Suboptimal Distributed LQR Design for Physically Coupled Systems ⋆

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Abstract: In this paper, we propose a distributed LQR control method, applicable to physically coupled systems, with a general-type cost function. Thanks to a novel suboptimal but distributed cost-to-go matrix update that enforces block-diagonality, the LQR gain matrix is structured, making the overall control scheme distributed.
The proposed control scheme is totally distributed, scalable, and has self-tuning capabilities. The theoretical properties, including the stability of the so-obtained sub-optimal control scheme, are investigated. A case study is finally shown to illustrate the potentialities of the current scheme.

Keywords: linear systems, distributed control, linear quadratic regulators, block-diagonal upper bound approximation, asymptotic stability

1. INTRODUCTION

In the last decades, the complexity of engineering systems and the connectivity between plants (e.g., stemming from production systems to process and manufacturing plants) have continuously and dramatically increased. Some notable examples of new generation complex and large-scale systems are current power networks (Resende and Peças Lopes, 2011) (including smart grids and distributed generation systems), environmental monitoring systems (Tennina et al., 2011), large-scale chemical plants (Farina et al., 2016), large-scale irrigation and hydraulic networks (Cantoni et al., 2007), and fleets of multi-agent autonomous (possibly cooperating) vehicles (Cortés et al., 2004).

The widespread of new generation plants has posed significant challenges in the design and the development of dedicated control systems (Šiljak, 1991; Lunze, 1992). This includes dramatic challenges in the design and implementation of monitoring, fault detection, state estimation, and control algorithms. The centralized paradigm relies upon the assumption that the control scheme is integrated in a monolithic computing unit and is based upon a reliable communication network allowing for a fast and synchronous data exchange with all the system actuators and sensors.
This paradigm shows significant pitfalls. Indeed, as the system scale increases, the control algorithm becomes, on one side, prohibitively complex and both computationally and communication-wise demanding. On the other hand, it exhibits increasing fragilities with respect to model inaccuracies, system changes, and communication issues.

To provide solutions to these challenges, distributed methods have been developed over the years, both for unconstrained (Šiljak, 1991; Lunze, 1992) and constrained (Maestre and Negenborn, 2014) systems. Distributed algorithms rely upon the assumption that the system under control can be regarded as a set of interacting subsystems, and that a local control unit - having both computational and communication capabilities - can be integrated in each subsystem. Among the main challenges and opportunities of distributed approaches, the following have been subject of particular attention: (i) local control algorithms should be integrated in the local computing units and should provide stability properties, robustly for any possible system configuration and possibly in absence of communication; (ii) thanks to communication and data exchange between local control units, cooperation and system-wide optimality should be sought for; (iii) distributed and automatic self-design capabilities should be conferred, not only to make the design procedure perfectly distributed and scalable, but also to make the system adaptive to system changes, including plug-and-play events.

In this paper we focus our attention on linear quadratic regulators (LQR), which have been thoroughly studied in the past, especially in the centralized framework (Bertsekas, 2017). Only few notable works have addressed the design of distributed LQRs, proposing suboptimal solutions, e.g., (Borrelli and Keviczky, 2008; Jiao et al., 2019;

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Vlahakis and Halikias, 2018; Wang et al., 2016; Zhang et al., 2015; Deshpande et al., 2012). However, the existing works have been developed under the assumption that, although the overall control problem (which relies upon the solution to an optimization one) accounts for the states and the inputs of all subsystems in an integrated fashion, the subsystems under control are physically decoupled, in the sense that their dynamics are independent with each other.

In this paper, we propose a distributed LQR control method, applicable to physically coupled systems, with general-type cost functions. In the proposed method, gains are computed according to the usual equation of LQR. On the other hand, we propose a suboptimal but distributed cost-to-go matrix update that enforces block-diagonality. Thank to this, the possibly time-varying LQR gain matrix is structured, making the overall control scheme distributed. More precisely, neighbor-to-neighbor bidirectional communication is required for cost-to-go matrix update and for computing the control action to take, by each subsystem, at local level.

Therefore, the proposed control scheme is totally distributed, scalable, and has self-tuning capabilities, since an actual design procedure is not required, being implicitly carried out through the cost-to-go matrix update. Another remarkable fact about the proposed scheme is that it automatically provides a block-diagonal Lyapunov function, explicitly required by the majority of available distributed model predictive control algorithms (Maestre and Negenborn, 2014).

The theoretical properties, including the stability of the so-obtained sub-optimal control scheme, are investigated. Also, a case study is illustrated to show the potentialities of the current scheme.

This paper is organized as follows. In Section II, we give the problem formulation and in Section III, we demonstrate the stability and performance of the proposed distributed controller. In Section IV, simulations are conducted to verify the effectiveness the proposed controller. Conclusion remarks are made in Section V.

Notation: Through out the paper, all the matrices and vectors are assumed to have compatible dimensions. $\|\cdot\|$ denotes the vector 2-norm or the matrix induced 2-norm. A' denotes the transpose of A. A^{-1} denotes the inverse of A. $[A_{ij}]$ denotes a matrix whose ij-th element (block) is A_{ij}. $\text{diag}\{A_i\}$ denotes a diagonal (block diagonal) matrix whose diagonal element (block) is A_i. $[A]_{ij}$ denotes the ij-th element of A. I denotes the identity matrix. 0 zero denotes the matrix with all elements to be zero. We say that a matrix X fulfills a structural constraint if certain blocks of X are required to be zero, i.e., $X_{ij} = 0$ for some i, j. Moreover, we say that two matrices X, Y have the same information structure, if $X_{ij} = 0$ implies $Y_{ij} = 0$ and vice versa.

2. PROBLEM FORMULATION

Consider N physically coupled systems, each described by

$$x_i(t+1) = A_{ii}x_i(t) + \sum_{j\neq i} A_{ij}x_j(t) + B_i u_i(t),$$

for $i = 1, \ldots, N$, where $x_i(t) \in \mathbb{R}^{n_i}$ and $u_i(t) \in \mathbb{R}^{m_i}$ are the state and control input for the i-th system, and $\sum_{j\neq i} A_{ij}x_j(t)$ denotes the physical coupling. The interconnection among systems can be described by a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V}$ is the node set and the $\mathcal{E}$ is the edge set. Each node $i \in \mathcal{V}$ represents a system, and the edge $(i, j) \in \mathcal{E}$ exists if and only if $A_{ij} \neq 0$. The in-neighbors' set $\mathcal{N}_i^{\text{in}}$ of agent i is defined as $\{j : (i, j) \in \mathcal{E}\}$. The out-neighbors' set $\mathcal{N}_i^{\text{out}}$ of agent i is defined as $\{j : (j, i) \in \mathcal{E}\}$. The sets $\mathcal{N}_i^{\text{in}}$ and $\mathcal{N}_i^{\text{out}}$ may contain agent i itself if $A_{ii} \neq 0$. The adjacency matrix $\mathcal{A} \in \mathbb{R}^{N\times N}$ for the graph $\mathcal{G}$ is defined by the elements: $[\mathcal{A}]_{ij} = 1$ if $A_{ij} \neq 0$ and $[\mathcal{A}]_{ij} = 0$ if $A_{ij} = 0$.

We can write the collective system in a compact form as

$$x(t+1) = Ax(t) + Bu(t),$$

where $x = [x_1', \ldots, x_N']'$, $A = [A_{ij}]$, $B = \text{diag}\{B_i\}$, $u = [u_1', \ldots, u_N']'$. In this paper, we are interested in the infinite-horizon LQR control problem, i.e., to design a control law $u(t)$ such that the following LQR performance is minimized

$$J = \sum_{t=0}^{\infty} x(t)'Qx(t) + u(t)'Ru(t),$$

where $Q = [Q_{ij}] \geq 0$, $R = \text{diag}\{R_i\} \geq 0$. It is known that the optimal centralized control law is given by $u(t) = K^*x(t)$ with

$$K^* = -(B'S^*B + R)^{-1}B'S^*A, \tag{1}$$

where S^* is the solution to the following Riccati equation

$$S^* = A'S^*A + Q - A'S^*B(B'S^*B + R)^{-1}B'S^*A.$$

Moreover, the optimal cost is given by

$$J^* = x(0)'S^*x(0).$$

However, computing the control law u_i from (1) could require the knowledge of x_j for all $j = 1, \ldots, N$. When the number N of subsystems is high, such a centralized controller becomes prohibitive, both in terms of computation and communication requirements.

In this paper, we will study distributed solutions where the computation of u_i requires only system i's neighbors' states. In other words, we would like to design a controller gain K with the same information structure as $\mathcal{A}$. As shown in (Rotkowitz and Lall, 2006), solving optimization problems with controller information structure constraints can be extremely difficult. To circumvent this issue, we will develop a suboptimal approach.

It is clear from (1) that, the reason why K^* does not satisfy the information structure constraint is because S^* might be a full matrix. If S^* was block diagonal, K^* would have the same information structure as $\mathcal{A}$. Therefore, in this paper, we propose a suboptimal controller design, where S^* is approximated by a block diagonal matrix $P(t)$ to generate the controller gain $K(t)$. Moreover, we require that $P(t)$ is an upper bound to the optimal cost, with which we can guarantee the stability of closed-loop system by studying the boundedness of $P(t)$.

3. MAIN RESULTS

In this section, we first propose the controller design and then analyze the asymptotic stability and performance of the closed-loop system.

3.1 Controller Design

Our controller is designed as $u(t) = K(t)x(t)$ with

$$K(t) = -(B'P(t)B + R)^{-1}B'P(t)A, \qquad (2)$$

where $P(t) = \mathrm{diag}\{P_i(t)\}$ is calculated from the following iteration for $t = 0, 1, 2, \ldots$ and $i = 1, \ldots, N$

$$P_i(t+1) = \sum_{j\in\mathcal{N}_i^{\mathrm{out}}} A'_{ji}P_j^F(t)A_{ji} + Q_i + \sum_{j\in\mathcal{N}_i^{\mathrm{out}}, j\neq i} \|Q_{ij}\|I + \sum_{k\in\mathcal{N}_i^{\mathrm{out}}} \|A'_{ki}\|\|P_k^F(t)\|(\sum_{j\in\mathcal{N}_k^{\mathrm{in}}, j\neq i}\|A_{kj}\|)I, \qquad (3)$$

with

$$P_j^F(t) = P_j(t) - P_j(t)B_j(B'_jP_j(t)B_j + R_j)^{-1}B'_jP_j(t). \qquad (4)$$

From the collective control law (2), the control law for agent i is given by

$$u_i(t) = \sum_{j\in\mathcal{N}_i^{\mathrm{in}}} K_{ij}(t)x_j(t),$$

where

$$K_{ij}(t) = -(B'_iP_i(t)B_i + R_i)^{-1}B'_iP_i(t)A_{ij}.$$

Remark 1. Since $P(t)$ is block diagonal, as previously described, the control law u_i depends only on the states $x_j, j \in \mathcal{N}_i^{\mathrm{in}}$, which must be measured and made available to the controller of system i through a communication network.

Remark 2. For computing $P_i(t)$, we require information from systems that are one-hop away from system i. Indeed, (i) the term $\sum_{j\in\mathcal{N}_i^{\mathrm{out}}} A'_{ji}P_j^F(t)A_{ji}$ in (3) can be updated using the matrices P_j and A_{ji} from the out-neighbors of agent i; (ii) the term

$$\sum_{k\in\mathcal{N}_i^{\mathrm{out}}} \|A'_{ki}\|\|P_k^F(t)\|(\sum_{j\in\mathcal{N}_k^{\mathrm{in}}, j\neq i}\|A_{kj}\|)I$$

in (3) depends on matrices P_k, A_{ki}, A_{kj}. P_k and A_{ki} can be obtained from the out-neighbor k of agent i. Moreover, A_{kj} can also be obtained from the out-neighbor k of agent i since agent k has access to the coupling matrix A_{kj} for $j \in \mathcal{N}_k^{\mathrm{in}}$.

Remark 3. The proposed control law is a variant of the value iteration in approximate dynamic programming, see (Bertsekas, 2017). In value iteration, the following Centralized Riccati Iteration (CRI) is used to approximate the optimal cost S^*

$$S(t+1) = A'S(t)A + Q - A'S(t)B(B'S(t)B + R)^{-1}B'S(t)A. \qquad (5)$$

In contrast, we propose to use a block diagonal matrix $P(t)$ generated from (3) and (4) to approximate S^*. Moreover, similar to the value iteration, the control law $K(t)$ is obtained from the Bellman equation assuming the optimal cost is the approximation $P(t)$, i.e.,

$$K(t) = \arg\min_K x(t)'Qx(t) + x(t)'KRKx(t) + (Ax(t) + BKx(t))'P(t)(Ax(t) + BKx(t)).$$

In the following, we will show that if certain conditions are satisfied, $K(t)$ converges to a stabilizing gain.

3.2 Asymptotic Stability of the Closed-loop System

Assume that, in (5), $S(t)$ is replaced by $P(t)$. Then, from (5), the CRI gives $A'P^F(t)A + Q$, which, unfortunately, is not block-diagonal due to the coupling among subsystems. In contrast, the proposed iteration (3) remains to be block-diagonal, which guarantees a distributed implementation of the controller $K(t)$. Moreover, the next lemma shows that the block-diagonal matrix $P(t)$ obtained through (3) is an upper bound to the result of CRI, which is essential for characterizing the stability property of the closed-loop system.

Lemma 4. In the iteration (3), one has

$$P(t+1) \geq A'P^F(t)A + Q,$$

where $P^F(t) = \mathrm{diag}\{P_i^F(t)\}$.

The following lemma is needed for the proof, which is the Gersgorin theorem for block matrices.

Lemma 5. (Theorem 6.3 of (Varga, 2010)). Consider $A = [A_{ij}]$. Let

$$G_i = \sigma(A_{ii}) \cup \{\lambda \notin \sigma(A_{ii}) : (\|(\lambda I - A_{ii})^{-1}\|)^{-1} \leq \sum_{j\neq i}\|A_{ij}\|\},$$

where $\sigma(\cdot)$ denotes the spectrum of a matrix. Then

$$\sigma(A) \in \cup_i G_i.$$

The proof of Lemma 4 is given below.

Proof. Since

$$[A'P^F(t)A]_{ij} = \sum_k A'_{ki}P_k^F(t)A_{kj},$$

we have that

$$[P(t+1) - AP^F(t)A' - Q]_{ij} = \begin{cases} \sum_{l\neq i}\sum_k \|A'_{ki}\|\|P_k^F(t)\|\|A_{kl}\|I + \sum_{l\neq i}\|Q_{il}\| & j = i \\ -\sum_k A'_{ki}P_k^F(t)A_{kj} - Q_{ij} & j \neq i \end{cases}$$

In view of the Lemma 5, the eigenvalues of $P(t+1) - A'P^F(t)A - Q$ are in the region

$$\cup_i \left\{\lambda : \left|\lambda - \sum_{j\neq i}\left(\sum_k \|A'_{ki}\|\|P_k^F(t)\|\|A_{kj}\|I + \|Q_{ij}\|\right)\right| \leq \sum_{j\neq i}\left\|-\sum_k A'_{ki}P_k^F(t)A_{kj} - Q_{ij}\right\|\right\},$$

which is included in the right half complex plain. Since $P(t+1) - A'P^F(t)A - Q$ is symmetric, we know $P(t+1) - A'P^F(t)A - Q \geq 0$, which finishes the proof. ■

In this section, we analyze the asymptotic stabilizing properties of the proposed controller. The following assumption is needed.

Assumption 6. A_{ii} is invertible for all i.

Under Assumption (6), we can show that if we initialize $P_i(0)$ appropriately and suitable conditions on the coupling between subsystems are satisfied, the matrices $P(t)$ and $K(t)$ converge to constant values $\bar{P}$ and $\bar{K}$. Moreover, $\bar{K}$ is stabilizing. The result is stated as follows.

Theorem 7. Initialize $P_i(0) = 0$ for all i. Let $F_i = A_{ii} + B_iK_i$ with $K_i \in \mathbb{R}^{m_i \times n_i}$, $F = \text{diag}\{\|F_i\|^2\}$ and define the matrix Γ as

$$[\Gamma]_{ij} = \begin{cases} 1 + \sum_{k\neq i} \|A_{ii}^{-1}\|^2\|A_{ii}\|\|A_{ik}\| & j = i, \\ \|A_{jj}^{-1}A_{ji}\|^2 + \sum_{k\neq i} \|A_{jj}^{-1}\|^2\|A_{ji}\|\|A_{jk}\| & j \neq i. \end{cases}$$

If there exist K_i for $i = 1, \ldots, N$ such that

$$\rho(F\Gamma) < 1, \tag{6}$$

then

$$\lim_{t\to-\infty} P(t) = \bar{P}, \quad \lim_{t\to-\infty} K(t) = \bar{K}$$

and $\bar{K}$ is stabilizing.

To prove Theorem 7, we first show monotonicity and boundedness of the $P(t)$ iteration. The monotonicity property is stated in the following lemma.

Lemma 8. Let $P_i^A(t) \in \mathbb{R}^{n_i\times n_i}$ and $P_i^B(t) \in \mathbb{R}^{n_i\times n_i}$ be two positive semidefinite matrices. Let $P_i^A(t+1)$ and $P_i^B(t+1)$ be the matrices produced by (3) and (4) when selecting $P_i(t) = P_i^A(t)$ and $P_i(t) = P_i^B(t)$, respectively. Suppose $P_i^A(t) > P_i^B(t)$ for all i. Then $P_i^A(t+1) > P_i^B(t+1)$ for all i.

Proof. From the definition of $P_i(t+1)$ in (3), we only need to prove that $A'_{ji}P_j^F(t)A_{ji}$ and $\|A'_{ki}\|\|P_k^F(t)\|\|A_{kj}\|$ are monotonic with respect to $P_i(t)$. The monotonicity of $A'_{ji}P_j^F(t)A_{ji}$ with respect to $P_i(t)$ follows from Lemma 1.c in (Kar et al., 2012). Therefore, we only need to prove the monotonicity of $\|A'_{ki}\|\|P_k^F(t)\|\|A_{kj}\|$ with respect to $P_i(t)$. Let $P_k^{AF}(t)$ denote the matrix $P_k^F(t)$ when $P_k(t) = P_k^A(t)$. Assume $P_k^{BF}(t)$ is defined similarly. Assume $P_k^A > P_k^B$, since $P_k^F = (P_k^{-1} + B_kR_k^{-1}B'_k)^{-1}$, we have $P_k^{AF} > P_k^{BF}$. Therefore,

$$\|A'_{ki}\|\|P_k^{AF}(t)\|\|A_{kj}\| > \|A'_{ki}\|\|P_k^{BF}(t)\|\|A_{kj}\|,$$

which means $\|A'_{ki}\|\|P_k^F(t)\|\|A_{kj}\|$ is monotonic with respect to $P_i(t)$. The proof is completed. ■

Next, we will show the boundedness of the $P(t)$ iteration. The result is stated in the following lemma.

Lemma 9. If (6) holds, the sequence of matrices $P(t)$ generated from (3) is bounded for all t.

Proof. From the definition of $P_i(t+1)$, we have that

$$P_i(t+1) = P_i^L(t+1) + \Delta_i(t+1) + S_i(t+1) + \tilde{Q}_i,$$

where

$$\Delta_i(t+1) = \sum_{j\neq i} A'_{ji}P_j^F(t)A_{ji},$$

$$S_i(t+1) = \sum_{j\neq i}\sum_k \|A'_{ki}\|\|P_k^F(t)\|\|A_{kj}\|I.$$

$$P_i^L(t+1) = A'_{ii}P_i^F(t)A_{ii}, \quad \tilde{Q}_i = Q_i + \sum_{j\neq i}\|Q_{ij}\|I.$$

Since A_{ii} is invertible, we have that

$$P_i^F(t) = (A'_{ii})^{-1}P_i^L(t+1)A_{ii}^{-1}.$$

Therefore, we obtain

$$\Delta_i(t+1) = \sum_{j\neq i} A'_{ji}(A'_{jj})^{-1}P_j^L(t+1)A_{jj}^{-1}A_{ji},$$

$$S_i(t+1) = \sum_{j\neq i}\sum_k \|A'_{ki}\|\|(A'_{kk})^{-1}P_k^L(t+1)A_{kk}^{-1}\|\|A_{kj}\|I,$$

which further implies

$$\begin{aligned} &P_i^L(t+1) = A'_{ii}P_i^F(t)A_{ii} \\ &= (A_{ii} + B_iK_{ii}(t))'P_i(t)(A_{ii} + B_iK_{ii}(t)) + K_{ii}(t)'R_iK_{ii}(t) \\ &\overset{(a)}{\leq} (A_{ii} + B_iK_i)'P_i(t)(A_{ii} + B_iK_i) + K'_iR_iK_i \\ &= F'_iP_i(t)F_i + K'_iR_iK_i \\ &= F'_i(P_i^L(t) + \Delta_i(t) + S_i(t))F_i + K'_iR_iK_i + F'_i\tilde{Q}_iF_i \\ &= F'_i(P_i^L(t) + \Delta_i(t) + S_i(t))F_i + \delta_i, \end{aligned} \tag{7}$$

where (a) follows from the fact that $K_{ii}(t)$ minimizes $(A_{ii} + B_iK_i)'P_i(t)(A_{ii} + B_iK_i) + K'_iR_iK_i$ for any K_i and $\delta_i = K'_iR_iK_i + F'_i\tilde{Q}_iF_i$.

Since

$$\begin{aligned} \|\Delta_i(t)\| &= \|\sum_{j\neq i} A'_{ji}(A'_{jj})^{-1}P_j^L(t)A_{jj}^{-1}A_{ji}\| \\ &\leq \sum_{j\neq i}\|A_{jj}^{-1}A_{ji}\|^2\|P_j^L(t)\| \end{aligned}$$

and

$$\begin{aligned} \|S_i(t)\| &\leq \sum_{j\neq i}\sum_k \|A_{kk}^{-1}\|^2\|A_{ki}\|\|A_{kj}\|\|P_k^L(t)\| \\ &= \sum_j\sum_{k\neq i}\|A_{jj}^{-1}\|^2\|A_{ji}\|\|A_{jk}\|\|P_j^L(t)\|, \end{aligned}$$

where the last equation is obtained via the swap of index j, k, from (7), we have

$$\begin{aligned} &\|P_i^L(t+1)\| \\ &\leq \|F_i\|^2\Bigg(\|P_i^L(t)\| + \sum_{j\neq i}\|A_{jj}^{-1}A_{ji}\|^2\|P_j^L(t)\| \\ &\quad + \sum_j\sum_{k\neq i}\|A_{jj}^{-1}\|^2\|A_{ji}\|\|A_{jk}\|\|P_j^L(t)\|\Bigg) + \|\delta_i\| \\ &= \|F_i\|^2(\Gamma_{ii}\|P_i^L(t)\| + \sum_{j\neq i}\Gamma_{ij}\|P_j^L(t)\|) + \|\delta_i\|. \end{aligned}$$

Therefore, we have that

$$\|P^L(t+1)\| \leq F\Gamma\|P^L(t)\| + \|\delta\|,$$

where with a slight abuse of notion $\|P^L\| = [\|P_1^L\|, \ldots, \|P_N^L\|]'$ and $\|\delta\| = [\|\delta_1\|, \ldots, \|\delta_N\|]'$. Therefore, if $\rho(F\Gamma) < 1$, we have that $P_i^L(t+1)$ is bounded, which further implies the boundedness of $P_i(t+1)$. The proof is completed. ■

Based on the above results, the proof of Theorem 7 is given below.

Proof. Since $P_i(1) = \tilde{Q}_i > P_i(0) = 0$, in view of Lemma 8, we can show that $P_i(t+1) > P_i(t)$ for all t by induction. Therefore $P(t)$ is monotonically increasing with respect to time t. Moreover, since $\rho(F\Gamma) < 1$, $P(t)$ is bounded from Lemma 9. Therefore $P(t)$ and $K(t)$ converges to some constant value $\bar{P}, \bar{K}$ as $t \to \infty$. Besides, in view of Lemma 4, $\bar{P}, \bar{K}$ should satisfy that

$$\bar{P} > (A + B\bar{K})'\bar{P}(A + B\bar{K}).$$

Therefore, $\bar{K}$ is stabilizing. ■

Remark 10. Verifying the stability conditions in Theorem 7 requires to solve the nonlinear optimization problem $\min_{K_i} \rho(F\Gamma)$. An alternative is to manually select K_i and then decide whether the condition $\rho(F\Gamma) < 1$ is satisfied. A first heuristic method of selecting K_i is to let K_i solves the optimization problem $\min_{K_i} \|A_{ii} + B_i K_i\|$, which can be cast into the following LMI problem

$$\min_{K_i,\gamma} \gamma$$
$$\text{s.t.} \quad \begin{bmatrix} \gamma I & (A_{ii} + B_i K_i)' \\ (A_{ii} + B_i K_i) & I \end{bmatrix} \geq 0.$$

The motivation is that if $\|A_{ii} + B_i K_i\| = 0$ for all i, $\rho(F\Gamma) = 0 < 1$. Therefore, we are motivated to select K_i to make $\|A_{ii} + B_k K_i\|$ as small as possible and expect that $\rho(F\Gamma) < 1$. Another heuristic method is to run the iteration (3) for a sufficiently long time, use the final $P_i(t)$ to construct $K_i = K_{ii}(t)$ with $K_{ii}(t) = -(B_i' P_i(t) B_i + R_i)^{-1} B_i' P_i(t) A_{ii}$ and use such K_i for verifying $\rho(F\Gamma) < 1$. The motivation is that if $P_i(t)$ converges and $\bar{K}$ is stabilizing, we can expect that the finite horizon approximation $K_{ii}(t)$ fulfills the stability condition $\rho(F\Gamma) < 1$.

4. SIMULATIONS

We consider a system composed of $N = 6$ subsystems. For $i \in \{1, 2, ..., 6\}$, we set

$$A_{ii} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & -0.9 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and for $i, j \in \{1, \ldots, 6\}$, let

$$A_{ij} = \begin{cases} \text{diag}(\alpha, -\alpha) & \text{if } |i - j| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$. For the LQR control problem, we set the symmetric matrices $Q_{ii} = R_i = I$ for $i = 1, \ldots, 6$ and $Q_{ij} = 0$ for $i \neq j$. For each system, we compare its behaviour when using (i) K^*, the static gain from the infinite horizon centralized LQR controller (1), (ii) $K(t)$ as proposed in our paper, (iii) $K_d(t)$ obtained by the dualisation of the Partition-Based Distributed Kalman Filter from (Farina and Carli, 2018), which is given by $K_d(t) = [K_{d,ij}(t)]$ with

$$K_{d,ij}(t) = -(B_i' P_i(t) B_i + R_i)^{-1} B_i P_i(t) A_{ij}, \tag{8}$$
$$P_i(t+1) = \sum_{j \in \mathcal{N}_i^{\text{out}}} \tilde{A}_{ji}' P_j^F(t) \tilde{A}_{ji} + Q_i, \tag{9}$$

where $P_j^F(t)$ is given by (4); $\tilde{A}_{ji} = \sqrt{|\mathcal{N}_j^{\text{in}}|} A_{ji}$ and $|\mathcal{N}_j^{\text{in}}|$ is the cardinality of $\mathcal{N}_j^{\text{in}}$.

In Figure 1, we show the response for the first subsystem when $\alpha = 0.4$, $T = 100$ and initial conditions of each subsystem are $x = [100, -50]'$. We plot the system response of the first subsystem using K^* (see (1)), $K(t)$ and $K_d(t)$, where it can be seen that the temporal response converges to $[0, 0]'$ when using K^* and $K(t)$ and diverges when using $K_d(t)$. Besides, the response with K^* and $K(t)$ are close to each other, which demonstrates the effectiveness of our proposed controller. Responses of other subsystems are not shown since they have a similar evolution.

Figure 2 shows the spectral radius of $A + BK$ with respect to α when let K equal to K^*, $K(T)$ and $K_d(T)$, after $T = 100$ iterations of (3) and (9), respectively. In the considered situations, we can verify that the gains $K(t)$ and $K_d(t)$ converge to constant values after $T = 100$ iterations. From Figure 2, it is clear that in spite of the coupling between subsystems, the overall system remains stable when using our proposed controller $K(t)$. In contrast, if the controller $K_d(t)$ is used, the coupling α should be sufficiently small (in this case, $\alpha < 0.36$) in order to preserve stability of the closed-loop system.

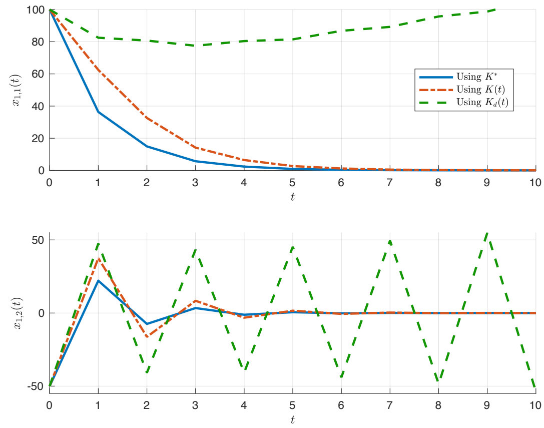

Fig. 1. Temporal response of the first subsystem state variables $x_{1,1}(t)$ and $x_{1,2}(t)$ under different controllers.

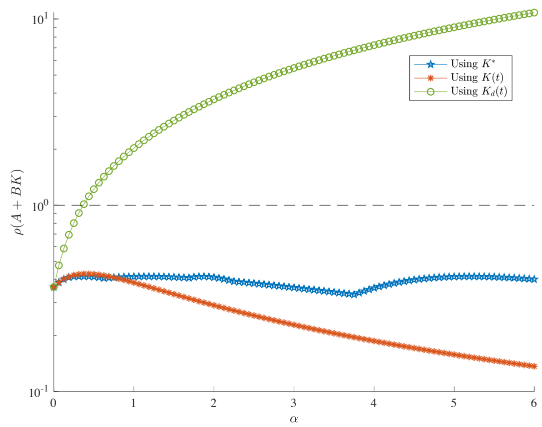

Fig. 2. Values of $\rho(A + BK)$ as a function of α under different controllers.

We verify Theorem 7, by corroborating that if $\rho(F\Gamma) < 1$ then the closed-loop system is stable. In order to obtain $F\Gamma$, we calculate Γ from Theorem 7, and $F = diag\{\|A_{ii} + B_i K_i\|^2\}$ using $K_i = K_{ii}(T)$ obtained after $T = 100$ iterations of (3). Figure 3 presents $\rho(F\Gamma)$ for $\alpha \in [0, 6]$. As $\rho(F\Gamma) < 1$, from Theorem 7, we can conclude that the asymptotic control gain $\bar{K}$ is stabilizing, which is also reflected in Figure 2.

Finally, we evaluate the finite-horizon cost $\sum_{t=0}^{T} x'(t)Qx(t) + u'(t)Ru(t)$ when using the three different control laws for $\alpha = \{0.1, 0.2, 0.3\}$. Table 1 summarises the obtained results, for the finite-horizon LQR with $T = 100$. As it is expected, the performance of our controller $K(t)$ is suboptimal but improves over $K_d(t)$.

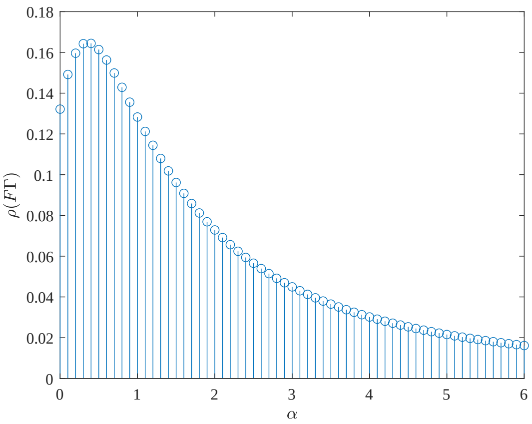

Fig. 3. Values of $\rho(F\Gamma)$ as a function of α.

Table 1. Cost-to-go of the T-step finite-horizon LQR performance, for different coupling values and different controllers

	K^*	$K(t)$	$K_d(t)$
$\alpha = 0.1$	5.47×10^4	7.27×10^4	7.54×10^4
$\alpha = 0.2$	7.76×10^4	11.3×10^4	14.8×10^4
$\alpha = 0.3$	10.6×10^4	17.0×10^4	45.2×10^4

5. CONCLUSIONS

This paper studies distributed LQR control design for physically coupled systems. Different from other contributions available in the literature, we do not assume any structures for the state penalty matrix in the LQR performance index. We propose a distributed control design and provides an upper bound to its LQR performance. Moreover, we study the asymptotic performance of the closed-loop system and show that if certain conditions are satisfied, asymptotic stability is guaranteed. Further research will consider the development of distributed output feedback controllers with stability guarantees by combining the partitioned Kalman filter in (Farina and Carli, 2018) with the distributed LQR scheme proposed in this paper.

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