

Self-Adjointness criterion for operators in Fock spaces

Marco Falconi*

IRMAR and Centre Henri Lebesgue; Université de Rennes I

Campus de Beaulieu, 263 avenue du Général Leclerc

CS 74205, 35042 RENNES Cedex

(Dated: December 4, 2014)

In this paper we provide a criterion of essential self-adjointness for operators in the tensor product of a separable Hilbert space and a Fock space. The class of operators we consider may contain a self-adjoint part, a part that preserves the number of Fock space particles and a non-diagonal part that is at most quadratic with respect to the creation and annihilation operators. The hypotheses of the criterion are satisfied in several interesting applications.

Keywords: Essential Self-Adjointness, Fock Spaces, Interacting Quantum Field Theories, Nelson Hamiltonian, Pauli-Fierz Hamiltonian.

1. INTRODUCTION

Let $\mathcal{H}_1, \mathcal{H}_2$ be separable Hilbert spaces. We consider the following space:

$$(1) \quad \mathcal{H} = \mathcal{H}_1 \otimes \Gamma_s(\mathcal{H}_2);$$

where $\Gamma_s(\mathcal{H})$ is the symmetric Fock space based on \mathcal{H} [see 7, 9, 24, for mathematical presentations of Fock spaces and second quantization]. The symmetric structure of the Fock space does not play a role in the argument: in principle it is possible to formulate the same criterion for anti-symmetric Fock spaces $\mathcal{H}_1 \otimes \Gamma_a(\mathcal{H}_2)$. We focus on symmetric spaces, the corresponding antisymmetric results should be deduced without effort.

We are interested in proving a criterion of essential self-adjointness for densely defined operators of the form:

$$(2) \quad H = H_{01} \otimes 1 + 1 \otimes H_{02} + H_I;$$

with suitable assumptions on H_{01}, H_{02} and H_I . Operators based on these spaces and with such structure are crucial in physics, to describe the quantum dynamics of interacting particles and fields.

* marco.falconi@univ-rennes1.fr

Self-adjointness of operators in Fock spaces has been widely studied, in particular in the context of Constructive Quantum Field Theory [e.g. 13–15, 21, 25, 26] and Quantum Electrodynamics [e.g. 1, 4–6, 16, 18, 20, 22, 27]. A variety of advanced tools has been utilized, for even “simple” systems present technical difficulties to overcome: many questions still remain unsolved.

In some favourable situations, however, it is possible to take advantage of the peculiar structure of the Fock space and prove essential self-adjointness with almost no effort. The idea first appeared in a paper by Ginibre and Velo [13]; and the author utilized it in [2, 11] for the Nelson model with cut off: essential self-adjointness can be proved with less assumptions than using the Kato-Rellich Theorem (and that becomes particularly significant in dimension two), see Section 4.2. Another remarkable application is the Pauli-Fierz Hamiltonian describing particles coupled with a radiation field. For general coupling constants, essential self-adjointness has been first proved in a probabilistic setting, using stochastic integration [17, 18]. In this paper we prove the same result directly in Section 4.3, applying the criterion formulated in Assumptions A_0 , A_I and Theorem 3.1.

In the literature, self-adjointness of operators in Fock spaces has been studied using various tools of functional analysis: the Kato-Rellich and functional integration arguments mentioned above are two examples, as well as the Nelson commutator theorem [10]. For each particular system, a strategy is utilized ad hoc: the more complicated is the correlation between \mathcal{H}_1 and $\Gamma_s(\mathcal{H}_2)$, the more difficult is the strategy. We realized that, if we take suitable advantage of the fibered structure of the Fock space, the type of interaction between the spaces is not so relevant. This was a strong motivation to study the problem from a general perspective. Due to the variety of possible applications, an effort has been made to formulate the necessary assumptions in a general form. Roughly speaking, the essential requirement is that the part of H_I that does not commute with the number operator of $\Gamma_s(\mathcal{H}_2)$ is at most quadratic with respect to the creation and annihilation operators. As anticipated, the space \mathcal{H}_1 does not play a particular role, as long as H_I behaves sufficiently well with respect to H_{01} .

1. Paper organization.

In Section 1.2 we introduce the notation, and recall some basic definitions of operators in Fock spaces. In Section 2 we formulate the necessary assumptions on the operator H . In Section 3 we prove the criterion. In Section 4 we outline some of the most interesting

applications. Finally in Section 5 we give some conclusive remarks, and an extension of the criterion to semi-bounded quartic operators.

2. Definitions and notations.

- Let \mathcal{H} be a separable Hilbert space. Then the symmetric Fock space $\Gamma_s(\mathcal{H})$ is defined as the direct sum:

$$\Gamma_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s n},$$

where $\mathcal{H}^{\otimes_s n}$ is the n -fold symmetric tensor product of \mathcal{H} , and $\mathcal{H}^{\otimes_s 0} := \mathbb{C}$.

- Let $h : \mathcal{H} \supseteq D(h) \rightarrow \mathcal{H}$ be a densely defined self-adjoint operator on a separable Hilbert space \mathcal{H} . Its second quantization $d\Gamma(h)$ is the self-adjoint operator on $\Gamma_s(\mathcal{H})$ defined by

$$d\Gamma(h)|_{D(h)^{\otimes_s n}} = \sum_{k=1}^n 1 \otimes \cdots \otimes \underbrace{h}_k \otimes \cdots \otimes 1.$$

Let u be a unitary operator on \mathcal{H} . We define $\Gamma(u)$ to be the unitary operator on $\Gamma_s(\mathcal{H})$ given by

$$\Gamma(u)|_{\mathcal{H}^{\otimes_s n}} = \bigotimes_{k=1}^n u.$$

If e^{ith} is a group of unitary operators on \mathcal{H} , $\Gamma(e^{ith}) = e^{itd\Gamma(h)}$.

- $N := d\Gamma(1)$ the number operator of $\Gamma_s(\mathcal{H}_2)$.
- $H_0 := H_{01} \otimes 1 + 1 \otimes H_{02}$; the free Hamiltonian.
- If X is a self-adjoint operator on a Hilbert space, we denote by $D(X)$ its domain, by $q_X(\cdot, \cdot)$ the form associated with X and by $Q(X)$ the form domain.
- Let \mathcal{H} be a Hilbert space; $\{\mathcal{H}^{(j)}\}_{j \in \mathbb{N}}$ a collection of disjoint subspaces of \mathcal{H} ; X an operator densely defined on \mathcal{H} . We say that $\{\mathcal{H}^{(j)}\}_{j \in \mathbb{N}}$ is invariant for X if $\forall j \in \mathbb{N}$, X maps $D(X) \cap \mathcal{H}^{(j)} \rightarrow \mathcal{H}^{(j)}$, and $D(X) \cap \mathcal{H}^{(j)}$ is dense in $\mathcal{H}^{(j)}$.
- Let \mathcal{H} be a Hilbert space; $\{\mathcal{H}^{(j)}\}_{j \in \mathbb{N}}$ a collection of disjoint closed subspaces of \mathcal{H} such that $\bigoplus_{j \in \mathbb{N}} \mathcal{H}^{(j)} = \mathcal{H}$. Then we call the collection complete, and we define the dense

subset $f_0(\mathcal{K}^{(\cdot)})$ of \mathcal{K} as:

$$(3) \quad f_0(\mathcal{K}^{(\cdot)}) = \left\{ \phi \in \mathcal{K}, \exists n \in \mathbb{N} \text{ s.t. } \phi \in \bigoplus_{j=0}^n \mathcal{K}^{(j)} \right\}.$$

Also, we denote by $\mathbb{1}_j(\mathcal{K}^{(\cdot)})$ the orthogonal projection on $\mathcal{K}^{(j)}$, by $\mathbb{1}_{\leq n}(\mathcal{K}^{(\cdot)})$ the orthogonal projection on $\bigoplus_{j=0}^n \mathcal{K}^{(j)}$.

- Let $\mathcal{K} \ni f, g$ be two elements of a separable Hilbert space. We define the creation $a^*(f)$ and annihilation $a(f)$ operators on $\Gamma_s(\mathcal{K})$ by their action on n -fold tensor products (with $a(f)\phi_0 = 0$ for any $\phi_0 \in \mathcal{K}^{\otimes_s 0} = \mathbb{C}$):

$$\begin{aligned} a(f)g^{\otimes n} &= \sqrt{n} \langle f, g \rangle_{\mathcal{K}} g^{\otimes(n-1)} \\ a^*(f)g^{\otimes n} &= \sqrt{n+1} f \otimes_s g^{\otimes n}. \end{aligned}$$

They extend to densely defined closed operators and are adjoint of each other: we denote again by $a^\#(f)$ their closures. For any $f \in \mathcal{K}$, $D(a^*(f)) = D(a(f))$ with

$$D(a(f)) = \left\{ \phi \in \Gamma_s(\mathcal{K}) : \sum_{n=0}^{\infty} (n+1) \|\langle f(x), \phi_{n+1}(x, X_n) \rangle_{\mathcal{K}(x)}\|_{\mathcal{H}^{\otimes_s n}(X_n)}^2 < +\infty \right\},$$

where $\phi_{n+1} = \phi|_{\mathcal{H}^{\otimes_s n+1}}$; also $D(a(f)) \supset D(d\Gamma(1)^{1/2})$, $D(a(f)) \supset f_0(\mathcal{K}^{(\cdot)})$. They satisfy the Canonical Commutation Relations $[a(f_1), a^*(f_2)] = \langle f_1, f_2 \rangle_{\mathcal{K}}$ on suitable domains (e.g. $f_0(\mathcal{K}^{(\cdot)})$).

- We decompose $\Gamma_s(\mathcal{H}_2)$ in its subspaces with fixed number of particles as usual: $\forall n \in \mathbb{N}$, define $\mathcal{H}_2^{(n)} := \mathcal{H}_2^{\otimes_s n}$, with the convention $\mathcal{H}_2^{(0)} = \mathbb{C}$. Then $\{\mathcal{H}_2^{(n)}\}_{n \in \mathbb{N}}$ is a complete collection of closed disjoint subspaces of $\Gamma_s(\mathcal{H}_2)$ invariant for N .
- Let X be an operator on \mathcal{H} . We say that X is diagonal if $\{\mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}\}_{n \in \mathbb{N}}$ is invariant for X ; X is non-diagonal if for all $n \in \mathbb{N}$ and $\phi \in D(X) \cap \mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}$, $X\phi \notin \mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}$.

2. ASSUMPTIONS ON H

In this section we discuss Assumptions **A₀** and **A_I(A'_I)**. In Section 4 below they are checked in concrete examples.

We recall that our Hilbert space \mathcal{H} has the form

$$\mathcal{H} = \mathcal{H}_1 \otimes \Gamma_s(\mathcal{H}_2);$$

while the operator is

$$H = H_{01} \otimes 1 + 1 \otimes H_{02} + H_I .$$

We separate the assumptions on H_0 from the ones on H_I , to improve readability. On H_I we require either Assumption \mathbf{A}_I or Assumption \mathbf{A}'_I . In \mathbf{A}_I the non-diagonal part of H_I can be more singular: that restricts the diagonal part to be at most quadratic in the creation and annihilation operators. In \mathbf{A}'_I on the other hand is assumed more regularity on the non-diagonal part of H_I , allowing for a more singular diagonal part.

Assumption \mathbf{A}_0 . H_{01} and H_{02} are semi-bounded self-adjoint operators. We denote respectively by $-M_1$ and $-M_2$ their lower bounds. Furthermore, $\forall t \in \mathbb{R}$, $\{\mathcal{H}_2^{(n)}\}_{n \in \mathbb{N}}$ is invariant for $e^{itH_{02}}$.

This is quite natural. In physical systems the Hamiltonian is often split in a part describing the free dynamics (usually a self-adjoint and positive unbounded operator), and an interaction part. The invariance of the n -particles subspaces is also a usual feature of free quantum theories: let h_{02} be a semi-bounded self-adjoint operator on the one-particle space \mathcal{H}_2 ; then the second quantization $d\Gamma(h_{02})$ is self-adjoint, and the group $\Gamma(e^{ith_{02}})$ generated by it satisfies the assumption.

Assumption \mathbf{A}_I . H_I is a symmetric operator on \mathcal{H} , with a domain of definition $D(H_I)$ such that $D(H_0) \cap D(H_I)$ is dense in \mathcal{H} . Furthermore $\forall \phi \in Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}$,

$$(4) \quad H_I \phi \in \bigoplus_{i=-2}^2 \mathcal{H}_1 \otimes \mathcal{H}_2^{(n+i)} .$$

Also, H_I satisfies the following bound: $\forall n \in \mathbb{N} \exists C > 0$ such that $\forall \psi \in \mathcal{H}$, $\forall \phi \in Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}$:

$$(5) \quad |\langle \psi, H_I \phi \rangle_{\mathcal{H}}|^2 \leq C^2 \sum_{i=-2}^2 \|\psi_{n+i}\|_{\mathcal{H}_1 \otimes \mathcal{H}_2^{(n+i)}}^2 \left[(n+1)^2 \|\phi\|_{\mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}}^2 + (n+1) \left(q_{H_{01} \otimes 1}(\phi, \phi) \right. \right. \\ \left. \left. + q_{1 \otimes H_{02}}(\phi, \phi) + (|M_1| + |M_2| + 1) \|\phi\|_{\mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}}^2 \right) \right] ;$$

where we define $\psi_n := 1 \otimes \mathbb{1}_n(\mathcal{H}_2^{(\cdot)})\psi$.

Consider Assumption \mathbf{A}_I . First of all, H_I has to be sufficiently regular, i.e. relatively bounded by H_0 (in some sense) when restricted to the subspaces $\mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}$. Essentially, we require that H_I is at most quadratic in the annihilation and creation operators, as reflected by the n -dependence in (5).

Assumption \mathbf{A}'_I . H_I is a symmetric operator on \mathcal{H} , with a domain of definition $D(H_I)$ such that $D(H_0) \cap D(H_I)$ is dense in \mathcal{H} . Furthermore $\forall \phi \in Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}$,

$$(6) \quad H_I \phi \in \bigoplus_{i=-2}^2 \mathcal{H}_1 \otimes \mathcal{H}_2^{(n+i)}.$$

Also, $H_I = H_{diag} + H_2$ with the following properties:

i) H_{diag} is diagonal; H_2 is non-diagonal.

ii) H_{diag} satisfies the following bound. $\forall n \in \mathbb{N} \exists C(n) > 0$ such that $\forall \psi \in \mathcal{H}, \forall \phi \in Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}$:

$$(7) \quad |\langle \psi, H_{diag} \phi \rangle_{\mathcal{H}}|^2 \leq C^2(n) \|\psi_n\|_{\mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}}^2 \left(q_{H_{01} \otimes 1}(\phi, \phi) + q_{1 \otimes H_{02}}(\phi, \phi) + (|M_1| + |M_2| + 1) \|\phi\|_{\mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}}^2 \right).$$

iii) H_2 satisfies the following bound. $\forall n \in \mathbb{N} \exists C > 0$ such that $\forall \psi \in \mathcal{H}, \forall \phi \in \mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}$:

$$(8) \quad |\langle \psi, H_2 \phi \rangle_{\mathcal{H}}| \leq C(n+1) \|\phi\|_{\mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}} \sum_{\substack{i=-2 \\ i \neq 0}}^2 \|\psi_{n+i}\|_{\mathcal{H}_1 \otimes \mathcal{H}_2^{(n+i)}}.$$

Assumption \mathbf{A}'_I is similar to Assumption \mathbf{A}_I . However since the non-diagonal quadratic part H_2 is more regular than before, we can be less demanding on the diagonal part H_{diag} : it has still to be bounded in a suitable sense by H_0 , but it can be non-quadratic with respect to the creation and annihilation operators.

Remark 2.1. In some applications, there is a decomposition of \mathcal{H}_1 invariant for H . For example, it may happen that \mathcal{H}_1 is also a Fock space but H leaves invariant each sector with fixed number of particles. In this situation, we can prove essential self-adjointness with little less regularity on the assumptions. In particular, Assumption \mathbf{A}_I would be changed in:

H_I is a symmetric operator on \mathcal{H} , with a domain of definition $D(H_I)$ such that $D(H_0) \cap D(H_I)$ is dense in \mathcal{H} . Furthermore there exists a complete collection $\{\mathcal{H}_1^{(j)} \otimes \Gamma_s(\mathcal{H}_2)\}_{j \in \mathbb{N}}$ invariant for H_0 and H_I such that: $\forall \phi \in Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1^{(j)} \otimes \mathcal{H}_2^{(n)}$,

$$H_I \phi \in \bigoplus_{i=-2}^2 \mathcal{H}_1^{(j)} \otimes \mathcal{H}_2^{(n+i)}.$$

Also, H_I satisfies the following bound: $\forall j, n \in \mathbb{N} \exists C(j) > 0$ such that $\forall \psi \in \mathcal{H}, \forall \phi \in$

$$Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1^{(j)} \otimes \mathcal{H}_2^{(n)}:$$

$$|\langle \psi, H_I \phi \rangle_{\mathcal{H}}|^2 \leq C^2(j) \sum_{i=-2}^2 \|\psi_{j,n+i}\|_{\mathcal{H}_1^{(j)} \otimes \mathcal{H}_2^{(n+i)}}^2 \left[(n+1)^2 \|\phi\|_{\mathcal{H}_1^{(j)} \otimes \mathcal{H}_2^{(n)}}^2 + (n+1) \left(q_{H_{01} \otimes 1}(\phi, \phi) + q_{1 \otimes H_{02}}(\phi, \phi) + (|M_1| + |M_2| + 1) \|\phi\|_{\mathcal{H}_1^{(j)} \otimes \mathcal{H}_2^{(n)}}^2 \right) \right];$$

where we define $\psi_{j,n} := \mathbb{1}_j(\mathcal{H}_1^{(\cdot)}) \otimes \mathbb{1}_n(\mathcal{H}_2^{(\cdot)})\psi$.

Theorem 3.1 would then read:

Assume \mathbf{A}_0 and $\mathbf{A}_I(\mathbf{A}'_I)$. Then H is essentially self adjoint on $D(H_{01} \otimes 1) \cap D(H_{02} \otimes 1) \cap f_0(\mathcal{H}_1^{(\cdot)} \otimes \mathcal{H}_2^{(\cdot)})$.

3. DIRECT PROOF OF SELF-ADJOINTNESS

In this section we present the criterion of essential self-adjointness. The strategy is to prove that $\text{Ran}(H \pm i)$ is dense in \mathcal{H} , by an argument of reductio ad absurdum. As already discussed, the non-diagonal part of H_I is at most quadratic with respect to the annihilation and creation operators of $\Gamma_s(\mathcal{H}_2)$, and that plays a crucial role in the proof. We prove Theorem 3.1 assuming \mathbf{A}_I ; the other case being analogous.

Theorem 3.1. *Assume \mathbf{A}_0 and $\mathbf{A}_I(\mathbf{A}'_I)$. Then H is essentially self adjoint on $D(H_{01} \otimes 1) \cap D(H_{02} \otimes 1) \cap \mathcal{H}_1 \otimes f_0(\mathcal{H}_2^{(\cdot)})$.*

Proof. Let $\psi \in \mathcal{H}$, $z \in \mathbb{C}$ with $\text{Im}z \neq 0$. Suppose that $\forall \phi \in D(H_{01} \otimes 1) \cap D(1 \otimes H_{02}) \cap \mathcal{H}_1 \otimes f_0(\mathcal{H}_2^{(\cdot)})$:

$$(9) \quad \langle \psi, (H - z)\phi \rangle_{\mathcal{H}} = 0.$$

Then it suffices to show that $\psi = 0$. This is done in few steps. Let $n \in \mathbb{N}$ and $\phi_n \in D(H_{01} \otimes 1) \cap D(1 \otimes H_{02}) \cap \mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}$. For all $n \in \mathbb{N}$, the space $Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}$ with the scalar product:

$$(10) \quad \langle \cdot, \cdot \rangle_{\mathcal{X}_n} = q_{H_{01} \otimes 1}(\cdot, \cdot) + q_{1 \otimes H_{02}}(\cdot, \cdot) + (|M_1| + |M_2| + 1) \langle \cdot, \cdot \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}}$$

is complete, and therefore a Hilbert space. We denote it by \mathcal{X}_n . Then (9) together with Assumption \mathbf{A}_0 imply, since $\phi_n \in D(H_{01} \otimes 1) \cap D(1 \otimes H_{02})$:

$$(11) \quad \langle \psi_n, \phi_n \rangle_{\mathcal{X}_n} = (z + |M_1| + |M_2| + 1) \langle \psi_n, \phi_n \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}} - \langle \psi, H_I \phi_n \rangle_{\mathcal{H}}.$$

Use bound (7) and then Riesz's Lemma on \mathcal{X}_n : it follows that $\psi_n \in Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}$ for any $n \in \mathbb{N}$.

Let $\phi \in Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1 \otimes f_0(\mathcal{H}_2^{(\cdot)})$. Then $\exists \{\phi^{(\alpha)}\}_{\alpha \in \mathbb{N}}$ such that $\forall \alpha \in \mathbb{N}$, $\phi^{(\alpha)} \in D(H_{01} \otimes 1) \cap D(1 \otimes H_{02}) \cap \mathcal{H}_1 \otimes f_0(\mathcal{H}_2^{(\cdot)})$; and $\forall n \in \mathbb{N}$, $\phi_n^{(\alpha)} \rightarrow \phi_n$ in the topology induced by $\|\cdot\|_{\mathcal{X}_n}$. Furthermore $\forall \alpha \in \mathbb{N}$:

$$(12) \quad \langle \psi, (H - z)\phi^{(\alpha)} \rangle_{\mathcal{H}} = 0 .$$

Since $\psi_n \in Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1 \otimes \mathcal{H}_2^{(n)}$, we can take the limit of (12) and obtain, $\forall \phi \in Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1 \otimes f_0(\mathcal{H}_2^{(\cdot)})$:

$$(13) \quad q_{H_{01} \otimes 1}(\psi, \phi) + q_{1 \otimes H_{02}}(\psi, \phi) + \langle \psi, H_I \phi \rangle_{\mathcal{H}} = z \langle \psi, \phi \rangle_{\mathcal{H}} .$$

Hence we can choose $\phi = \psi_{\leq n} := 1 \otimes \mathbb{1}_{\leq n}(\mathcal{H}_2^{(\cdot)})\psi$ in (13). Then, using Assumption \mathbf{A}_0 and taking the imaginary part we obtain:

$$(14) \quad \text{Im}(z) \langle \psi_{\leq n}, \psi_{\leq n} \rangle = \text{Im}(\langle \psi - \psi_{\leq n}, H_I \psi_{\leq n} \rangle) .$$

Now, by Assumption \mathbf{A}_I (the equality holds on the suitable domain):

$$H_I(1 \otimes \mathbb{1}_{\leq n}(\mathcal{H}_2^{(\cdot)})) = (1 \otimes \mathbb{1}_{\leq n+2}(\mathcal{H}_2^{(\cdot)}))H_I(1 \otimes \mathbb{1}_{\leq n}(\mathcal{H}_2^{(\cdot)})) .$$

Furthermore $1 \otimes \mathbb{1}_{\leq n+2}(\mathcal{H}_2^{(\cdot)})(\psi - \psi_{\leq n}) = \psi_{n+1} \oplus \psi_{n+2}$. Then Equation (14) becomes:

$$\text{Im}(z) \langle \psi_{\leq n}, \psi_{\leq n} \rangle = \sum_{i=1}^2 \text{Im}(\langle \psi_{n+i}, H_I \psi_{\leq n} \rangle) .$$

Using the symmetry of H_I , and (4) we obtain:

$$(15) \quad \text{Im}(z) \langle \psi_{\leq n}, \psi_{\leq n} \rangle = \text{Im}(\langle \psi_{n+2}, H_I \psi_n \rangle + \langle \psi_{n+1}, H_I \psi_n \rangle + \langle \psi_{n+1}, H_I \psi_{n-1} \rangle) .$$

Now bound (15) using (5); then we obtain $\forall n \in \mathbb{N}$:

$$(16) \quad |\text{Im}z| \sum_{i=0}^n \|\psi_i\|^2 \leq C \left[\|\psi_{n+1}\| \left((n+1)(\|\psi_n\| + \|\psi_{n-1}\|) + \sqrt{n+1}(\|\psi_n\|_{\mathcal{X}_n} + \|\psi_{n-1}\|_{\mathcal{X}_{n-1}}) \right) \right. \\ \left. + \|\psi_{n+2}\| \left((n+1)\|\psi_n\| + \sqrt{n+1}\|\psi_n\|_{\mathcal{X}_n} \right) \right] \\ \leq 2C(n+1) \left[\sum_{i_1=0}^2 \|\psi_{n+i_1}\|^2 + \sum_{i_2=-1}^0 (n+1)^{-1} \|\psi_{n+i_2}\|_{\mathcal{X}_{n+i_2}}^2 \right] .$$

For all $\alpha > 0$ define:

$$S := \sum_{n=0}^{\infty} \|\psi_n\|^2 ; S_{\alpha} := \sum_{n=0}^{\infty} (n+\alpha)^{-1} \|\psi_n\|_{\mathcal{X}_n}^2 .$$

$\psi \in \mathcal{H}$, hence S is finite. We prove that also S_α is finite. Using equation (13) with $\phi = \psi_n$ we obtain, for all $n \in \mathbb{N}$:

$$(17) \quad (n + \alpha)^{-1} \|\psi_n\|_{\mathcal{X}_n}^2 = (n + \alpha)^{-1} (z + |M_1| + |M_2| + 1) \|\psi_n\|^2 - (n + \alpha)^{-1} \langle \psi, H_I \psi_n \rangle.$$

Now, we can use bound (5) on $(n + \alpha)^{-1} |\langle \psi, H_I \psi_n \rangle|$, obtaining

$$(18) \quad \begin{aligned} (n + \alpha)^{-2} |\langle \psi, H_I \psi_n \rangle_{\mathcal{H}}|^2 &\leq C^2 (n + \alpha)^{-2} \sum_{i=-2}^2 \|\psi_{n+i}\|^2 \left[(n + 1)^2 \|\psi_n\|^2 + (n + 1) \|\psi_n\|_{\mathcal{X}_n}^2 \right] \\ &\leq C^2(\alpha) \sum_{i=-2}^2 \|\psi_{n+i}\|^2 \left[\|\psi_n\|^2 + (n + \alpha)^{-1} \|\psi_n\|_{\mathcal{X}_n}^2 \right], \end{aligned}$$

for some $C(\alpha) > 0$. The only terms we need to deal with are $(n + \alpha)^{-1} \|\psi_{n+i}\|^2 \|\psi_n\|_{\mathcal{X}_n}^2$. We use the fact that for any $\varepsilon, a, b > 0$, $ab \leq \frac{1}{2}(\varepsilon a^2 + \frac{1}{\varepsilon} b^2)$, obtaining

$$(19) \quad (n + \alpha)^{-1} \|\psi_{n+i}\|^2 \|\psi_n\|_{\mathcal{X}_n}^2 \leq \frac{1}{2} \left(\varepsilon (n + \alpha)^{-2} \|\psi_n\|_{\mathcal{X}_n}^4 + \frac{1}{\varepsilon} \|\psi_{n+i}\|^4 \right).$$

Combining (19) with (18), and applying to Equation (17), we obtain the following bound: for all $\varepsilon, \alpha > 0$, $\exists C(\alpha, \varepsilon) > 0$ such that

$$(20) \quad (n + \alpha)^{-1} \|\psi_n\|_{\mathcal{X}_n}^2 \leq C(\alpha, \varepsilon) \sum_{i=-2}^2 \|\psi_{n+i}\|^2 + \varepsilon (n + \alpha)^{-1} \|\psi_n\|_{\mathcal{X}_n}^2.$$

Fix $\varepsilon < 1$, then for all $\alpha > 0$, $\exists C(\alpha) > 0$ such that $\forall \bar{n} \in \mathbb{N}$:

$$(21) \quad \sum_{n=0}^{\bar{n}} (n + \alpha)^{-1} \|\psi_n\|_{\mathcal{X}_n}^2 \leq C(\alpha) S;$$

uniformly in \bar{n} . Then we can take the limit $\bar{n} \rightarrow \infty$ and obtain $S_\alpha < \infty$.

Remark. The bound of Equation (20) could seem to follow from an implicit smallness condition on the interaction H_I . As it will become clearer with the examples of Section 4, it is not the case. Roughly speaking, Assumption \mathbf{A}_I allows for interaction parts that are at most as singular as $(H_0 + |M_1| + |M_2|)^{1/2} (N + 1)^{1/2}$.

Now return to Equation (16). There exists $n^* \in \mathbb{N}$ such that $\forall n \geq n^*$:

$$\frac{1}{2} S \leq \sum_{i=0}^n \|\psi_i\|^2 \leq S.$$

Hence summing in $n^* \leq n \leq \bar{n}$ on both sides of (16) we obtain for all $\bar{n} > n^*$:

$$\frac{1}{2} S \sum_{n=n^*}^{\bar{n}} (n + 1)^{-1} \leq \sum_{n=n^*}^{\bar{n}} (n + 1)^{-1} \sum_{i=0}^n \|\psi_i\|^2 \leq 2 \frac{C}{|\operatorname{Im} z|} (3S + S_1 + S_2).$$

The bound on the right hand side is uniform in \bar{n} : that is absurd, unless $S = S_1 = S_2 = 0 \Leftrightarrow \psi = 0$. \square

Once essential self-adjointness is established, it is possible to give the following characterization of the domain of self-adjointness $D(H)$.

Proposition 3.2. *Assume A_0 and $A_I(A'_I)$. If exists K self-adjoint operator with domain $D(K)$ such that:*

i) $D(H_0) \cap D(K)$ is dense in \mathcal{H} ; $\mathcal{H}_1 \otimes f_0(\mathcal{H}_2^{(\cdot)})$ is dense in $D(K)$.

ii) There exists $0 < \varepsilon < 1$ such that $\exists C(\varepsilon) > 0, \forall \phi \in D(H_0) \cap D(K)$:

$$(22) \quad \|H_I \phi\| \leq \varepsilon \|H_0 \phi\| + C(\varepsilon)(\|K \phi\| + \|\phi\|) .$$

Then $D(H) \cap D(K) = D(H_0) \cap D(K)$.

Proof. Using bound (22), we have $\forall \phi \in D(H_0) \cap D(K)$:

$$(23) \quad \|H \phi\| \leq (\varepsilon + 1)\|H_0 \phi\| + C(\varepsilon)(\|K \phi\| + \|\phi\|) .$$

Then $D(H) \supseteq D(H_0) \cap D(K)$. Now let $\phi \in D(H) \cap D(K)$: using (22)

$$(24) \quad \|H_0 \phi\| \leq \varepsilon \|H_0 \phi\| + \|H \phi\| + C(\varepsilon)(\|K \phi\| + \|\phi\|) ;$$

since $\varepsilon < 1$, $D(H_0) \supseteq D(H) \cap D(K)$. □

4. APPLICATIONS

It is possible to apply Theorem 3.1 in several situations of mathematical and physical interest. We present and discuss some of them in this section; not before a brief discussion of the “boundaries” of Theorem 3.1: it may be interesting to see how its proof fails when we consider operators that are more than quadratic in the annihilation/creation operators; and to define a quadratic operator that is not sufficiently regular for Assumption $A_I(A'_I)$ to hold. According to this purpose, we will consider simple toy models on $\Gamma_s(\mathbb{C})$. We denote by $a^\#$ the corresponding annihilation/creation operators.

Let's consider a simple trilinear Hamiltonian on $\Gamma_s(\mathbb{C})$:

$$H_3 = a^* a + a^* a^* a^* + a a a .$$

The free part is $H_0 = a^* a$, and the interaction part is $H_I = a^* a^* a^* + a a a$. Assumption A_0 is satisfied, and Assumption A'_I is slightly modified: i now ranges from -3 to 3 , and bounds (7) and (8) are replaced by the simple bound:

$$|\langle \psi, H_I \phi \rangle|_{\mathcal{H}} \leq C(n+1)^{3/2} \|\phi\|_{\mathcal{H}^{(n)}} (\|\psi_{n+3}\|_{\mathcal{H}^{(n+3)}} + \|\psi_{n-3}\|_{\mathcal{H}^{(n-3)}}) .$$

The proof of Theorem 3.1 carries on, almost unchanged, up to Equation (16) that would now read

$$|\operatorname{Im}z| \sum_{i=0}^n \|\psi_i\|^2 \leq C(n+1)^{3/2} \|\psi_{n+3}\|^2 .$$

However if we now take the sum in n from n^* to \bar{n} (where n^* is such that $\frac{1}{2}\|\psi\|^2 \leq \sum_{i=0}^n \|\psi_i\|^2 \leq \|\psi\|^2$ for all $n \geq n^*$) we *cannot* conclude that $\|\psi\|$ must be zero, because the series $\sum_{n=0}^{\infty} (n+1)^{-3/2}$ converges. Hence the proof fails, and analogously would fail for any higher order polynomial of the annihilation/creation operators.

On the other hand, we introduce now a quadratic model for which Assumption $A_I(A'_I)$ fails to hold, and thus Theorem 3.1 cannot be applied. For the following operator on $L^2(\mathbb{R}) \otimes \Gamma_s(\mathbb{C})$ Assumption A_I is satisfied:

$$H_{\partial a} = -\partial_x^2 + a^*a - i\partial_x(a^* + a) + a^*a^* + aa ,$$

where $H_0 = -\partial_x^2 + a^*a$ and $H_I = -i\partial_x(a^* + a) + a^*a^* + aa$. If, however, the derivative operator is coupled with the quadratic term

$$H_{\partial aa} = -\partial_x^2 + a^*a - i\partial_x(a^*a^* + aa) ,$$

$A_I(A'_I)$ is no longer satisfied. The interaction in this case would be of type $H_0^{1/2}N$, and therefore too singular: Theorem 3.1 *does not hold* for $H_{\partial aa}$.

Throughout the section we will adopt the following notations, in addition to the ones of Section 1.2. Let \mathcal{H} a Hilbert space; we denote by $\mathcal{L}(\mathcal{H})$ the set of bounded operators on \mathcal{H} and by $|\cdot|_{\mathcal{L}(\mathcal{H})}$ the operator norm. It is also useful to define the annihilation/creation operator valued distributions $a^\#(x)$, $x \in \mathbb{R}^d$. Let $f \in L^2(\mathbb{R}^d)$, $a^\#(f)$ the annihilation/creation operators on $\Gamma_s(L^2(\mathbb{R}^d))$. Then the operator valued distributions $a^\#(x)$ acting on $L^2(\mathbb{R}^d)$, with values on $\Gamma_s(L^2(\mathbb{R}^d))$, are defined by:

$$(a^*, f) \equiv \int_{\mathbb{R}^d} a^*(x)f(x)dx := a^*(f) ; (a, f) \equiv \int_{\mathbb{R}^d} a(x)\bar{f}(x)dx := a(f) .$$

They satisfy the commutation relations (inherited by the CCR) $[a(x), a^*(y)] = \delta(x - y)$.

1. Hamiltonians of identical bosons.

The criterion applies to operators in the Fock space $\Gamma_s(\mathcal{H})$, for any separable Hilbert space \mathcal{H} . Simply choose $\mathcal{H}_1 \equiv \mathbb{C}$ and $\mathcal{H}_2 \equiv \mathcal{H}$; then $\mathbb{C} \otimes \Gamma_s(\mathcal{H}) \approx \Gamma_s(\mathcal{H})$ up to an unitary isomorphism.

An example is given by the following class of operators. Let $\mathcal{H} = L^2(\mathbb{R}^d)$; h_0 a positive self adjoint operator on $L^2(\mathbb{R}^d)$ (the one-particle free Hamiltonian). Furthermore, let $V_1 \in L^2(\mathbb{R}^d)$, $V_2, V_3 \in L^2(\mathbb{R}^{2d})$, with $V_2 = \bar{V}_2$, and $V_4(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}$, such that $V_4(x) = V_4(-x)$ and $V_4(h_0 + 1)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^d))$. Consider

$$(25) \quad H = d\Gamma(h_0) + \int_{\mathbb{R}^d} \left(V_1(x)a^*(x) + \bar{V}_1(x)a(x) \right) dx + \int_{\mathbb{R}^{2d}} \left(V_2(x,y)a^*(x)a(y) + V_3(x,y)a^*(x)a^*(y) + \bar{V}_3(x,y)a(x)a(y) \right) dx dy + \frac{1}{2} \int_{\mathbb{R}^{2d}} V_4(x-y)a^*(x)a^*(y)a(x)a(y) dx dy .$$

We make the following identifications: $H_{01} \equiv 0$, $H_{02} \equiv d\Gamma(h_0)$, $H_{diag} \equiv \int (V_4 a^* a^* a a + V_2 a^* a)$, $H_2 \equiv \int (V_1 a^* + \bar{V}_1 a) + \int (V_3 a^* a^* + \bar{V}_3 a a)$. Assumption **A₀** is trivial to verify; and Assumption **A'₁** follows from standard estimates on Fock space: let $\psi \in \Gamma_s(L^2(\mathbb{R}^d))$, $\phi_n \in L^2_s(\mathbb{R}^{nd}) \cap Q(d\Gamma(h_0))$, $n \in \mathbb{N}$, then

$$(26) \quad |\langle \psi, H_{diag} \phi_n \rangle| \leq \left(n \|V_2\|_2 \|\phi_n\| + |V_4(h_0 + 1)^{-1/2}|_{\mathcal{L}(L^2(\mathbb{R}^d))} (n^{3/2} \|(d\Gamma(h_0))^{1/2} \phi_n\| + n^2 \|\phi_n\|) \right) \|\psi_n\| ;$$

$$(27) \quad |\langle \psi, H_2 \phi_n \rangle| \leq 2 \left(\sqrt{n+1} \|V_1\|_2 + (n+1) \|V_3\|_2 \right) \|\phi_n\| \sum_{\substack{i=-2 \\ i \neq 0}}^2 \|\psi_{n+i}\| .$$

Hence we can apply Theorem 3.1; and prove essential self-adjointness of H in $D(d\Gamma(h_0)) \cap f_0(L^2(\mathbb{R}^d)^{(\cdot)})$. We can also apply Proposition 3.2 with $K \equiv N^3$, i.e. $D(H) \cap D(N^3) = D(d\Gamma(h_0)) \cap D(N^3)$. Observe that if $d = 3$, the well-known many body Hamiltonian with Coulomb pair interaction

$$H_C = d\Gamma(-\Delta) \pm \frac{1}{2} \int_{\mathbb{R}^6} \frac{1}{|x-y|} a^*(x)a^*(y)a(x)a(y) dx dy ,$$

is just the special case $h_0 = -\Delta$, $V_1 = V_2 = V_3 = 0$ and $V_4 = \pm|x|^{-1}$.

2. Nelson-type Hamiltonians.

We consider now the dynamics of different species of particles (or fields) interacting. A typical example is the Nelson Hamiltonian. It was introduced in a rigorous way by Nelson [20] to describe nucleons in a meson field, and studied by several authors [e.g. 1, 8, 10, 12].

Let $\mathcal{H} = L^2(\mathbb{R}^{pd}) \otimes \Gamma_s(L^2(\mathbb{R}^d))$: the first space corresponds to n non-relativistic particles; the second to a scalar relativistic field. Let ω be a positive self-adjoint operator on $L^2(\mathbb{R}^d)$ (the dispersion relation of the relativistic field), $V \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}_+)$ an external potential acting on the particles. The interaction between the particles and the field is linear in the creation and annihilation operators $a^\#$ corresponding to the field. Let $v : \mathbb{R}^{2d} \rightarrow \mathbb{C}$ such that

- $(1 - \Delta_x)^{-1/2} \|v(x, \cdot)\|_{L^2_{(k)}(\mathbb{R}^d)}^2 (1 - \Delta_x)^{-1/2} \in \mathcal{L}(L^2_{(x)}(\mathbb{R}^d));$
- for all $k \in \mathbb{R}^d$, $v(x, k)(1 - \Delta_x)^{-1/2} \in \mathcal{L}(L^2_{(x)}(\mathbb{R}^d))$, with $|v(x, \cdot)(1 - \Delta_x)^{-1/2}|_{\mathcal{L}(L^2_{(x)}(\mathbb{R}^d))} \in L^2_{(k)}(\mathbb{R}^d).$

Then we define the Nelson Hamiltonian:

$$(28) \quad H_N = \left(\sum_{i=1}^p -\Delta_{x_i} + V(x_i) \right) \otimes 1 + 1 \otimes d\Gamma(\omega) + \sum_{i=1}^p a^*(v(x_i, \cdot)) + a(v(x_i, \cdot)).$$

The function v describes the coupling between the particles and the relativistic field. The assumptions above imply that it has a good behaviour both for high and small momenta; in particular in three-dimensions it acts as an UV cutoff function.

Remark. The model of Nelson [20] was much more specific: $d = 3$, $\omega(k) = \sqrt{k^2 + \mu^2}$ with $\mu > 0$, $V = 0$ and $v(x, k) = \lambda(2\pi)^{-3/2}(2\omega(k))^{-1/2}e^{-ik \cdot x} \mathbf{1}_{|\cdot| \leq \sigma}(k)$ with $\lambda, \sigma > 0$. With these assumptions, $v \in L^\infty(\mathbb{R}^3, L^2(\mathbb{R}^3))$, $\omega^{-1/2}v \in L^\infty(\mathbb{R}^3, L^2(\mathbb{R}^3))$; then H_N (the Nelson model with UV cut off) is self-adjoint by the Kato-Rellich Theorem. However, if we consider $d = 2$ and $\mu = 0$ (massless relativistic field), the Kato-Rellich Theorem is not applicable because $\omega^{-1/2}v \notin L^\infty(\mathbb{R}^2, L^2(\mathbb{R}^2))$ due to an infrared divergence. Instead assumptions \mathbf{A}_0 and \mathbf{A}'_I are still satisfied, thus Theorem 3.1 can be used.

In order to check Assumptions \mathbf{A}_0 and \mathbf{A}_I on (28), we make the (straightforward) identifications: $\mathcal{H}_1 \equiv L^2(\mathbb{R}^{pd})$, $\mathcal{H}_2 \equiv L^2(\mathbb{R}^d)$, $H_{01} \equiv \sum_i -\Delta_{x_i} + V(x_i)$, $H_{02} \equiv d\Gamma(\omega)$, $H_I \equiv \sum_i a^*(v(x_i, \cdot)) + a(v(x_i, \cdot))$. We do not need to introduce a decomposition of \mathcal{H}_1 . Assumption \mathbf{A}_0 is satisfied: for all $V \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}_+)$, $-\Delta + V(\cdot)$ is a positive self-adjoint operator, and the vectors with fixed number of particles are invariant for the evolution associated with the positive self-adjoint operator $d\Gamma(\omega)$. Furthermore, since $H_{01} \otimes 1$ and $1 \otimes H_{02}$ are positive self-adjoint commuting operators, H_0 is a positive self-adjoint operator with domain $D(H_0) = D(H_{01} \otimes 1) \cap D(1 \otimes H_{02})$. Assumption \mathbf{A}_I is also satisfied by usual estimates: $\forall \psi \in \mathcal{H}$, $\forall \phi_n \in L^2(\mathbb{R}^{pd}) \otimes L^2_s(\mathbb{R}^{nd}) \cap Q(H_{01} \otimes 1)$, $n \in \mathbb{N}$,

$$(29) \quad |\langle \psi, H_I \phi_n \rangle| \leq \sqrt{2p}(2\sqrt{n}) \|v(x, \cdot)(1 - \Delta_x)^{-1/2}\|_{\mathcal{L}(L^2_{(x)}(\mathbb{R}^d))} \|L^2_{(k)}\| + \|(1 - \Delta_x)^{-1/2} \|v(x, \cdot)\|_{L^2_{(k)}}^2 (1 - \Delta_x)^{-1/2}\|_{\mathcal{L}(L^2_{(x)}(\mathbb{R}^d))}^{1/2} \left(\left\| \left(\sum_{i=1}^p -\Delta_{x_i} \right)^{1/2} \phi_n \right\| + \sqrt{p} \|\phi_n\| \right) \sum_{\substack{i=-1 \\ i \neq 0}}^1 \|\psi_{n+i}\|.$$

Then H_N is essentially self-adjoint on $D(H_0) \cap f_0(L^2(\mathbb{R}^{pd}) \otimes L^2(\mathbb{R}^d)^{(\cdot)})$.

Let $H_N|_s$ be the restriction of H_N to $L_s^2(\mathbb{R}^{pd}) \otimes \Gamma_s(L^2(\mathbb{R}^d))$. It is possible to extend $H_N|_s$ to $\Gamma_s(L^2(\mathbb{R}^d)) \otimes \Gamma_s(L^2(\mathbb{R}^d))$ in the following way. Define

$$(30) \quad \tilde{H}_N = d\Gamma(-\Delta + V) \otimes 1 + 1 \otimes d\Gamma(\omega) + \int_{\mathbb{R}^d} \psi^*(x) (a^*(v(x, \cdot)) + a(v(x, \cdot))) \psi(x) dx ,$$

where $\psi^\#$ are the creation and annihilation operators corresponding to the first Fock space. Then $H_N|_s$ and \tilde{H}_N agree on the p -particle sector $L_s^2(\mathbb{R}^{pd}) \otimes \Gamma_s(L^2(\mathbb{R}^d))$ of $\Gamma_s(L^2(\mathbb{R}^d)) \otimes \Gamma_s(L^2(\mathbb{R}^d))$. The self-adjointness of \tilde{H}_N still follows from Theorem 3.1 using the bound (29): it is sufficient to choose for $\mathcal{H}_1 \equiv \Gamma_s(L^2(\mathbb{R}^d))$ the decomposition in finite particle vectors $\{\mathcal{H}_1^{(j)} \otimes \Gamma_s(\mathcal{H}_2)\}_{j \in \mathbb{N}} \equiv \{L_s^2(\mathbb{R}^{jd}) \otimes \Gamma_s(\mathcal{H}_2)\}_{j \in \mathbb{N}}$. Let $H_0 \equiv d\Gamma(-\Delta + V) \otimes 1 + 1 \otimes d\Gamma(\omega)$, then the domain of essential self-adjointness for \tilde{H}_N is $D(H_0) \cap f_0(L^2(\mathbb{R}^d)^{(\cdot)} \otimes L^2(\mathbb{R}^d)^{(\cdot)})$. Let N_1 and N_2 be the number operators corresponding to the first and second Fock space respectively. Then applying Proposition 3.2 we also obtain $D(\tilde{H}_N) \cap D(N_1^2 + N_2^2) = D(H_0) \cap D(N_1^2 + N_2^2)$.

3. Pauli-Fierz Hamiltonian.

The last example considered is an operator describing the dynamics of rigid charges and their radiation field interacting. The model was introduced by Pauli and Fierz [23], and has been extensively studied by a mathematical standpoint. See Spohn [27, and references thereof contained] for a detailed presentation.

Let $\mathcal{H}^{(spin)} = (\otimes^p \mathbb{C}^{2^{\lfloor \frac{d}{2} \rfloor}}) \otimes L^2(\mathbb{R}^{pd}) \otimes \Gamma_s(\mathbb{C}^{d-1} \otimes L^2(\mathbb{R}^d))$, $\mathcal{H} = L^2(\mathbb{R}^{pd}) \otimes \Gamma_s(\mathbb{C}^{d-1} \otimes L^2(\mathbb{R}^d))$: the first space corresponds to p spin- $\frac{1}{2}$ particles, the second to spinless particles. Let $\chi \in L^2(\mathbb{R}^d)$, $V \in L_{loc}^2(\mathbb{R}^{pd}, \mathbb{R}_+)$, $\omega = |k|$, $m_j > 0$, $q_j \in \mathbb{R}$ for all $j = 1, \dots, p$. Furthermore, let $e_\lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that for almost all $k \in \mathbb{R}^d$, $k \cdot e_\lambda(k) = 0$ and $e_\lambda(k) \cdot e_{\lambda'}(k) = \delta_{\lambda\lambda'}$ for all $\lambda, \lambda' = 1, \dots, d-1$. Then we define the electromagnetic vector potential in the Coulomb gauge as

$$(31) \quad A(x) = \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} e_\lambda(k) \left(a_\lambda^*(k) \chi(k) e^{ik \cdot x} + a_\lambda(k) \bar{\chi}(k) e^{ik \cdot x} \right) dk ;$$

where $a_\lambda^\#$ are the creation and annihilation operators of $\Gamma_s(\mathbb{C}^{d-1} \otimes L^2(\mathbb{R}^d))$ satisfying the canonical commutation relations $[a_\lambda(k), a_{\lambda'}^*(k')] = \delta_{\lambda\lambda'} \delta(k - k')$; the (spinless) Pauli-Fierz Hamiltonian on \mathcal{H} is then

$$(32) \quad H_{PF} = \sum_{j=1}^p \frac{1}{2m_j} (-i\nabla_j \otimes 1 + q_j A(x_j))^2 + V(x_1, \dots, x_p) \otimes 1 + 1 \otimes \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \omega(k) a_\lambda^*(k) a_\lambda(k) dk .$$

The function χ plays the role of an ultraviolet cut off in the interaction, and is usually interpreted as the Fourier transform of the particles' charge distribution. Let $\{\sigma^{(\mu)}\}_{\mu=1}^d$ the $2^{\lfloor \frac{d}{2} \rfloor} \times 2^{\lfloor \frac{d}{2} \rfloor}$

matrices satisfying $\sigma^{(\mu)}\sigma^{(\nu)} + \sigma^{(\nu)}\sigma^{(\mu)} = 2\delta_{\mu\nu}\text{Id}$. Also, denote by $\sigma_j^{(\mu)}$, $j = 1, \dots, p$ the operator on $(\otimes^p \mathbb{C}^{2^{\lfloor \frac{d}{2} \rfloor}})$ acting as $\sigma^{(\mu)}$ on the j -th space of the tensor product. Then the spin- $\frac{1}{2}$ Pauli-Fierz Hamiltonian on $\mathcal{H}^{(spin)} = (\otimes^p \mathbb{C}^{2^{\lfloor \frac{d}{2} \rfloor}}) \otimes \mathcal{H}$ can be written as:

$$(33) \quad H_{PF}^{(spin)} = 1 \otimes H_{PF} + \frac{i}{2} \sum_{j=1}^p q_j \sum_{1 \leq \mu < \nu \leq d} \sigma_j^{(\mu)} \sigma_j^{(\nu)} \otimes \left(\partial_j^{(\mu)} A^{(\nu)}(x_j) - \partial_j^{(\nu)} A^{(\mu)}(x_j) \right);$$

where $A^{(\mu)}(x)$ is the μ -th component of the vector $A(x)$.

The quadratic form corresponding to the Pauli-Fierz Hamiltonian is bounded from below, so it is possible to define at least one self-adjoint extension by means of the Friedrichs Extension Theorem. This type of information is not completely satisfactory, since infinitely many extensions may exist, each one dictating a different dynamics for the system. For small values of the ratios q_j^2/m_j between charge and mass of the particles, and if $\chi, \chi/\sqrt{\omega} \in L^2(\mathbb{R}^d)$, a unique self-adjoint extension is given by KLMN Theorem. For arbitrary values of the ratios q_j^2/m_j , it is possible to prove essential self-adjointness of both H_{PF} and $H_{PF}^{(spin)}$ (for the spin operator we need in addition $\omega\chi \in L^2(\mathbb{R}^d)$) by means of Theorem 3.1, under the sole assumption $\chi \in L^2(\mathbb{R}^d)$. As discussed in Section 1, an analogous result (on a slightly different domain) has been obtained with an argument of functional integration by Hiroshima [18]. If the dependence on x of $A(x)$ is more general, functional integration methods may not be applicable; however Theorem 3.1 still holds.

In the following discussion we will focus on a simplified model, for the sake of clarity. Assumptions **A₀** and **A_I** are checked on H_{PF} with $p = 1$, $m = 1/2$ and $q = -1$, i.e.: $\mathcal{H} \equiv L^2(\mathbb{R}^d) \otimes \Gamma_s(\mathbb{C}^{d-1} \otimes L^2(\mathbb{R}^d))$ and

$$(34) \quad H \equiv (i\nabla_x \otimes 1 + A(x))^2 + V(x) \otimes 1 + 1 \otimes \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \omega(k) a_\lambda^*(k) a_\lambda(k) dk.$$

Observe that, since we are in the Coulomb gauge, $\nabla_x \cdot A(x) = 0$ hence $[-i\nabla_x \otimes 1, A(x)] = 0$ on a suitable dense domain. Rewrite H in the following form, to identify the free and interaction parts:

$$(35) \quad H = (-\Delta_x + V(x)) \otimes 1 + 1 \otimes \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \omega(k) a_\lambda^*(k) a_\lambda(k) dk + 2iA(x) \cdot (\nabla_x \otimes 1) + A^2(x).$$

We identify $H_{01} \equiv -\Delta + V$, $H_{02} \equiv \sum_{\lambda} \int_{\mathbb{R}^d} \omega a_\lambda^* a_\lambda$ and $H_I \equiv 2iA \cdot (\nabla \otimes 1) + A^2$. Assumption **A₀** is satisfied, as in the Nelson model (28) above. For the interaction part, we have the following

bounds: $\forall \psi \in \mathcal{H}, \forall \phi_n \in L^2(\mathbb{R}^d) \otimes (\mathbb{C}^{d-1} \otimes L^2(\mathbb{R}^d))^{\otimes n} \cap Q(H_{01} \otimes 1), n \in \mathbb{N}$,

$$(36) \quad |\langle \psi, A(x) \cdot (\nabla_x \otimes 1) \phi_n \rangle| \leq \sqrt{2(d-1)} \|\chi\|_2 \sqrt{n+1} \|(|\nabla_x| \otimes 1) \phi_n\| \sum_{\substack{i=-1 \\ i \neq 0}}^1 \|\psi_{n+i}\| ;$$

$$|\langle \psi, A^2(x) \phi_n \rangle| \leq 2(d-1) \|\chi\|_2 (n+1) \|\phi_n\| \sum_{i=-2}^2 \|\psi_{n+i}\| .$$

Hence Assumption \mathbf{A}_I is satisfied. Then H is essentially self-adjoint on $D(H_0) \cap f_0(L^2(\mathbb{R}^d) \otimes (\mathbb{C}^{d-1} \otimes L^2(\mathbb{R}^d))^{\cdot})$.

Remark. Neither non-negativity of the Pauli-Fierz operator nor smallness of the coupling constant are necessary to prove essential self-adjointness by means of Theorem 3.1. Using operator methods (commutator estimates), self-adjointness of H_{PF} with $V = 0$ has been proved for general coupling constants in [16], but the non-negativity was needed to associate a unique self-adjoint operator to the quadratic form. Theorem 3.1 relies on different assumptions, and takes advantage of the fibered structure of the Fock space: boundedness from below of the operator is, in general, not necessary. In fact, the Hamiltonians considered in Sections 4.1 and 4.2 are possibly unbounded from below, as well as the following extension (37) of the Pauli-Fierz Hamiltonian to infinite degrees of freedom (for the particles). As outlined in Section 5, if we assume boundedness from below, Theorem 3.1 can be extended to operators quartic in the creation/annihilation operators (see Assumptions $\mathbf{B}_H, \mathbf{B}_I$ and Theorem 5.1).

Let $m_j = 1/2, q_j = -1$ and $V = \sum_{i=1}^p V_{ext}(x_i) + \sum_{i < j} V_{pair}(x_i - x_j)$ such that $V_{ext} \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}_+)$, $V_{pair}(x) = V_{pair}(-x)$ and $V_{pair}(1 - \Delta)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^d))$. Under these assumptions define $H_{PF}|_s$ as the restriction of (32) to $L^2_s(\mathbb{R}^{pd}) \otimes \Gamma_s(\mathbb{C}^{d-1} \otimes L^2(\mathbb{R}^d))$. The physical interpretation is a system of p identical bosonic charges subjected to an external potential, interacting via pair interaction and with their radiation field. As we did for the Nelson model in (30), we can extend $H_{PF}|_s$ to $\Gamma_s(L^2(\mathbb{R}^d)) \otimes \Gamma_s(\mathbb{C}^{d-1} \otimes L^2(\mathbb{R}^d))$:

$$(37) \quad \tilde{H}_{PF} = \int_{\mathbb{R}^d} \psi^*(x) \left\{ (i\nabla_x \otimes 1 + A(x))^2 + V_{ext}(x) \right\} \psi(x) dx + \frac{1}{2} \int_{\mathbb{R}^{2d}} V_{pair}(x-y) \psi^*(x) \psi^*(y) \psi(x) \psi(y) dx dy + 1 \otimes \sum_{\lambda=1}^{d-1} \int_{\mathbb{R}^d} \omega(k) a_\lambda^*(k) a_\lambda(k) dk .$$

We would like to prove essential self-adjointness by means of Theorem 3.1. Identify $H_{01} \equiv \int \psi^*(-\Delta + V_{ext})\psi; H_{02} \equiv \sum_\lambda \int_{\mathbb{R}^d} \omega a_\lambda^* a_\lambda; H_I \equiv \int \psi^*(2iA \cdot (\nabla \otimes 1) + A^2)\psi + \frac{1}{2} \int V_{pair} \psi^* \psi^* \psi \psi; \text{ and } \{\mathcal{H}_1^{(j)} \otimes \Gamma_s(\mathcal{H}_2)\}_{j \in \mathbb{N}} \equiv \{L^2_s(\mathbb{R}^{jd}) \otimes \Gamma_s(\mathbb{C}^{d-1} \otimes L^2(\mathbb{R}^d))\}_{j \in \mathbb{N}}$. Then Assumptions \mathbf{A}_0 and \mathbf{A}_I are satisfied using bounds analogous to (36) and (26) (for V_{pair}), for each fixed $j \in \mathbb{N}$. Hence \tilde{H}_{PF} is essentially self-adjoint on $D(H_{01} \otimes 1) \cap D(1 \otimes H_{02}) \cap f_0(L^2(\mathbb{R}^d)^{\cdot}) \otimes (\mathbb{C}^{d-1} \otimes L^2(\mathbb{R}^d))^{\cdot}$.

5. CONCLUSIVE REMARKS

The examples of the preceding section are not exhaustive: we focused on them because of their relevance in physical and mathematical literature. The application to operators on curved space-time, or to anti-symmetric systems may also lead to results of interest.

The Assumptions A_0 , A_I and A'_I are easy to check: in the examples above follow from basic estimates of creation and annihilation operators. The proof of Theorem 3.1 itself is not complicated, and relies on the direct sum decomposition of $\Gamma_s(\mathcal{H}_2)$ and the structure of the interaction with respect to the latter. Hence this criterion gives, in our opinion, a simple yet powerful tool to prove essential self-adjointness in Fock spaces, tailored to take maximum advantage of their structure.

If we assume that H is bounded from below, we can take inspiration from Masson and McClary [19] and extend our criterion to accommodate quartic operators. The modified assumptions and theorem would then read:

Assumption B_H . H is a densely defined symmetric operator on $\mathcal{H} = \mathcal{H}_1 \otimes \Gamma_s(\mathcal{H}_2)$ bounded from below. H_{01} and H_{02} are self-adjoint operators bounded from below such that $\forall t \in \mathbb{R}$, $\{\mathcal{H}_2^{(n)}\}_{n \in \mathbb{N}}$ is invariant for $e^{itH_{02}}$.

Assumption B_I . H_I is a symmetric operator on \mathcal{H} , with a domain of definition $D(H_I)$ such that $D(H_0) \cap D(H_I)$ is dense in \mathcal{H} . Furthermore exists a complete collection $\{\mathcal{H}_1^{(j)} \otimes \Gamma_s(\mathcal{H}_2)\}_{j \in \mathbb{N}}$ invariant for H_0 and H_I such that: $\forall \phi \in Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1^{(j)} \otimes \mathcal{H}_2^{(n)}$,

$$(38) \quad H_I \phi \in \bigoplus_{i=-4}^4 \mathcal{H}_1^{(j)} \otimes \mathcal{H}_2^{(n+i)}.$$

Also, H_I satisfies the following bound: $\forall j, n \in \mathbb{N} \exists C(j) > 0$ such that $\forall \psi \in \mathcal{H}$, $\forall \phi \in Q(H_{01} \otimes 1) \cap Q(1 \otimes H_{02}) \cap \mathcal{H}_1^{(j)} \otimes \mathcal{H}_2^{(n)}$:

$$(39) \quad |\langle \psi, H_I \phi \rangle_{\mathcal{H}}|^2 \leq C^2(j) \sum_{i=-4}^4 \|\psi_{j,n+i}\|_{\mathcal{H}_1^{(j)} \otimes \mathcal{H}_2^{(n+i)}}^2 \left[(n+1)^4 \|\phi\|_{\mathcal{H}_1^{(j)} \otimes \mathcal{H}_2^{(n)}}^2 + (n+1)^2 \left(q_{H_{01} \otimes 1}(\phi, \phi) + q_{1 \otimes H_{02}}(\phi, \phi) + (|M_1| + |M_2| + 1) \|\phi\|_{\mathcal{H}_1^{(j)} \otimes \mathcal{H}_2^{(n)}}^2 \right) \right].$$

Theorem 5.1. Assume B_H and B_I . Then H is essentially self adjoint on $D(H_{01} \otimes 1) \cap D(H_{02} \otimes 1) \cap f_0(\mathcal{H}_1^{(\cdot)} \otimes \mathcal{H}_2^{(\cdot)})$.

Remark. An attempt to extend the results of [19] can be found in [3]. Theorem 5.1 is a generalization of both: it can be applied to more singular situations and a more general class of spaces.

The proof of Theorem 3.1 can be adapted to Theorem 5.1, making use of the inferior bound for H . We remark that Assumption B_H , by itself, implies that H has at least one self-adjoint extension: it may be tricky to prove for general operators. Theorem 5.1 essentially states that for regular enough quartic interactions, existence of a particular self-adjoint extension (the Friedrichs one) is equivalent to its uniqueness. It may have interesting applications in CQFT: e.g. the d -dimensional (bounded from below) Y_d and $(\lambda\varphi(x)^4)_d$ models with cut offs have interactions that are at most quartic and regular. It is our hope that the ideas utilized in this paper could contribute to improve the mathematical insight on interacting quantum field theories, and could be developed to study self-adjointness of more singular systems.

ACKNOWLEDGMENTS

This work has been supported by the Centre Henri Lebesgue (programme “Investissements d’avenir” — ANR-11-LABX-0020-01). The author would like to thank Giorgio Velo, that has suggested to him the idea of a direct proof of self-adjointness on Fock spaces. Also, he would like to thank Zied Ammari and Francis Nier for precious advices and stimulating discussions during the redaction of the paper.

-
- [1] Z. Ammari. Asymptotic completeness for a renormalized nonrelativistic Hamiltonian in quantum field theory: the Nelson model. *Math. Phys. Anal. Geom.*, 3(3):217–285, 2000. ISSN 1385-0172. doi:10.1023/A:1011408618527. URL <http://dx.doi.org/10.1023/A:1011408618527>.
 - [2] Z. Ammari and M. Falconi. Wigner measures approach to the classical limit of the Nelson model: Convergence of dynamics and ground state energy. *J. Stat. Phys.*, 157(2):330–362, 10 2014. doi: 10.1007/s10955-014-1079-7. URL <http://dx.doi.org/10.1007/s10955-014-1079-7>.
 - [3] A. Arai. A theorem on essential self-adjointness with application to hamiltonians in nonrelativistic quantum field theory. *J. Math. Phys.*, 32(8):2082–2088, 1991. URL <http://dx.doi.org/10.1063/1.529178>.
 - [4] V. Bach, J. Fröhlich, and I. M. Sigal. Quantum electrodynamics of confined nonrelativistic particles. *Adv. Math.*, 137(2):299–395, 1998. ISSN 0001-8708. doi:10.1006/aima.1998.1734. URL <http://dx.doi.org/10.1006/aima.1998.1734>.

- [5] V. Bach, J. Fröhlich, and I. M. Sigal. Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field. *Comm. Math. Phys.*, 207(2):249–290, 1999. ISSN 0010-3616. doi:10.1007/s002200050726. URL <http://dx.doi.org/10.1007/s002200050726>.
- [6] V. Bach, J. Fröhlich, I. M. Sigal, and A. Soffer. Positive commutators and the spectrum of Pauli-Fierz Hamiltonian of atoms and molecules. *Comm. Math. Phys.*, 207(3):557–587, 1999. ISSN 0010-3616. doi:10.1007/s002200050737. URL <http://dx.doi.org/10.1007/s002200050737>.
- [7] J. M. Cook. The mathematics of second quantization. *Proc. Nat. Acad. Sci. U. S. A.*, 37:417–420, 1951. ISSN 0027-8424.
- [8] J. Dereziński and C. Gérard. Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians. *Rev. Math. Phys.*, 11(4):383–450, 1999. ISSN 0129-055X. doi:10.1142/S0129055X99000155. URL <http://dx.doi.org/10.1142/S0129055X99000155>.
- [9] J. Dereziński and C. Gérard. *Mathematics of quantization and quantum fields*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2013. ISBN 978-1-107-01111-3. doi:10.1017/CBO9780511894541. URL <http://dx.doi.org/10.1017/CBO9780511894541>.
- [10] J. Dereziński and V. Jakšić. Spectral theory of Pauli-Fierz operators. *J. Funct. Anal.*, 180(2):243–327, 2001. ISSN 0022-1236. doi:10.1006/jfan.2000.3681. URL <http://dx.doi.org/10.1006/jfan.2000.3681>.
- [11] M. Falconi. *Classical limit of the Nelson model*. PhD thesis, Dottorato di Ricerca in Matematica XXIV ciclo, Università di Bologna, 2012.
- [12] C. Gérard, F. Hiroshima, A. Panati, and A. Suzuki. Infrared problem for the Nelson model on static space-times. *Comm. Math. Phys.*, 308(2):543–566, 2011. ISSN 0010-3616. doi:10.1007/s00220-011-1289-7. URL <http://dx.doi.org/10.1007/s00220-011-1289-7>.
- [13] J. Ginibre and G. Velo. Renormalization of a quadratic interaction in the Hamiltonian formalism. *Comm. Math. Phys.*, 18:65–81, 1970. ISSN 0010-3616.
- [14] J. Glimm. Yukawa coupling of quantum fields in two dimensions. I. *Comm. Math. Phys.*, 5:343–386, 1967. ISSN 0010-3616.
- [15] J. Glimm and A. Jaffe. *Collected papers. Vol. 2*. Birkhäuser Boston, Inc., Boston, MA, 1985. ISBN 0-8176-3272-7. Constructive quantum field theory. Selected papers, Reprint of articles published 1968–1980.
- [16] D. Hasler and I. Herbst. On the self-adjointness and domain of Pauli-Fierz type Hamiltonians. *Rev. Math. Phys.*, 20(7):787–800, 2008. ISSN 0129-055X. doi:10.1142/S0129055X08003389. URL <http://dx.doi.org/10.1142/S0129055X08003389>.
- [17] F. Hiroshima. Essential self-adjointness of translation-invariant quantum field models for arbitrary coupling constants. *Comm. Math. Phys.*, 211(3):585–613, 2000. ISSN 0010-3616. doi:10.1007/s002200050827. URL <http://dx.doi.org/10.1007/s002200050827>.
- [18] F. Hiroshima. Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling

- constants. *Ann. Henri Poincaré*, 3(1):171–201, 2002. ISSN 1424-0637. doi:10.1007/s00023-002-8615-8. URL <http://dx.doi.org/10.1007/s00023-002-8615-8>.
- [19] D. Masson and W. K. McClary. On the self-adjointness of the $(g(x)\phi^4)_2$ hamiltonian. *Comm. Math. Phys.*, 21(1):71–74, 1971. URL <http://projecteuclid.org/euclid.cmp/1103857260>.
- [20] E. Nelson. Interaction of nonrelativistic particles with a quantized scalar field. *J. Math. Phys.*, 5(9):1190–1197, 1964. doi:10.1063/1.1704225. URL <http://link.aip.org/link/?JMP/5/1190/1>.
- [21] E. Nelson. A quartic interaction in two dimensions. In *Mathematical Theory of Elementary Particles (Proc. Conf., Dedham, Mass., 1965)*, pages 69–73. M.I.T. Press, Cambridge, Mass., 1966.
- [22] T. Okamoto and K. Yajima. Complex scaling technique in nonrelativistic massive QED. *Ann. Inst. H. Poincaré Phys. Théor.*, 42(3):311–327, 1985. ISSN 0246-0211. URL http://www.numdam.org/item?id=AIHPB_1985__42_3_311_0.
- [23] W. Pauli and M. Fierz. Zur theorie der emission langwelliger lichtquanten. *Il Nuovo Cimento*, 15(3):167–188, 1938.
- [24] M. Reed and B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press, New York, 1975.
- [25] L. Rosen. A $\lambda\phi^{2n}$ field theory without cutoffs. *Comm. Math. Phys.*, 16:157–183, 1970. ISSN 0010-3616.
- [26] I. Segal. Construction of non-linear local quantum processes. I. *Ann. of Math. (2)*, 92:462–481, 1970. ISSN 0003-486X.
- [27] H. Spohn. *Dynamics of charged particles and their radiation field*. Cambridge University Press, Cambridge, 2004. ISBN 0-521-83697-2. doi:10.1017/CBO9780511535178. URL <http://dx.doi.org/10.1017/CBO9780511535178>.