Wigner measures approach to the classical limit of the Nelson model: Convergence of dynamics and ground state energy.

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(Dated: May 27, 2015)

We consider the classical limit of the Nelson model, a system of stable nucleons interacting with a meson field. We prove convergence of the quantum dynamics towards the evolution of the coupled Klein-Gordon-Schrödinger equation. Also, we show that the ground state energy level of N nucleons, when N is large and the meson field approaches its classical value, is given by the infimum of the classical energy functional at a fixed density of particles. Our study relies on a recently elaborated approach for mean field theory and uses Wigner measures.

1. INTRODUCTION

The Nelson model refers to a quantum dynamical system describing a nucleon field interacting with a meson scalar Bose field. When an ultraviolet cut off is put in the interaction, the Hamiltonian becomes a self-adjoint operator and so the quantum dynamics is well defined. In the early sixties, E. Nelson showed that the quantum dynamics of this system exists even when the ultraviolet cut off is removed [see 22]. It is indeed one of the simplest examples in non-relativistic quantum field theory (QFT) where renormalization is needed and successfully performed using only basic tools of functional analysis.

Over the past two decades, there has been considerable effort devoted to the study of the Nelson model that have led to a thorough investigation of its spectral and scattering properties [see 1, 7, 8, 10, 12, 14–16, 21, 23, 26, to mention but a few]. However, the fact that the Nelson Hamiltonian is a Wick quantization of a classical Hamiltonian system is quite often neglected except in few references [13, 18]. We believe that the study of the classical limit of such quantum dynamical systems is a significant question leading to an unexplored phase-space point of view in QFT. This for sure will enrich the subject and may also provide some insight on some of the remaining open problems.

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In this article, we neglect the spin and isospin for nucleons, so we are considering a scalar Yukawa field theory. We also suppose that an ultraviolet cut off is imposed on the interaction. We prove two main results stated in Theorem 1.1 and Theorem 1.2, namely:

(i) Convergence of the quantum dynamics towards the classical evolution of the coupled Klein-Gordon-Schrödinger equation.

(ii) Convergence of the ground state energy level of N nucleons to the infimum of the classical energy functional with fixed density of particles, when N tends to infinity and the Bose field approaches its classical limit.

There are basically two schemes for proving (i): either one studies the propagation of states or those of observables. The latter strategy being very difficult for systems with unconserved number of particles, we rely on the first scheme. To establish (i), we follow indeed a Wigner measures approach, recently elaborated in [3–6] for the purpose of mean field limit in many-body theory. This method turns out to be quite general and flexible. It can be adapted to quantum electrodynamics (QED) and relativistic quantum field theory (QFT) and it gives a fair description of the propagation of general states in the classical limit (see Theorem 1.1). Actually, the convergence (i) is known in the particular case of coherent states [see 13, 18] by Hepp's method [20] which relies on the special structure of those states. The result in Theorem 1.1 says that the convergence of the Neslon quantum dynamics towards the classical one has nothing to do with any particular structure or choice of states but it is rather a general (Bohr) quantum-classical correspondence principle for a system with an infinite number of degrees of freedom. In this sense, Theorem 1.1 is more general and provides a better understanding of the classical limit of Nelson Hamiltonians.

In addition to the fact that the Wigner measures approach gives a stronger convergence result compared to the coherent state method, it also proves to be a powerful tool for tackling variational questions of type (ii). Indeed, asymptotic properties of a given minimizing sequence can be derived by looking to its Wigner measures and it turns out that some a priori information on these Wigner measures are crucial in the proof of Theorem 1.2. However, both our results give only the limit of quantum quantities in terms of their classical approximations and provide no error bound on the difference. This is of course an interesting question, among several others, and it is beyond the scope of this article. Actually, our work is also meant to stimulate further investigations and to underline some open problems. For instance, removal of the momentum cutoff and drop of the confining potential as well as time asymptotics and scattering theory within the classical limit are quite interesting open questions. We believe indeed that our work provides a basis for further developments on the above-mentioned problems.

The phase space of the theory is $\mathscr{Z} := L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$, and we consider the symmetric Fock space $\mathscr{H} := \Gamma_s(\mathscr{Z}) \sim \Gamma_s(L^2(\mathbb{R}^d)) \otimes \Gamma_s(L^2(\mathbb{R}^d))$. We denote by $\psi^\#$ the annihilation and creation of the non-relativistic particles (nucleons), by $a^\#$ the annihilation and creation of the relativistic meson field. We recall that for each $\varepsilon \in (0, \bar{\varepsilon})$, with $\bar{\varepsilon} > 0$ fixed once and for all, we choose the algebra:

$$[\psi(x_1), \psi^*(x_2)] = \varepsilon\delta(x_1 - x_2) \ , \quad [a(k_1), a^*(k_2)] = \varepsilon\delta(k_1 - k_2) \ .$$

This fixes the scaling so that each $\psi^\#$ and $a^\#$ behaves like $\sqrt{\varepsilon}$. For instance, the second quantization operators $d\Gamma(\cdot) = \int_{\mathbb{R}^d} a^*(k)(\,\cdot\,)a(k)dk$ or $\int_{\mathbb{R}^d} \psi^*(x)(\,\cdot\,)\psi(x)dx$ scale like ε. This is also the case for the number operators $N_1 = d\Gamma(1) \otimes 1, N_2 = 1 \otimes d\Gamma(1)$ and $N = N_1 + N_2$. The Weyl operators are $W(\xi) = W(\xi_1) \otimes W(\xi_2)$, for $\xi = \xi_1 \oplus \xi_2 \in \mathscr{Z}$, with $W(\xi_1) = e^{i\frac{\psi^*(\xi_1)+\psi(\xi_1)}{\sqrt{2}}}$ and $W(\xi_2) = e^{i\frac{a^*(\xi_2)+a(\xi_2)}{\sqrt{2}}}$ being the Weyl operators on $\Gamma_s(L^2(\mathbb{R}^d))$.

In the Fock representation of these canonical commutation relations, the Nelson Hamiltonian takes the form:

$$H = d\Gamma(-\frac{\Delta}{2M} + V) \otimes 1 + 1 \otimes d\Gamma(\omega) + \int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}} \psi^*(x)\big(a^*(k)e^{-ik\cdot x} + a(k)e^{ik\cdot x}\big)\psi(x)dkdx \ ;$$

where $\omega(k) = \sqrt{k^2 + m_0^2}$ and $m_0 \geq 0$. Here m_0 and M are respectively the meson and nucleon mass at rest. It is useful to split H in a free part H_0, and an interaction part H_I, with:

$$\begin{aligned} H_0 &= d\Gamma(-\frac{\Delta}{2M} + V) \otimes 1 + 1 \otimes d\Gamma(\omega) \ , \\ H_I &= \int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}} \psi^*(x)\big(a^*(k)e^{-ik\cdot x} + a(k)e^{ik\cdot x}\big)\psi(x)dkdx \ . \end{aligned}$$

We assume the potential $V(x)$ to be in $L^2_{loc}(\mathbb{R}^d, \mathbb{R}_+)$, so that $-\Delta + V$ is a positive self-adjoint operator on $L^2(\mathbb{R}^d)$, by Kato inequality, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. The main assumption we require on the cut off function χ is that $\omega^{-1/2}\chi \in L^2(\mathbb{R}^d)$. This is enough to define H as self-adjoint operator (see Proposition 2.5). To recapitulate, we assume throughout the article the assumption

(**A**) $$V \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}_+) \text{ and } \omega^{-1/2}\chi \in L^2(\mathbb{R}^d) \, .$$

Actually, the Nelson Hamiltonian is a Wick quantization of the classical energy functional

$$h(z_1 \oplus z_2) = \langle z_1, -\frac{\Delta}{2M} + V\, z_1\rangle + \langle z_2, \omega(k) z_2\rangle + \int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}} |z_1|^2(x)\big(\bar{z}_2(k)e^{-ik\cdot x} + z_2(k)e^{ik\cdot x}\big)dkdx \ .$$

The Hamiltonian h describes the coupled Klein-Gordon-Schrödinger system with an Yukawa type interaction subject to a momentum cut off. With the assumption (**A**), the related Cauchy problem is well posed in $\mathscr{Z}$ (see Propositions 2.7 and 2.8).

The main point in the proof of (i) is to understand the propagation of normal states on the Fock space $\mathscr{H}$ with the appropriate scaling. The idea is to encode the oscillations of any family of states with respect to the semiclassical parameter ε by classical quantities, namely probability measures on the phase space (Wigner measures). Then (i) can be restated as the propagation of these measures along the classical flow of the Klein-Gordon-Schrödinger equation.

We say that a Borel probability measure μ on $\mathscr{Z}$ is a Wigner measure of a family of normal states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ on $\mathscr{H}$ if there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ in $(0,\bar\varepsilon)$, such that $\varepsilon_k\to 0$ and for any $\xi\in\mathscr{Z}$,

$$\lim_{k\to\infty}\mathrm{Tr}[\varrho_{\varepsilon_k}W(\xi)]=\int_{\mathscr{Z}} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}\,d\mu(z)\,, \tag{1}$$

where $W(\xi)$ refers to the Weyl operator on the Fock space $\mathscr{H}$ which depends on ε_k (here $\mathrm{Re}\langle\cdot,\cdot\rangle$ is the real part of the scalar product on $\mathscr{Z}$). We denote the set of all Wigner measures of a given family of states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ by $\mathscr{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon))$. It was proved in [4] that the assumption

$$\exists\delta>0,\exists C>0,\forall\varepsilon\in(0,\bar\varepsilon)\quad \mathrm{Tr}[\varrho_\varepsilon N^\delta]<C\,,$$

ensures that the set of Wigner measures $\mathscr{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon))$ is non empty. Notice that the $\mathrm{Tr}[\,\cdot\,]$ is understood as $\sum_{i=0}^\infty\lambda_i\langle\varphi_i,N^\delta\varphi_i\rangle$, where $\{\varphi_i\}_{i\in\mathbb{N}}$ is an O.N.B. of eigenvectors of ϱ_ε associated to the eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$.

The Nelson Hamiltonian H has a fibred structure with respect to the number of nucleons. So, it can be written as $H=\oplus_{n=0}^\infty H_{|L_s^2(\mathbb{R}^{nd})\otimes\Gamma_s(L^2(\mathbb{R}^d))}$ where $L_s^2(\mathbb{R}^{nd})$ denotes the space of symmetric square integrable functions (see Section 2). It also turns out that H is unbounded from below while $H_{|L_s^2(\mathbb{R}^{nd})\otimes\Gamma_s(L^2(\mathbb{R}^d))}$ is bounded from below.
Under the aforementioned assumptions on the potential V and the cut off function χ, we are in position to precisely state our two main results.

Theorem 1.1. *Assume that* (**A**) *holds. Let* $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ *be a family of normal states on the Hilbert space* $\mathscr{H}$ *satisfying the assumption:*

$$\exists\delta>0,\exists C>0,\forall\varepsilon\in(0,\bar\varepsilon)\quad \mathrm{Tr}[\varrho_\varepsilon(H_0+N+1)^\delta]<C. \tag{2}$$

Then for any $t\in\mathbb{R}$ *the set of Wigner measures associated with the family* $(e^{-i\frac{t}{\varepsilon}H}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H})_{\varepsilon\in(0,\bar\varepsilon)}$ *is*

$$\mathscr{M}(e^{-i\frac{t}{\varepsilon}H}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H},\varepsilon\in(0,\bar\varepsilon))=\{\Phi(t,0)_\#\mu_0,\mu_0\in\mathscr{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon))\}\,,$$

where $\Phi(t,0)_{\#}\mu_0$ *is the push forward of* μ_0 *by the classical flow* $\Phi(t,s)$ *of the coupled Klein-Gordon-Schrödinger equation* (13) *well defined and continuous on* $\mathscr{X}$ *by Propositions 2.7 and 2.8.*

Theorem 1.2. *Assume that* (**A**) *holds and additionally* $m_0 > 0$ *and* V *is a confining potential, i.e.:* $\lim_{|x|\to\infty} V(x) = +\infty$*. Then the ground state energy of the restricted Nelson Hamiltonian has the following limit, for any* $\lambda > 0$*,*

$$\lim_{\varepsilon\to 0, n\varepsilon=\lambda^2} \inf \sigma(H_{|L^2_s(\mathbb{R}^{dn})\otimes\Gamma_s(L^2(\mathbb{R}^d))}) = \inf_{||z_1||_{L^2(\mathbb{R}^d)}=\lambda} h(z_1 \oplus z_2)\,, \tag{3}$$

where the infimum on the right hand side is taken over all $z_1 \in D(\sqrt{\frac{-\Delta}{2M} + V})$ *and* $z_2 \in D(\omega^{1/2})$ *with the constraint* $||z_1||_{L^2(\mathbb{R}^d)} = \lambda$*.*

The proof of Theorem 1.1 is given in Sections 3 and 4 and uses the properties of the quantum and classical dynamics proved in Section 2. It is rather lengthy, so for reader's convenience we outline its key arguments below. The proof of Theorem 1.2, given in Section 5, relies on an upper bound derived by using coherent states localized around the infimum of the classical energy and a lower bound resulting from the a priori information on the Wigner measures of a given minimizing sequence. So, we conclude that these measures are a fortiori concentrated around the infimum of the classical energy.

Proof of Theorem 1.1:

Our goal is to identify the Wigner measures of the evolved state $\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H}$ given in Theorem 1.1. However, instead of considering $\varrho_\varepsilon(t)$, we work in the interaction representation with

$$\tilde{\varrho}_\varepsilon(t) = e^{i\frac{t}{\varepsilon}H_0}\varrho_\varepsilon(t)e^{-i\frac{t}{\varepsilon}H_0}\,.$$

By doing so, we require less regularity on the state ϱ_ε and it is still easy to recover Wigner measures of $\varrho_\varepsilon(t)$ from those of $\tilde{\varrho}_\varepsilon(t)$. The main point now is that Wigner measures of the latter states are determined through all possible "limit points", when $\varepsilon \to 0$, of the map

$$\xi \mapsto \mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t)W(\xi)\Big]\,. \tag{4}$$

Despite its apparent simplicity, there is no straightforward way to compute such limit explicitly. Moreover, uniqueness of Wigner measures at each time t is not guarantied even if it is assumed at the initial time $t = 0$ (i.e.: the map (4) may have several limit points though it has one single limit at $t = 0$). To overcome the last difficulty, we use a diagonal extraction (or Ascoli type) argument which implies that for any sequence $(\tilde{\varrho}_{\varepsilon_n})_{n\in\mathbb{N}}$, $\varepsilon_n \to 0$, we can extract a subsequence $(\tilde{\varrho}_{\varepsilon_{n_k}})_{k\in\mathbb{N}}$ such that for each time, $t \in \mathbb{R}$, $(\tilde{\varrho}_{\varepsilon_{n_k}}(t))_{k\in\mathbb{N}}$ admits a unique Wigner measure denoted by $\tilde{\mu}_t$.

The next step is to observe that (4) satisfies a dynamical equation which when $\varepsilon \to 0$ leads to a well behaved classical dynamical equation on the inverse-Fourier transform of the Wigner measures $\tilde{\mu}_t$. By integrating with respect to appropriate trial functions, we obtain a natural transport (Liouville) equation satisfied by $\tilde{\mu}_t$. Therefore, it is possible to identify the measures $\tilde{\mu}_t$ if we can prove that such transport equation has a unique solution for each data $\tilde{\mu}_0$ given by the push-forward of $\tilde{\mu}_0$ by the corresponding classical dynamics. To sum up, the outline of the proof goes as follows:

1) We justify the integral (or Duhamel) formula

$$\mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t)W(\xi)\Big] = \mathrm{Tr}\Big[\varrho_\varepsilon W(\xi)\Big] + \frac{i}{\varepsilon}\int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)[H_I, W(\tilde{\xi}(s))]\Big]ds\,,$$

in Proposition 3.5 for states ϱ_ε satisfying a strong regularity condition, namely that it belongs to the space $\mathcal{T}^1_\varepsilon$ given in Definition 3.1.

2) By explicit computation and taking care of domain problems, we show in Proposition 3.9 that

$$\mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t)W(\xi)\Big] = \mathrm{Tr}\Big[\varrho_\varepsilon W(\xi)\Big] + \sum_{j=0}^{2}\varepsilon^j\int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s))B_j(\tilde{\xi}(s))\Big]ds\,; \tag{5}$$

where $B_j(\tilde{\xi}(s))$ are operators given in *(22)-(24)*.

3) There is no loss of generality if we assume that $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ has a single Wigner measure μ_0. Moreover, we prove as explained before that from any sequence $\varepsilon_n \to 0$ we can extract a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ such that $(\tilde{\varrho}_{\varepsilon_{n_k}}(t))_{k\in\mathbb{N}}$ has a single Wigner measure $\tilde{\mu}_t$ for each time $t\in\mathbb{R}$ (see Subsection 4 b).

4) Letting $\varepsilon_{n_k}\to 0$ in (5) and using some elementary ε-uniform estimates proved in Section 2 with some Wigner measures properties; we show in Proposition 4.10 that

$$\tilde{\mu}_t(e^{i\sqrt{2}\mathrm{Re}\langle\xi,\,\cdot\,\rangle}) = \mu_0(e^{i\sqrt{2}\mathrm{Re}\langle\xi,\,\cdot\,\rangle}) + i\sqrt{2}\int_0^t \tilde{\mu}_s\left(e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}\mathrm{Re}\langle\xi,\mathscr{V}_s(z)\rangle\right)\,ds;$$

with a velocity vector field $\mathscr{V}_s(z)$ defined by (16).

5) In Proposition 4.11, we show that $t\in\mathbb{R}\mapsto\tilde{\mu}_t$ is a weakly narrowly continuous map valued on probability measures satisfying the transport equation

$$\partial_t\tilde{\mu}_t + \nabla^T\left(\mathscr{V}_t\tilde{\mu}_t\right) = 0\,,$$

understood in the weak sense,

$$\int_{\mathbb{R}}\int_{\mathscr{Z}}\left(\partial_t f + \mathrm{Re}\langle\nabla f,\mathscr{V}_t\rangle\right)\,d\tilde{\mu}_t dt = 0\,.$$

6) To identify the measures $\tilde{\mu}_t$ we rely on an argument worked out in finite dimension by Ambrosio et al. [2] for the purpose of optimal transport theory and extended in [3] to an infinite dimensional Hilbert space setting. This yields, in Proposition 4.12, the result of Theorem 1.1 but under a strong assumption on $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \in \cap_{\delta>0}\mathcal{T}^\delta \cap \mathcal{S}^1$, given in Definition 4.1.

7) To complete the proof, we use an approximation argument allowing to extend the previous result to states satisfying the weak assumption (2) in Theorem 1.1 (see Section 4 e).

2. DYNAMICS, QUANTUM AND CLASSICAL.

In this section we provide informations on the dynamics of the Nelson model with cut off and its classical counterpart. Most results are proved in detail in [13], in the case $d = 3$; and such results extend immediately to any dimension. We will briefly outline the proofs here, for the reader's convenience.

a. Quantum system.

The phase space of the theory is $\mathscr{Z} := L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$, and we construct the Fock space $\mathscr{H} := \Gamma_s(\mathscr{Z}) \sim \Gamma_s(L^2(\mathbb{R}^d)) \otimes \Gamma_s(L^2(\mathbb{R}^d))$. The Nelson Hamiltonian H as well as the annihilation-creation of the nucleon field $\psi^\#$ and the meson field $a^\#$ are recalled in the introduction. As mentioned in the introduction, it is useful to set:

$$
\begin{aligned}
H_{01} &:= d\Gamma_1(-\frac{\Delta}{2M} + V) = \frac{1}{2M}\int_{\mathbb{R}^d} (\nabla\psi)^*(x)\nabla\psi(x)dx + \int_{\mathbb{R}^d} V(x)\psi^*(x)\psi(x)dx \ , \\
H_{02} &:= d\Gamma_2(\omega) = \int_{\mathbb{R}^d} \omega(k)a^*(k)a(k)dk \ , \\
H_0 &:= H_{01} + H_{02} \ , \\
H_I &:= \int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}}\psi^*(x)\big(a^*(k)e^{-ik\cdot x} + a(k)e^{ik\cdot x}\big)\psi(x)dkdx \ .
\end{aligned}
$$

We remark that, with our assumption (**A**) on V, H_0 is a positive self-adjoint operator on $\Gamma_s(\mathscr{Z})$ with its natural domain $D(H_0)$.

Let $N_1 = d\Gamma_1(1) = \int_{\mathbb{R}^d} \psi^*(x)\psi(x)dx$ and $N_2 = d\Gamma_2(1) = \int_{\mathbb{R}^d} a^*(k)a(k)dk$ be the number operators. Since H commutes with N_1 it is natural to split the Fock space into sectors with a fixed number of non-relativistic particles; hence we define the subspace

$$
\mathscr{H}_n := L^2_s(\mathbb{R}^{nd}) \otimes \Gamma_s(L^2(\mathbb{R}^d)) \ . \tag{6}
$$

Here $L^2_s(\mathbb{R}^{nd})$ denotes the space of symmetric square integrable functions (i.e.: $\phi(x_1,\cdots,x_n) = \phi(x_{\sigma_1},\cdots,x_{\sigma_n})$ for any permutation σ). By definition, we have:

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n \ .$$

Given an operator X on $\mathscr{H}$, we call X_n its restriction to the subspace $\mathscr{H}_n$. The restriction on $\mathscr{H}_n$ of the operator $a(f)$, $f \in L^2(\mathbb{R}^d)$, can be extended to any function $f \in L^\infty(\mathbb{R}^{nd}, L^2(\mathbb{R}^d))$; we will denote $a(f)_n$ again by $a(f)$ if no confusion arises.

Lemma 2.1. *i) Let $f \in L^\infty(\mathbb{R}^{nd}, L^2(\mathbb{R}^d))$ such that $\omega^{-1/2} f \in L^\infty(\mathbb{R}^{nd}, L^2(\mathbb{R}^d))$. Then for any $\phi \in D(H_{02}^{1/2}) \cap \mathscr{H}_n$:*

$$\|a(f)\phi\|^2 \leq \|\omega^{-1/2} f\|^2_{L^\infty(\mathbb{R}^{nd},L^2(\mathbb{R}^d))} \|H_{02}^{1/2}\phi\|^2 \tag{7}$$

$$\|a^*(f)\phi\|^2 \leq \|\omega^{-1/2} f\|^2_{L^\infty(\mathbb{R}^{nd},L^2(\mathbb{R}^d))} \|H_{02}^{1/2}\phi\|^2 + \varepsilon \|f\|^2_{L^\infty(\mathbb{R}^{nd},L^2(\mathbb{R}^d))} \|\phi\|^2 \ . \tag{8}$$

ii) Let $f \in L^\infty(\mathbb{R}^{nd}, L^2(\mathbb{R}^d))$. Then for any $\phi \in D(N_2^{1/2}) \cap \mathscr{H}_n$:

$$\|a(f)\phi\| \leq \|f\|_{L^\infty(\mathbb{R}^{nd},L^2(\mathbb{R}^d))} \|N_2^{1/2}\phi\| \tag{9}$$

$$\|a^*(f)\phi\| \leq \|f\|_{L^\infty(\mathbb{R}^{nd},L^2(\mathbb{R}^d))} \|(N_2+\varepsilon)^{1/2}\phi\| \ . \tag{10}$$

The proof of this lemma is standard and follows by means of a direct calculation on $\mathscr{H}_n$ (see e.g. [18]).

Corollary 2.2. *Let χ such that $\omega^{-1/2}\chi \in L^2(\mathbb{R}^d)$. Then for any $\phi \in D(N_1^2 + N_2)$:*

$$\|H_I\phi\| \leq \|\omega^{-1/2}\chi\|_2 \|(N_1^2 + N_2 + \varepsilon)\phi\| \ .$$

We have now all the ingredients to prove the essential self-adjointness of H.

Proposition 2.3 (self-adjointness/Kato perturbation)**.** *Assume that (**A**) holds. Furthermore, let χ such that $\omega^{-1}\chi \in L^2(\mathbb{R}^d)$. Then H is self-adjoint on $\mathscr{H}$ with domain:*

$$D(H) = \{\phi \in \mathscr{H}; \forall n \in \mathbb{N}, \phi_n := \phi\big|_{\mathscr{H}_n} \in D(H_0) \cap \mathscr{H}_n \ \textit{and} \ \sum_{n=0}^{\infty} \|H_n\phi_n\|^2 < \infty\} \ .$$

Proof. The operator H_n is self-adjoint on $\mathscr{H}_n$ with domain $D(H_0) \cap \mathscr{H}_n$ since, by Lemma 2.1, $(H_I)_n$ is a Kato perturbation of $(H_0)_n$. Furthermore, the small constant in the perturbation is independent of n, hence we can define the self-adjoint extension of H as the direct sum $\bigoplus_{n=0}^{\infty} H_n$ [see 13, Proposition IV.1]. □

Remark 2.4. It is usual to assume $\chi(k)$ to be a characteristic function $1_{\{|k|\leq\kappa\}}(k)$, for some $\kappa > 0$. If $m_0 = 0$ and $\chi = 1_{\{|k|\leq\kappa\}}$, then for all $d \geq 3$, $\omega^{-1/2}\chi \in L^2(\mathbb{R}^d)$ and $\omega^{-1}\chi \in L^2(\mathbb{R}^d)$; hence H is self-adjoint. However, if $d = 2$ then $\omega^{-1}\chi \notin L^2(\mathbb{R}^2)$. With a different approach, we can relax the requirement on χ and prove essential self-adjointness of H under the sole assumption (**A**).

Define $\mathscr{F}_0 \subset \Gamma_s(\mathscr{Z})$ to be subspace of finite particle vectors of $\Gamma_s(\mathscr{Z})$ (i.e.: vectors with finite number of nucleons and mesons).

Proposition 2.5 (self-adjointness/direct proof)**.** *Assume that* (**A**) *holds. Then H is essentially self-adjoint on $D(H_0) \cap \mathscr{F}_0$. We denote the self-adjointness domain of H as $D(H)$.*

Proof. Let $\phi \in \Gamma_s(\mathscr{Z})$. Then we denote by ϕ_{n_1,n_2} its restriction to $L^2_s(\mathbb{R}^{n_1 d}) \otimes L^2_s(\mathbb{R}^{n_2 d})$. Define the orthogonal projector $P_{\nu_1,\nu_2} \in \mathcal{L}(\Gamma_s(\mathscr{Z}))$, $\nu_1, \nu_2 \in \mathbb{N}$ by:

$$(P_{\nu_1,\nu_2}\phi)_{n_1,n_2} = \begin{cases} \phi_{\nu_1,n_2} & \text{if } n_1 = \nu_1 \text{ and } n_2 \leq \nu_2 \\ 0 & \text{otherwise} \end{cases} .$$

The operator H is symmetric; we will prove that $(\zeta - H)(D(H_0) \cap \mathscr{F}_0)$ is dense in $\Gamma_s(\mathscr{Z})$ for all $\zeta \in \mathbb{C}$ with $\mathrm{Im}\zeta \neq 0$. Let ζ such that $\mathrm{Im}\zeta \neq 0$; consider $\eta \in \Gamma_s(\mathscr{Z})$ such that for any $\phi \in D(H_0) \cap \mathscr{F}_0$:

$$\langle \eta, (\zeta - H)\phi \rangle = 0 \; . \tag{11}$$

If equation (11) holds only for $\eta = 0$, then $(\zeta - H)(D(H_0) \cap \mathscr{F}_0)$ is dense in $\Gamma_s(\mathscr{Z})$. Equation (11) also implies:

$$\langle \eta, H_0\phi \rangle = \zeta \langle \eta, \phi \rangle - \langle \eta, H_I \phi \rangle \; .$$

Let $n_1, n_2 \in \mathbb{N}$; we choose $\phi_{n_1,n_2} \in D(H_0|_{n_1,n_2})$ as ϕ ($H_0|_{n_1,n_2}$ is the restriction of H_0 to $L^2_s(\mathbb{R}^{n_1 d}) \otimes L^2_s(\mathbb{R}^{n_2 d})$). Then

$$\langle \eta_{n_1,n_2}, H_0 \phi_{n_1,n_2} \rangle = \zeta \langle \eta_{n_1,n_2}, \phi_{n_1,n_2} \rangle - \varepsilon \sum_{j=1}^{n_1} \Big(\langle \big(a(\omega^{-1/2}\chi e^{-ik\cdot x_j})\eta \big)_{n_1,n_2}, \phi_{n_1,n_2} \rangle + \langle \big(a^*(\omega^{-1/2}\chi e^{-ik\cdot x_j})\eta \big)_{n_1,n_2}, \phi_{n_1,n_2} \rangle \Big) \; .$$

Hence

$$|\langle \eta_{n_1,n_2}, H_0\phi_{n_1,n_2} \rangle| \leq \Big(|\zeta| \|\eta_{n_1,n_2}\| + \varepsilon^{3/2} n_1 (n_2+1)^{1/2} \|\omega^{-1/2}\chi\|_2 (\|\eta_{n_1,n_2-1}\| + \|\eta_{n_1,n_2+1}\|) \Big) \|\phi_{n_1,n_2}\| \; . \tag{12}$$

Since $\eta \in \Gamma_s(\mathscr{L})$, (12) implies $\eta_{n_1,n_2} \in D(H_0|_{n_1,n_2})$ for all $n_1, n_2 \in \mathbb{N}$. Then $P_{\nu_1,\nu_2}\eta \in D(H_0) \cap \mathscr{F}_0$, for all $\nu_1, \nu_2 \in \mathbb{N}$. Consider now equation (11); since it holds for all $\phi \in D(H_0) \cap \mathscr{F}_0$, we can choose $\phi = P_{\nu_1,\nu_2}\eta$. Then:

$$\mathrm{Im}\langle \eta, H_0 P_{\nu_1,\nu_2}\eta\rangle = (\mathrm{Im}\zeta)\|P_{\nu_1,\nu_2}\eta\|^2 - \varepsilon \mathrm{Im}\Bigl(\sum_{j=1}^{\nu_1}\langle \eta, \bigl(a^*(\omega^{-1/2}\chi e^{-ik\cdot x_j}) + a(\omega^{-1/2}\chi e^{-ik\cdot x_j})\bigr)P_{\nu_1,\nu_2}\eta\rangle\Bigr) \; .$$

P_{ν_1,ν_2} commutes with H_0, hence we obtain:

$$(\mathrm{Im}\zeta)\|P_{\nu_1,\nu_2}\eta\|^2 = \varepsilon \sum_{j=1}^{\nu_1} \mathrm{Im}\langle (1-P_{\nu_1,\nu_2})\eta, \bigl(a^*(\omega^{-1/2}\chi e^{-ik\cdot x_j}) + a(\omega^{-1/2}\chi e^{-ik\cdot x_j})\bigr)P_{\nu_1,\nu_2}\eta\rangle \; .$$

Now we use the following two facts: $a(f)P_{\nu_1,\nu_2} = P_{\nu_1,\nu_2-1}a(f)$, and $P_{\nu_1,\nu_2-1}(1-P_{\nu_1,\nu_2}) = 0$. Then:

$$\begin{aligned}(\mathrm{Im}\zeta)\|P_{\nu_1,\nu_2}\eta\|^2 &= \varepsilon \sum_{j=1}^{\nu_1} \mathrm{Im}\langle P_{\nu_1,\nu_2+1}(1-P_{\nu_1,\nu_2})\eta, a^*(\omega^{-1/2}\chi e^{-ik\cdot x_j})P_{\nu_1,\nu_2}\eta\rangle \\ &= \varepsilon \sum_{j=1}^{n_1} \mathrm{Im}\langle a(\omega^{-1/2}\chi e^{-ik\cdot x_j})\eta_{\nu_1,\nu_2+1}, \eta_{\nu_1,\nu_2}\rangle \; .\end{aligned}$$

Taking the absolute value we obtain:

$$|\mathrm{Im}\zeta|\,\|P_{\nu_1,\nu_2}\eta\|^2 \leq \varepsilon^{3/2}\nu_1(\nu_2+1)^{1/2}\|\omega^{-1/2}\chi\|_2\|\eta_{\nu_1,\nu_2+1}\|\,\|\eta_{\nu_1,\nu_2}\| \; ,$$

hence

$$\frac{1}{(\nu_2+1)^{1/2}}\sum_{n_2=0}^{\nu_2}\|\eta_{\nu_1,n_2}\|^2 \leq \varepsilon^{3/2}\frac{\nu_1}{|\mathrm{Im}\zeta|}\|\omega^{-1/2}\chi\|_2\frac{1}{2}\bigl(\|\eta_{\nu_1,\nu_2+1}\|^2 + \|\eta_{\nu_1,\nu_2}\|^2\bigr) \; .$$

We define now:

$$S := \sum_{n_2=0}^{\infty}\|\eta_{\nu_1,n_2}\|^2 = \|P_{\nu_1,\infty}\eta\|^2 \; ;$$

where $P_{\nu_1,\infty}$ is the orthogonal projector on $\mathscr{H}_{\nu_1}$. Then exists a $\bar{\nu}_2$ such that for all $\nu_2 \geq \bar{\nu}_2$:

$$\frac{1}{2}S \leq \sum_{n_2=0}^{\nu_2}\|\eta_{\nu_1,n_2}\|^2 \leq S \; .$$

So for all $\nu_2 \geq \bar{\nu}_2$:

$$\frac{1}{(\nu_2+1)^{1/2}}S \leq \varepsilon^{3/2}\frac{\nu_1}{|\mathrm{Im}\zeta|}\|\omega^{1/2}\chi\|_2\bigl(\|\eta_{\nu_1,\nu_2+1}\|^2 + \|\eta_{\nu_1,\nu_2}\|^2\bigr) \; ;$$

taking now the sum in ν_2 it becomes:

$$S\sum_{\nu_2=\bar{\nu}_2}^{\bar{\nu}_2'}\frac{1}{(\nu_2+1)^{1/2}} \leq 2S\varepsilon^{3/2}\frac{\nu_1}{|\mathrm{Im}\zeta|}\|\omega^{-1/2}\chi\|_2 \; ,$$

for all $\bar{\nu}_2' \geq \bar{\nu}_2$. If $S \neq 0$, we have an absurd, since $\sum_{\nu_2\geq\bar{\nu}_2}(\nu_2+1)^{-1/2}$ is divergent. It follows that $(\forall \nu_1 \in \mathbb{N}, P_{\nu_1,\infty}\eta = 0) \Leftrightarrow \eta = 0$. □

Finally, we describe some properties of H and the corresponding evolution $e^{-i\frac{t}{\varepsilon}H}$ in mapping domains of particular operators in $\mathscr{H}$.

Proposition 2.6. *Assume that* (**A**) *holds. Then:*

i) $D(H_0) \cap D(N_1^2 + N_2) \subseteq D(H)$.

ii) $D(H) \cap D(N_1^2 + N_2) \subseteq D(H_0)$.

iii) Let $\delta \in \mathbb{R}$, $t \in \mathbb{R}$ *and* $m_\delta(\varepsilon) := \max\{2+\varepsilon, 1+(1+\varepsilon)^\delta\}$. *Then for any* $\phi \in \mathscr{H}$*:*

$$\|(N_1^2 + N_2 + \varepsilon)^\delta e^{-i\frac{t}{\varepsilon}H}(N_1^2 + N_2 + \varepsilon)^{-\delta}\phi\| \leq e^{m_\delta(\varepsilon)\sqrt{\varepsilon}|\delta||t|\|\omega^{-1/2}\chi\|_2}\|\phi\| \ .$$

Proof. i) From $H = H_0 + H_I$ we obtain $\|H\phi\| \leq \|H_0\phi\| + \|\omega^{-1/2}\chi\|_2\|(N_1^2+N_2+\varepsilon)\phi\|$ by Lemma 2.1.
ii) From $H_0 = H - H_I$ we obtain $\|H_0\phi\| \leq \|H\phi\| + \|\omega^{-1/2}\chi\|_2\|(N_1^2 + N_2 + \varepsilon)\phi\|$.
iii) We define, for $\delta < -1/2$, $M(t) := \|(N_1^2+N_2+\varepsilon)^\delta e^{-i\frac{t}{\varepsilon}H}\phi\|$. The result is then an application of Gronwall's lemma on $\mathscr{H}_n$, taking the derivative on a suitable domain. The result is then extended, by density, to all vectors of $\mathscr{H}$. Interpolating between $\delta = -1$ and $\delta = 0$ we obtain the result for all $\delta \leq 0$; by duality we conclude the proof for all real δ [see 13, Proposition IV.2]. □

b. Classical system.

In this part we are concerned with the following partial differential equation on the phase space $\mathscr{Z} = L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$:

$$\left\{\begin{aligned} &i\partial_t z_1 = \Big(-\frac{1}{2M}\Delta + V\Big)z_1 + \Big(\int_{\mathbb{R}^d}\frac{\chi(k)}{\sqrt{\omega(k)}}\big(\bar{z}_2(k)e^{-ik\cdot x} + z_2(k)e^{ik\cdot x}\big)dk\Big)z_1 \\ &i\partial_t z_2 = \omega z_2 + \omega^{-1/2}\chi\int_{\mathbb{R}^d}e^{-ik\cdot x}\bar{z}_1(x)z_1(x)dx \end{aligned}\right. . \tag{13}$$

This system describes a coupled Klein-Gordon ($m_0 > 0$)/Wave ($m_0 = 0$)-Schrödinger equation; it is the classical dynamics limit of the Nelson model. In this form the second equation does not seem a Klein-Gordon/Wave equation, however rewriting it for $A := \int_{\mathbb{R}^d}\frac{\chi(k)}{\sqrt{\omega(k)}}\big(\bar{z}_2(k)e^{-ik\cdot x} + z_2(k)e^{ik\cdot x}\big)dk$, we obtain the more usual form: $(\Box + m_0^2)A = -(2\pi)^{-d/2}\mathcal{F}^{-1}(\chi) * |z_1|^2$.

In the case where the ultraviolet cut off is removed (i.e.: $\chi = 1$), we obtain a coupled system with an Yukawa interaction. This latter PDE has attracted a lot of attention, see e.g. [9, 17, 19, 24].

Proposition 2.7. *Assume* (**A**) *holds; and let* $\mathscr{Z} \ni z^0 := z_1^0 \oplus z_2^0$. *Then the Cauchy problem:*

$$\begin{cases} i\partial_t z_1 = \Big(-\frac{1}{2M}\Delta + V\Big) z_1 + \Big(\int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} \big(\bar{z}_2(k) e^{-ik\cdot x} + z_2(k) e^{ik\cdot x}\big) dk\Big) z_1 \\ i\partial_t z_2 = \omega z_2 + \omega^{-1/2}\chi \int_{\mathbb{R}^d} e^{-ik\cdot x} \bar{z}_1(x) z_1(x) dx \end{cases} \qquad \begin{cases} z_1(s) = z_1^0 \\ z_2(s) = z_2^0 \end{cases}$$

admits an unique global solution in $\mathscr{C}^0(\mathbb{R}, \mathscr{Z})$.

Proof. Local existence is proved by means of a fixed point argument. This solution is then extended globally using the conservation of $\|z_1\|_2$ [see 13, Proposition III.1]. □

Define now the flow $\Phi(t,s)$ on $\mathscr{Z}$ as:

$$\Phi(t,s)z(s) := \begin{pmatrix} e^{-i(t-s)(-\frac{\Delta}{2M}+V)} & 0 \\ 0 & e^{-i(t-s)\omega} \end{pmatrix} z(s) - i \int_s^t \begin{pmatrix} e^{-i(t-\tau)(-\frac{\Delta}{2M}+V)} & 0 \\ 0 & e^{-i(t-\tau)\omega} \end{pmatrix} \begin{pmatrix} \Phi_1(z(\tau)) \\ \Phi_2(z(\tau)) \end{pmatrix} d\tau \,, \tag{14}$$

with $z(\tau)$, $\tau \in [s,t]$, the $\mathscr{C}^0(\mathbb{R}, \mathscr{Z})$-solution of the Cauchy problem of Proposition 2.7, and

$$\Phi_1(z(t)) := \Big(\int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} \big(\bar{z}_2(t,k) e^{-ik\cdot x} + z_2(t,k) e^{ik\cdot x}\big) dk\Big) z_1(t,x)$$
$$\Phi_2(z(t)) := \omega^{-1/2}(k)\chi(k) \int_{\mathbb{R}^d} e^{-ik\cdot x} \bar{z}_1(t,x) z_1(t,x) dx \,.$$

The Klein-Gordon-Schrödinger equation is a Hamiltonian system and therefore (13) can be written in a more compact way, namely

$$i\partial_t z = \partial_{\bar{z}} h(z) \,, \tag{15}$$

with $h(z), z \in \mathscr{Z}$, the classical hamiltonian given by $h(z) = h_0(z) + h_I(z)$; with

$$h_0(z) = \langle z_1, (-\frac{\Delta}{2M} + V) z_1 \rangle + \langle z_2, \omega(k) z_2 \rangle \,,$$
$$h_I(z) = \int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}} |z_1|^2(x) \big(\bar{z}_2(k) e^{-ik\cdot x} + z_2(k) e^{ik\cdot x}\big) dk dx \,.$$

Define the free flow

$$\Phi_0(t,s) = \Phi_0(t-s) = \begin{pmatrix} e^{-i(t-s)(-\frac{\Delta}{2M}+V)} & 0 \\ 0 & e^{-i(t-s)\omega} \end{pmatrix} .$$

The classical field equation (15) can be written on the interaction representation:

$$\partial_t \tilde{z} = \mathscr{V}_s(\tilde{z}) = -i\Phi_0(-t)\partial_{\bar{z}} h_I(\Phi_0(t)\tilde{z}) \,; \tag{16}$$

with the (velocity) vector field $\mathscr{V}_s$ continuous on $\mathscr{Z}$ and satisfying the estimate:

$$\|\mathscr{V}_s(z)\|_{\mathscr{Z}} \le 2\|\omega^{-1/2}\chi\|_2 \|z_1\|_2 \big(\|z_1\|_2 + \|z_2\|_2\big) \,. \tag{17}$$

Proposition 2.8. *Assume* (**A**) *holds. Then for all* $t, s \in \mathbb{R}$, $\Phi(t,s)$ *given by* (14) *is the well defined global continuous flow on* $\mathscr{Z} = L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ *of the Klein-Gordon-Schrödinger equation* (13)*.*

Proof. Let $z(s)$, $s \in \mathbb{R}$, be in $\mathscr{Z}$ and $z(t)$ be the unique global $\mathscr{C}^0(\mathbb{R}, \mathscr{Z})$-solution of the Cauchy problem of Proposition 2.7. Then $\mathscr{Z} \ni z(t) = \Phi(t,s)z(s)$. □

3. TRACE OF STATES.

First of all we recall some definitions. Let ϱ_ε be a positive trace class operator with $\mathrm{Tr}[\varrho_\varepsilon] = 1$ (the conditions it have to satisfy will be discussed later); then we define

$$\varrho_\varepsilon(t) := e^{-i\frac{t}{\varepsilon}H}\varrho_\varepsilon e^{+i\frac{t}{\varepsilon}H} \; ,$$
$$\tilde{\varrho}_\varepsilon(t) := e^{+i\frac{t}{\varepsilon}H_0}\varrho_\varepsilon(t) e^{-i\frac{t}{\varepsilon}H_0} \; .$$

We denote by $\mathcal{L}^1(\mathscr{H})$ the space of trace class operators on $\mathscr{H}$. Also, let $\mathscr{Z} \ni \xi = \xi_1 \oplus \xi_2$. Then we define the Weyl operator

$$W(\xi) := e^{i\frac{\psi^*(\xi_1)+\psi(\xi_1)}{\sqrt{2}}} \otimes e^{i\frac{a^*(\xi_2)+a(\xi_2)}{\sqrt{2}}} = W(\xi_1) \otimes W(\xi_2) \; .$$

We have used here the representation of $\Gamma_s(\mathscr{Z})$ as the tensor product $\Gamma_s(L^2(\mathbb{R}^d)) \otimes \Gamma_s(L^2(\mathbb{R}^d))$; we will use freely the more suitable representation of the two, depending on the context. Finally, let $\mathscr{Z} \ni z = z_1 \oplus z_2$, and $\Phi_0(t)$ be the free flow on $\mathscr{Z}$, defined above. Then we have

$$\tilde{z}(s) = \Phi_0(s) z = \begin{pmatrix} e^{-is(-\frac{\Delta}{2M}+V)} & 0 \\ 0 & e^{-is\omega} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \; .$$

In this section we give a rigorous derivation of the following formula, crucial in the analysis of the limit $\varepsilon \to 0$:

$$\mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t) W(\xi)\Big] = \mathrm{Tr}\Big[\varrho_\varepsilon W(\xi)\Big] + \frac{i}{\varepsilon}\int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)[H_I, W(\tilde{\xi}(s))]\Big] ds \; . \tag{18}$$

Furthermore, we will give a characterization of the terms in the commutator $[H_I, W(\tilde{\xi}(s))]$.

Remark. The estimates on this section are made more precise than what we need, for a possible derivation of a quantitative rate of convergence.

a. Derivation of the integral formula.

For convenience, let $\boxed{T := N_1^2 + N_2 + 1}$ and $\boxed{S := H_0 + T}$. Then we make the following definition:

Definition 3.1 ($\mathcal{S}_\varepsilon^\delta$, $\mathcal{T}_\varepsilon^\delta$)**.** Let $\varrho_\varepsilon \in \mathcal{L}^1(\mathscr{H})$, $\varepsilon > 0, \delta \in \mathbb{R}$. Then

$$\varrho_\varepsilon \in \mathcal{S}_\varepsilon^\delta \Leftrightarrow |\varrho_\varepsilon|_{\mathcal{S}_\varepsilon^\delta} := |\varrho_\varepsilon S^\delta|_{\mathcal{L}^1(\mathscr{H})} < +\infty \; ;$$
$$\varrho_\varepsilon \in \mathcal{T}_\varepsilon^\delta \Leftrightarrow |\varrho_\varepsilon|_{\mathcal{T}_\varepsilon^\delta} := |\varrho_\varepsilon T^\delta|_{\mathcal{L}^1(\mathscr{H})} < +\infty \; .$$

Define now the subspace $\mathscr{Z}_1 \subset \mathscr{Z}$ as:

(19) $$\mathscr{Z}_1 := \{\mathscr{Z} \ni z = z_1 \oplus z_2 : z_1 \in H^2(\mathbb{R}^d) \text{ and } \omega z_2 \in L^2(\mathbb{R}^d)\} \; .$$

In order to prove (18) we need some preparatory results proved in Appendix A. The Corollary A.2, adapted to our spaces $\mathscr{Z}$ and $\Gamma_s(\mathscr{Z})$, becomes:

Lemma 3.2. *i) Let $\xi \in \mathscr{Z}_1$. Then $S^{-1}W(\xi)S \in \mathcal{L}(\mathscr{H})$. Furthermore, there exists $C > 0$ such that*

$$|S^{-1}W(\xi)S|_{\mathcal{L}(\mathscr{H})} \leq C\Big(1 + \varepsilon\|\xi\|_{\mathscr{Z}_1} + \varepsilon^2\|\xi\|_{\mathscr{Z}_1}^2 + \varepsilon^3\|\xi_1\|_2^3 + \varepsilon^4\|\xi_1\|_2^4\Big) \; .$$

ii) Let $\xi \in \mathscr{Z}$. Then for any $\delta \in \mathbb{R}$, $T^{-\delta}W(\xi)T^\delta \in \mathcal{L}(\mathscr{H})$. Furthermore, there exists $C(\delta, \|\xi\|_{\mathscr{Z}}) > 0$ such that

$$|T^{-\delta}W(\xi)T^\delta|_{\mathcal{L}(\mathscr{H})} \leq C(\delta, \|\xi\|_{\mathscr{Z}})(1 + O(\varepsilon)) \; .$$

If $\delta = 1$, there exists $C > 0$ such that

$$|T^{-1}W(\xi)T|_{\mathcal{L}(\mathscr{H})} \leq C\Big(1 + \varepsilon\|\xi\|_{\mathscr{Z}} + \varepsilon^2\|\xi\|_{\mathscr{Z}}^2 + \varepsilon^3\|\xi_1\|_2^3 + \varepsilon^4\|\xi_1\|_2^4\Big) \; .$$

Next we consider the operators $T^{-\delta}e^{-i\frac{t}{\varepsilon}H}T^\delta$ and $S^{-1}e^{-i\frac{t}{\varepsilon}H}S$.

Lemma 3.3. *Let $\delta \in \mathbb{R}$. Then for all $t \in \mathbb{R}$, $T^{-\delta}e^{-i\frac{t}{\varepsilon}H}T^\delta \in \mathcal{L}(\mathscr{H})$. Furthermore, there exists $C(\delta, t, \|\omega^{-1/2}\chi\|_2) > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$:*

$$|T^{-\delta}e^{-i\frac{t}{\varepsilon}H}T^\delta|_{\mathcal{L}(\mathscr{H})} \leq C(\delta, t, \|\omega^{-1/2}\chi\|_2) \; .$$

Proof. Let $\delta \in \mathbb{N}$. We recall that, for any $a \geq 0$, $n_1, n_2 \in \mathbb{N}$:

$$(n_1^2 + n_2 + a)^\delta \leq (1 + 2^\delta\tilde{a})(n_1^2 + n_2)^\delta + a^\delta \; ,$$

where $\tilde{a} = \max\{a, a^{\delta-1}\}$. Hence, using Proposition 2.6, we obtain

$$\begin{aligned}\|(N_1^2 + N_2 + 1)^\delta e^{-i\frac{t}{\varepsilon}H}\phi\| \leq (1 + 2^\delta)\|(N_1^2 + N_2)^\delta e^{-i\frac{t}{\varepsilon}H}\phi\| + \|\phi\| \leq (1 + 2^\delta)e^{m_\delta(\varepsilon)\sqrt{\varepsilon}|\delta||t|\|\omega^{-1/2}\chi\|_2} \\ \|(N_1^2 + N_2 + \varepsilon)^\delta\phi\| + \|\phi\| \leq (1 + 2^\delta)(1 + 2^\delta\tilde{\varepsilon})e^{m_\delta(\varepsilon)\sqrt{\varepsilon}|\delta||t|\|\omega^{-1/2}\chi\|_2}\|(N_1^2 + N_2 + 1)^\delta\phi\| \\ + \Big((1 + 2^\delta)\varepsilon^\delta e^{m_\delta(\varepsilon)\sqrt{\varepsilon}|\delta||t|\|\omega^{-1/2}\chi\|_2} + 1\Big)\|\phi\| \; .\end{aligned}$$

Then for all $\delta \in \mathbb{Z}$:

$$|T^{-\delta} e^{-i\frac{t}{\varepsilon}H} T^{\delta}|_{\mathcal{L}(\mathscr{H})} \leq \Big(1 + (1+2^{|\delta|})\big(1 + 2^{|\delta|}\max\{\varepsilon, \varepsilon^{|\delta|-1}\} + \varepsilon^{|\delta|}\big) e^{m_\delta(\varepsilon)\sqrt{\varepsilon}|\delta||t|\|\omega^{-1/2}\chi\|_2}\Big) . \tag{20}$$

The result is extended by interpolation to all $\delta \in \mathbb{R}$. □

Lemma 3.4. *For all $t \in \mathbb{R}$, $S^{-1} e^{-i\frac{t}{\varepsilon}H} S \in \mathcal{L}(\mathscr{H})$.*

Proof. Let $\phi_1, \phi_2 \in \mathscr{H}$. Then using Lemma 3.3 we obtain:

$$\begin{aligned} |\langle \phi_1, S^{-1} e^{i\frac{t}{\varepsilon}H} S\phi_2\rangle| = |\langle S(H+T)^{-1}(H+T)e^{-i\frac{t}{\varepsilon}H} S^{-1}\phi_1, \phi_2\rangle| &\leq |\langle S(H+T)^{-1} e^{-i\frac{t}{\varepsilon}H} HS^{-1}\phi_1, \phi_2\rangle| \\ &+ \Big(1 + (9+9\varepsilon)e^{(2+\varepsilon)\sqrt{\varepsilon}|t|\|\omega^{-1/2}\chi\|_2}\Big)\|TS^{-1}\phi_1\|\|S(H+T)^{-1}\phi_2\| \\ &\leq \Big(\|\omega^{-1/2}\chi\|_2 + 1 + (9+9\varepsilon)e^{(2+\varepsilon)\sqrt{\varepsilon}|t|\|\omega^{-1/2}\chi\|_2}\Big)\|\phi_1\|\|\phi_2\| . \end{aligned}$$

□

We are now ready to prove the integral formula (18).

Proposition 3.5. *Assume that* (**A**) *holds; and let $\xi \in \mathscr{Z}$, $\tilde{\xi}(s) = \Phi_0(s)\xi$. Then for all $\varrho_\varepsilon \in \mathcal{T}^1_\varepsilon$:*

$$\mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t) W(\xi)\Big] = \mathrm{Tr}\Big[\varrho_\varepsilon W(\xi)\Big] + \frac{i}{\varepsilon}\int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)[H_I, W(\tilde{\xi}(s))]\Big] ds .$$

Proof. The formula is proved for ξ in $\mathscr{Z}_1$; and for $\varrho_\varepsilon \in \mathcal{S}^1_\varepsilon$. The result is then extended by density ($\mathcal{S}^1_\varepsilon$ is dense in $\mathcal{T}^1_\varepsilon$ in the $\mathcal{L}^1(\mathscr{H})$ topology).

If we are able to differentiate, in t, $\mathrm{Tr}[\tilde{\varrho}_\varepsilon(t)W(\xi)]$ we are done. Consider then, for all $t, s \in \mathbb{R}$:

$$\begin{aligned} \mathrm{Tr}\Big[\big(\tilde{\varrho}_\varepsilon(t) - \tilde{\varrho}_\varepsilon(s)\big)W(\xi)\Big] &= \mathrm{Tr}\Big[\big(e^{i\frac{t}{\varepsilon}H_0}e^{-i\frac{t}{\varepsilon}H} - e^{i\frac{s}{\varepsilon}H_0}e^{-i\frac{s}{\varepsilon}H}\big)\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H}e^{-i\frac{t}{\varepsilon}H_0}W(\xi)\Big] \\ &+ \mathrm{Tr}\Big[e^{i\frac{s}{\varepsilon}H_0}e^{-i\frac{s}{\varepsilon}H}\varrho_\varepsilon\big(e^{i\frac{t}{\varepsilon}H}e^{-i\frac{t}{\varepsilon}H_0} - e^{i\frac{s}{\varepsilon}H}e^{-i\frac{s}{\varepsilon}H_0}\big)W(\xi)\Big] \\ &= \mathrm{Tr}\Big[\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H}e^{-i\frac{t}{\varepsilon}H_0}W(\xi)\big(e^{i\frac{t}{\varepsilon}H_0}e^{-i\frac{t}{\varepsilon}H} - e^{i\frac{s}{\varepsilon}H_0}e^{-i\frac{s}{\varepsilon}H}\big)\Big] \\ &+ \mathrm{Tr}\Big[e^{i\frac{s}{\varepsilon}H_0}e^{-i\frac{s}{\varepsilon}H}\varrho_\varepsilon\big(e^{i\frac{t}{\varepsilon}H}e^{-i\frac{t}{\varepsilon}H_0} - e^{i\frac{s}{\varepsilon}H}e^{-i\frac{s}{\varepsilon}H_0}\big)W(\xi)\Big] \\ = \mathrm{Tr}\Big[\varrho_\varepsilon SS^{-1}e^{i\frac{t}{\varepsilon}H}SS^{-1}e^{-i\frac{t}{\varepsilon}H_0}SS^{-1}W(\xi)SS^{-1}&\big(e^{i\frac{t}{\varepsilon}H_0}e^{-i\frac{t}{\varepsilon}H} - e^{i\frac{s}{\varepsilon}H_0}e^{-i\frac{s}{\varepsilon}H}\big)\Big] \\ + \mathrm{Tr}\Big[e^{i\frac{s}{\varepsilon}H_0}e^{-i\frac{s}{\varepsilon}H}\varrho_\varepsilon SS^{-1}&\big(e^{i\frac{t}{\varepsilon}H}e^{-i\frac{t}{\varepsilon}H_0} - e^{i\frac{s}{\varepsilon}H}e^{-i\frac{s}{\varepsilon}H_0}\big)W(\xi)\Big] . \end{aligned}$$

Every operation is justified since $\varrho_\varepsilon \in \mathcal{L}^1(\mathscr{H})$ and $e^{-i\frac{t}{\varepsilon}H}, e^{-i\frac{t}{\varepsilon}H_0}, W(\xi) \in \mathcal{L}(\mathscr{H})$. Now recall that $\varrho_\varepsilon \in \mathcal{S}^1_\varepsilon$ and also $S^{-1}e^{-i\frac{t}{\varepsilon}H}S, S^{-1}e^{-i\frac{t}{\varepsilon}H_0}S, S^{-1}W(\xi)S \in \mathcal{L}(\mathscr{H})$ by Lemmas 3.2 and 3.4 and since S commutes with H_0. Then we can just look at the limits:

$$\begin{aligned} \lim_{s\to t}\frac{1}{t-s}S^{-1}\big(e^{i\frac{t}{\varepsilon}H_0}e^{-i\frac{t}{\varepsilon}H} - e^{i\frac{s}{\varepsilon}H_0}e^{-i\frac{s}{\varepsilon}H}\big) &= -S^{-1}e^{i\frac{t}{\varepsilon}H_0}H_I e^{-i\frac{t}{\varepsilon}H} \\ \lim_{s\to t}\frac{1}{t-s}S^{-1}\big(e^{i\frac{t}{\varepsilon}H}e^{-i\frac{t}{\varepsilon}H_0} - e^{i\frac{s}{\varepsilon}H}e^{-i\frac{s}{\varepsilon}H_0}\big) &= S^{-1}e^{i\frac{t}{\varepsilon}H}H_I e^{-i\frac{t}{\varepsilon}H_0} . \end{aligned}$$

The convergence is intended in the strong topology, and we have used Stone's theorem. The result is finally obtained using the fact that

$$e^{-i\frac{t}{\varepsilon}H_0}W(\xi)e^{i\frac{t}{\varepsilon}H_0} = W(\tilde{\xi}(t)) \; .$$

□

b. The commutator $[H_I, W(\tilde{\xi}(s))]$.

Now, once the integral formula is proved, we want to give an explicit form of the commutator $[H_I, W(\tilde{\xi}(s))]$, in particular with respect to the dependence on ε, since we are interested in the limit $\varepsilon \to 0$.

Lemma 3.6. *For all $\delta \in \mathbb{R}$ and $t \in \mathbb{R}$: $\varrho_\varepsilon \in \mathcal{T}_\varepsilon^\delta \Leftrightarrow \varrho_\varepsilon(t) \in \mathcal{T}_\varepsilon^\delta$.*

Proof. The free Hamiltonian H_0 commutes with T, hence the result is a direct application of Lemma 3.3. □

The next lemma can be proved using a general result on Wick quantized operators [see 4], or with a strategy similar to the one used in Lemma A.1.

Lemma 3.7. *On $D(T)$ the following equality holds strongly for any $\xi \in \mathscr{Z}$:*

$$\begin{aligned} B_\varepsilon(\xi) :&= W^*(\xi)H_I W(\xi) \\ &= \int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}} \big(\psi^*(x) - i\frac{\varepsilon}{\sqrt{2}}\bar{\xi}_1(x)\big)\Big(\big(a^*(k) - i\frac{\varepsilon}{\sqrt{2}}\bar{\xi}_2(k)\big)e^{-ik\cdot x} \\ &+ \big(a(k) + i\frac{\varepsilon}{\sqrt{2}}\xi_2(k)\big)e^{ik\cdot x}\Big)\big(\psi(x) + i\frac{\varepsilon}{\sqrt{2}}\xi_1(x)\big)dxdk \; . \end{aligned}$$

Corollary 3.8. *For all $\varrho_\varepsilon \in \mathcal{T}_\varepsilon^1$ and $s \in \mathbb{R}$:*

$$\mathrm{Tr}\Big[\varrho_\varepsilon(s)[H_I, W(\tilde{\xi}(s))]\Big] = \mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s))\big(B_\varepsilon(\tilde{\xi}(s)) - H_I\big)\Big] \; .$$

Now we would like to write

$$\frac{i}{\varepsilon}(B_\varepsilon(\tilde{\xi}(s)) - H_I) = \sum_{j=0}^{r} \varepsilon^j B_j(\tilde{\xi}(s)) \tag{21}$$

for some $r \in \mathbb{N}$. This can be easily done, with $r = 2$, obtaining:

$$\begin{aligned} B_0(\xi) = \frac{i}{\sqrt{2}} \int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}} \Big[&\psi^*(x)\big(a^*(k)e^{-ik\cdot x} + a(k)e^{ik\cdot x}\big)\xi_1(x) - \psi(x)\big(a^*(k)e^{-ik\cdot x} + a(k) \\ &e^{ik\cdot x}\big)\bar{\xi}_1(x) + \psi^*(x)\psi(x)\big(\bar{\xi}_2 e^{-ik\cdot x} - \xi_2 e^{ik\cdot x}\big)\Big]dxdk \; ; \end{aligned} \tag{22}$$

$$B_1(\xi) = \frac{1}{2}\int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}}\Big[\psi^*(x)\xi_1(x)\big(\bar{\xi}_2(k)e^{-ik\cdot x} - \xi_2(k)e^{ik\cdot x}\big) + \psi(x)\bar{\xi}_1(x)\big(\xi_2(k)e^{ik\cdot x} - \bar{\xi}_2(k) e^{-ik\cdot x}\big) + \big(a^*(k)e^{-ik\cdot x} + a(k)e^{ik\cdot x}\big)\bar{\xi}_1(x)\xi_1(x)\Big]dxdk\ ; \tag{23}$$

$$B_2(\xi) = \frac{i}{2\sqrt{2}}\int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}}\bar{\xi}_1(x)\xi_1(x)\big(\xi_2(k)e^{ik\cdot x} - \bar{\xi}_2(k)e^{-ik\cdot x}\big)dxdk\ . \tag{24}$$

We can sum up these results in the following proposition:

Proposition 3.9. *Assume* (**A**) *holds; and let* $\xi \in \mathscr{Z}$. *Then for all* $\varrho_\varepsilon \in \mathcal{T}^1_\varepsilon$:

$$\mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t)W(\xi)\Big] = \mathrm{Tr}\Big[\varrho_\varepsilon W(\xi)\Big] + \sum_{j=0}^{2}\varepsilon^j\int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s))B_j(\tilde{\xi}(s))\Big]ds\ ; \tag{25}$$

where the $B_j(\tilde{\xi}(s))$ *are given in* (22)-(24).

Finally, we give a bound on B_1 and B_2, and a more detailed characterization of B_0 (since it is the term which will have non zero limit when $\varepsilon \to 0$). We start with the bound on B_1 and B_2:

Proposition 3.10. *Assume that* (**A**) *holds; and let* $\xi \in \mathscr{Z}$, $s \in [0,t]$, $t \in \mathbb{R}$. *Then for any* $\varrho_\varepsilon \in \mathcal{T}^1_\varepsilon$, *there exists* $C(s, \|\omega^{-1/2}\chi\|_2) > 0$ *such that for all* $\varepsilon \in (0,\bar{\varepsilon})$:

$$\Big|\sum_{j=1}^{2}\varepsilon^j\int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s))B_j(\tilde{\xi}(s))\Big]ds\Big| \leq \varepsilon\Big(1 + \varepsilon\|\xi\|_{\mathscr{Z}} + (\varepsilon+\varepsilon^2)\|\xi\|^2_{\mathscr{Z}} + \varepsilon^3\|\xi_1\|_2^3 + (\varepsilon^2+\varepsilon^4)\|\xi_1\|_2^4\Big)\int_0^t C(s,\|\omega^{-1/2}\chi\|_2)ds\ |\varrho_\varepsilon|_{\mathcal{T}^1_\varepsilon}\ .$$

Proof. We have that:

$$\Big|\sum_{j=1}^{2}\varepsilon^j\int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s))B_j(\tilde{\xi}(s))\Big]ds\Big| \leq \sum_{j=1}^{2}\varepsilon^j\int_0^t\Big|\mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s))B_j(\tilde{\xi}(s))\Big]\Big|ds$$
$$\leq \sum_{j=1}^{2}\varepsilon^j\int_0^t |T^{-1}B_j(\tilde{\xi}(s))|_{\mathcal{L}(\mathscr{H})}|T^{-1}W(\tilde{\xi}(s))T|_{\mathcal{L}(\mathscr{H})}|T^{-1}e^{i\frac{s}{\varepsilon}H}T|_{\mathcal{L}(\mathscr{H})}|\varrho_\varepsilon|_{\mathcal{T}^1_\varepsilon}ds\ .$$

Now, since $\|\tilde{\xi}(s)\|_{\mathscr{Z}} = \|\xi\|_{\mathscr{Z}}$, we obtain:

$$\sum_{j=1}^{2}\varepsilon^j|T^{-1}B_j(\tilde{\xi}(s))|_{\mathcal{L}(\mathscr{H})} \leq 2\varepsilon\|\omega^{-1/2}\chi\|_2\Big(2\|\xi_1\|_2(\frac{1}{2}\|\xi_1\|_2 + \|\xi_2\|_2) + \varepsilon\|\xi_1\|_2^2\|\xi_2\|_2\Big)\ .$$

Also, using Lemmas 3.2 and 3.3:

$$|T^{-1}W(\tilde{\xi}(s))T|_{\mathcal{L}(\mathscr{H})} \leq C\Big(1 + \varepsilon\|\xi\|_{\mathscr{Z}} + \varepsilon^2\|\xi\|^2_{\mathscr{Z}} + \varepsilon^3\|\xi_1\|_2^3 + \varepsilon^4\|\xi_1\|_2^4\Big)\ ;$$
$$|T^{-1}e^{-i\frac{s}{\varepsilon}H}T^1|_{\mathcal{L}(\mathscr{H})} \leq C(\delta=1,s,\|\omega^{-1/2}\chi\|_2)(1+O(\varepsilon))\ .$$

Hence we conclude the proof by choosing a suitable constant $C(s,\|\omega^{-1/2}\chi\|_2)$. □

Now we analyse in detail B_0. We write it as:

$$B_0 := B_{-,-} + B_{-,+} + B_{+,-} + B_{+,+} + B_{+-,0} \ ,$$

with

$$B_{-,-}(\tilde{\xi}(s)) = \int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}} \psi(x) a(k) e^{ik\cdot x} \bar{\tilde{\xi}}_1(s,x) dx dk \ ,$$

$$B_{-,+}(\tilde{\xi}(s)) = \int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}} \psi(x) a^*(k) e^{-ik\cdot x} \bar{\tilde{\xi}}_1(s,x) dx dk \ ,$$

$$B_{+,+}(\tilde{\xi}(s)) = \big(B_{-,-}(\tilde{\xi}(s))\big)^* \ ,$$

$$B_{+,-}(\tilde{\xi}(s)) = \big(B_{-,+}(\tilde{\xi}(s))\big)^* \ ,$$

$$B_{+-,0}(\tilde{\xi}(s)) = \int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}} \psi^*(x)\psi(x) \Big(\tilde{\xi}_2(s,k) e^{ik\cdot x} - \bar{\tilde{\xi}}_2(s,k) e^{-ik\cdot x}\Big) dx dk \ .$$

We want to interpret these operators as the Wick quantization of some symbol on $\mathscr{Z}$. A detailed description of Wick quantization in Fock space is given in [4]. We can write a symbol $b(z)$, $z \in \mathscr{Z}$, corresponding to the product of q creation and p annihilation operators, as a sesquilinear form on $(\mathscr{Z}^{\otimes_s^q}) \times (\mathscr{Z}^{\otimes_s^p})$. Hence we associate with it an operator $\tilde{b}$ from $\mathscr{Z}^{\otimes_s^p}$ into $\mathscr{Z}^{\otimes_s^q}$. A special role is played by the symbols with compact $\tilde{b}$ (we will call them compact symbols), since their Wick quantization can be approximated by some Weyl or Anti-Wick quantization with an $O(\varepsilon)$ error.

Apart from $B_{+-,0}$, all the operators written above turn out to have finite rank (compact) symbol, as stated in the following proposition.

Proposition 3.11. *Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^d)$. Then the following statements are true for any $s \in \mathbb{R}$:*

i) $B_{-,-}(\tilde{\xi}(s)) = b_{-,-}(z)^{Wick}$ *with* $b_{-,-}(z) = \langle \tilde{b}_{-,-}(\tilde{\xi}(s)), (z)^{\otimes_2}\rangle_{\mathscr{Z}^{\otimes_2}}$. *Furthermore*

$$\langle \tilde{b}_{-,-}(\tilde{\xi}(s)),\, \cdot\,\rangle_{\mathscr{Z}^{\otimes_2}} = \sum_{i,j\in\mathbb{N}} \langle \omega^{-1/2}(k)\chi(k)e^{ik\cdot x}\tilde{\xi}_1(s,x), e_i\otimes e_j\rangle_{L^2(\mathbb{R}^{2d})} \langle \left(\begin{smallmatrix} e_i \\ 0\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 \\ e_j\end{smallmatrix}\right) , \,\cdot\,\rangle_{\mathscr{Z}^{\otimes_2}}$$

is a finite rank operator from $\mathscr{Z}^{\otimes_2}$ *to* $\mathbb{C}$ *(since* $\mathbb{C}$ *is spanned by a single vector).*

ii) $B_{+,+}(\tilde{\xi}(s)) = b_{+,+}(z)^{Wick}$ *with* $b_{+,+}(z) = \langle (z)^{\otimes_2}, \tilde{b}_{+,+}(\tilde{\xi}(s))(z)^{\otimes_0}\rangle_{\mathscr{Z}^{\otimes_2}}$. *Furthermore*

$$\tilde{b}_{+,+}(\tilde{\xi}(s)) = \sum_{i,j\in\mathbb{N}} \langle e_i\otimes e_j, \omega^{-1/2}(k)\chi(k)e^{-ik\cdot x}\tilde{\xi}_1(s,x)\rangle_{L^2(\mathbb{R}^{2d})} \left(\begin{smallmatrix} e_i \\ 0\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 \\ e_j\end{smallmatrix}\right)$$

is a finite rank operator from $\mathbb{C}$ *to* $\mathscr{Z}^{\otimes_2}$.

iii) $B_{-,+}(\tilde{\xi}(s)) = b_{-,+}(z)^{Wick}$ *with* $b_{-,+}(z) = \langle z, \tilde{b}_{-,+}(\tilde{\xi}(s))z\rangle_{\mathscr{Z}}$. *Furthermore*

$$\tilde{b}_{-,+}(\tilde{\xi}(s)) = \sum_{i,j\in\mathbb{N}} \langle \omega^{-1/2}(k)\chi(k)e^{-ik\cdot x}\tilde{\xi}_1(s,x), e_i \otimes e_j\rangle_{L^2(\mathbb{R}^{2d})}|0\oplus \bar{e}_j\rangle\langle e_i \oplus 0|$$

is a Hilbert-Schmidt operator on $\mathscr{Z}$.

iv) $B_{+,-}(\tilde{\xi}(s)) = b_{+,-}(z)^{Wick}$ *with* $b_{+,-}(z) = \langle z, \tilde{b}_{+,-}(\tilde{\xi}(s))z\rangle_{\mathscr{Z}}$. *Furthermore*

$$\tilde{b}_{+,-}(\tilde{\xi}(s)) = \sum_{i,j\in\mathbb{N}} \langle e_i \otimes e_j, \omega^{-1/2}(k)\chi(k)e^{ik\cdot x}\tilde{\xi}_1(s,x)\rangle_{L^2(\mathbb{R}^{2d})}|e_i\oplus 0\rangle\langle 0 \oplus \bar{e}_j|$$

is a Hilbert-Schmidt operator on $\mathscr{Z}$.

Proof. It is very easy to see that the Wick quantization of these symbols is the corresponding operator on $\mathscr{H}$ (formally we substitute each $z_1^{\#}$ with $\psi^{\#}$ and each $z_2^{\#}$ with $a^{\#}$, in normal ordering).

Also, since the sum in i,j is convergent, $\tilde{b}_{+,+}(\tilde{\xi}(s))$ is a vector of $\mathscr{Z}^{\otimes_2}$, hence a finite rank operator. Finally,

$$\mathrm{Tr}_{\mathscr{Z}}\Big[\tilde{b}_{-,+}(\tilde{\xi}(s))^*\tilde{b}_{-,+}(\tilde{\xi}(s))\Big] \leq \|\xi_1\|_2^2\|\omega^{-1/2}\chi\|_2^2 \;.$$

For $b_{+,-}$ we obtain an analogous bound. □

The operator $B_{+-,0}$ can be seen as the second quantization of a multiplication operator, hence its symbol is not compact. In order to make it compact we need to use a regularization scheme. We define the symbol $b_{+-,0}(z)$ as $b_{+-,0}(z) = \langle z, \tilde{b}_{+-,0}(\tilde{\xi}(s))z\rangle_{\mathscr{Z}}$ with

$$\tilde{b}_{+-,0}(\tilde{\xi}(s)) = \begin{pmatrix} f(\tilde{\xi}_2(s)) & 0 \\ 0 & 0 \end{pmatrix} \;,$$

$$f(\tilde{\xi}_2(s),x) = \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}}\Big(\tilde{\xi}_2(s,k)e^{ik\cdot x} - \bar{\tilde{\xi}}_2(s,k)e^{-ik\cdot x}\Big)dk \;.$$

Since for all $s\in\mathbb{R}$, $\omega^{-1/2}\chi, \tilde{\xi}_2(s) \in L^2(\mathbb{R}^d)$, then $f(\tilde{\xi}_2(s)) \in L^\infty(\mathbb{R}^d)$, and $\lim_{|x|\to\infty} f(\tilde{\xi}_2(s)) = 0$. We would like to use the following compactness criterion [see e.g. 11, 25].

Proposition 3.12. *Let* $f, g \in L^\infty(\mathbb{R}^d)$ *such that*

$$\lim_{|x|\to\infty} f(x) = 0 \;,\; \lim_{|\kappa|\to\infty} g(\kappa) = 0 \;.$$

Also, let $g(i\partial_x)$ *be the operator acting as:*

$$g(i\partial_x)u(x) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-i\kappa\cdot x}g(\kappa)\check{u}(\kappa)d\kappa \;.$$

Then the operator $g(i\partial_x)f(x)$ *on* $L^2(\mathbb{R}^d)$ *is compact.*

Definition 3.13 ($g_m(i\partial_x)$)**.** Let $\{g_m\}_{m\in\mathbb{N}}$ be a family of functions in $L^\infty(\mathbb{R}^d)$, decaying to zero at infinity, satisfying the following properties:

i) For all $m \in \mathbb{N}$; $0 \leq g_m(x) \leq 1$, for all $x \in \mathbb{R}^d$.

ii) $g_m(x) \to 1$ pointwise when $m \to \infty$.

iii) For all $a, b > 0$, there exists $C(a) > 0$ such that for all $m \in \mathbb{N}\backslash\{0\}$: $\|(1+a\kappa^b)^{-1}(1-g_m(\kappa))\|_\infty \leq C(a)m^{-b}$.

Then the operators $g_m(i\partial_x)$ will compactify $f(\tilde{\xi}_2(s), x)$ in the sense of Proposition 3.12. Furthermore they will behave suitably in the limit $\varepsilon \to 0$.

Example. Let $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $g = 1$ if $|x| \leq 1$, $g = 0$ if $|x| \geq 2$ and $0 \leq g \leq 1$ if $1 \leq |x| \leq 2$. Define $g_m(x) := g(x/m)$. Then $\{g_m\}_{m\in\mathbb{N}}$ satisfies Definition 3.13.

Consider now $\mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s))B_{+-,0}(\tilde{\xi}(s))\Big]$, we can write it as:

$$\mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s))B_{+-,0}(\tilde{\xi}(s))\Big] = \mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s))d\Gamma\Big(\Big(\begin{smallmatrix} g_m(i\partial_x)f(\tilde{\xi}_2(s),x) & 0 \\ 0 & 0\end{smallmatrix}\Big)\Big)\Big] + \mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s)) d\Gamma\Big(\Big(\begin{smallmatrix} \big(1-g_m(i\partial_x)\big)f(\tilde{\xi}_2(s),x) & 0 \\ 0 & 0\end{smallmatrix}\Big)\Big)\Big]\ .$$

The first term on the right hand side has a now compact symbol; and thanks to the assumptions on $\{g_m\}_{m\in\mathbb{N}}$ we can make the second small when $m \to \infty$. A precise statement is given in the next lemma, proved with the aid of Proposition A.3 of Appendix A.

Lemma 3.14. *Let $\xi \in \mathscr{Z}_1$, $s \in [0,t]$, $t \in \mathbb{R}$. Then for any $\varrho_\varepsilon \in \mathcal{S}^1_\varepsilon$ and $\bar{\varepsilon} > 0$, there exists $C(s, \|\omega^{-1/2}\chi\|_2) > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$:*

$$\Big|\mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s))d\Gamma\Big(\Big(\begin{smallmatrix} \big(1-g_m(i\partial_x)\big)f(\tilde{\xi}_2(s),x) & 0 \\ 0 & 0\end{smallmatrix}\Big)\Big)\Big]\Big| \leq C(s, \|\omega^{-1/2}\chi\|_2)\|\xi_2\|_2\Big(1+\varepsilon\|\xi\|_{\mathscr{Z}_1} +\varepsilon^2\|\xi\|^2_{\mathscr{Z}_1} + \varepsilon^3\|\xi_1\|_2^3 + \varepsilon^4\|\xi_1\|_2^4\Big)\frac{1}{m}|\varrho_\varepsilon|_{\mathcal{S}^1_\varepsilon}\ .$$

Proof. The proof is done splitting the trace in parts as usual:

$$\Big|\mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s))d\Gamma\Big(\Big(\begin{smallmatrix} \big(1-g_m(i\partial_x)\big)f(\tilde{\xi}_2(s),x) & 0 \\ 0 & 0\end{smallmatrix}\Big)\Big)\Big]\Big| \leq |S^{-1}e^{i\frac{s}{\varepsilon}}S|_{\mathcal{L}(\mathscr{H})}|S^{-1}W(\tilde{\xi}(s))S|_{\mathcal{L}(\mathscr{H})} |S^{-1}d\Gamma_1\big((1-g_m(i\partial_x))f(\tilde{\xi}_2(s),x)\big)|_{\mathcal{L}(\mathscr{H})}|\varrho_\varepsilon|_{\mathcal{S}^1_\varepsilon}\ ;$$

where $d\Gamma_1(f) = \int_{\mathbb{R}^d} f(x)\psi^*(x)\psi(x)dx$. The first two terms of the right hand side are bounded by Lemmas 3.4 and 3.2 respectively; for the third one we use Proposition A.3 as follows:

$$\begin{aligned}|S^{-1}d\Gamma_1\big((1-g_m(i\partial_x))f(\tilde{\xi}_2(s))\big)|_{\mathcal{L}(\mathscr{H})} &\leq \big|\big(d\Gamma_1(1-\frac{\Delta}{2M})+1\big)^{-1}d\Gamma_1\big((1-g_m(i\partial_x))f(\tilde{\xi}_2(s))\big)\big|_{\mathcal{L}(\mathscr{H})} \\ &\leq (1+\sqrt{2})\big|\big(1+i\partial_x/\sqrt{2M}\big)^{-1}(1-g_m(i\partial_x))f(\tilde{\xi}_2(s))\big|_{\mathcal{L}(L^2(\mathbb{R}^d))} \\ &\leq (1+\sqrt{2})\|\big(1+\kappa/\sqrt{2M}\big)^{-1}(1-g_m(\kappa))\|_\infty\|f(\tilde{\xi}_2(s))\|_\infty \leq C(1+\sqrt{2})\|\omega^{-1/2}\chi\|_2\|\xi_2\|_2\frac{1}{m}\ ;\end{aligned}$$

where the last inequality follows from definition 3.13 of $\{g_m\}$. Defining the suitable global constant $C(s, \|\omega^{-1/2}\chi\|_2)$ we conclude the proof. □

4. THE CLASSICAL LIMIT $\varepsilon \to 0$.

Up to this point we have analysed the time evolved state $\varrho_\varepsilon(t)$ at fixed $\varepsilon \in (0, \bar{\varepsilon})$, now we will focus on the limit $\varepsilon \to 0$. First we will introduce and discuss the results we need about the convergence of states to Wigner measures; then study the limit of the integral equation (18).

a. Wigner measures.

In the classical limit, the density matrix ϱ_ε behaves like a weak distribution, or probability measure, on the phase space $\mathscr{Z}$. We give a brief introduction to infinite dimensional semiclassical analysis and detailed results can be found in [3–6]. Here we present the results we need most, adapted to our setting.

Definition 4.1 ($\mathcal{S}^\delta$, $\mathcal{T}^\delta$). Let $\bar{\varepsilon} > 0$, $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \in \mathcal{L}^1(\mathscr{H})$ a family of normal states and $\delta \in \mathbb{R}$. Then

$$(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \in \mathcal{S}^\delta \Leftrightarrow \exists C(\delta,\bar{\varepsilon}) > 0, |(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}|_{\mathcal{S}^\delta} := \sup_{\varepsilon\in(0,\bar{\varepsilon})} |\varrho_\varepsilon S^\delta|_{\mathcal{L}^1(\mathscr{H})} \leq C(\delta,\bar{\varepsilon}) \;;$$

$$(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \in \mathcal{T}^\delta \Leftrightarrow \exists C(\delta,\bar{\varepsilon}) > 0, |(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}|_{\mathcal{T}^\delta} := \sup_{\varepsilon\in(0,\bar{\varepsilon})} |\varrho_\varepsilon T^\delta|_{\mathcal{L}^1(\mathscr{H})} \leq C(\delta,\bar{\varepsilon}) \;.$$

We remark that if $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \in \mathcal{S}^\delta$(respectively $\mathcal{T}^\delta$), then $\varrho_\varepsilon \in \mathcal{S}^\delta_\varepsilon$(respectively $\mathcal{T}^\delta_\varepsilon$) for all $\varepsilon \in (0,\bar{\varepsilon})$; furthermore the bound of $|\varrho_\varepsilon|_{\mathcal{S}^\delta_\varepsilon(\mathcal{T}^\delta_\varepsilon)}$ is independent of ε. With this definition, we are ready to introduce the Wigner measures; the following result holds for general symmetric Fock spaces over a separable Hilbert space, and it is proved in [4, Theorem 6.2].

Proposition 4.2. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \in \bigcup_{\delta>0} \mathcal{T}^\delta$, i.e. there exists $\bar{\delta} > 0$ such that $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \in \mathcal{T}^{\bar{\delta}}$. Then for every sequence $(\varepsilon_n)_{n\in\mathbb{N}} \in (0,\bar{\varepsilon})$ with $\lim_{n\to\infty}\varepsilon_n = 0$, there exists a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a Borel probability measure μ on $\mathscr{Z}$ associated with $(\varrho_{\varepsilon_{n_k}})_{k\in\mathbb{N}}$ characterized by:*

$$\lim_{k\to\infty} \mathrm{Tr}\Big[\varrho_{\varepsilon_{n_k}} W(\xi)\Big] = \int_{\mathscr{Z}} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle} d\mu(z) \;,\; \forall\xi \in \mathscr{Z} \;.$$

Furthermore μ satisfies the following property:

$$\int_{\mathscr{Z}} \big(\|z_1\|_2^2 + \|z_2\|_2 + 1\big)^{2\bar{\delta}} d\mu(z) \leq |(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}|_{\mathcal{T}^{\bar{\delta}}} < +\infty \;. \tag{26}$$

Definition 4.3. The set of Wigner measures associated with $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)} \in \bigcup_{\delta>0} \mathcal{T}^\delta$ is denoted by

$$\mathscr{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon)) \; .$$

In general, $\mathscr{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon))$ is not constituted by a single element; however for each countable sequence $\varepsilon_n \to 0$ we can extract a subsequence $(\varepsilon_{n_k})_k$ such that $\mathscr{M}(\varrho_{\varepsilon_{n_k}}, (\varepsilon_{n_k})_{k\in\mathbb{N}}) = \{\mu\}$; hence we can suppose without loss of generality that $\mathscr{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon)) = \{\mu\}$.

Remark 4.4. Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)} \in \bigcup_{\delta>0} \mathcal{T}^\delta$; with associated measure μ. Then, using Lemma 3.2 and Weyl's relation, for any $\xi \in \mathscr{Z}$, $(\varrho_\varepsilon W(\xi))_{\varepsilon\in(0,\bar\varepsilon)}$ has an associated (complex) measure μ_ξ with

$$d\mu_\xi(z) = e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle} d\mu(z) \; .$$

We refer the reader to [4] for further informations on Wigner measures of general trace class operators.

The convergence of ρ_ε holds with a large class of operators (under suitable conditions); in particular with Wick quantized polynomials with compact symbol. The precise statement is the following: Let $\mathcal{P}^\infty_{p,q}(\mathscr{Z})$ be the compact polynomial symbols of degree p in z and q in $\bar z$; define $\mathcal{P}^\infty_{alg}(\mathscr{Z}) = \bigoplus^{alg}_{p,q\in\mathbb{N}} \mathcal{P}^\infty_{p,q}(\mathscr{Z})$. Then the following proposition holds:

Proposition 4.5. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)} \in \bigcap_{\delta\geq 0} \mathcal{T}^\delta$ such that $\mathscr{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon)) = \{\mu\}$. Then for any $b \in \mathcal{P}^\infty_{alg}(\mathscr{Z})$:*

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\Big[\varrho_\varepsilon\, b^{Wick}\Big] = \int_{\mathscr{Z}} b(z) d\mu(z) \; .$$

Remark 4.6. Since we have only operators bounded by T, we can relax the hypothesis of Proposition 4.5 to states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)} \in \mathcal{T}^1$. In this case, the result is true for compact polynomial symbols $b \in \mathcal{P}^\infty_{alg}(\mathscr{Z})$ such that $T^{-1/2} b^{Wick} T^{-1/2}$ is bounded uniformly in $\varepsilon \in [0,\bar\varepsilon]$.

b. Subsequence extraction for all times.

We would like to apply proposition 4.5 to the integral formula (25) and obtain an integral equation for the measure μ_t associated with $\varrho_\varepsilon(t)$. In order to do that we need to be able to extract the same converging subsequence at any time $t \in \mathbb{R}$. This is what we prove in the next proposition; preceded by a preparatory lemma.

Lemma 4.7. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)} \in \mathcal{T}^1$. Then $\tilde G_\varepsilon(t,\xi) := \mathrm{Tr}\Big[\tilde\varrho_\varepsilon(t) W(\xi)\Big]$ is uniformly equicontinuous with respect to $\varepsilon \in (0,\bar\varepsilon)$ on bounded subsets of $\mathbb{R}\times\mathscr{Z}$.*

Proof. Let $\varepsilon \in (0,\bar\varepsilon)$. We split $|\tilde G_\varepsilon(t,\xi) - \tilde G_\varepsilon(s,\eta)| \leq X_1 + X_2$, with

$$X_1 := |\tilde G_\varepsilon(t,\xi) - \tilde G_\varepsilon(s,\xi)| \ , \ X_2 := |\tilde G_\varepsilon(s,\xi) - \tilde G_\varepsilon(s,\eta)| \ .$$

Using Proposition 3.9, Lemma 3.3, (20) and the fact that $B_j(\tilde\xi(\tau))$ is bounded uniformly in τ and $\varepsilon \in (0,\bar\varepsilon)$ for $j = 0,1,2$, we obtain for some $C_1(\bar\varepsilon, \|\xi\|_{\mathscr{Z}}), C_2(\bar\varepsilon, \|\xi\|_{\mathscr{Z}}) > 0$:

$$X_1 = \Big|\sum_{j=0}^{2} \varepsilon^j \int_s^t \mathrm{Tr}\Big[\varrho_\varepsilon(\tau) W(\tilde\xi(\tau)) B_j(\tilde\xi(\tau))\Big] ds\Big| \leq C_1 |e^{C_2|t|} - e^{C_2|s|}| \ .$$

Consider now X_2; using Weyl's relation and the fact that $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)} \in \mathcal{T}^1$ we obtain, for some $C_3(s,\bar\varepsilon) > 0$:

$$\begin{aligned} X_2 &\leq |(W(\eta)W^*(\xi) - 1)T^{-1}|_{\mathcal{L}(\mathscr{H})} |\tilde\varrho_\varepsilon(s)|_{\mathcal{T}^1} \leq C_3 |(e^{i\frac{\varepsilon}{2}\mathrm{Im}\langle\eta,\xi\rangle} W(\eta-\xi) - 1)T^{-1}|_{\mathcal{L}(\mathscr{H})} \\ &\leq C_3 \Big(|e^{i\frac{\varepsilon}{2}\mathrm{Im}\langle\eta,\xi\rangle} - 1| + |(W(\eta-\xi) - 1)T^{-1}|_{\mathcal{L}(\mathscr{H})} \Big) \ . \end{aligned}$$

Now, we use the following bound for the first term:

$$|e^{i\frac{\varepsilon}{2}\mathrm{Im}\langle\eta,\xi\rangle} - 1| = |e^{i\frac{\varepsilon}{2}\mathrm{Im}\langle\eta-\xi,\xi\rangle} - 1| \leq 2\bar\varepsilon \|\xi\| e^{\frac{\bar\varepsilon}{2}\|\xi\|(\|\eta\|+\|\xi\|)} \|\eta - \xi\|;$$

and for the second:

$$|(W(\eta-\xi) - 1)T^{-1}|_{\mathcal{L}(\mathscr{H})} \leq \Big| \int_0^1 W(\lambda(\eta-\xi)) \varphi(\eta-\xi) T^{-1} d\lambda \Big|_{\mathcal{L}(\mathscr{H})} \leq \sqrt{2}\|\xi - \eta\| \ ;$$

where $\sqrt{2}\varphi(z) = \big(\psi^*(z_1) + \psi(z_1) + a^*(z_2) + a(z_2)\big)$. Finally we obtain

$$X_2 \leq C_3 \Big(2\bar\varepsilon \|\xi\| e^{\frac{\bar\varepsilon}{2}\|\xi\|(\|\eta\|+\|\xi\|)} + \sqrt{2} \Big) \|\xi - \eta\| \ .$$

Now, choose a bounded subset $I = [-T_0, T_0] \times \{z, \|z\| \leq R\}$, $T_0, R > 0$. Then there exist $C_1, C_2, C_3 > 0$ that depend only on T_0, R and $\bar\varepsilon$ such that for all $(t,\xi),(s,\eta) \in I$:

$$|\tilde G_\varepsilon(t,\xi) - \tilde G_\varepsilon(s,\eta)| \leq C_1 \big| e^{C_2|t|} - e^{C_2|s|} \big| + C_3 \|\xi - \eta\| \ . \tag{27}$$

□

Proposition 4.8. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)} \in \mathcal{T}^\delta$, $\delta \geq 1$. Then for any sequence $(\varepsilon_n)_{n\in\mathbb{N}} \in (0,\bar\varepsilon)$, converging to zero, there exists a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a family of Borel measures $(\tilde\mu_t)_{t\in\mathbb{R}}$ on $\mathscr{Z}$ such that for all $t \in \mathbb{R}$:*

$$\mathscr{M}(\tilde\varrho_{\varepsilon_{n_k}}(t), k \in \mathbb{N}) = \{\tilde\mu_t\} \ .$$

Furthermore for any $T_0 \geq 0$ there exists $C(T_0) > 0$ such that for all $t \in [-T_0, T_0]$:

$$\int_{\mathscr{Z}} \big(\|z_1\|_2^2 + \|z_2\|_2 + 1\big)^{2\delta} d\tilde\mu_t(z) < C(T_0) \ . \tag{28}$$

Proof. Recall that for any $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \in \mathcal{T}^\delta \subset \mathcal{T}^1$, $(\tilde{\varrho}_\varepsilon(t))_{\varepsilon\in(0,\bar{\varepsilon})} \in \mathcal{T}^1$ for all $t \in \mathbb{R}$ (using Lemma 3.3 and the fact that H_0 commutes with T). The field $\mathbb{R}$ is separable, so we can consider a dense countable set $D \subset \mathbb{R}$. Let $(\varepsilon_n)_{n\in\mathbb{N}} \in (0,\bar{\varepsilon})$. We can choose, by a diagonal extraction argument, a *single* subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ such that we can apply Proposition 4.2 and obtain, for any $t_j \in D$:

$$\lim_{k\to\infty} \mathrm{Tr}\Big[\tilde{\varrho}_{\varepsilon_{n_k}}(t_j)W(\xi)\Big] = \int_{\mathscr{Z}} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}\, d\tilde{\mu}_{t_j}(z) =: \tilde{G}_0(t_j,\xi)\,.$$

Also, since $0 \le \mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t_j)W(\xi)\Big] \le 1$ holds for any $\varepsilon \in (0,\bar{\varepsilon})$, then $0 \le \tilde{G}_0(t_j,\xi) \le 1$. Now we can use Lemma 4.7 and obtain for all $t_j, t_l \in D$:

$$|\tilde{G}_{\varepsilon_{n_k}}(t_j,\xi) - \tilde{G}_{\varepsilon_{n_k}}(t_l,\xi)| \le C_1\big|e^{C_2|t_j|} - e^{C_2|t_l|}\big|\,,$$

uniformly in ε_{n_k}, then we can take the limit $k \to \infty$ and obtain

$$|\tilde{G}_0(t_j,\xi) - \tilde{G}_0(t_l,\xi)| \le C_1\big|e^{C_2|t_j|} - e^{C_2|t_l|}\big|\,.$$

Let $t \in \mathbb{R}$; choose $(t_i)_{i\in\mathbb{N}} \in D$, such that $t_i \to t$, when $i \to \infty$. Then $(\tilde{G}_0(t_i,\xi))_{i\in\mathbb{N}}$ is a Cauchy sequence and we can define

$$\tilde{G}_0(t,\xi) := \lim_{i\to\infty} \tilde{G}_0(t_i,\xi)\,.$$

For all $t \in \mathbb{R}$, $\tilde{G}_0(t,\,\cdot\,)$ is a norm continuous normalised function of positive type which satisfies

$$|\tilde{G}_0(t,\xi) - \tilde{G}_0(s,\eta)| \le C_1\big|e^{C_2|t|} - e^{C_2|s|}\big| + C_3\|\xi-\eta\|\,, \tag{29}$$

on any bounded subset of $\mathbb{R}\times\mathscr{Z}$, for some positive constants C_1, C_2 and C_3.

Hence it is the characteristic function of a weak distribution $\tilde{\mu}_t$ on $\mathscr{Z}$, and for all $t \in \mathbb{R}$:

$$\lim_{k\to\infty} \mathrm{Tr}\Big[\tilde{\varrho}_{\varepsilon_{n_k}}(t)W(\xi)\Big] = \int_{\mathscr{Z}} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}\, d\tilde{\mu}_t(z)\,.$$

Furthermore $\tilde{\mu}_t$ are Borel probability measures since they are Wigner measures of $(\tilde{\varrho}_{\varepsilon_{n_k}}(t))_{k\in\mathbb{N}} \in \mathcal{T}^\delta$. The bound (28) comes from (26) and Lemma 3.3. □

Corollary 4.9. *The following statements are true:*

i) Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \in \mathcal{T}^1$. Then for any sequence $(\varepsilon_n)_{n\in\mathbb{N}} \in (0,\bar{\varepsilon})$ converging to zero, there exists a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a family of Borel measures $(\mu_t)_{t\in\mathbb{R}}$ on $\mathscr{Z}$ such that for all $t \in \mathbb{R}$:

$$\mathscr{M}(\varrho_{\varepsilon_{n_k}}(t), k \in \mathbb{N}) = \{\mu_t\}\,.$$

ii) Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)} \in \mathcal{T}^1$, $\xi \in \mathscr{Z}$*. Then for any sequence* $(\varepsilon_n)_{n\in\mathbb{N}} \in (0,\bar\varepsilon)$ *converging to zero, there exists a subsequence* $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ *and a family of Borel measures* $(\mu_{t,\xi})_{t\in\mathbb{R}}$ *on* $\mathscr{Z}$ *such that for all* $t \in \mathbb{R}$*:*

$$\mathscr{M}(\varrho_{\varepsilon_{n_k}}(t)W(\tilde\xi(t)), k \in \mathbb{N}) = \{\mu_{t,\xi}\} \; ;$$

furthermore:

$$d\mu_{t,\xi}(z) = e^{i\sqrt{2}\mathrm{Re}\langle\tilde\xi(t),z\rangle} d\mu_t(z) \; .$$

Proof. i) follows easily since for any $\varrho \in \mathcal{L}^1(\mathscr{H})$ and $\xi \in \mathscr{Z}$: $\mathrm{Tr}\Big[\tilde\varrho(t)W(\xi)\Big] = \mathrm{Tr}\Big[\varrho(t)W(\tilde\xi(t))\Big]$.
ii) is a consequence of Remark 4.4. □

c. Integral formula in the limit $\varepsilon \to 0$.

We have all the ingredients to calculate the limit $\varepsilon \to 0$ of the integral equation (25).

Proposition 4.10. *Assume that* (**A**) *holds and let* $\xi \in \mathscr{Z}$, $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)} \in \mathcal{S}^1$*. Then for any sequence* $(\varepsilon_n)_{n\in\mathbb{N}} \in (0,\bar\varepsilon)$ *such that* $\lim_{n\to\infty}\varepsilon_n = 0$*; there exists a subsequence* $(\varepsilon_{n_k})_{k\in\mathbb{N}}$*, and a family* $(\mu_t)_{t\in\mathbb{R}}$ *of Borel probability measures on* $\mathscr{Z}$ *such that for all* $t \in \mathbb{R}$*:*

1. $\mathscr{M}(\varrho_{\varepsilon_{n_k}}(t), k \in \mathbb{N}) = \{\mu_t\}$ *and* $\mathscr{M}(\tilde\varrho_{\varepsilon_{n_k}}(t), k \in \mathbb{N}) = \{\tilde\mu_t = \Phi_0(-t)_\#\mu_t)\}$.
2. $\tilde\mu_t$ *satisfies the following integral equation:*

$$\tilde\mu_t(e^{i\sqrt{2}\mathrm{Re}\langle\xi,\,\cdot\,\rangle}) = \mu_0(e^{i\sqrt{2}\mathrm{Re}\langle\xi,\,\cdot\,\rangle}) + i\sqrt{2}\int_0^t \tilde\mu_s\left(e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}\mathrm{Re}\langle\xi,\mathscr{V}_s(z)\rangle\right)\, ds; \tag{30}$$

with the (velocity) vector field $\mathscr{V}_s(z) = -i\Phi_0(-t)\partial_{\bar z}h_I(\Phi_0(t)z)$ *for all* $z \in \mathscr{Z}$*.*

Proof. The first point is just a restatement of Corollary 4.9. The second is proved starting from the integral equation (25) and assuming $\xi \in \mathscr{Z}_1$. Fix the subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ such that we can associate a measure μ_t to $(\varrho_{\varepsilon_{n_k}}(t))_{k\in\mathbb{N}}$ for all times. Then in (25) the left hand side and the first term in the right hand side converge by virtue of Proposition 4.8, and its Corollary 4.9. The terms involving B_1 and B_2 converge to zero in absolute value by Proposition 3.10, since $(\varrho_{\varepsilon_{n_k}})_{k\in\mathbb{N}} \in \mathcal{T}^1 \supset \mathcal{S}^1$. It remains to consider the B_0 term. If we split it as described in Section 3 b, we see that the $B_{-,-}$, $B_{-,+}$, $B_{+,-}$ and $B_{+,+}$ terms converge by means of Proposition 4.5 (applied to the state $\varrho_{\varepsilon_{n_k}}(s)W(\tilde\xi(s)))$, since they have compact symbols.

We have to be more careful with the $B_{+-,0}$ term, and use the regularization scheme introduced in Definition 3.13. Consider:

$$\left|\mathrm{Tr}\left[\varrho_\varepsilon(s)W(\tilde{\xi}(s))B_{+-,0}(\tilde{\xi}(s))\right] - \int_{\mathscr{Z}} e^{i\sqrt{2}\langle\tilde{\xi}(s),z\rangle_{\mathscr{Z}}}\langle z_1, f(\tilde{\xi}_2(s))z_1\rangle_{L^2(\mathbb{R}^d)}d\mu_s(z)\right| .$$

Define now $B^m_{+-,0}(\tilde{\xi}(s)) := d\Gamma_1\big(g_m(i\partial_x)f(\tilde{\xi}_2(s),x)\big)$ to be the regularized operator with compact symbol. Then we obtain:

$$\begin{aligned}\left|\mathrm{Tr}\left[\varrho_\varepsilon(s)W(\tilde{\xi}(s))B_{+-,0}(\tilde{\xi}(s))\right] - \int_{\mathscr{Z}} e^{i\sqrt{2}\langle\tilde{\xi}(s),z\rangle_{\mathscr{Z}}}\langle z_1, f(\tilde{\xi}_2(s))z_1\rangle_2 d\mu_s(z)\right| \leq \Big|\mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}(s))\\ B^m_{+-,0}(\tilde{\xi}(s))\Big] - \int_{\mathscr{Z}} e^{i\sqrt{2}\langle\tilde{\xi}(s),z\rangle_{\mathscr{Z}}}\langle z_1, g_m(i\partial_x)f(\tilde{\xi}_2(s))z_1\rangle_2 d\mu_s(z)\Big|\\ +\left|\mathrm{Tr}\left[\varrho_\varepsilon(s)W(\tilde{\xi}(s))d\Gamma_1\big((1-g_m(i\partial_x))f(\tilde{\xi}_2(s),x)\big)\right]\right|\\ +\left|\int_{\mathscr{Z}} e^{i\sqrt{2}\langle\tilde{\xi}(s),z\rangle_{\mathscr{Z}}}\langle z_1, (1-g_m(i\partial_x))f(\tilde{\xi}_2(s))z_1\rangle_2 d\mu_s(z)\right| .\end{aligned}$$

The first term on the right hand side goes to zero by virtue of Proposition 4.5; the second goes to zero when $m \to \infty$ by Lemma 3.14.

Finally consider the last term. By definition, $|(1 - g_m(i\partial_x))|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq 1$ uniformly in m. Furthermore, $f(\tilde{\xi}_2(s), \cdot\,) \in L^\infty(\mathbb{R}^d)$. Hence

$$\left|e^{i\sqrt{2}\langle\tilde{\xi}(s),z\rangle_{\mathscr{Z}}}\langle z_1, (1-g_m(i\partial_x))f(\tilde{\xi}_2(s))z_1\rangle_2\right| \leq \|z_1\|_2^2 ,$$

that is integrable with respect to μ_s by virtue of Proposition 4.2. Then we can apply dominated convergence theorem and prove that the term goes to zero when $m \to \infty$, since $(1 - g_m(i\partial_x)) \to 0$ strongly as an operator of $L^2(\mathbb{R}^d)$.

Once the integral formula (30) is proved for $\xi \in \mathscr{Z}_1$, the extension for all $\xi \in \mathscr{Z}$ is straightforward since $\mathscr{V}_s$ satisfies the estimate (17) and a dominated convergence theorem applies thanks to the estimate (28). □

d. Transport equation and uniqueness

Proposition 4.10 shows that Wigner measures $\tilde{\mu}_t$ of propagated normal states $\tilde{\varrho}_\varepsilon(t)$ satisfy the integral equation (30). Actually, this can be written as a Liouville (continuity) equation with respect to the classical Hamiltonian of the Klein-Gordon-Schrödinger system. Proving uniqueness of solutions of the latter equation implies that the measure $\tilde{\mu}_t$ is the push forward of μ_0 (the Wigner measure at time $t = 0$) by the classical flow $\Phi(t, 0)$ which is a well defined continuous map on $\mathscr{Z}$ by Proposition 2.8.

One of our concerns is the regularity with respect to time of the curve $t \mapsto \tilde{\mu}_t$ as a map valued on $\mathscr{P}(\mathscr{Z})$, the space of Borel probability measures over $\mathscr{Z}$. For our purpose, the most appropriate topology on $\mathscr{P}(\mathscr{Z})$ is the weakly narrowly convergence topology which is described below. Let $(e_n)_{n\in\mathbb{N}}$ be a Hilbert basis of $\mathscr{Z}$. In the following, we endow $\mathscr{Z} = L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ by the distance $d_\omega(z_1, z_2) = \sqrt{\sum_{n\in\mathbb{N}} \frac{|\langle z_1 - z_2, e_n\rangle|^2}{(1+n)^2}}$. It is not difficult to see that the topology of $(\mathscr{Z}, d_w)$ coincides with the weak topology on bounded sets. We say that a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathscr{P}(\mathscr{Z})$ weakly narrowly converges to $\mu \in \mathscr{P}(\mathscr{Z})$ if

$$\forall f \in \mathscr{C}_b(\mathscr{Z}, d_w), \quad \lim_{n\to\infty} \int_{\mathscr{Z}} f(z)\, d\mu_n = \int_{\mathscr{Z}} f(z)\, d\mu\,,$$

where $\mathscr{C}_b(\mathscr{Z}, d_w)$ denotes the space of all bounded continuous real-valued functions on $(\mathscr{Z}, d_w)$. In practice, it is more convenient to use cylindrical functions in order to check weak narrow continuity properties. We recall that a function $f : \mathscr{Z} \to \mathbb{R}$ is in the cylindrical Schwartz space $\mathcal{S}_{cyl}(\mathscr{Z})$ if there exists a finite rank orthogonal projection $\wp$ on $\mathscr{Z}$ and a function $g : \wp\mathscr{Z} \to \mathbb{R}$ in the Schwartz space $\mathcal{S}(\wp\mathscr{Z})$ such that

$$\forall z \in \mathscr{Z}, \quad f(z) = g(\wp z)\,.$$

In the same way, if $g \in \mathcal{C}_0^\infty(\wp\mathscr{Z})$ we can define the space of smooth cylindrical functions of compact support $\mathcal{C}_{0,cyl}^\infty(\mathscr{Z})$. We caution the reader that neither $\mathcal{S}_{cyl}(\mathscr{Z})$ nor $\mathcal{C}_{0,cyl}^\infty(\mathscr{Z})$ possess a vector space structure. The Fourier transform of $f \in \mathcal{S}_{cyl}(\mathscr{Z})$, based on a finite dimensional subspace $\wp\mathscr{Z}$, is

$$\mathcal{F}[f](\xi) = \int_{\wp\mathscr{Z}} e^{-2i\pi \mathrm{Re}\langle \xi\,,\, z\rangle_{\mathscr{Z}}} f(z)\; dL_\wp(z)\,, \tag{31}$$

where $dL_\wp(z)$ denotes the Lebesgue measure on $\wp\mathscr{Z}$ and the inverse formula is

$$f(z) = \int_{\wp\mathscr{Z}} e^{2i\pi \mathrm{Re}\langle \xi\,,\, z\rangle_{\mathscr{Z}}} \mathcal{F}[f](\xi)\; dL_\wp(\xi)\,.$$

Proposition 4.11. *Assume that* (**A**) *holds and that* $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ *are Wigner measures of the family* $(\tilde{\varrho}_\varepsilon(t)))_{\varepsilon\in(0,\bar{\varepsilon})} \in \mathcal{S}^1$ *provided by Proposition 4.10. Then the map* $t \in \mathbb{R} \mapsto \tilde{\mu}_t$ *is weakly narrowly continuous and satisfies the transport equation*

$$\partial_t \tilde{\mu}_t + \nabla^T \left(\mathscr{V}_t \tilde{\mu}_t\right) = 0\,, \tag{32}$$

in the weak sense,

$$\forall f \in \mathcal{C}_{0,cyl}^\infty(\mathbb{R}\times\mathscr{Z})\,, \quad \int_{\mathbb{R}} \int_{\mathscr{Z}} \left(\partial_t f + \mathrm{Re}\langle \nabla f, \mathscr{V}_t\rangle\right)\; d\tilde{\mu}_t dt = 0\,. \tag{33}$$

Proof. For any $f \in \mathcal{S}_{cyl}(\mathscr{Z})$, based on $\wp\mathscr{Z}$ with $\wp$ a finite rank orthogonal projection, Fubini's theorem gives

$$\int_{\mathscr{Z}} f(z)\, d\tilde{\mu}_t(z) = \int_{\wp\mathscr{Z}} \mathcal{F}[f](\xi)\, \tilde{\mu}_t(e^{2i\pi \mathrm{Re}\langle \xi\,,\, z\rangle_{\mathscr{Z}}})\, dL_{\wp}(z)\,,$$

where $\mathcal{F}$ is the Fourier transform (31). Hence, by the estimate (29) (with $\eta = \xi$) and the decay at infinity of $\mathcal{F}[f]$ the map $t \mapsto \int_{\mathscr{Z}} f(z)\, d\tilde{\mu}_t(z)$ is continuous for any $f \in \mathcal{S}_{cyl}(\mathscr{Z})$. Now, the bound $\int_{\mathscr{Z}} ||z||^2_{\mathscr{Z}}\, d\tilde{\mu}_t(z) \le C(T_0)$ (proved in Proposition 4.8) and [2, Lemma 5.1.12-f)] guaranties the weak narrow continuity of the curve $t \mapsto \tilde{\mu}_t$.
The transport equation (32) follows by integrating (30) against $\mathcal{F}[g](\xi)\, dL_{\wp}(z)$ for any $g \in \mathcal{C}^\infty_{0,cyl}(\mathscr{Z})$ based on $\wp\mathscr{Z}$. So, we obtain

$$\int_{\mathscr{Z}} g(z)\, d\tilde{\mu}_t(z) = \int_{\mathscr{Z}} g(z)\, d\tilde{\mu}_0(z) + 2i\pi \int_0^t \int_{\wp\mathscr{Z}} \tilde{\mu}_s\left(\mathrm{Re}\langle \xi, \mathscr{V}_s(z)\rangle\right) \mathcal{F}[g](\xi)\, dL_{\wp\mathscr{Z}}(\xi)\, ds\,.$$

By Fubini's theorem and properties of finite dimensional Fourier transform, the identity

$$\int_{\mathscr{Z}} g(z)\, d\tilde{\mu}_t(z) = \int_{\mathscr{Z}} g(z)\, d\tilde{\mu}_0(z)\rangle + \int_0^t \int_{\mathscr{Z}} \mathrm{Re}\langle \nabla g(z), \mathscr{V}_s(z)\rangle\, d\tilde{\mu}_s(z)\, ds\,, \tag{34}$$

holds true with $\nabla g(z)$ the differential of $g : \mathscr{Z} \to \mathbb{R}$ (here $\mathscr{Z}$ is considered as a real Hilbert space with the scalar product $\mathrm{Re}\langle\cdot,\cdot\rangle$). We observe that for any $g \in \mathcal{S}_{cyl}(\mathscr{Z})$ the r.h.s. of (34) is $\mathcal{C}^1(\mathbb{R})$. Hence, a differentiation with respect to t gives

$$\partial_t \left(\int_{\mathscr{Z}} g(z)\, d\tilde{\mu}_t(z) \right) - \int_{\mathscr{Z}} \mathrm{Re}\langle \nabla g(z), \mathscr{V}_t(z)\rangle\, d\tilde{\mu}_t(z) = 0\,.$$

Thus, multiplying the above relation by $\varphi(t)$, with $\varphi \in \mathcal{C}_0^\infty(\mathbb{R},\mathbb{R})$, and integrating by part proves (33) for $f(t,z) = \varphi(t) g(z)$. We conclude by observing that any $f \in \mathcal{C}^\infty_{0,cyl}(\mathbb{R}\times\mathscr{Z})$, $f(t,z) = g(t,\wp z)$ with $g \in \mathcal{C}_0^\infty(\mathbb{R}\times\wp\mathscr{Z})$ can be approximated by a sequence $\big(g_n(\wp\,\cdot\,,\,\cdot)\big)_{n\in\mathbb{N}}$ in $\mathcal{C}_0^\infty(\mathbb{R}) \overset{alg}{\otimes} \mathcal{C}_0^\infty(\wp\mathscr{Z})$. □

Proposition 4.12. *Assume that* (**A**) *holds. Let* $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \in \cap_{\delta>0}\mathcal{T}^\delta \cap \mathcal{S}^1$ *and admits a unique Wigner measure* μ_0*. Then for any time* $t \in \mathbb{R}$*, the family* $(\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon})_{\varepsilon\in(0,\bar{\varepsilon})}$ *admits a unique Wigner measure* $\mu_t = \Phi(t,0)_{\#}\mu_0$*, where* Φ *is the flow of the Klein-Gordon-Schrödinger system defined on* $\mathscr{Z}$ *by Proposition 2.8.*

Proof. Proposition 4.10 and Proposition 4.11 say that for any sequence $(\varepsilon_n)_{n\in\mathbb{N}} \in (0,\bar{\varepsilon})$ such that $\lim_{n\to\infty}\varepsilon_n = 0$; there exists a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$, and a family of Wigner measures $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ of $(\tilde{\varrho}_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ which are Borel probability measures on $\mathscr{Z}$ satisfying the transport equation (32)-(33)

for all $t \in \mathbb{R}$ with initial datum μ_0 a time $t = 0$. Now, we apply [3, Proposition C.8] in order to conclude that such transport equation (32) admits a unique solution given by

$$\Phi_0(t)_{\#}\tilde{\mu}_t = \Phi(t,0)_{\#}\mu_0 \text{ i.e. } \mu_t = \Phi(t,0)_{\#}\mu_0 \,.$$

The assumptions to be checked are:

(i) For all $T > 0$,

$$\int_{-T}^{T} \left(\int_{\mathscr{Z}} ||\mathscr{V}_t(z)||^2_{\mathscr{Z}} \, d\tilde{\mu}_t(z) \right)^{1/2} dt < \infty \,.$$

This holds true by (17) and the a priori estimate (28).

(ii) The map $t \in \mathbb{R} \mapsto \tilde{\mu}$ is continuous with respect to the Wasserstein distance W_2. Indeed, [3, Proposition C1] shows that a weakly narrowly continuous curve satisfying a transport equation with a Borel velocity field satisfying (i) is continuous with respect to the Wasserstein distance. □

e. Propagation for general states

The extension of Proposition 4.12 to general states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfying the assumption (2) of Theorem 1.1 follows by a general approximation argument introduced in [6] and briefly sketched below. We recall that $S = H_0 + T$ and $T = N_1^2 + N_2 + 1$. Suppose that for some $\delta > 0$ there exists $C_\delta > 0$ such that

(35) $$\forall \varepsilon \in (0, \bar{\varepsilon}), \quad |S^{\delta/4} \varrho_\varepsilon S^{\delta/4}|_{\mathcal{L}^1(\mathscr{H})} \leq C_\delta \,.$$

Let $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ in a neighbourhood of 0 and $\chi_R(x) = \chi(\frac{x}{R})$. Then the family of normal states

$$\varrho_{\varepsilon,R} = \frac{\chi_R(S)\varrho_\varepsilon\chi_R(S)}{\mathrm{Tr}\,[\chi_R(S)\varrho_\varepsilon\chi_R(S)]}$$

approximate ϱ_ε as $R \to \infty$. Notice that $\varrho_{\varepsilon,R}$ is well defined for R sufficiently large for all $\varepsilon \in (0, \bar{\varepsilon})$. Actually, thanks to the assumption (2),

$$|\varrho_\varepsilon(t) - \varrho_{\varepsilon,R}(t)|_{\mathcal{L}^1(\mathscr{H})} = |\varrho_\varepsilon - \varrho_{\varepsilon,R}|_{\mathcal{L}^1(\mathcal{H})} \leq \nu(R)$$

where $\varrho_{\varepsilon,R}(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon} \varrho_{\varepsilon,R} \, e^{i\frac{t}{\varepsilon}H_\varepsilon}$ and $\nu(R)$ is independent of ε with $\lim_{R\to\infty} \nu(R) = 0$. Now, it is easy to see that for any $R \in (0, \infty)$ the family of states $(\varrho_{\varepsilon,R})_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfies the assumptions of Proposition 4.12 except the uniqueness of the Wigner measure at time $t = 0$. However, up to extracting a sequence which a priori depends on R, we can suppose that $\mathscr{M}(\varrho_{\varepsilon_n,R}, n \in \mathbb{N}) = \{\mu_{0,R}\}$ and $\mathscr{M}(\varrho_{\varepsilon_n}, n \in \mathbb{N}) = \{\mu_0\}$. Thus, we obtain

$$\forall t \in \mathbb{R}\,, \quad \mathcal{M}(\varrho_{\varepsilon_n}(t), n \in \mathbb{N}) = \{\Phi(t,0)_{\#}\mu_{0,R}\}\ .$$

For each $t \in \mathbb{R}$, we can again extract a subsequence, which may depend on t, such that

$$\mathcal{M}(\varrho_{\varepsilon_n}(t), n \in \mathbb{N}) = \{\mu_t\}\ .$$

Now, [6, Proposition 2.10] implies

$$\int_{\mathscr{Z}} |\mu_t - \Phi(t,0)_{\#}\mu_{0,R}| \leq \liminf_{n\to\infty} |\varrho_{\varepsilon_n}(t) - \varrho_{\varepsilon_n,R}(t)|_{\mathcal{L}^1(\mathcal{H})} \leq \nu(R)\,, \qquad \text{and}$$
$$\int_{\mathscr{Z}} |\mu_0 - \mu_{0,R}| \leq \liminf_{n\to\infty} |\varrho_{\varepsilon_n} - \varrho_{\varepsilon_n,R}|_{\mathcal{L}^1(\mathcal{H})} \leq \nu(R)\,,$$

where the left hand side denotes the total variation of the signed measures $\mu_t - \Phi(t,0)_{\#}\mu_{0,R}$ and $\mu_0 - \mu_{0,R}$. Therefore, we obtain

$$\int_{\mathscr{Z}} |\mu_t - \Phi(t,0)_{\#}\mu_0| \leq \int_{\mathscr{Z}} |\mu_t - \Phi(t,0)_{\#}\mu_{0,R}| + \int_{\mathscr{Z}} |\mu_{0,R} - \mu_0| \leq 2\nu(R)\,,$$

since the total variation of $\Phi(t,0)_{\#}\mu_{0,R} - \Phi(t,0)_{\#}\mu_0$ and $\mu_{0,R} - \mu_0$ are equal. Letting $R \to \infty$ implies $\mu_t = \Phi(t,0)_{\#}\mu_0$. Thus, the argument above shows

$$\mu_t \in \mathcal{M}(\varrho_\varepsilon(t), \varepsilon \in (0,\bar{\varepsilon})) \Leftrightarrow (\mu_t = \Phi(t,0)_{\#}\mu_0, \mu_0 \in \mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})))\ .$$

It is easy to see that the assumption (2) implies (35). This ends the proof of Theorem 1.1.

5. GROUND STATE ENERGY LIMIT

In this section we give the proof of Theorem 1.2. We recall that we assume (**A**), $m_0 > 0$ and suppose that V is a confining potential (i.e.: $\lim_{|x|\to\infty} V(x) = +\infty$). The classical energy functional related to the Klein-Gordon-Schrödinger system is given by $h(z) = h_0(z) + h_I(z)$ where

$$h_0(z) = \langle z_1, (-\frac{\Delta}{2M} + V)z_1\rangle + \langle z_2, \omega(k) z_2\rangle\,, \quad z = z_1 \oplus z_2 \in D((\frac{-\Delta}{2M} + V)^{1/2}) \oplus D(\omega^{1/2})\,,$$

is the quadratic positive part while $h_I(z)$ is the nonlinear regular one given by

$$h_I(z) = \int_{\mathbb{R}^{2d}} \frac{\chi(k)}{\sqrt{\omega(k)}} |z_1|^2(x) \big(\bar{z}_2(k) e^{-ik\cdot x} + z_2(k) e^{ik\cdot x}\big) dk dx\ , \quad z = z_1 \oplus z_2 \in \mathscr{Z}\ .$$

Actually, the simple inequality $|h_I(z)| \leq 2||z_1||_2^2 ||\frac{\chi}{\sqrt{\omega}}||_2 ||z_2||_2$ holds true as well as the scaling $h(\lambda z) = \lambda^2 h_0(z) + \lambda^3 h_I(z)$ for any $\lambda \in \mathbb{R}$. Therefore, the functional h is unbounded from below whenever χ is different from zero. However, the Nelson Hamiltonian preserves the number of nucleons and the ground state energy of $H_{|\mathscr{H}_n}$ is bounded from below (here $\mathscr{H}_n = L^2_s(\mathbb{R}^{dn}) \otimes \Gamma_s(L^2(\mathbb{R}^d))$ and $L^2_s(\mathbb{R}^{dn})$ is the space of symmetric square integrable functions). This means classically that the Klein-Gordon-Schrödinger system preserves the L^2 norm of z_1 and h is bounded from below under the constraint $||z_1||_2 = \lambda$ with λ fixed.

Lemma 5.1. *Assume* (**A**) *and* $m_0 > 0$. *Then, for any* $\lambda > 0$,

$$\inf_{||z_1||_2=\lambda} h(z_1 \oplus z_2) > -\infty\,.$$

Proof. A phase space translation shows for $z = z_1 \oplus z_2$ such that $||z_1||_2 = \lambda$ that the energy functional can be written as

$$h(z) = \langle z_1, (-\frac{\Delta}{2M} + V)z_1\rangle + \int_{\mathbb{R}^d} \langle \frac{z_2}{\lambda} + \lambda\frac{e^{-ikx}}{\omega^{3/2}}\chi, \omega(k)\,(\frac{z_2}{\lambda} + \lambda\frac{e^{-ikx}}{\omega^{3/2}}\chi)\,\rangle\, |z_1|^2(x)dx - \lambda^4||\frac{\chi}{\omega}||_2^2\,.$$

Observe that $\frac{z_2}{\lambda} + \lambda\frac{e^{-ikx}}{\omega^{3/2}}\chi$ belongs to $\omega^{-1/2}L^2(\mathbb{R}^d)$, so that all the terms make sense. Hence, the quantitative bound $h(z) \geq -\lambda^4||\frac{\chi}{\omega}||_2^2$ holds true. □

a. Upper bound

The upper bound is very simple to prove. It follows by an appropriate choice of trial functions (coherent type states) for the quantum energy.

Lemma 5.2. *Let* $\lambda > 0$. *Then for any* $\varepsilon \in (0, \bar{\varepsilon})$ *and* $n \in \mathbb{N}$ *such that* $n\varepsilon = \lambda^2$,

$$\inf \sigma(H_{|\mathscr{H}_n}) \leq \inf_{||z_1||_2=\lambda} h(z_1 \oplus z_2)\,. \tag{36}$$

Proof. Take for $\lambda > 0$, $z_1 \in C_0^\infty(\mathbb{R}^d)$ such that $||z_1||_2 = \lambda$ and $z_2 \in D(\omega)$, the coherent vector

$$C(z_1 \oplus z_2) = (\frac{z_1}{\lambda})^{\otimes n} \otimes W(\frac{\sqrt{2}}{i\varepsilon} z_2)\Omega\,,$$

with $\Omega = (1, 0 \cdots)$ the vacuum vector of the Fock space $\Gamma_s(L^2(\mathbb{R}^d))$. It is easy to check that $C(z_1 \oplus z_2)$ belongs to the domain $D(H_{|\mathscr{H}_n}) = D(H_{0|\mathscr{H}_n})$ since $(\frac{z_1}{\lambda})^{\otimes n}$ is in $D(d\Gamma(-\frac{\Delta}{2M} + V))$ and $W(\frac{\sqrt{2}}{i\varepsilon} z_2)\Omega$ is in $D(d\Gamma(\omega))$. Using the fact $n\varepsilon = \lambda^2$, an explicit computation yields

$$\langle C(z_1 \oplus z_2), H_{|\mathscr{H}_n} C(z_1 \oplus z_2)\rangle = h(z_1 \oplus z_2)\,.$$

□

b. Lower bound

The lower bound proof is more elaborated and uses an a priori information on Wigner measures of minimizing sequences. It is convenient to work with

$$\mathscr{D} = C_0^\infty(\mathbb{R}^{nd}) \otimes_{alg} (\mathscr{F} \cap D(d\Gamma(\omega))) \,,$$

where $\mathscr{F}$ denotes the dense subspace of finite particles vectors of the Fock space $\Gamma_s(L^2(\mathbb{R}^d))$.

Lemma 5.3. *Let $\lambda > 0$. There exists a normalized minimizing sequence $(\Psi^{(n)})_{n\in\mathbb{N}}$ in $\mathscr{D}$, such that for all $\varepsilon \in (0, \bar\varepsilon)$, $n\varepsilon = \lambda$,*

$$\langle \Psi^{(n)}, H_{|\mathscr{H}_n} \Psi^{(n)} \rangle \leq \frac{1}{n} + \inf \sigma(H_{|\mathscr{H}_n}) \,. \tag{37}$$

Proof. Remember that the Kato-Rellich theorem applies for $H_{|\mathscr{H}_n}$. Therefore $D(H_{|\mathscr{H}_n}) = D(H_{0|\mathscr{H}_n})$ and since $\mathscr{D}$ is a core for $H_{0|\mathscr{H}_n}$ then it is also a core for $H_{|\mathscr{H}_n}$. Thus, one can construct a normalized sequence in $\mathscr{D}$ satisfying the inequality (37) since

$$\inf \sigma(H_{|\mathscr{H}_n}) = \inf_{||\Psi^{(n)}||=1, \Psi^{(n)}\in\mathscr{D}} \langle \Psi^{(n)}, H_{|\mathscr{H}_n} \Psi^{(n)} \rangle \,.$$

□

Lemma 5.4. *Let $(\Psi^{(n)})_{n\in\mathbb{N}}$ be a minimizing sequence as in Lemma 5.3. We can assume that $(\Psi^{(n)})_{n\in\mathbb{N}}$ has a unique Wigner measure μ. Then for any $R > 0$,*

$$\lim_{n\to\infty} \langle \Psi^{(n)}, d\Gamma(1_{|x|\leq R}) \otimes 1\, \Psi^{(n)} \rangle = \int_{\mathscr{Z}} \langle z_1, 1_{|x|\leq R}\, z_1 \rangle \, d\mu(z) \,.$$

Proof. Proposition 4.2 ensures the existence of Wigner measures for $(\Psi^{(n)})_{n\in\mathbb{N}}$ since

$$\langle \Psi^{(n)}, N\, \Psi^{(n)} \rangle \leq \lambda^2 + \langle \Psi^{(n)}, H_{0|\mathscr{H}_n}\, \Psi^{(n)} \rangle \,,$$

and the right hand side is uniformly bounded with respect to $n \in \mathbb{N}$. Moreover, by extracting a subsequence we can always assume that $(\Psi^{(n)})_{n\in\mathbb{N}}$ has a unique Wigner measure.
Let $\tilde\chi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \tilde\chi(x) \leq 1$, $\tilde\chi(x) = 1$ if $|x| \leq 1$ and $\tilde\chi(x) = 0$ if $|x| \geq 2$. Let $\tilde\chi_\kappa(x) = \tilde\chi(\frac{x}{\kappa})$, for $\kappa > 0$.

$$\left| \lambda^2 \langle \Psi^{(n)}, 1_{|x_1|\leq R}\, \Psi^{(n)} \rangle - \int_{\mathscr{Z}} \langle z_1, 1_{|x|\leq R}\, z_1 \rangle \, d\mu(z) \right| \leq \left| \lambda^2 \langle \Psi^{(n)}, 1_{|x_1|\leq R} \,[1 - \tilde\chi_\kappa(D_{x_1}^2)]\, \Psi^{(n)} \rangle \right| \tag{38}$$

$$+ \Big| \lambda^2 \langle \Psi^{(n)}, 1_{|x_1|\leq R}\, \tilde\chi_\kappa(D_{x_1}^2)\, \Psi^{(n)} \rangle \tag{39}$$

$$- \int_{\mathscr{Z}} \langle z_1, 1_{|x|\leq R} \tilde\chi_\kappa(D_x^2)\, z_1 \rangle \, d\mu(z) \Big| \tag{40}$$

$$+ \left| \int_{\mathscr{Z}} \langle z_1, 1_{|x|\leq R} [\tilde\chi_\kappa(D_x^2) - 1]\, z_1 \rangle \, d\mu(z) \right| \tag{41}$$

The first term in right hand side can be estimated by

$$\left|\langle \Psi^{(n)}, 1_{|x_1|\leq R}\,[1-\tilde{\chi}_\kappa(D^2_{x_1})]\,\Psi^{(n)}\rangle\right| \leq ||(1+D^2_{x_1})^{\frac{1}{2}}\,\Psi^{(n)}||\;||[1-\tilde{\chi}_\kappa(D^2_x)](1+D^2_x)^{-\frac{1}{2}}||. \tag{42}$$

So, the left hand side of (42) tend to zero, uniformly with respect to $R>0$, when $\kappa\to\infty$. Now, observe that the operator $1_{|x|\leq R}\,\tilde{\chi}_\kappa(D^2_x)$ is compact. Then by Proposition 4.5 and Remark 4.6, we get for all $\kappa>0$

$$\begin{aligned}\lim_{n\to\infty}\lambda^2\langle\Psi^{(n)}, 1_{|x_1|\leq R}\,\tilde{\chi}_\kappa(D^2_{x_1})\,\Psi^{(n)}\rangle &= \lim_{n\to\infty}\langle\Psi^{(n)}, d\Gamma(1_{|x|\leq R}\,\tilde{\chi}_\kappa(D^2_{x_1}))\otimes 1\,\Psi^{(n)}\rangle\\ &= \int_{\mathscr{Z}}\langle z_1, 1_{|x|\leq R}\,\tilde{\chi}_\kappa(D^2_x)\,z_1\rangle\,d\mu(z)\,.\end{aligned}$$

Since $\tilde{\chi}_\kappa(D^2_x)$ converges strongly to 1, we see by dominated convergence theorem that

$$\lim_{\kappa\to\infty}\int_{\mathscr{Z}}\langle z_1, 1_{|x|\leq R}[\tilde{\chi}_\kappa(D^2_x)-1]\,z_1\rangle\,d\mu(z)=0\,.$$

Hence an $\eta/3$-argument proves the limit. □

Lemma 5.5. *Let $\lambda>0$ and $(\Psi^{(n)})_{n\in\mathbb{N}}$ be a minimizing sequence as in Lemma 5.3. Then there exists $C>0$ such that for any $R>0$ and any $n\in\mathbb{N}$, $n\varepsilon=\lambda^2$,*

$$\langle\Psi^{(n)}, d\Gamma(1_{|x|\leq R})\otimes 1\,\Psi^{(n)}\rangle \geq \lambda^2-\frac{C}{C(R)}$$

with $C(R)=\inf\{V(x), |x|>R\}$.

Proof. Remark that

$$\begin{aligned}\lambda^2 &= \langle\Psi^{(n)}, N_1\,\Psi^{(n)}\rangle\\ &= \langle\Psi^{(n)}, d\Gamma(1_{|x|\leq R})\otimes 1\,\Psi^{(n)}\rangle + \langle\Psi^{(n)}, d\Gamma(1_{|x|>R})\otimes 1\,\Psi^{(n)}\rangle.\end{aligned}$$

Using Lemma 2.1, one can see that $H_{I|\mathscr{H}_n}$ is bounded by $H_{02}^{1/2}$ uniformly in $n\in\mathbb{N}$, in the operator sense. Hence, $(\langle\Psi^{(n)}, H_{02}+H_I\,\Psi^{(n)}\rangle)_{n\in\mathbb{N}}$ is bounded from below and since $\Psi^{(n)}$ is a minimizing sequence there exists $C>0$ such that $\langle\Psi^{(n)}, d\Gamma(V(x))\otimes 1\,\Psi^{(n)}\rangle\leq C$. Using the inequality $d\Gamma(V(x))\geq C(R)d\Gamma(1_{|x|>R})$, one obtains

$$\begin{aligned}\langle\Psi^{(n)}, d\Gamma(1_{|x|\leq R})\otimes 1\,\Psi^{(n)}\rangle &= \lambda^2-\langle\Psi^{(n)}, d\Gamma(1_{|x|>R})\otimes 1\,\Psi^{(n)}\rangle\\ &\geq \lambda^2-\frac{C}{C(R)}\,.\end{aligned}$$

□

Lemma 5.6. *Let $(\Psi^{(n)})_{n\in\mathbb{N}}$ be a minimizing sequence as in Lemma 5.3. Then any Wigner measure $\mu\in\mathscr{M}(\psi^{(n)}, n\in\mathbb{N}, \varepsilon n=\lambda^2)$ is supported on $S(0,\lambda)\times L^2(\mathbb{R}^d)$ where $S(0,\lambda)$ is the sphere of $L^2(\mathbb{R}^d)$ of radius λ.*

Proof. Observe that $\langle \Psi^{(n)}, N_1^k \Psi^{(n)} \rangle = \lambda^{2k}$ for all $k \in \mathbb{N}$. Hence, [4, theorem 6.2] shows that μ is supported on $B(0,\lambda) \times L^2(\mathbb{R}^d)$ where $B(0,\lambda)$ is the ball in $L^2(\mathbb{R}^d)$ of radius λ centered at the origin. Using Lemma 5.5 and Lemma 5.4, we obtain for any $R > 0$

$$\int_{B(0,\lambda)\times L^2(\mathbb{R}^d)} ||z_1||_2^2 \, d\mu(z) \geq \int_{B(0,\lambda)\times L^2(\mathbb{R}^d)} \langle z_1, 1_{|x|\leq R} z_1 \rangle \, d\mu(z) \geq \lambda^2 - \frac{C}{C(R)}.$$

Recall that $\lim_{|x|\to\infty} V(x) = +\infty$ so that $C(R) \to \infty$ when $R \to \infty$. □

Lemma 5.7. *For any $\lambda > 0$,*

$$\liminf_{n\to\infty, n\varepsilon=\lambda^2} \inf \sigma(H_{|\mathscr{H}_n}) \geq \inf_{||z_1||_2=\lambda} h(z_1 \oplus z_2)\,.$$

Proof. Let $(\Psi^{(n)})_{n\in\mathbb{N}}$ be a minimizing sequence as in Lemma 5.3. Recall that the annihilation distribution $a(k), k \in \mathbb{R}^d$ is a well defined operator $a(.) : \mathscr{F} \to L^2\left(\mathbb{R}^d, \Gamma_s(L^2(\mathbb{R}^d))\right)$. A direct computation, using symmetry and Fubini, gives

$$\begin{aligned}\theta(n) := \langle \Psi^{(n)}, H_{02} + H_{I|\mathscr{H}_n} \Psi^{(n)} \rangle &= \int_{\mathbb{R}^d} ||a(k)\Psi^{(n)}||^2_{\Gamma_s(L^2(\mathbb{R}^d))} \, \omega(k) \, dk \\ &+ \lambda^2 \int_{\mathbb{R}^d\times\mathbb{R}^{dn}} \frac{e^{ikx_1}}{\sqrt{\omega(k)}} \chi(k) \, \left(\langle \Psi^{(n)}, a(k)\Psi^{(n)} \rangle_{\Gamma_s(L^2(\mathbb{R}^d))} + hc \right) dk dx\,.\end{aligned}$$

Therefore, we can write

$$\theta(n) = \int_{\mathbb{R}^d\times\mathbb{R}^{dn}} \omega(k) || \left(a(k) + \lambda^2 \frac{e^{-ikx_1}}{\omega(k)^{3/2}} \chi(k) \right) \Psi^{(n)} ||^2_{\Gamma_s(L^2(\mathbb{R}^d))} \, dx dk - \lambda^4 ||\frac{\chi}{\omega}||_2^2\,. \tag{43}$$

Taking any cut off function $0 \leq \tilde{\chi} \leq 1$. Let Q_κ be a sequence of positive finite rank operators such that $0 \leq Q_\kappa \leq \tilde{\chi}\omega$ and $Q_\kappa \overset{w}{\to} \tilde{\chi}\omega$. Let $\{e_\alpha\}_{\alpha\in\mathbb{N}}$ be an O.N.B of $L^2(\mathbb{R}^d)$ so that $Q_\kappa = \sum_{\alpha=0}^r t_\alpha |e_\alpha\rangle\langle e_\alpha|$ (for simplicity the dependence on κ is omitted). Expanding all the integrals and sums in (43), then using $Q_\kappa \leq \tilde{\chi}\omega$, one proves

$$\begin{aligned}\theta(n) \geq \langle \Psi^{(n)}, 1 \otimes d\Gamma(Q_\kappa)\Psi^{(n)} \rangle &+ \sum_{\alpha=0}^r t_\alpha \langle a(e_\alpha)\Psi^{(n)}, d\Gamma(\frac{\widehat{\chi \bar{e}_\alpha}}{\omega^{3/2}}) \otimes 1 \Psi^{(n)} \rangle + hc \\ &+ \lambda^2 \sum_{\alpha=0}^r t_\alpha \langle \Psi^{(n)}, d\Gamma(|\frac{\widehat{\chi \bar{e}_\alpha}}{\omega^{3/2}}|^2) \otimes 1 \Psi^{(n)} \rangle - \lambda^4 ||\frac{\chi}{\omega}||_2^2\,.\end{aligned}$$

The right hand side is the expectation value of a Wick operator with symbol given by

$$\Theta(z) = \langle z_2, Q_\kappa z_2 \rangle + \int_{\mathbb{R}^d} \left(\langle z_2, Q_\kappa \frac{\chi e^{-ikx}}{\omega^{3/2}} \rangle + hc \right) |z_1(x)|^2 dx + \lambda^2 \sum_{\alpha=0}^r t_\alpha \langle z_1, |\frac{\widehat{\chi \bar{e}_\alpha}}{\omega^{3/2}}|^2 z_1 \rangle - \lambda^4 ||\frac{\chi}{\omega}||_2^2\,.$$

In this symbol some monomials have non "compact kernels" (see the discussion in Section 4 a). So, using the same approximation scheme as in Definition 3.13 and Lemma 3.14, we show

$$\begin{aligned}\theta(n) \geq \langle \Psi^{(n)}, 1 \otimes d\Gamma(Q_\kappa)\Psi^{(n)} \rangle &+ \sum_{\alpha=0}^r t_\alpha \langle a(e_\alpha)\Psi^{(n)}, d\Gamma(\frac{\widehat{\chi \bar{e}_\alpha}}{\omega^{3/2}} g_m(i\partial_x)) \otimes 1 \Psi^{(n)} \rangle + hc \\ &+ \lambda^2 \sum_{\alpha=0}^r t_\alpha \langle \Psi^{(n)}, d\Gamma(|\frac{\widehat{\chi \bar{e}_\alpha}}{\omega^{3/2}}|^2 g_m(i\partial_x)) \otimes 1 \; \Psi^{(n)} \rangle - \lambda^4 ||\frac{\chi}{\omega}||_2^2 + O(m^{-1})\,,\end{aligned}$$

with an error uniform in $n \in \mathbb{N}$. Now, the point is that the right hand side is an expectation value of a Wick quantization with compact kernel symbol. We can apply the same argument as in Proposition 4.5 and Remark 4.6. Therefore, we obtain

$$\liminf_{n\to\infty} \theta_n \geq \int_{\mathscr{Z}} \Theta_m(z)\, d\mu(z)\,,$$

where μ is the Wigner measure of the sequence $(\Psi^{(n)})_{n\in\mathbb{N}}$ and

$$\Theta_m(z) = \langle z_2, Q_\kappa z_2\rangle + \sum_{\alpha=0}^{r} t_\alpha \left(\langle z_2, e_\alpha\rangle \langle z_1, \frac{\widehat{\chi \bar{e}_\alpha}}{\omega^{3/2}} g_m(i\partial_x) z_1\rangle + hc \right)$$
$$+\lambda^2 \langle z_1, |\frac{\widehat{\chi \bar{e}_\alpha}}{\omega^{3/2}}| g_m(i\partial_x) z_1\rangle - \lambda^4 ||\frac{\chi}{\omega}||_2^2\,.$$

We can remove, by dominated convergence, the cut off g_m and let $\kappa \to \infty$. So we obtain

$$\liminf_{n\to\infty} \theta_n \geq \int_{\mathscr{Z}} \langle z_2, \omega z_2\rangle + h_I(z)\, d\mu(z)\,,$$

Now, a similar argument of approximation from below gives

$$\langle \Psi^{(n)}, d\Gamma(\frac{-\Delta}{2M} + V) \otimes 1 \Psi^{(n)}\rangle \geq \langle \Psi^{(n)}, d\Gamma(\tilde{\chi}(\frac{-\Delta}{2M} + V)) \otimes 1 \Psi^{(n)}\rangle\,,$$

where $\tilde{\chi}$ is a cut off function, $\tilde{\chi}(x) = x$ on $|x| \leq 1$, so that $\tilde{\chi}(\frac{-\Delta}{2M} + V)$ is a compact operator. Applying Proposition 4.5, we get

$$\liminf_{n\to\infty, n\varepsilon=\lambda^2} \langle \Psi^{(n)}, H_{|\mathscr{H}_n} \Psi^{(n)}\rangle \geq \int_{\mathscr{Z}} h(z) d\mu(z).$$

Therefore, we obtain

$$\inf_{||z_1||=\lambda} h(z) \leq \int_{S(0,\lambda)\times L^2(\mathbb{R}^d)} h(z)\, d\mu(z) \leq \liminf_{n\to\infty, n\varepsilon=\lambda^2} \langle \Psi^{(n)}, H_{|\mathscr{H}_n} \Psi^{(n)}\rangle \leq \liminf_{n\to\infty, n\varepsilon=\lambda^2} \inf \sigma(H_{|\mathscr{H}_n})\,,$$

since by Lemma 5.6 the Wigner measure μ is supported on the sphere of radius λ. □

Thus, Lemma 5.7 and Lemma 5.2 imply Theorem 1.2.

Remark 5.8. It is not difficult to show that the infimum of the classical energy h, under the constraint $||z_1||_2 = \lambda$, is actually a minimum.

Appendix A: Estimates on Fock space.

We provide some technical results used throughout the paper and proved here for general Hilbert spaces.

Lemma A.1. *Let $\mathscr{Y}$ be an Hilbert space, $\Gamma_s(\mathscr{Y})$ the corresponding symmetric Fock space (with $a^{\#}$, N, $W(\xi)$ the annihilation/creation, number and Weyl operators respectively).*

Let y be a positive self-adjoint operator on $\mathscr{Y}$ with domain $D(y)$; and let $d\Gamma(y)$ be the second quantization of y, with form domain $D(Y^{1/2})$. Then for all $\xi \in D(y^{1/2})$, and $\phi_1, \phi_2 \in D(Y^{1/2})$:

$$\langle \phi_1, W^*(\xi) d\Gamma(y) W(\xi) \phi_2 \rangle = \langle \phi_1, \big(d\Gamma(y) + \frac{i\varepsilon}{\sqrt{2}} (a^*(y\xi) - a(y\xi)) + \frac{\varepsilon^2}{2} \langle \xi, y\xi \rangle_{\mathscr{Y}} \big) \phi_2 \rangle \ .$$

Proof. Let $\xi \in D(y^{1/2})$ be fixed, let $\phi_1, \phi_2 \in D(N)$. Furthermore, let $(y_m)_{m \in \mathbb{N}} \in \mathcal{L}(\mathscr{Y})$ be a sequence of bounded operators that converges strongly to y on $D(y)$, with $y_m \leq y$ for all m. Then we define, for all $\lambda \in \mathbb{R}$,

$$M(\lambda) := \langle \phi_1, W(\lambda\xi) \big(d\Gamma(y_m) + \frac{i\lambda\varepsilon}{\sqrt{2}} (a^*(y_m\xi) - a(y_m\xi)) + \frac{\lambda^2 \varepsilon^2}{2} \langle \xi, y_m\xi \rangle_{\mathscr{Y}} \big) W^*(\lambda\xi) \phi_2 \rangle \ .$$

We remark that for every $\delta \geq 0$ the Weyl operator maps $D(N^\delta)$ into itself. Taking the derivative in λ, we obtain

$$\frac{d}{d\lambda} M(\lambda) = \langle W^*(\lambda\xi)\phi_1, i\big[\varphi(\lambda\xi) \ , \ d\Gamma(y_m) + \frac{i\lambda\varepsilon}{\sqrt{2}} (a^*(y_m\xi) - a(y_m\xi))\big] W^*(\lambda\xi)\phi_2 \rangle + \langle W^*(\lambda\xi)\phi_1, \\ \big(\frac{i\varepsilon}{\sqrt{2}} (a^*(y_m\xi) - a(y_m\xi)) + \lambda\varepsilon^2 \langle \xi, y_m\xi \rangle_{\mathscr{Y}} \big) W^*(\lambda\xi)\phi_2 \rangle = 0 \ .$$

Hence for all $\phi_1, \phi_2 \in D(N)$ we obtain, by $M(0) = M(1)$, for all $m \in \mathbb{N}$:

$$\text{(A1)} \quad \langle \phi_1, W^*(\xi) d\Gamma(y_m) W(\xi) \phi_2 \rangle = \langle \phi_1, \big(d\Gamma(y_m) + \frac{i\varepsilon}{\sqrt{2}} (a^*(y_m\xi) - a(y_m\xi)) + \frac{\varepsilon^2}{2} \langle \xi, y_m\xi \rangle_{\mathscr{Y}} \big) \phi_2 \rangle \ .$$

Choose now $\phi_1 = \phi_2 = \phi \in D(Y^{1/2}) \cap D(N)$. Then

$$\langle \phi, W^*(\xi) d\Gamma(y_m) W(\xi) \phi \rangle \leq \| d\Gamma(y)^{1/2} \phi \|^2 + \sqrt{2}\varepsilon \| y^{1/2}\xi \|_{\mathscr{Y}} \| d\Gamma(y)^{1/2}\phi \| \|\phi\| + \frac{\varepsilon^2}{2} \| y^{1/2}\xi \|_{\mathscr{Y}}^2 \|\phi\| \ .$$

By monotone convergence theorem, the left hand side converges to $\langle \phi, W^*(\xi) d\Gamma(y) W(\xi) \phi \rangle$ when $m \to \infty$, since $d\Gamma(y)$ is a closed operator. The result extends by density to all $\phi \in D(Y^{1/2})$; so the Weyl operator W maps the form domain of $d\Gamma(y)$ into itself. Then for all $\phi_1, \phi_2 \in D(Y^{1/2}) \cap D(N)$, we can take the limit $m \to \infty$ in (A1). The result is then extended by density to all $\phi_1, \phi_2 \in D(Y^{1/2})$. □

Corollary A.2. *i) Let $\xi \in D(y)$. Then $(d\Gamma(y) + 1)^{-1} W(\xi) (d\Gamma(y) + 1) \in \mathcal{L}(\Gamma_s(\mathscr{Y}))$. Furthermore, there exists $C(\|y\xi\|_{\mathscr{Y}}, \|\xi\|_{\mathscr{Y}}) > 0$ independent of ε such that:*

$$|(d\Gamma(y) + 1)^{-1} W(\xi) (d\Gamma(y) + 1)|_{\mathcal{L}(\Gamma_s(\mathscr{Y}))} \leq C(\|y\xi\|_{\mathscr{Y}}, \|\xi\|_{\mathscr{Y}})(1 + O(\varepsilon)) \ .$$

ii) Let y be a positive bounded *operator and let $\xi \in \mathscr{Y}$. Then for any $\delta_1 > 0$ and $\delta_2 \in \mathbb{R}$, $(d\Gamma(y)^{\delta_1}+1)^{-\delta_2}W(\xi)(d\Gamma(y)^{\delta_1}+1)^{\delta_2} \in \mathcal{L}(\Gamma_s(\mathscr{Y}))$. Furthermore, there exists a constant $C(\delta_1,\delta_2,\|\xi\|_{\mathscr{Y}},|y|_{\mathcal{L}(\mathscr{Y})}) > 0$ independent of ε such that:*

$$|(d\Gamma(y)^{\delta_1}+1)^{-\delta_2}W(\xi)(d\Gamma(y)^{\delta_1}+1)^{\delta_2}|_{\mathcal{L}(\Gamma_s(\mathscr{Y}))} \leq C(\delta_1,\delta_2,\|\xi\|_{\mathscr{Y}},|y|_{\mathcal{L}(\mathscr{Y})})(1+O(\varepsilon))\ .$$

The following proposition is a useful adaptation of [3, Lemmas B.4 and B.6]:

Proposition A.3. *Let $\mathscr{Y}$ be an Hilbert space, $\Gamma_s(\mathscr{Y})$ the corresponding symmetric Fock space.*

Let y_1, y_2 be two operators on $\mathscr{Y}$ such that $(y_2+1)^{-1}y_1 \in \mathcal{L}(\mathscr{Y})$. Then $(d\Gamma(y_2^*y_2+1)+1)^{-1}d\Gamma(y_1) \in \mathcal{L}(\Gamma_s(\mathscr{Y}))$, with:

$$|(d\Gamma(y_2^*y_2+1)+1)^{-1}d\Gamma(y_1)|_{\mathcal{L}(\Gamma_s(\mathscr{Y}))} \leq (1+\sqrt{2})|(y_2+1)^{-1}y_1|_{\mathcal{L}(\mathscr{Y})}\ .$$

Proof. Let $\phi_1,\phi_2 \in D(d\Gamma(y_1))$. Then ($y(j)$ is the operator acting on the j-th variable):

$$\begin{aligned}|\langle\phi_1, d\Gamma(y_1)\phi_2\rangle| &\leq \sum_n |\varepsilon\langle\phi_{1n}, \sum_{j=1}^n y_1(j)\phi_{2n}\rangle| \leq \sum_n |\varepsilon n\langle\phi_{1n}, (y_2(1)+1)(y_2(1)+1)^{-1}y_1(1)\phi_{2n}\rangle| \\ &\leq |(y_2+1)^{-1}y_1|_{\mathcal{L}(\mathscr{Y})}\sum_n \|\phi_{2n}\|\big(\|\varepsilon n\phi_{1n}\| + \|\varepsilon n y_2(1)\phi_{1n}\|\big)\ .\end{aligned}$$

However, we have that:

$$\begin{aligned}\|\varepsilon n y_2(1)\phi_{1n}\|^2 &= \langle\phi_{1n}, \varepsilon^2 n^2 y_2^*(1)y_2(1)\phi_{1n}\rangle = \langle\phi_{1n}, d\Gamma(1)d\Gamma(y_2^*y_2)\phi_{1n}\rangle \leq \frac{1}{2}\langle\phi_{1n}, \Big(\big(d\Gamma(1)\big)^2 \\ &\quad + \big(d\Gamma(y_2^*y_2)\big)^2\Big)\phi_{1n}\rangle \leq \frac{1}{2}\Big(\|d\Gamma(1)\phi_{1n}\|^2 + \|d\Gamma(y_2^*y_2)\phi_{1n}\|^2\Big)\ .\end{aligned}$$

Hence, we obtain for any $\phi_1,\phi_2 \in \Gamma_s(\mathscr{Y})$:

$$\begin{aligned}|\langle\phi_1, (d\Gamma(y_2^*y_2+1)+1)^{-1}d\Gamma(y_1)\phi_2\rangle| &\leq (1+\sqrt{2})|(y_2+1)^{-1}y_1|_{\mathcal{L}(\mathscr{Y})}\sum_n \|\phi_{1n}\|\,\|\phi_{2n}\| \\ &\leq (1+\sqrt{2})|(y_2+1)^{-1}y_1|_{\mathcal{L}(\mathscr{Y})}\|\phi_1\|\,\|\phi_2\|\ .\end{aligned}$$

□

ACKNOWLEDGMENTS

The second author has been supported by the Centre Henri Lebesgue (programme "Investissements d'avenir" — ANR-11-LABX-0020-01).

The article is published in the Journal of Statistical Physics. The final publication is available at Springer.

[1] A. Abdesselam and D. Hasler. Analyticity of the ground state energy for massless Nelson models. *Comm. Math. Phys.*, 310(2):511–536, 2012. ISSN 0010-3616. doi:10.1007/s00220-011-1407-6. URL http://dx.doi.org/10.1007/s00220-011-1407-6.

[2] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005. ISBN 978-3-7643-2428-5; 3-7643-2428-7.

[3] Z. Ammari and F. Nier. Mean field propagation of infinite dimensional Wigner measures with a singular two-body interaction potential. *Annali della Scuola Normale Sup.*, to appear.

[4] Z. Ammari and F. Nier. Mean field limit for bosons and infinite dimensional phase-space analysis. *Ann. Henri Poincaré*, 9(8):1503–1574, 2008. ISSN 1424-0637. doi:10.1007/s00023-008-0393-5. URL http://dx.doi.org/10.1007/s00023-008-0393-5.

[5] Z. Ammari and F. Nier. Mean field limit for bosons and propagation of Wigner measures. *J. Math. Phys.*, 50(4):042107, 16, 2009. ISSN 0022-2488. doi:10.1063/1.3115046. URL http://dx.doi.org/10.1063/1.3115046.

[6] Z. Ammari and F. Nier. Mean field propagation of Wigner measures and BBGKY hierarchies for general bosonic states. *J. Math. Pures Appl. (9)*, 95(6):585–626, 2011. ISSN 0021-7824. doi:10.1016/j.matpur.2010.12.004. URL http://dx.doi.org/10.1016/j.matpur.2010.12.004.

[7] A. Arai. Ground state of the massless Nelson model without infrared cutoff in a non-Fock representation. *Rev. Math. Phys.*, 13(9):1075–1094, 2001. ISSN 0129-055X. doi:10.1142/S0129055X01000934. URL http://dx.doi.org/10.1142/S0129055X01000934.

[8] V. Bach, J. Fröhlich, and I. M. Sigal. Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field. *Comm. Math. Phys.*, 207(2):249–290, 1999. ISSN 0010-3616. doi:10.1007/s002200050726. URL http://dx.doi.org/10.1007/s002200050726.

[9] A. Bachelot. Problème de Cauchy pour des systèmes hyperboliques semi-linéaires. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(6):453–478, 1984. ISSN 0294-1449. URL http://www.numdam.org/item?id=AIHPC_1984__1_6_453_0.

[10] V. Betz, F. Hiroshima, J. Lőrinczi, R. A. Minlos, and H. Spohn. Ground state properties of the Nelson Hamiltonian: a Gibbs measure-based approach. *Rev. Math. Phys.*, 14(2):173–198, 2002. ISSN 0129-055X. doi:10.1142/S0129055X02001119. URL http://dx.doi.org/10.1142/S0129055X02001119.

[11] J. Dereziński and C. Gérard. *Scattering theory of classical and quantum N-particle systems*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997. ISBN 3-540-62066-4.

[12] J. Dereziński and C. Gérard. Asymptotic completeness in quantum field theory. Massive

Pauli-Fierz Hamiltonians. *Rev. Math. Phys.*, 11(4):383–450, 1999. ISSN 0129-055X. doi:10.1142/S0129055X99000155. URL http://dx.doi.org/10.1142/S0129055X99000155.

[13] M. Falconi. Classical limit of the Nelson model with cutoff. *J. Math. Phys.*, 54(1):012303, 30, 2013. ISSN 0022-2488. doi:10.1063/1.4775716. URL http://dx.doi.org/10.1063/1.4775716.

[14] J. Fröhlich, M. Griesemer, and B. Schlein. Asymptotic completeness for Compton scattering. *Comm. Math. Phys.*, 252(1-3):415–476, 2004. ISSN 0010-3616. doi:10.1007/s00220-004-1180-x. URL http://dx.doi.org/10.1007/s00220-004-1180-x.

[15] V. Georgescu, C. Gérard, and J. S. Møller. Spectral theory of massless Pauli-Fierz models. *Comm. Math. Phys.*, 249(1):29–78, 2004. ISSN 0010-3616. doi:10.1007/s00220-004-1111-x. URL http://dx.doi.org/10.1007/s00220-004-1111-x.

[16] C. Gérard, F. Hiroshima, A. Panati, and A. Suzuki. Infrared problem for the Nelson model on static space-times. *Comm. Math. Phys.*, 308(2):543–566, 2011. ISSN 0010-3616. doi:10.1007/s00220-011-1289-7. URL http://dx.doi.org/10.1007/s00220-011-1289-7.

[17] J. Ginibre and G. Velo. Long range scattering and modified wave operators for the wave-Schrödinger system. *Ann. Henri Poincaré*, 3(3):537–612, 2002. ISSN 1424-0637. doi:10.1007/s00023-002-8627-4. URL http://dx.doi.org/10.1007/s00023-002-8627-4.

[18] J. Ginibre, F. Nironi, and G. Velo. Partially classical limit of the Nelson model. *Ann. Henri Poincaré*, 7 (1):21–43, 2006. ISSN 1424-0637. doi:10.1007/s00023-005-0240-x. URL http://dx.doi.org/10.1007/s00023-005-0240-x.

[19] N. Hayashi and W. von Wahl. On the global strong solutions of coupled Klein-Gordon-Schrödinger equations. *J. Math. Soc. Japan*, 39(3):489–497, 1987. ISSN 0025-5645. doi:10.2969/jmsj/03930489. URL http://dx.doi.org/10.2969/jmsj/03930489.

[20] K. Hepp. The classical limit for quantum mechanical correlation functions. *Comm. Math. Phys.*, 35: 265–277, 1974. ISSN 0010-3616.

[21] J. S. Møller. The translation invariant massive Nelson model. I. The bottom of the spectrum. *Ann. Henri Poincaré*, 6(6):1091–1135, 2005. ISSN 1424-0637. doi:10.1007/s00023-005-0234-8. URL http://dx.doi.org/10.1007/s00023-005-0234-8.

[22] E. Nelson. Interaction of nonrelativistic particles with a quantized scalar field. *J. Mathematical Phys.*, 5:1190–1197, 1964. ISSN 0022-2488.

[23] A. Pizzo. One-particle (improper) states in Nelson's massless model. *Ann. Henri Poincaré*, 4(3): 439–486, 2003. ISSN 1424-0637. doi:10.1007/s00023-003-0136-6. URL http://dx.doi.org/10.1007/s00023-003-0136-6.

[24] A. Shimomura. Scattering theory for the coupled Klein-Gordon-Schrödinger equations in two space dimensions. II. *Hokkaido Math. J.*, 34(2):405–433, 2005. ISSN 0385-4035.

[25] B. Simon. *Trace ideals and their applications*, volume 120 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2005. ISBN 0-8218-3581-5.

[26] H. Spohn. Ground state of a quantum particle coupled to a scalar Bose field. *Lett. Math. Phys.*, 44(1):

9–16, 1998. ISSN 0377-9017. doi:10.1023/A:1007473300274. URL http://dx.doi.org/10.1023/A:1007473300274.