0,T;V)}^4 \le c\|h\|_{L^2(0,T;H)}^4.
\end{aligned} \tag{4.12}$$

Furthermore, owing to (2.7), (2.10), (2.11), Hölder's inequality, and (**H5**), we get

$$\begin{aligned}
&\int_0^t \left\| \Big(P(\varphi^h) - P(\bar\varphi)\Big)\Big((\sigma^h - \bar\sigma) - (\mu^h - \mu)\Big) \right\|_H^2\\
&\le c\int_{Q_t} |\varphi^h - \bar\varphi|^2(|\sigma^h - \bar\sigma|^2 + |\mu^h - \mu|^2)\\
&\le c\int_0^t \|\varphi^h(s) - \bar\varphi(s)\|_4^2(\|\sigma^h(s) - \bar\sigma(s)\|_4^2 + \|\mu^h(s) - \mu(s)\|_4^2)\,ds\\
&\le c\int_0^t \|\varphi^h - \bar\varphi\|_V^2\Big(\|\sigma^h - \bar\sigma\|_V^2 + \|\mu^h - \mu\|_V^2\Big) \le c\,\|h\|_{L^2(0,T;H)}^4,
\end{aligned} \tag{4.13}$$

where in the third line we have applied the Sobolev embedding of $V \subset L^4(\Omega)$. Moreover, from (**H5**), (2.7) and (2.9), we obtain

$$\int_0^t \left\| P'(\bar\varphi)(\bar\sigma - \bar\mu)\psi \right\|_H^2 \le c\int_{Q_t} \Big(|\bar\sigma|^2 + |\bar\mu|^2\Big)|\psi|^2$$

$$\leq c \int_0^t \left(\|\bar{\sigma}\|_V^2 + \|\bar{\mu}\|_V^2 \right) \|\psi\|_V^2 \leq c \int_0^t \|\psi\|_V^2. \tag{4.14}$$

Finally, thanks to (4.11), Hölder's inequality, (2.7), (2.10), (2.11), and to the Sobolev embeddings, we have

$$\begin{aligned}
&\int_0^t \left\| (\bar{\sigma} - \bar{\mu}) R_2^h (\varphi^h - \bar{\varphi})^2 \right\|_H^2 \leq c \int_{Q_t} (|\bar{\sigma}|^2 + |\bar{\mu}|^2) |\varphi^h - \bar{\varphi}|^4 \\
&\leq c \int_0^t \left(\|\bar{\sigma}(s)\|_6^2 + \|\bar{\mu}(s)\|_6^2 \right) \|\varphi^h(s) - \bar{\varphi}(s)\|_6^4 \, ds \\
&\leq c \int_0^t \left(\|\bar{\sigma}\|_V^2 + \|\bar{\mu}\|_V^2 \right) \|\varphi^h - \bar{\varphi}\|_V^4 \leq c \|h\|_{L^2(0,T;H)}^4.
\end{aligned} \tag{4.15}$$

Now, we start with the actual estimates.

First estimate First, we add to both sides of (4.7) the term ψ, then we multiply (4.6) by ζ, this new second equation by $-\partial_t \psi$, and (4.8) by χ. Adding the resulting equations and integrating over Q_t, we get

$$\begin{aligned}
&\frac{\alpha}{2} \int_\Omega |\zeta(t)|^2 + \int_{Q_t} |\nabla \zeta|^2 + \frac{1}{2} \int_\Omega |\psi(t)|^2 + \frac{1}{2} \int_\Omega |\nabla \psi(t)|^2 + \beta \int_{Q_t} |\partial_t \psi|^2 \\
&+ \int_\Omega |\chi(t)|^2 + \int_{Q_t} |\nabla \chi|^2 = \int_{Q_t} \Theta \zeta - \int_{Q_t} F''(\bar{\varphi}) \, \psi \, \partial_t \psi \\
&- \int_{Q_t} R_1^h (\varphi^h - \bar{\varphi})^2 \, \partial_t \psi + \int_{Q_t} \psi \, \partial_t \psi - \int_{Q_t} \Theta \chi,
\end{aligned}$$

where the last five integrals of the right-hand side are denoted by $I_1, \ldots, I_5$, in this order. Simply using (2.9), (2.29), and (4.12), we deduce

$$\begin{aligned}
|I_2| + |I_3| + |I_4| &\leq 3\delta \int_{Q_t} |\partial_t \psi|^2 + c_\delta \int_{Q_t} |\psi|^2 + c_\delta \int_{Q_t} |R_1^h (\varphi^h - \bar{\varphi})^2|^2 \\
&\leq 3\delta \int_{Q_t} |\partial_t \psi|^2 + c_\delta \int_{Q_t} |\psi|^2 + c_\delta \|h\|_{L^2(0,T;H)}^4.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
|I_1| \leq &\ \Big| \int_{Q_t} P(\bar{\varphi})(\chi - \zeta) \, \zeta + \left(P(\varphi^h) - P(\bar{\varphi}) \right) \left((\sigma^h - \bar{\sigma}) - (\mu^h - \mu) \right) \zeta \\
&+ P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu}) \, \psi \, \zeta + (\bar{\sigma} - \bar{\mu}) R_2^h (\varphi^h - \bar{\varphi})^2 \, \zeta \Big| \\
\leq &\ c \int_{Q_t} |\chi|^2 + c \int_{Q_t} |\zeta|^2 + c \int_0^t \|\psi\|_V^2 + c \|h\|_{L^2(0,T;H)}^4,
\end{aligned}$$

owing to the Young inequality, (**H5**), (2.7), (2.9), and the estimates (4.13)–(4.15). The last term I_5 is treated the same way, while it is referred to the variable χ instead of ζ.

Choosing $0 < \delta < \beta/3$, we can apply the Gronwall lemma in order to realize that

$$\|\zeta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\psi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|\chi\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c\|h\|^2_{L^2(0,T;H)}.$$

Second estimate Accounting for the previous estimate, by comparison in (4.7), we easily conclude that

$$\|\Delta\psi\|_{L^2(0,T;H)} \leq c\|h\|^2_{L^2(0,T;H)}.$$

Third estimate To recover the stated regularity, let us reformulate Eqs. (4.6) and (4.8) as follows

$$\alpha\partial_t\zeta - \Delta\zeta = \Theta - \partial_t\varphi := g_1, \quad \text{and} \quad \partial_t\chi - \Delta\chi = \Theta := g_2.$$

Accounting for (4.16), we realize that both the forcing terms g_1 and g_2 have been already estimated in $L^2(0, T; H)$. Moreover, owing to the smoothness of the initial conditions (4.9), the parabolic regularity theory (see, e.g., [25]) gives

$$\|\zeta\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} + \|\chi\|_{H^1(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W)} \leq c\|h\|^2_{L^2(0,T;H)}.$$

This proves (4.5), that is the Fréchet differentiability of $\mathcal{S}$. □

At this point Corollary 2.7 immediately follows from (2.20) by direct calculations.

4.4 Adjoint Problem

The last part of our work regards the improvement of (2.21) by dealing with the system (2.22)–(2.26). In fact, our aim is to prove Theorem 2.8.

Proof of Theorem 2.8 *Existence* As in the proof of Theorem 2.4, we apply a Faedo-Galerkin scheme based on a basis $\{w_j\}_j \subset W$, and we again refer to $\mathcal{W}_n$ as to the space generated by the first n eigenvectors. We look for approximated solutions of the form

$$q_n(x,t) = \sum_{k=1}^n a_k^n(t)w_k(x),$$
$$p_n(x,t) = \sum_{k=1}^n b_k^n(t)w_k(x), \quad r_n(x,t) = \sum_{k=1}^n c_k^n(t)w_k(x),$$

which satisfies, for a.a. $t \in (0, T)$ the following problem

$$\beta(\partial_t q_n, v) + (-\partial_t p_n, v) - (\nabla q_n, \nabla v) - (F''(\bar\varphi)q_n, v) + (P'(\bar\varphi)(\bar\sigma - \bar\mu)(r_n - p_n), v) = (b_1(\bar\varphi - \varphi_Q), v) \quad \text{for all } v \in \mathcal{W}_n, \tag{4.16}$$

$$(q_n, v)-\alpha(\partial_t p_n, v)+(\nabla p_n, \nabla v) + (P(\bar\varphi)(p_n - r_n), v) = 0 \quad \text{for all } v \in \mathcal{W}_n, \tag{4.17}$$

$$(-\partial_t r_n, v) + (\nabla r_n, \nabla v) + (P(\bar\varphi)(r_n - p_n), v) = (b_3(\bar\sigma - \sigma_Q), v) \quad \text{for all } v \in \mathcal{W}_n, \tag{4.18}$$

$$p_n(T) - \beta q_n(T) = \mathbb{P}\big(b_2(\bar\varphi(T) - \varphi_\Omega)\big), \quad \alpha p_n(T) = 0,$$
$$r_n(T) = \mathbb{P}\big(b_4(\bar\sigma(T) - \sigma_\Omega)\big), \tag{4.19}$$

where $\mathbb{P}$ represents the orthogonal projection in H onto $\mathcal{W}_n$. Arguing as before, we can easily conclude that the backward-in-time problem (4.16)–(4.19) admits a unique solution triple that satisfies the following regularity $(q_n, p_n, r_n) \in \big(W^{1,\infty}(0, T; \mathcal{W}_n)\big)^3$. So, to ensure the existence of the adjoint problem, we need to provide some a priori estimates independent of n in order to apply standard compactness arguments and motivate rigorously the passage to the limit as $n \nearrow +\infty$.

As for the notation, we again adopt the convention used in the proof of Theorem 2.4.

First estimate First, we add to both sides of (4.17) the term p. Then, we test (4.16) by $-q$, this new second equation by $-\partial_t p$, and (4.18) by r. Finally, we add these equations and integrate over $\Omega \times [t, T] =: Q_t^T$ and by parts to find the following identity

$$\begin{aligned}
&\frac{\beta}{2}\int_\Omega |q(t)|^2 + \int_{Q_t^T} \partial_t p\, q + \int_{Q_t^T} |\nabla q|^2 - \int_{Q_t^T} \partial_t p\, q + \alpha \int_{Q_t^T} |\partial_t p|^2 \\
&\quad + \frac{1}{2}\int_\Omega |\nabla p(t)|^2 + \frac{1}{2}\int_\Omega |p(t)|^2 + \frac{1}{2}\int_\Omega |r(t)|^2 + \int_{Q_t^T} |\nabla r|^2 \\
&= \frac{1}{2}\int_\Omega |r(T)|^2 + \frac{\beta}{2}\int_\Omega |q(T)|^2 + \frac{1}{2}\int_\Omega |\nabla p(T)|^2 + \frac{1}{2}\int_\Omega |p(T)|^2 \\
&\quad + \int_{Q_t^T} P'(\bar\varphi)(\bar\sigma - \bar\mu)(r - p)\, q - \int_{Q_t^T} F''(\bar\varphi) q^2 - \int_{Q_t^T} b_1(\bar\varphi - \varphi_Q) q \\
&\quad + \int_{Q_t^T} b_3(\bar\sigma - \sigma_Q) r - \int_{Q_t^T} P(\bar\varphi)(r - p) r - \int_{Q_t^T} P(\bar\varphi)(r - p)\, \partial_t p - \int_{Q_t^T} p\, \partial_t p.
\end{aligned}$$

Let us note that two terms cancel out and that the first four integrals of the right-hand side can be explicitly written using (4.19) and are bounded due to (**H1**)–(**H2**). Let us call, in the order, $I_1, \ldots, I_7$ the other terms. Using (2.9) and (**H2**), we have

$$|I_2| + |I_3| + |I_4| \le c + c\int_{Q_t^T} |q|^2 + c\int_{Q_t^T} |r|^2.$$

In addition, (2.9) and (2.29) yield that

$$\begin{aligned}
|I_5| + |I_6| + |I_7| &\le 2\delta \int_{Q_t^T} |\partial_t p|^2 + c\int_{Q_t^T} |r|^2 + c_\delta \int_{Q_t^T} |P(\bar\varphi)(r - p)|^2 \\
&\quad + c_\delta \int_{Q_t^T} |p|^2 \le 2\delta \int_{Q_t^T} |\partial_t p|^2 + c_\delta \int_{Q_t^T} |r|^2 + c_\delta \int_{Q_t^T} |p|^2.
\end{aligned}$$

Finally, we obtain from (**H5**), (2.7), (2.9), (2.29), the Sobolev embeddings, and the Hölder inequality that

$$
\begin{aligned}
|I_1| &\leq c \int_{Q_t^T} |P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu})q|^2 + c \int_{Q_t^T} |r - p|^2 \\
&\leq c \int_{Q_t^T} (|\bar{\sigma}|^2 + |\bar{\mu}^2|)|q||q| + c \int_{Q_t^T} |r|^2 + c \int_{Q_t^T} |p|^2 \\
&\leq c \int_t^T (\|\bar{\sigma}\|_6^2 + \|\bar{\mu}\|_6^2)\|q\|_6\|q\|_2 + c \int_{Q_t^T} |r|^2 + c \int_{Q_t^T} |p|^2 \\
&\leq \frac{1}{2} \int_{Q_t^T} (|q|^2 + |\nabla q|^2) + c \int_t^T (\|\bar{\sigma}\|_V^4 + \|\bar{\mu}\|_V^4)\|q\|_2^2 + c \int_{Q_t^T} |r|^2 + c \int_{Q_t^T} |p|^2 \\
&\leq \frac{1}{2} \int_{Q_t^T} |\nabla q|^2 + c \int_{Q_t^T} |r|^2 + c \int_{Q_t^T} |p|^2 + c \int_{Q_t^T} |q|^2.
\end{aligned}
$$

We now fix $0 < \delta < \alpha/2$, and applying the backward in time Gronwall lemma, we infer that

$$
\|q_n\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|p_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|r_n\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c.
$$

Second estimate We test (4.17) by Δp. Using the Young inequality and the previous estimate it is quite easy to realize that

$$
\|\Delta p_n\|_{L^2(0,T;H)} \leq c,
$$

whence, from elliptic regularity, we infer that

$$
\|p_n\|_{L^2(0,T;W)} \leq c.
$$

Third estimate We now test (4.16) by $\partial_t q$. Integrating over $\Omega \times [t, T]$ and by parts, we obtain that

$$
\begin{aligned}
&\beta \int_{Q_t^T} |\partial_t q|^2 + \frac{1}{2} \int_\Omega |\nabla q(t)|^2 = \frac{1}{2} \int_\Omega |\nabla q(T)|^2 + \int_{Q_t^T} \partial_t p \, \partial_t q \\
&\quad - \int_{Q_t^T} P'(\bar{\varphi})(\bar{\sigma} - \bar{\mu})(r - p) \, \partial_t q + \int_{Q_t^T} F''(\bar{\varphi})q \, \partial_t q + \int_{Q_t^T} b_1(\bar{\varphi} - \varphi_Q) \, \partial_t q,
\end{aligned}
$$

and we denote by $I_1, \ldots, I_4$ the last four summands on the right-hand side. Note that the first term on the right-hand side is finite by (4.19) and (**H2**). A simple application of (2.9) and of the Young inequality show that

$$
|I_1| + |I_3| + |I_4| \leq c_\delta + 3\delta \int_{Q_t^T} |\partial_t q|^2 + c_\delta \int_{Q_t^T} |\partial_t p|^2 + c_\delta \int_{Q_t^T} |q|^2.
$$

Furthermore, owing to the Young inequality and to the Sobolev embeddings, we also have that

$$|I_2| \le c \int_{Q_t^T} (|\bar{\sigma}| + |\bar{\mu}|)(|r| + |p|)|\partial_t q| \le c \int_t^T (\|\bar{\sigma}\|_6 + \|\bar{\mu}\|_6)(\|r\|_3 + \|p\|_3)\|\partial_t q\|_2$$
$$\le \delta \int_{Q_t^T} |\partial_t q|^2 + c_\delta \int_t^T (\|\bar{\sigma}\|_V^2 + \|\bar{\mu}\|_V^2)(\|r\|_V^2 + \|p\|_V^2),$$

where all the terms on the right-hand side of both these inequalities have been already estimated above. Therefore, fixing $0 < \delta < \beta/4$, we conclude

$$\|q_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c.$$

Fourth estimate Arguing exactly as above, by testing (4.18) by $-\partial_t r$, we also infer that

$$\|r_n\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \le c.$$

Fifth estimate Moreover, accounting for the above estimates and the Young inequality, by taking $-\Delta r$ and Δq as test functions in (2.22) and (2.24), respectively, we can easily deduce that

$$\|q_n\|_{L^2(0,T;W)} + \|r_n\|_{L^2(0,T;W)} \le c.$$

Conclusion of the proof It follows from the above a priori estimates that there exist functions (q, p, r) such that, possibly for some subsequence which is again indexed by n, the following convergences

$$q_n \to q, \quad p_n \to p, \quad r_n \to r \quad \text{weakly star in}$$
$$H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W)$$

hold. Moreover, by continuous embedding, also

$$q_n \to q, \quad p_n \to p, \quad r_n \to r \quad \text{weakly in } C^0([0, T]; V).$$

It is now a standard matter to verify that (q, p, r) is in fact a solution to the system (2.22)–(2.26) satisfying (2.27).

Uniqueness As before we denote $q := q_1 - q_2,\ p := p_1 - p_2,\ r := r_1 - r_2$, where (q_i, p_i, r_i), $i = 1, 2$, are two solutions to (2.22)–(2.26). If we consider the system obtained by subtracting the corresponding equations each others, we can repeat the argument of the existence and realize that $q = p = r = 0$. □

4.5 Final Necessary Condition

We are now in the position to eliminate ϑ and ρ from (2.21). This procedure automatically leads to (2.28) and prove Theorem 2.9.

Proof to Theorem 2.9 Comparing (2.21) with (2.28), we realize that it sufficies to show that

$$\int_Q rh = b_1 \int_Q (\bar\varphi - \varphi_Q)\vartheta + b_2 \int_\Omega (\bar\varphi(T) - \varphi_\Omega)\vartheta(T) \\ + b_3 \int_Q (\bar\sigma - \sigma_Q)\rho + b_4 \int_\Omega (\bar\sigma(T) - \sigma_\Omega)\rho(T), \tag{4.20}$$

where ϑ and ρ solve the linearized system (2.13)–(2.17) with $h = v - \bar u$. Indeed, if this equality are satisfied (2.28) directly follows by (2.21) by a mere substitution. In this direction, owing to (2.13)–(2.17), we have that the following equalities are satisfied:

$$\begin{aligned} 0 &= \int_Q q\,[\eta - \beta\partial_t\vartheta + \Delta\vartheta - F''(\bar\varphi)\,\vartheta], \\ 0 &= \int_Q p\,[\alpha\,\partial_t\eta + \partial_t\vartheta - \Delta\eta - P'(\bar\varphi)(\bar\sigma - \bar\mu)\,\vartheta - P(\bar\varphi)(\rho - \eta)], \\ 0 &= \int_Q r\,[\partial_t\rho - \Delta\rho + P'(\bar\varphi)(\bar\sigma - \bar\mu)\,\vartheta + P(\bar\varphi)(\rho - \eta) - h]. \end{aligned}$$

Hence, summing the above equalities and integrating by parts, we realize that

$$\begin{aligned} 0 = {} & \int_Q q\eta + \beta\int_Q \partial_t q\,\vartheta - \beta\int_\Omega \vartheta(T)q(T) + \int_Q \Delta q\,\vartheta - \int_Q F''(\bar\varphi)\,q\,\vartheta - \alpha\int_Q \partial_t p\,\eta \\ & + \alpha\int_\Omega p(T)\eta(T) - \int_Q \partial_t p\,\vartheta + \int_\Omega p(T)\vartheta(T) - \int_Q \Delta p\,\eta - \int_Q P'(\bar\varphi)(\bar\sigma - \bar\mu)\,p\,\vartheta \\ & - \int_Q P(\bar\varphi)p\,\rho + \int_Q P(\bar\varphi)p\,\eta - \int_Q \partial_t r\,\rho + \int_\Omega r(T)\rho(T) - \int_Q \Delta r\,\rho \\ & + \int_Q P'(\bar\varphi)(\bar\sigma - \bar\mu)r\,\vartheta + \int_Q P(\bar\varphi)r\,\rho - \int_Q P(\bar\varphi)r\,\eta - \int_Q r\,h, \end{aligned}$$

where, after the time integration, only the final conditions are remained since the initial value of the linearized variables are all zero by (2.17). Moreover, in the integration by parts of the terms with the Laplacian, we also account for the homogeneous Neumann boundary conditions (2.16). Therefore, by rearranging the above equality we get

$$\begin{aligned} \int_Q r\,h = {} & \int_Q [q - \alpha\partial_t p - \Delta p + P(\bar\varphi)(p - r)]\,\eta \\ & + \int_Q [\beta\partial_t q - \partial_t p + \Delta q - F''(\bar\varphi)q + P'(\bar\varphi)(\bar\sigma - \bar\mu)(r - p)]\,\vartheta \\ & + \int_Q [-\partial_t r - \Delta r + P(\bar\varphi)(r - p)]\,\rho \\ & + \int_\Omega [-\beta\vartheta(T)q(T) + \alpha\eta(T)p(T) + p(T)\vartheta(T) + r(T)\rho(T)], \end{aligned}$$

and invoking the adjoint system (2.22)–(2.26), this latter reduces to

$$\int_Q rh = b_1 \int_Q (\bar{\varphi} - \varphi_Q)\vartheta + b_2 \int_\Omega (\bar{\varphi}(T) - \varphi_\Omega)\vartheta(T)$$
$$+ b_3 \int_Q (\bar{\sigma} - \sigma_Q)\rho + b_4 \int_\Omega (\bar{\sigma}(T) - \sigma_\Omega)\rho(T),$$

which is the equality we were looking for. □

Remark 4.1 Let us slightly digress to point out a mathematical issue. The choice of the tracking type cost functional (1.1) is essentially led by the model interpretation. Indeed, from a mathematical point of view, only little rearrangements are needed to treat the more general version

$$\widehat{\mathcal{J}}(\varphi, \mu, \sigma, u) := \mathcal{J}(\varphi, \sigma, u) + \frac{b_5}{2}\|\mu - \mu_Q\|^2_{L^2(Q)} + \frac{b_6}{2}\|\mu(T) - \mu_\Omega\|^2_{L^2(\Omega)},$$

in which all the variables appear. At this stage, we understood the natural requirements on the constants and on the targets that are necessary to give sense to these lines. As the necessary condition is concerned, we expect something like (2.21) in which the following additional terms on its left-hand side

$$b_5 \int_Q (\bar{\mu} - \mu_Q)\eta + b_6 \int_\Omega (\bar{\mu}(T) - \mu_\Omega)\eta(T)$$

occur. Moreover, the adjoint system will read exactly as (2.22)–(2.26), but instead of (2.23) and (2.25) we should have

$$q - \alpha\partial_t p - \Delta p + P(\bar{\varphi})(p - r) = b_5(\bar{\mu} - \mu_Q) \quad \text{in } Q,$$
$$\text{and} \quad \alpha p(T) = b_6(\bar{\mu}(T) - \mu_\Omega) \quad \text{in } \Omega,$$

respectively. About the existence result, note that the presence of this new term on the right-hand side of (2.23) does not add difficulties since it can be easily handled by the Young inequality. In fact, only straightforward modifications are needed to extend the proof of Theorem 2.8 to this general framework. In a similar way also the new final condition can be handled.

To conclude, let us mention that for forthcoming contributions, it will be interesting to couple our study for the control problem **(CP)** with asymptotic analysis as α and β go to zero. Of course, this would require less generality for the potentials, in order to handle the passage to the limit, as pointed out in [4,5,9].

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