Quantum Markov chains associated with open quantum random walks

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Abstract

In this paper we construct (nonhomogeneous) quantum Markov chains associated with open quantum random walks. The quantum Markov chain, like the classical Markov chain, is a fundamental tool for the investigation of the basic properties of the underlying dynamics such as reducibility/irreducibility, recurrence/transience, accessibility, ergodicity, etc. So, the quantum Markov chain machinery opens many new features of the dynamics. On the other hand, as will be shown in this paper, the open quantum random walks serves as a very interesting nontrivial model for which one can construct the associated quantum Markov chains. Here, after constructing the quantum Markov chain associated with the open quantum random walks, we focus on the discussion of the reducibility and irreducibility of open quantum random walks via the corresponding quantum Markov chains. Particularly we show that the concept of reducibility/irreducibility of open quantum random walks in this approach is equivalent to the one previously done by Carbone and Pautrat. We provide with some examples. We see also that the classical Markov chains can be reconstructed as quantum Markov chains.

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1   Introduction

The purpose of this paper is to construct the quantum Markov chains (QMCs hereafter) associated with the open quantum random walks (OQRWs) and investigate some interesting properties. Here we focus on the reducibility and irreducibility of QMCs for OQRWs.

The OQRWs were introduced by Attal, *et al.* in [13, 14, 15] to model the quantum random walks. In particular, the OQRWs were developed to formulate the dissipative quantum computing algorithms and dissipative quantum state preparation. In that paper the authors introduced the concept of quantum trajectories. This is a repeated process of completely positive mapping on a state (an evolution of OQRW, see the next section for the detail) and a measurement of the position. By this they constructed a (classical) Markov chain. Using this Markov chain, Attal *et al.* established a central limit theorem for the asymptotic behavior of the OQRWs [13].

Recently the dynamical behavior of OQRWs drew many interests and some works have been done for the ergodicity, hitting times, recurrence, reducibility, etc, of OQRWs [17, 18, 21, 22]. In [18], Dhahri and Mukhamedov constructed the QMCs for the OQRWs and investigated recurrence and accessibility. The approach by the QMCs for some (quantum) dynamical systems serves as a fundamental and strong tool in the investigation of further properties of the underlying dynamics. Otherwise, it would be hard to understand the properties as much as done by the method. As we will see in the sequel, the reducibility and irreducibility properties of OQRWs can be successfully treated, though this property itself has been studied without appealing to the QMCs [17]. We emphasize, however, that many further properties of the dynamics can be shown if we follow the QMC approach [18].

On the other hand the QMC was introduced by Accardi [1, 2, 3] and further developed [6, 7], and has found several applications. See e.g., [4, 5, 9, 10, 11] and references therein. Here the model of OQRWs is a nontrivial dynamical system to

which the QMC approach can be applied. The main ingredient for the QMC is the transition expectation, which is a completely positive map and it is a quantum version of the transition matrix for the classical Markov chains [6, 7]. See Section 3 for the details. Accardi and Koroliuk, after defining the QMC, developed the quantum versions of reducibility and irreducibility, accessibility, recurrence and transience [6, 7]. In this paper we adopt the construction of QMCs for OQRWs introduced in [18] with some modifications. A remarkable point in our construction is that we have introduced the sub-Markovian transition expectations, contrasting to the fact that it is generally required to have Markovianity for the transition expectations. The Markovianity is recovered when we talk together with the initial conditions and the transition expectations. It seems that this approach is necessary when we try to recover the original dynamics. Another typical notice in our construction is that we have considered the nonhomogeneous quantum Markov chains instead of homogeneous ones. This is also necessary to recover the original dynamics unless we start with an initial state which is invariant under the dynamics. After constructing the QMCs associated with OQRWs, we study the reducibility and irreducibility of the OQRWs in the language of the constructed QMCs. We give some sufficient conditions for reducibility/irreducibility providing with some examples. We separately show that the classical Markov chains are reconstructed by the quantum Markov chains and the classical reducibility/irreducibility can be studied by the language of QMCs.

Let us briefly overview the contents of this paper. In Section 2, we recall the definition of OQRWs as defined in [14]. Section 3 summarizes the construction of QMCs. Section 4 is the main part of this paper. We construct the nonhomogeneous QMCs associated with OQRWs using (sub-Markovian) transition expectations. We then develop a characterization for the reducibility/irreducibility (Theorem 4.12) and give some sufficient conditions for reducibility (Theorem 4.13) and irreducibility (Theorem 4.15). Section 5 is devoted to the examples. We construct some examples of reducible and irreducible OQRWs in 1-dimensional integer lattice. We also investigate the relation with classical Markov chains. In Subsection 5.2 we construct a QMC for a given classical Markov chain. We show that our construction is natural in the sense that it realizes the original classical Markov chain. We then compare the reducibility and irreducibility properties viewed in quantum and classical Markov chains. Finally, in the Appendix we compare with the previous results on the reducibility/irreducibility for OQRWs studied by Carbone and Pautrat [17].

In fact, it turns out that the concepts of reducibility/irreducibility of OQRWs given in [17] and in the present paper are equivalent.

2   Open quantum random walks

In this section we briefly introduce the open quantum random walks.

Let $\mathcal{K}$ be a separable Hilbert space with an orthonormal basis $\{|i\rangle\}_{i \in \Lambda}$ indexed by the vertices of some graph Λ. Here the set Λ of vertices may be finite or countably infinite. Let $\mathcal{H}$ be another separable Hilbert space, which will describe the degrees of freedom given at each point of Λ. We consider the space $\mathcal{H} \otimes \mathcal{K}$.

For each pair $i, j \in \Lambda$ we give a bounded linear operator B_j^i on $\mathcal{H}$. This operator stands for the effect of passing from j to i. We assume that for each j

$$\sum_i B_j^{i*} B_j^i = I, \tag{2.1}$$

where the series is strongly convergent to the identity operator I. This constraint means that the sum of all the effects leaving site j is I. We dilate the operators B_j^i on $\mathcal{H}$ as operators on $\mathcal{H} \otimes \mathcal{K}$ by defining

$$M_j^i = B_j^i \otimes |i\rangle\langle j|.$$

The operators M_j^i encodes exactly the idea that while passing from j to i on the space, the effect is the operator B_j^i on $\mathcal{H}$. By (2.1), it is easy to see that

$$\sum_{i,j} M_j^{i*} M_j^i = I. \tag{2.2}$$

Using the operators $\{M_j^i\}_{i,j}$, define a completely positive map on $\mathcal{I}_1(\mathcal{H} \otimes \mathcal{K})$, the ideal of trace class operators, by:

$$\mathcal{M}(\rho) = \sum_i \sum_j M_j^i \rho M_j^{i*}. \tag{2.3}$$

We consider density matrices on $\mathcal{H} \otimes \mathcal{K}$ of the particular form

$$\rho = \sum_i \rho_i \otimes |i\rangle\langle i|,$$

where for each $i \in \Lambda$, ρ_i is a positive definite trace class operator and satisfies $\sum_i \mathrm{Tr}(\rho_i) = 1$. For a given initial state of such form, the OQRW is defined by the

completely positive map $\mathcal{M}$:

$$\mathcal{M}(\rho) = \sum_i \left(\sum_j B_j^i \rho_j B_j^{i*} \right) \otimes |i\rangle\langle i|. \tag{2.4}$$

Hence a measurement of the position in $\mathcal{K}$ would give a probability $\sum_j \mathrm{Tr}(B_j^i \rho_j B_j^{i*})$ to find out the particle at site i. The OQRW is a repeated operation of the completely positive map $\mathcal{M}$. The two-step evolution, for instance, is of the form

$$\mathcal{M}^2(\rho) = \sum_i \sum_j \sum_k B_j^i B_k^j \rho_k B_k^{j*} B_j^{i*} \otimes |i\rangle\langle i|.$$

3 Quantum Markov chains

In this section we briefly recall the definitions of quantum Markov chains [6, 7, 18, 23] and (ir)reducibility [6, 7].

3.1 Quantum Markov chains

Let $\mathbb{Z}_+$ be the set of all nonnegative integers. Let $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{B}(h)$, the space of all bounded linear operators on a separable Hilbert space h. For any bounded $\Lambda \subset \mathbb{Z}_+$, let

$$\mathcal{A}_\Lambda := \bigotimes_{i \in \Lambda} \mathcal{A}_i, \quad \mathcal{A}_i = \mathcal{B}, \tag{3.1}$$

be the finite tensor product of von Neumann algebras and

$$\mathcal{A} := \bigotimes_{i \in \mathbb{Z}_+} \mathcal{A}_i \tag{3.2}$$

be the infinite tensor product of von Neumann algebras [16, 24]. Here we adopt the definition of infinite tensor product of von Neumann algebras introduced in ref. [16]. For each $i \in \mathbb{Z}_+$, let J_i be the embedding homomorphism

$$J_i : \mathcal{B} \hookrightarrow I_0 \otimes I_1 \otimes \cdots \otimes I_{i-1} \otimes \mathcal{B} \otimes I_{i+1} \otimes \cdots =: I_{i-1]} \otimes \mathcal{B} \otimes I_{[i+1}$$

defined by

$$J_i(a) = I_{i-1]} \otimes a \otimes I_{[i+1}, \quad \forall a \in \mathcal{B}.$$

For each $\Lambda \subset \mathbb{Z}_+$, we identify $\mathcal{A}_\Lambda$ as a subalgebra of $\mathcal{A}$. We denote $\mathcal{A}_{n]}$ the subalgebra of $\mathcal{A}$, generated by the first $(n+1)$ factors, i.e., by the elements of the form

$$a_{n]} = a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes I_{[n+1} = J_0(a_0) J_1(a_1) \cdots J_n(a_n)$$

with $a_0, a_1, \cdots, a_n \in \mathcal{B}$.

A bilinear map $\mathcal{E}$ from $\mathcal{B} \otimes \mathcal{B}$ to $\mathcal{B}$ is called a transition expectation if it is completely positive and sub-Markovian in the sense that [12]

$$\mathcal{E}(I \otimes I) \leq I. \qquad (3.3)$$

Remark 3.1 In the literature, it is required in general the Markovian property, i.e., $\mathcal{E}(I \otimes I) = I$, to define quantum Markov chains. The sub-Markovian condition (3.3) is definitely weaker than the Markovian condition. We emphasize, however, that when we apply the QMCs to special models, like the OQRWs of the present model, it is generally required to use sub-Markovian transition expectations in order to properly recover the original dynamics. Nonetheless, as will be seen in Definition 3.4, since the QMCs are always defined by a pair of initial states and transition expectations, we have a room to recover the Markovian property, and this really works in the present model. We therefore impose the Markovian property only when we speak together with initial states and transition expectations.

Given a sequence of transition expectations $(\mathcal{E}^{(n)})_{n \geq 0}$, for each $m \geq 0$ we will define a (unique) completely positive, sub-Markovian map $E_{m]} : \mathcal{A} \to \mathcal{A}_{m]}$. Since we have sub-Markovian transition expectations in general, we need some auxiliary preparation.

Lemma 3.2 *For each $n \geq 0$, there exists a (unique) nonnegative element, denoted by $\bar{b}(n) \in \mathcal{B}$, such that $\bar{b}(n) \leq I$ and*

$$\lim_{k \to \infty} \mathcal{E}^{(n)}(I \otimes \mathcal{E}^{(n+1)}(I \otimes \cdots \otimes \mathcal{E}^{(n+k)}(I \otimes I))) = \bar{b}(n).$$

In the case that the transition expectations $(\mathcal{E}^{(n)})_{n \geq 0}$ are Markovian, $\bar{b}(n) = I$.

Proof: The second statement is trivial. Define $a_k^{(n)} := \mathcal{E}^{(n)}(I \otimes \mathcal{E}^{(n+1)}(I \otimes \cdots \otimes \mathcal{E}^{(n+k)}(I \otimes I)))$. By (3.3), $\{a_k^{(n)}\}_{k \geq 0}$ is a sequence of positive decreasing operators on $\mathcal{B}$. Hence by Vigier's Theorem [27] it strongly converges to a nonnegative element, say $\bar{b}(n) \in \mathcal{B}$. $\square$

In order to define $E_{m]} : \mathcal{A} \to \mathcal{A}_{m]}$, first for an element $a_{n]} = a_0 \otimes \cdots \otimes a_n \otimes I_{[n+1} \in \mathcal{A}_{n]}$, $n \geq m$, we define

$$E_{m]}(a_{n]}) := a_0 \otimes \cdots \otimes a_{m-1} \otimes \mathcal{E}^{(m)}(a_m \otimes \mathcal{E}^{(m+1)}(a_{m+1} \otimes \cdots$$
$$\otimes \mathcal{E}^{(n)}(a_n \otimes \bar{b}(n+1)))). \qquad (3.4)$$

And for $a = a_0 \otimes a_1 \otimes \cdots \in \mathcal{A}$, we let

$$E_{m]}(a) := \lim_{n \to \infty} E_{m]}(a_{n]}). \qquad (3.5)$$

See [1, 2, 3, 12]. More precisely, the existence of the limit (3.5) can be shown by the following argument.

Lemma 3.3 *For any $a \in \mathcal{A}$, and $m \geq 0$, the limit (3.5) is well defined and $E_{m]}$ is completely positive.*

Proof: Recall that the infinite tensor product $\mathcal{A}$ of von Neumann algebras is in the sense introduced in [16]. For any $a \in \mathcal{A}$, by definition, $a = a_0 \otimes a_1 \otimes \cdots$ is the weak limit of the sequence

$$a_{n]} = a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes I_{[n+1}.$$

Define for each $n \geq m \geq 0$,

$$E_{m]}^{(n)}(a) := E_{m]}(a_{n]}), \quad a \in \mathcal{A}.$$

It is clear that for each fixed $m \geq 0$, as a finite composition of completely positive maps, $E_{m]}^{(n)}$ is completely positive for all $n \geq m$. By (3.3) (see also Lemma 3.2), we see that the sequence $(E_{m]}^{(n)})_{n \geq m}$ is weakly convergent. Therefore, the limit (3.5) is well defined, and as a weak limit of completely positive maps, $E_{m]}$ is also completely positive (see [19, Proposition 2.15]). $\square$

Suppose that a sequence of transition expectations $(\mathcal{E}^{(n)})_{n \geq 0}$ and a state ϕ_0 on $\mathcal{B}$ are given. We define a positive definite functional ϕ on $\mathcal{A}$ by

$$\phi(a) := \phi_0(E_{0]}(a)), \quad a \in \mathcal{A}. \qquad (3.6)$$

Notice that by (3.3) and Lemma 3.2, and from the definition of $E_{0]}$ in (3.4) and (3.5), ϕ is sub-Markovian, meaning that $\phi(I \otimes I \otimes \cdots) \leq 1$.

Definition 3.4 (i) A pair $(\phi_0, (\mathcal{E}^{(n)})_{n \geq 0})$ of a state ϕ_0 on $\mathcal{B}$ and a sequence of transition expectations $(\mathcal{E}^{(n)})_{n \geq 0}$ is called a Markov pair if the positive definite functional ϕ in (3.6) defines a state on $\mathcal{A}$, i.e., it is Markovian in the sense that

$$\phi(I \otimes I \otimes \cdots) = 1.$$

(ii) A Markov pair $(\phi_0, (\mathcal{E}^{(n)})_{n \geq 0})$, or alternatively the state ϕ in (3.6) defined by the pair, is called a nonhomogeneous QMC with initial state ϕ_0. When $\mathcal{E}^{(n)} = \mathcal{E}$ for all n, we say that the QMC is homogeneous.

Remark 3.5 The state ϕ in the Definition 3.4 was called a generalized Markov chain in [12].

We introduce a typical way of defining the transition expectations [8, 12]. Denote by Tr_i, $i = 1, 2$ the partial traces on $\mathcal{B} \otimes \mathcal{B}$ defined by

$$\mathrm{Tr}_1(a \otimes b) = \mathrm{Tr}(a)b, \quad \mathrm{Tr}_2(a \otimes b) = \mathrm{Tr}(b)a.$$

Let $\{K_i\}_{i \in \mathbb{Z}_+}$ be a set of Hilbert-Schmidt operators on $\mathcal{B} \otimes \mathcal{B}$ satisfying

$$\sum_i \|K_i\|^2 < \infty \text{ and } \sum_i \mathrm{Tr}_2(K_i K_i^*) \leq I. \tag{3.7}$$

Then a transition expectation is defined by [8, 12]

$$\mathcal{E}(a) := \sum_i \mathrm{Tr}_2(K_i a K_i^*), \quad a \in \mathcal{B} \otimes \mathcal{B}. \tag{3.8}$$

In this paper, the transition expectations of the type in (3.8) with suitably chosen operators $\{K_i\}$ will play a central role. We notice that in the literature, the equality was required in the equation (3.7) to define transition expectations satisfying the equality in (3.3). By relaxing it to an inequality as above, it will define a transition expectation which is sub-Markovian in the sense of (3.3). In the applications, like in the present model, the sub-Markovian property is natural. We remark also that Park and Shin computed the dynamical entropy of generalized QMC constructed by transition expectations of the type in (3.8) [25, 26].

3.2   Reducible and irreducible QMCs

In this subsection, we discuss the reducibility and irreducibility of QMCs.

   We introduce the notion of the reducibility of QMC [6, 7]. Given a projection $p \in \mathcal{B}$ and any $n \in \mathbb{Z}_+$, we denote

$$p_{[n} := I \otimes I \otimes \cdots I \otimes \overset{n\mathrm{th}}{p} \otimes p \otimes \cdots \in \mathcal{A}. \tag{3.9}$$

We define a subset of projections in $\mathcal{A}$ by

$$\mathcal{P}_0 := \{p_{[n} : p \in \mathcal{B}, \text{ a projection, } n \in \mathbb{Z}_+\}. \tag{3.10}$$

Definition 3.6 A quantum Markov chain is called reducible if there exists a non-trivial projection $p \in \mathcal{B}$ and $n_0 \in \mathbb{Z}_+$ such that

$$E_{0]}(p_{[n_0} a p_{[n_0}) = E_{0]}(a) \tag{3.11}$$

for all $a \in \mathcal{A}$. Otherwise it is called irreducible. Any projection satisfying (3.11) is called a reducing projection.

Remark 3.7 In the references [6, 7], the reducing projections are allowed to take much more general form. But here we will confine them to be of the forms in (3.10). It will be turned out that this is enough.

Theorem 3.8 *The QMC is reducible if and only if* $E_{0]}(I - p_{[n_0}) = 0$ *for some nontrivial projection* $p_{[n_0}$.

Proof: In the proof, for notational simplicity we just put p for $p_{[n_0}$. Suppose that p is a nontrivial projection such that $E_{0]}(I - p) = 0$. That is, $E_{0]}(p^\perp) = 0$. Since $E_{0]}$ is completely positive, by [19, Proposition 2.10] and using Lemma 3.2, it satisfies a Schwarz inequality:

$$E_{0]}(b)^* E_{0]}(b) \leq \|E_{0]}(I)\| \, E_{0]}(b^* b) = \|\bar{b}(0)\| E_{0]}(b^* b) \leq E_{0]}(b^* b).$$

Therefore,

$$E_{0]}(pap^\perp)^* E_{0]}(pap^\perp) \leq E_{0]}(p^\perp a^* pap^\perp) \leq E_{0]}(p^\perp a^* ap^\perp) \leq \|a\|^2 E_{0]}(p^\perp) = 0.$$

Thus $E_{0]}(pap^\perp) = 0$ and so $E_{0]}(p^\perp ap) = 0$. Similarly we have $E_{0]}(p^\perp ap^\perp) = 0$. Therefore we get

$$E_{0]}(a) = E_{0]}((p + p^\perp)a(p + p^\perp)) = E_{0]}(pap),$$

for all $a \in \mathcal{A}$. This means that the QMC is reducible. The converse trivially holds by taking $a = I$. $\square$

4 Quantum Markov chains associated with OQRWs

In this section, we construct QMCs associated with OQRWs. As mentioned in the Introduction, this is a slight modification of the one developed in [18]. We will construct a nonhomogeneous QMC, but in [18], a homogeneous QMC was considered. We will use notations from the previous section. In the sequel, we also use the density matrices as also for states (positive definite functions, in general), i.e.,

if ρ is a positive definite trace class operator in $\mathcal{B}$, then for any $a \in \mathcal{B}$, we write $\text{Tr}(\rho a)$ or $\rho(a)$ denoting the same value of the functional at a. Let us define some notations which will be used in the sequel. For $i, j \in \Lambda$, a path from i to j is any finite sequence $i_0, i_1, \cdots, i_l$ in Λ with $l \geq 1$, such that $i_0 = i$ and $i_l = j$. We denote such a path by $\pi(i_0, \cdots, i_l)$ and let $\mathcal{P}(i, j)$ be the set of all paths from i to j. For $\pi(i_0, \cdots, i_l)$ in $\mathcal{P}(i, j)$ we denote by $B_{\pi(i_0, \cdots, i_l)}$ the operator on $\mathcal{H}$:

$$B_{\pi(i_0, \cdots, i_l)} = B_{i_{l-1}}^{i_l} \cdots B_{i_0}^{i_1} = B_{i_{l-1}}^{j} \cdots B_i^{i_1}.$$

4.1   QMCs for OQRWs

Let $\mathcal{M}$ be an OQRW given by (2.3). We fix a density operator $\rho^{(0)} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ of the form

$$\rho^{(0)} = \sum_i \rho_i^{(0)} \otimes |i\rangle\langle i|,$$

where $\rho_i^{(0)} \geq 0$ and $\sum_i \text{Tr}(\rho_i^{(0)}) = 1$ for all i. For an initial state $\rho^{(0)}$, $\rho^{(n)} := \mathcal{M}^n(\rho^{(0)})$ is the state at time n. Then we can write

$$\rho^{(n)} = \sum_i \rho_i^{(n)} \otimes |i\rangle\langle i|. \tag{4.1}$$

We would like to remind the reader that starting with any initial state, even not of the block-diagonal form, after the evolution of OQRW the states result in the block-diagonal form as in (4.1) [14]. Therefore, it is natural and sufficient to consider also the observables of the block-diagonal form. So, define a subalgebra $\mathcal{B}_0 \subset \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ by

$$\mathcal{B}_0 = \{\sum_{i \in \Lambda} a(i) \otimes |i\rangle\langle i| : a(i) \in \mathcal{B}(\mathcal{H}) \text{ for all } i \in \Lambda \text{ and } \sum_i \|a(i)\| < \infty\}. \tag{4.2}$$

Let $\mathcal{B}$ be the von Neumann subalgebra of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ obtained by a weak closure of $\mathcal{B}_0$. We consider the algebra

$$\mathcal{A} = \bigotimes_{i \in \mathbb{Z}_+} \mathcal{A}_i$$

where $\mathcal{A}_i = \mathcal{B}$ for all $i \in \mathbb{Z}_+$. For each $n = 0, 1, 2, \cdots$, define the following operators

$$A_{ij}^{(n)} = \frac{1}{\text{Tr}(\rho_j^{(n)})^{1/2}}((\rho_j^{(n)})^{1/2} \otimes |i\rangle\langle j|), \quad i, j \in \Lambda,$$

$$K_{ij}^{(n)} = M_j^{i*} \otimes A_{ij}^{(n)}. \tag{4.3}$$

Here it is assumed $A_{ij}^{(n)} = 0$ if $\rho_j^{(n)} = 0$. Notice that by this convention, we can allow any kind of initial states $\rho^{(0)}$ so that $\rho_i^{(0)}$ might be zero for some $i \in \Lambda$. This is important when we recover the dynamics of OQRW itself by the QMC. See Proposition 4.5.

Proposition 4.1 *For each $n = 0, 1, 2, \cdots$,*

$$\mathrm{Tr}_2(\sum_{i,j} K_{ij}^{(n)} K_{ij}^{(n)*}) \leq I$$

holds.

Proof:

$$
\begin{aligned}
\mathrm{Tr}_2(\sum_{i,j} K_{ij}^{(n)} K_{ij}^{(n)*}) &= \sum_{j:\rho_j^{(n)} \neq 0} \sum_i \frac{\mathrm{Tr}(\rho_j^{(n)} \otimes |i\rangle\langle i|)}{\mathrm{Tr}(\rho_j^{(n)})} M_j^{i*} M_j^i \\
&= \sum_{j:\rho_j^{(n)} \neq 0} \sum_i B_j^{i*} B_j^i \otimes |j\rangle\langle j| \\
&= \sum_{j:\rho_j^{(n)} \neq 0} I_{\mathcal{H}} \otimes |j\rangle\langle j| \leq I.
\end{aligned}
$$

This proves the assertion. $\square$

By the above proposition we can define transition expectations.

Definition 4.2 (Transition expectations) For each $n = 0, 1, 2, \cdots$, and $x, y \in \mathcal{B}$, define

$$
\begin{aligned}
\mathcal{E}^{(n)}(x \otimes y) &:= \sum_{i,j} \mathrm{Tr}_2(K_{ij}^{(n)}(y \otimes x)K_{ij}^{(n)*}) \\
&= \sum_{j:\rho_j^{(n)} \neq 0} \sum_i \frac{\mathrm{Tr}(\rho_j^{(n)} \otimes |j\rangle\langle j|x)}{\mathrm{Tr}(\rho_j^{(n)})} M_j^{i*} y M_j^i. \quad (4.4)
\end{aligned}
$$

The above transition expectations are of the form in (3.8), but before taking a partial trace a transposition was applied, leading to the transpose transition expectation $\mathcal{E}^t$ of [18]. To say more, one may construct transition expectations by changing the roles of x and y in (4.4), which gives rise to define a new QMC. But it turns out that the present form is very convenient when we talk about the dynamics of OQRWs. See, e.g., Proposition 4.5. Using the above transition expectations, we define the

completely positive maps $E_{m]} : \mathcal{A} \to \mathcal{A}_{m]}$ by (3.5) and define a positive definite functional ρ on $\mathcal{A}$ like in (3.6):

$$\rho(a) := \rho^{(0)}(E_{0]}(a)), \quad a \in \mathcal{A}. \tag{4.5}$$

Before going further, we refine Lemma 3.2 for the present model by showing the following property. Recall the definition given in Lemma 3.2:

$$\bar{b}(n) = \lim_{k \to \infty} \mathcal{E}^{(n)}(I \otimes \mathcal{E}^{(n+1)}(I \otimes \cdots \otimes \mathcal{E}^{(n+k)}(I \otimes I))).$$

For a state of the form $\rho = \sum_i \rho_i \otimes |i\rangle\langle i|$, we let $\Lambda(\rho) := \{i \in \Lambda : \rho_i \neq 0\}$.

Lemma 4.3 *The operators $\{\bar{b}(n)\}_{n \geq 0}$ for the transition expectations of OQRWs satisfy the following properties.*

 (i) *For each $n \geq 0$ and $j \in \Lambda(\rho^{(n)})$, there exist strictly positive operators $\bar{b}(n, j) \in \mathcal{B}(\mathcal{H})$ such that*
$$\bar{b}(n) = \sum_{j \in \Lambda(\rho^{(n)})} \bar{b}(n, j) \otimes |j\rangle\langle j|.$$

 (ii) *For each $j \in \Lambda(\rho^{(n)})$, it holds that*

$$\sum_{i \in \Lambda(\rho^{(n+1)})} B_j^{i*} \bar{b}(n+1, i) B_j^i = \bar{b}(n, j).$$

 (iii) *For each $n \geq 0$ and $j \in \Lambda(\rho^{(n)})$,*

$$\mathrm{Tr}(\rho_j^{(n)} \bar{b}(n, j)) = \mathrm{Tr}(\rho_j^{(n)}).$$

Proof: (i) Define $a_k^{(n)} := \mathcal{E}^{(n)}(I \otimes \mathcal{E}^{(n+1)}(I \otimes \cdots \otimes \mathcal{E}^{(n+k)}(I \otimes I)))$. Then we have $\bar{b}(n) = \lim_{k \to \infty} a_k^{(n)}$. By directly computing with the definition (4.4) we get

$$\mathcal{E}^{(n+k)}(I \otimes I) = \sum_{j \in \Lambda(\rho^{(n+k)})} I_{\mathcal{H}} \otimes |j\rangle\langle j|.$$

Applying (4.4) once again, we get

$$\mathcal{E}^{(n+k-1)}(I \otimes \mathcal{E}^{(n+k)}(I \otimes I))$$
$$= \sum_{i_{n+k-1} \in \Lambda(\rho^{(n+k-1)})} \sum_{i_{n+k} \in \Lambda(\rho^{(n+k)})} B_{i_{n+k-1}}^{i_{n+k}*} B_{i_{n+k-1}}^{i_{n+k}} \otimes |i_{n+k-1}\rangle\langle i_{i+k-1}|.$$

Repeating this procedure, we get

$$a_k^{(n)} = \sum_{i_n \in \Lambda(\rho^{(n)})} b^{(n)}(i_n; k) \otimes |i_n\rangle\langle i_n|,$$

where

$$b^{(n)}(i_n; k) = \sum_{i_{n+1} \in \Lambda(\rho^{(n+1)})} \cdots \sum_{i_{n+k} \in \Lambda(\rho^{(n+k)})} B^*_{\pi(i_n, \cdots, i_{n+k})} B_{\pi(i_n, \cdots, i_{n+k})}.$$

By the property (2.1) we see that $\{b^{(n)}(i_n; k)\}_{k \geq 1}$ is a sequence of decreasing positive definite operators on $\mathcal{B}$. Thus by Vigier's Theorem [27] again, we see that the sequence converges strongly to a nonnegative element, say $\bar{b}(n, i_n)$ as $k \to \infty$. We thus get

$$\bar{b}(n) = \lim_{k \to \infty} a_k^{(n)} = \sum_{i_n \in \Lambda(\rho^{(n)})} \bar{b}(n, i_n) \otimes |i_n\rangle\langle i_n|.$$

The strict positivity of $\bar{b}(n, j)$ for $j \in \Lambda(\rho^{(n)})$ follows from (iii) whose proof does not use this property.

(ii) By the computations in (i), we see that for $j \in \Lambda(\rho^{(n)})$,

$$\sum_{i_{n+1} \in \Lambda(\rho^{(n+1)})} B_j^{i_{n+1}*} \bar{b}(n+1, i_{n+1}) B_j^{i_{n+1}}$$

$$= \lim_{k \to \infty} \sum_{i_{n+1} \in \Lambda(\rho^{(n+1)})} B_j^{i_{n+1}*} \left(\sum_{i_{n+2} \in \Lambda(\rho^{(n+2)})} \cdots \sum_{i_{n+k} \in \Lambda(\rho^{(n+k)})} B^*_{\pi(i_{n+1}, \cdots, i_{n+k})} B_{\pi(i_{n+1}, \cdots, i_{n+k})} \right) B_j^{i_{n+1}}$$

$$= \lim_{k \to \infty} \sum_{i_{n+1} \in \Lambda(\rho^{(n+1)})} \cdots \sum_{i_{n+k} \in \Lambda(\rho^{(n+k)})} B^*_{\pi(j, i_{n+1}, \cdots, i_{n+k})} B_{\pi(j, i_{n+1}, \cdots, i_{n+k})}$$

$$= \bar{b}(n, j).$$

(iii) We see again

$$\text{Tr}(\rho_j^{(n)} \bar{b}(n, j))$$

$$= \lim_{k \to \infty} \sum_{i_{n+1} \in \Lambda(\rho^{(n+1)})} \cdots \sum_{i_{n+k} \in \Lambda(\rho^{(n+k)})} \text{Tr}\left(\rho_j^{(n)} B^*_{\pi(j, i_{n+1}, \cdots, i_{n+k})} B_{\pi(j, i_{n+1}, \cdots, i_{n+k})} \right)$$

$$= \lim_{k \to \infty} \sum_{i_{n+1}} \cdots \sum_{i_{n+k}} \text{Tr}\left(\rho_j^{(n)} B^*_{\pi(j, i_{n+1}, \cdots, i_{n+k})} B_{\pi(j, i_{n+1}, \cdots, i_{n+k})} \right)$$

$$= \lim_{k \to \infty} \text{Tr}(\rho_j^{(n)})$$

$$= \text{Tr}(\rho_j^{(n)}).$$

Here in the third equality the relation (2.1) was used and the second equality can be shown by the following argument. Suppose, for example, $i_{n+l} \notin \Lambda(\rho^{(n+l)})$ for some $1 \leq l \leq k$. We claim that

$$\text{Tr}\left(\rho_j^{(n)} B^*_{\pi(j,i_{n+1},\cdots,i_{n+l},\cdots,i_{n+k})} B_{\pi(j,i_{n+1},\cdots,i_{n+l},\cdots,i_{n+k})}\right) = 0.$$

In fact,

$$\text{Tr}\left(\rho_j^{(n)} B^*_{\pi(j,i_{n+1},\cdots,i_{n+l},\cdots,i_{n+k})} B_{\pi(j,i_{n+1},\cdots,i_{n+l},\cdots,i_{n+k})}\right)$$

$$= \text{Tr}\left(B_{\pi(j,i_{n+1},\cdots,i_{n+l},\cdots,i_{n+k})} \rho_j^{(n)} B^*_{\pi(j,i_{n+1},\cdots,i_{n+l},\cdots,i_{n+k})}\right)$$

$$= \text{Tr}\left(B_{\pi(i_{n+l},\cdots,i_{n+k})} B_{\pi(j,i_{n+1},\cdots,i_{n+l})} \rho_j^{(n)} B^*_{\pi(j,i_{n+1},\cdots,i_{n+l})} B^*_{\pi(i_{n+l},\cdots,i_{n+k})}\right).$$

But using the definition of OQRW in (2.4) we see that

$$0 \leq B_{\pi(j,i_{n+1},\cdots,i_{n+l})} \rho_j^{(n)} B^*_{\pi(j,i_{n+1},\cdots,i_{n+l})}$$

$$\leq \sum_j \sum_{i_{n+1}} \cdots \sum_{i_{n+l-1}} B_{\pi(j,i_{n+1},\cdots,i_{n+l})} \rho_j^{(n)} B^*_{\pi(j,i_{n+1},\cdots,i_{n+l})}$$

$$= \rho_{i_{n+l}}^{(n+l)} = 0,$$

by the assumption that $i_{n+l} \notin \Lambda(\rho^{(n+l)})$. This proves the claim and the proof is completed. $\square$

In the sequel, by abuse of notations and to save the space, we use $\sum'_{i_l}$ for $\sum_{i_l \in \Lambda(\rho^{(l)})}$ whenever there is no danger of confusion.

Lemma 4.4 *For any* $a_{n]} = a_0 \otimes \cdots \otimes a_n \otimes I_{[n+1} \in \mathcal{A}_{n]}$, *we have*

$$E_{0]}(a_{n]}) \tag{4.6}$$

$$= \sum_{i_0}' \cdots \sum_{i_n}' \prod_{k=0}^n \frac{\text{Tr}(\rho_{i_k}^{(k)} \otimes |i_k\rangle\langle i_k|a_k)}{\text{Tr}(\rho_{i_k}^{(k)})} \left(B^*_{\pi(i_0,\cdots,i_n)} \bar{b}(n,i_n) B_{\pi(i_0,\cdots,i_n)} \otimes |i_0\rangle\langle i_0|\right).$$

Proof: Recall

$$E_{0]}(a_{n]}) = \mathcal{E}^{(0)}(a_0 \otimes \cdots \otimes \mathcal{E}^{(n)}(a_n \otimes \bar{b}(n+1))).$$

By definition (4.4) and Lemma 4.3 (i) and (ii), we see that

$$\mathcal{E}^{(n)}(a_n \otimes \bar{b}(n+1))$$

$$= \sum_{i_n}' \frac{\text{Tr}(\rho_{i_n}^{(n)} \otimes |i_n\rangle\langle i_n|a_n)}{\text{Tr}(\rho_{i_n}^{(n)})} \sum_{i_{n+1}}' B_{i_n}^{i_{n+1}*} \bar{b}(n+1,i_{n+1}) B_{i_n}^{i_{n+1}} \otimes |i_n\rangle\langle i_n|$$

$$= \sum_{i_n}' \frac{\text{Tr}(\rho_{i_n}^{(n)} \otimes |i_n\rangle\langle i_n|a_n)}{\text{Tr}(\rho_{i_n}^{(n)})} \bar{b}(n,i_n) \otimes |i_n\rangle\langle i_n|.$$

Now repeated application of (4.4) and Lemma 4.3 (i) gives the result. □

  The following proposition shows two important features of our definition. One is that for any initial state $\rho^{(0)}$, the pair $(\rho^{(0)}, (\mathcal{E}^{(n)})_{n \geq 0})$ is a Markov pair (see Corollary 4.6), in other words, ρ in (4.5) is a state on $\mathcal{A}$ and hence a QMC. The second one is that the QMCs associated with OQRWs naturally extend the classical Markov chains (see (5.4) in Subsection 5.2).

Proposition 4.5 *For any $x \in \mathcal{B}$,*

$$\rho(I \otimes \cdots \otimes I \otimes \overset{n\text{th}}{x} \otimes I \otimes \cdots) = \rho^{(n)}(x),$$

where $\rho^{(n)} = \mathcal{M}^n(\rho^{(0)})$.

Proof: Using the definition (3.4), we get

$$\rho(I \otimes \cdots \otimes I \otimes \overset{n\text{th}}{x} \otimes I \otimes \cdots)$$
$$= \rho^{(0)} \left(\mathcal{E}^{(0)}(I \otimes \cdots \otimes \mathcal{E}^{(n)}(x \otimes \bar{b}(n+1))) \right).$$

By Lemma 4.4 and Lemma 4.3 (iii), we have

$$\rho^{(0)} \left(\mathcal{E}^{(0)}(I \otimes \cdots \otimes \mathcal{E}^{(n)}(x \otimes \bar{b}(n+1))) \right)$$

$$= {\sum_{i_n}}' \frac{\text{Tr}(\rho_{i_n}^{(n)} \otimes |i_n\rangle\langle i_n|x)}{\text{Tr}(\rho_{i_n}^{(n)})} {\sum_{i_0}}' \cdots {\sum_{i_{n-1}}}' \text{Tr}\left(B_{\pi(i_0,\cdots,i_n)} \rho_{i_0}^{(0)} B^*_{\pi(i_0,\cdots,i_n)} \bar{b}(n, i_n) \right)$$

$$= {\sum_{i_n}}' \frac{\text{Tr}(\rho_{i_n}^{(n)} \otimes |i_n\rangle\langle i_n|x)}{\text{Tr}(\rho_{i_n}^{(n)})} \sum_{i_0} \cdots \sum_{i_{n-1}} \text{Tr}\left(B_{\pi(i_0,\cdots,i_n)} \rho_{i_0}^{(0)} B^*_{\pi(i_0,\cdots,i_n)} \bar{b}(n, i_n) \right)$$

$$= {\sum_{i_n}}' \frac{\text{Tr}(\rho_{i_n}^{(n)} \otimes |i_n\rangle\langle i_n|x)}{\text{Tr}(\rho_{i_n}^{(n)})} \text{Tr}\left(\rho_{i_n}^{(n)} \bar{b}(n, i_n) \right)$$

$$= {\sum_{i_n}}' \frac{\text{Tr}(\rho_{i_n}^{(n)} \otimes |i_n\rangle\langle i_n|x)}{\text{Tr}(\rho_{i_n}^{(n)})} \text{Tr}(\rho_{i_n}^{(n)})$$

$$= {\sum_{i_n}}' \text{Tr}(\rho_{i_n}^{(n)} \otimes |i_n\rangle\langle i_n|x)$$

$$= \rho^{(n)}(x).$$

The second and third equalities follow from the definition of OQRWs. The proof is completed. □

Corollary 4.6 *The pair $(\rho^{(0)}, (\mathcal{E}^{(n)})_{n \geq 0})$ is a Markov pair.*

Proof: It follows from Proposition 4.5 by taking $x = I$. $\square$

Definition 4.7 The pair $(\rho^{(0)}, (\mathcal{E}^{(n)})_{n \geq 0})$, or the state ρ in (4.5) is called the (non-homogeneous) QMC associated with the OQRW.

We remark that as will be noted in Subsection 5.2, the property in Proposition 4.5 is observed when the QMCs are applied to recover the classical Markov chains (see (5.4)), and this property was already observed in [6, 7].

Next we shortly discuss the invariant states for the QMCs.

Definition 4.8 (Invariant state) A state (density matrix) ω on $\mathcal{B}$ is called invariant to the QMC if

$$\mathrm{Tr}(\omega x) = \mathrm{Tr}(\omega \mathcal{E}^{(n)}(I \otimes x))$$

for all x and $n = 0, 1, 2, \cdots$.

This corresponds to the condition (2.3) of [25]. The following proposition shows that an invariant state ω to the Markov chain of a OQRW is an invariant state (density operator) with respect to $\mathcal{M}$.

Proposition 4.9 *A state $\omega = \sum_i \omega_i \otimes |i\rangle\langle i|$ is invariant to the QMC of OQRW if and only if $\sum_{i,j} \mathrm{Tr}_2(K_{ij}^{(n)^*} \omega \otimes I K_{ij}^{(n)}) = \omega$ for all $n \geq 0$, and in this case ω satisfies $\omega = \sum_{i,j} M_j^i \omega M_j^{i^*}$, that is, $\omega = \mathcal{M}(\omega)$. On the other hand, if $\omega = \mathcal{M}(\omega)$, the state ω is invariant to the QMC $(\rho^{(0)}, (\mathcal{E}^{(n)})_{n \geq 0})$ with $\rho^{(0)} = \omega$. In this case we have $\mathcal{E}^{(n)} = \mathcal{E}^{(0)}$ for all $n \geq 0$, i.e., the QMC is homogeneous.*

Proof: We have

$$\begin{aligned}
\mathrm{Tr}(\omega \mathcal{E}^{(n)}(I \otimes x)) &= \sum_{i,j} \mathrm{Tr}\big(\mathrm{Tr}_2((\omega \otimes I)(K_{ij}^{(n)}(x \otimes I)K_{ij}^{(n)^*}))\big) \\
&= \sum_{i,j} \tilde{\mathrm{Tr}}\big((\omega \otimes I)(K_{ij}^{(n)}(x \otimes I)K_{ij}^{(n)^*})\big) \\
&= \sum_{i,j} \mathrm{Tr}\big(\mathrm{Tr}_2\big(K_{ij}^{(n)^*}(\omega \otimes I)K_{ij}^{(n)}(x \otimes I)\big)\big) \\
&= \sum_{i,j} \mathrm{Tr}\big(\mathrm{Tr}_2\big(K_{ij}^{(n)^*}(\omega \otimes I)K_{ij}^{(n)}\big)x\big).
\end{aligned}$$

Thus $\text{Tr}(\omega \mathcal{E}^{(n)}(I \otimes x)) = \text{Tr}(\omega x)$ for all x if and only if $\sum_{i,j} \text{Tr}_2(K_{ij}^{(n)*} \omega \otimes I K_{ij}^{(n)}) = \omega$. By direct calculation, we have

$$
\begin{aligned}
\sum_{i,j} \text{Tr}_2(K_{ij}^{(n)*} \omega \otimes I K_{ij}^{(n)}) &= \sum_{j:\rho_j^{(n)} \neq 0} \sum_i M_j^i \omega M_j^{i*} \\
&= \sum_{j:\rho_j^{(n)} \neq 0} \sum_i B_j^i \omega_j B_j^{i*} \otimes |i\rangle\langle i|.
\end{aligned}
$$

Therefore $\sum_{i,j} \text{Tr}_2(K_{ij}^{(n)*} \omega \otimes I K_{ij}^{(n)}) = \omega$ if and only if

$$
\sum_i \omega_i \otimes |i\rangle\langle i| = \sum_i \left(\sum_{j:\rho_j^{(n)} \neq 0} B_j^i \omega_j B_j^{i*} \right) \otimes |i\rangle\langle i|. \tag{4.7}
$$

By taking trace to both sides of the above equation we get

$$
\begin{aligned}
1 &= \sum_i \sum_{j:\rho_j^{(n)} \neq 0} \text{Tr}\left(B_j^i \omega_j B_j^{i*} \right) \\
&= \sum_{j:\rho_j^{(n)} \neq 0} \text{Tr}(\omega_j).
\end{aligned}
$$

This means that $\omega_i = 0$ if $\rho_i^{(n)} = 0$ (for all $n \geq 0$). Thus (4.7) is written as

$$
\sum_i \omega_i \otimes |i\rangle\langle i| = \sum_i \sum_j B_j^i \omega_j B_j^{i*} \otimes |i\rangle\langle i| = \sum_i \mathcal{M}(\omega)_i \otimes |i\rangle\langle i|.
$$

We have therefore $\omega = \mathcal{M}(\omega)$.

Now conversely suppose $\omega = \mathcal{M}(\omega)$ and define a Markov pair $(\rho^{(0)}, (\mathcal{E}^{(n)})_{n \geq 0})$ with $\rho^{(0)} = \omega$. Then, since $\rho^{(n)} = \rho^{(0)} = \omega$ for all $n \geq 0$, it is a homogeneous QMC. Moreover, by (4.4)

$$
\begin{aligned}
\text{Tr}(\omega \mathcal{E}^{(n)}(I \otimes x)) &= \sum_{j:\omega_j \neq 0} \sum_i \text{Tr}(\omega M_j^{i*} x M_j^i) \\
&= \sum_{j:\omega_j \neq 0} \sum_i \text{Tr}(B_j^i \omega_j B_j^{i*} \otimes |i\rangle\langle i|x) \\
&= \sum_j \sum_i \text{Tr}(B_j^i \omega_j B_j^{i*} \otimes |i\rangle\langle i|x) \\
&= \text{Tr}(\mathcal{M}(\omega)x) = \text{Tr}(\omega x).
\end{aligned}
$$

Therefore, ω is invariant to the QMC $(\rho^{(0)}, (\mathcal{E}^{(n)})_{n \geq 0})$ with $\rho^{(0)} = \omega$. $\square$

4.2   Reducibility and irreducibility of QMCs for OQRWs

Recall the definition of reducibility and irreducibility of QMCs in Definition 3.6 with the projections in (3.9) and (3.10). When we consider the reducibility and irreducibility problems for QMCs associated with OQRWs, the possible reducing projections shall be of the form:

$$p_{[n} = I \otimes \cdots \otimes I \otimes \overset{n\text{th}}{p} \otimes p \otimes \cdots \in \mathcal{A} \text{ with } p = \sum_j p(j) \otimes |j\rangle\langle j| \in \mathcal{B}, \qquad (4.8)$$

where $p(j)$'s are projections on $\mathcal{H}$. Then we define

$$\mathcal{P}_0 := \{p_{[n} : p_{[n}, \text{ a projection of the form } (4.8), \ n \in \mathbb{Z}_+\}. \qquad (4.9)$$

We say that a QMC associated with an OQRW is reducible if there exists a non-trivial projection $p_{[n_0} \in \mathcal{P}_0$ satisfying (3.11) in Definition 3.6. Otherwise it is called irreducible.

Let $0 \le m \le n$ and $0 \le n_0 \le n$. Consider $a = a_0 \otimes a_1 \otimes \cdots \otimes a_m \otimes I_{[m+1} \in \mathcal{A}_{m]}$ and a projection $p_{[n_0,n]} := I \otimes \cdots \otimes I \otimes \overset{n_0\text{th}}{p} \otimes \cdots p \otimes I_{[n+1}$. Notice that $p_{[n_0} = \lim_{n\to\infty} p_{[n_0,n]}$. In order to compute $E_{0]}(p_{[n_0,n]} a p_{[n_0,n]})$, we let for the time being

$$p_l := \begin{cases} I, & 0 \le l \le n_0 - 1, \\ p, & n_0 \le l \le n. \end{cases} \qquad (4.10)$$

By Lemma 4.4 we get for $a = a_0 \otimes a_1 \otimes \cdots \otimes a_m \otimes I_{[m+1}, \ m \le n$,

$$\begin{aligned}
&E_{0]}\big(p_{[n_0,n]} a p_{[n_0,n]}\big) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.11) \\
&= {\sum_{i_0}}' \cdots {\sum_{i_n}}' \prod_{k=0}^{m} \frac{\text{Tr}(\rho_{i_k}^{(k)} p_k(i_k) a_k(i_k) p_k(i_k))}{\text{Tr}(\rho_{i_k}^{(k)})} \prod_{k=m+1}^{n} \frac{\text{Tr}(\rho_{i_k}^{(k)} p_k(i_k))}{\text{Tr}(\rho_{i_k}^{(k)})} \\
&\quad \times \big(B^*_{\pi(i_0,\cdots,i_n)} \bar{b}(n,i_n) B_{\pi(i_0,\cdots,i_n)} \otimes |i_0\rangle\langle i_0|\big),
\end{aligned}$$

where p_k's are given by (4.10). In particular, we have

$$\begin{aligned}
&E_{0]}\big(p_{[n_0,n]}\big) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.12) \\
&= {\sum_{i_0}}' \cdots {\sum_{i_n}}' \prod_{k=n_0}^{n} \frac{\text{Tr}(\rho_{i_k}^{(k)} p_k(i_k))}{\text{Tr}(\rho_{i_k}^{(k)})} \big(B^*_{\pi(i_0,\cdots,i_n)} \bar{b}(n,i_n) B_{\pi(i_0,\cdots,i_n)} \otimes |i_0\rangle\langle i_0|\big).
\end{aligned}$$

Lemma 4.10 *For $p_{[n_0} \in \mathcal{P}_0$, one has*

$$I - p_{[n_0} = \sum_{n \ge n_0}^{\infty} I \otimes \cdots I \otimes \overset{n_0 th}{p} \otimes \cdots \otimes p \otimes \overset{nth}{p^\perp} \otimes I_{[n+1} \qquad (4.13)$$

where $p^\perp = I - p$.

Proof: Let us adopt the notations in (4.10). We have

$$p_0^{\perp} \otimes I_{[1} + p_0 \otimes p_1^{\perp} \otimes I_{[2} = I - p_0 \otimes p_1 \otimes I_{[2}.$$

Continuing this procedure, we have

$$\sum_{k \geq 0}^{n} p_0 \otimes p_1 \otimes \cdots \otimes p_{k-1} \otimes p_k^{\perp} \otimes I_{[k+1} = I - p_0 \otimes p_1 \otimes \cdots \otimes p_n \otimes I_{[n+1}.$$

So taking the limit $n \to \infty$, and returning back the notations, we get (4.13). $\square$

Proposition 4.11 *Let $p_{[n_0} \in \mathcal{P}_0$. Then, $E_{0]}(I - p_{[n_0}) = 0$ if and only if $\rho_j^{(n)} p(j) = \rho_j^{(n)}$ for all $j \in \Lambda$ and $n \geq n_0$.*

Proof: If $E_{0]}(I - p_{[n_0}) = 0$ then by (4.13) we have

$$\sum_{n \geq n_0}^{\infty} E_{0]}(I \otimes \cdots I \otimes \overset{n_0 \text{th}}{p} \otimes \cdots \otimes p \otimes \overset{n \text{th}}{p^{\perp}} \otimes I_{[n+1}) = 0.$$

Therefore, by Lemma 4.4, we have for $n \geq n_0$,

$$E_{0]}(I \otimes \cdots I \otimes \overset{n_0 \text{th}}{p} \otimes \cdots \otimes p \otimes \overset{n \text{th}}{p^{\perp}} \otimes I_{[n+1}) \tag{4.14}$$
$$= \sideset{}{'}\sum_{i_0} \cdots \sideset{}{'}\sum_{i_n} \prod_{k=n_0}^{n-1} \frac{\mathrm{Tr}(\rho_{i_k}^{(k)} p(i_k))}{\mathrm{Tr}(\rho_{i_k}^{(k)})} \frac{\mathrm{Tr}(\rho_{i_n}^{(n)} p^{\perp}(i_n))}{\mathrm{Tr}(\rho_{i_n}^{(n)})} \Big(B_{\pi(i_0,\cdots,i_n)}^* \bar{b}(n, i_n) B_{\pi(i_0,\cdots,i_n)}$$
$$\otimes |i_0\rangle\langle i_0| \Big) = 0.$$

From this, we claim that $\mathrm{Tr}(\rho_j^{(n)} p(j)^{\perp}) = 0$ for all $n \geq n_0$ and $j \in \Lambda(\rho^{(n)})$. In fact, first we see that

$$0 = E_{0]}(I \otimes \cdots I \otimes \overset{n_0 \text{th}}{p^{\perp}} \otimes I_{[n_0+1})$$
$$= \sideset{}{'}\sum_{i_{n_0}} \frac{\mathrm{Tr}(\rho_{i_{n_0}}^{(n_0)} p^{\perp}(i_{n_0}))}{\mathrm{Tr}(\rho_{i_{n_0}}^{(n_0)})} \sideset{}{'}\sum_{i_0} \cdots \sideset{}{'}\sum_{i_{n_0-1}} \Big(B_{\pi(i_0,\cdots,i_{n_0})}^* \bar{b}(n_0, i_{n_0}) B_{\pi(i_0,\cdots,i_{n_0})} \otimes |i_0\rangle\langle i_0| \Big).$$

Since

$$\mathrm{Tr}\Big(\rho^{(0)} \sideset{}{'}\sum_{i_0} \cdots \sideset{}{'}\sum_{i_{n_0-1}} \Big(B_{\pi(i_0,\cdots,i_{n_0})}^* \bar{b}(n_0, i_{n_0}) B_{\pi(i_0,\cdots,i_{n_0})} \otimes |i_0\rangle\langle i_0| \Big) \Big)$$
$$= \mathrm{Tr}\Big(\rho_{i_{n_0}}^{(n_0)} \bar{b}(n_0, i_{n_0}) \Big)$$
$$= \mathrm{Tr}\Big(\rho_{i_{n_0}}^{(n_0)} \Big) > 0,$$

which follows by Lemma 4.4 (iii), the operator

$$\sideset{}{'}\sum_{i_0} \cdots \sideset{}{'}\sum_{i_{n_0-1}} \left(B^*_{\pi(i_0,\cdots,i_{n_0})} \overline{b}(n_0, i_{n_0}) B_{\pi(i_0,\cdots,i_{n_0})} \otimes |i_0\rangle\langle i_0| \right)$$

is positive. Thus we conclude that $\text{Tr}(\rho_j^{(n_0)} p(j)^\perp) = 0$ for $j \in \Lambda(\rho^{(n_0)})$. By induction and repeated use of (4.14) proves the claim. Now, since $\text{Tr}(\rho_j^{(n)} p(j)^\perp) = \text{Tr}(p(j)^\perp \rho_j^{(n)} p(j)^\perp) \geq 0$, $p(j)^\perp \rho_j^{(n)} p(j)^\perp = 0$ and so $\rho_j^{(n)} p(j)^\perp = 0$, or $\rho_j^{(n)} p(j) = \rho_j^{(n)}$ for all $n \geq n_0$ and $j \in \Lambda(\rho^{(n)})$, and hence for all $j \in \Lambda$.

On the other hand, if $\rho_j^{(n)} p(j) = \rho_j^{(n)}$ for all $n \geq n_0$ and $j \in \Lambda$, we get from (4.12) that

$$E_{0]}(I \otimes I \otimes \cdots) = E_{0]}(p_{[n_0,n]}) = \sideset{}{'}\sum_{i_0} \cdots \sideset{}{'}\sum_{i_n} \left(B^*_{\pi(i_0,\cdots,i_n)} \overline{b}(n, i_n) B_{\pi(i_0,\cdots,i_n)} \otimes |i_0\rangle\langle i_0| \right).$$

Therefore, we have $E_{0]}(I - p_{[n_0,n]}) = 0$. Taking the limit $n \to \infty$, we get $E_{0]}(I - p_{[n_0}) = 0$. The proof is completed. $\square$

Theorem 4.12 *The QMC associated with an OQRW is reducible with a reducing projection $p_{[n_0} \in \mathcal{P}_0$ if and only if $\rho_j^{(n)} p(j) = \rho_j^{(n)}$ for all $j \in \Lambda$ and $n \geq n_0$.*

Proof: The proof follows from Theorem 3.8 and Proposition 4.11. $\square$

Theorem 4.13 *Suppose that h is a nontrivial projection on $\mathcal{H}$ such that*

$$hB_\pi = B_\pi$$

for any path $\pi \in \mathcal{P}(i,j)$ for all $i, j \in \Lambda$. Then the QMC is reducible.

Proof: Define a projection $p \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ by $p := h \otimes I_\mathcal{K} = \sum_{j \in \Lambda} h \otimes |j\rangle\langle j|$ and consider $p_{[1} \in \mathcal{P}_0$. Then for $n \geq 1$,

$$\begin{aligned}
\rho_j^{(n)} h &= \sum_{i_0 \in \Lambda} \sum_{\pi \in \mathcal{P}(i_0,j)} B_\pi \rho_{i_0} B_\pi^* h \\
&= \sum_{i_0 \in \Lambda} \sum_{\pi \in \mathcal{P}(i_0,j)} B_\pi \rho_{i_0} B_\pi^* \\
&= \rho_j^{(n)}.
\end{aligned}$$

By Theorem 4.12, the QMC is reducible with a reducing projection $p_{[1}$. $\square$

Remark 4.14 (a) The condition $\rho_j^{(n)} p(j) = \rho_j^{(n)}$ for all $j \in \Lambda$ and $n \geq n_0$ in Proposition 4.11 is equivalent to $p(j)\rho_j^{(n)} p(j) = \rho_j^{(n)}$ for all $j \in \Lambda$ and $n \geq n_0$, which means that for each $j \in \Lambda$, the support of $\rho_j^{(n)}$ is in the range space of $p(j)$ for all $n \geq n_0$.

(b) By Theorem 4.13, if the range of B_j^i for all i, j belongs to the nontrivial subspace, that is, $hB_j^i = B_j^i$ for some nontrivial projection h, then the QMC is reducible.

Next we discuss some sufficient conditions for the irreducibility.

Theorem 4.15 *Suppose that the OQRW is such that $\rho_j^{(n)}/\text{Tr}(\rho_j^{(n)})$ is a faithful state on $\mathcal{B}(\mathcal{H})$ for all $j \in \Lambda$ and $n = 0, 1, 2, \cdots$. Then the QMC associated with this OQRW is irreducible.*

Proof: Suppose, on the contrary, that there is a nontrivial projection p on $\mathcal{H}$ and $p_{[n_0} \in \mathcal{P}_0$ is a reducing projection for the QMC. Then by Theorem 4.12 it follows that $\rho_j^{(n)} p(j) = \rho_j^{(n)}$ for all $j \in \Lambda$ and $n \geq n_0$. Since $\rho_j^{(n)}/\text{Tr}(\rho_j^{(n)})$ is a faithful state it must hold that $p(j)$ is the identity operator on $\mathcal{H}$, leading to a contradiction. $\square$
An example satisfying the conditions in the theorem will be considered in Subsection 5.2.

Remark 4.16 The reducibility and irreducibility of positive maps on the ideal of trace class operators (in Schrödinger representation), and equivalently, of positive maps on the operator algebras (in Heisenberg representation), was introduced in some literature, see for example, [17, 20]. Typically, the study of reducibility and irreducibility for OQRWs was investigated in [17]. It turns out that the concepts of reducibility and irreducibility for OQRWs defined in [17] and in this paper are equivalent. In the Appendix we will consider the equivalence in detail. Therefore, in particular, under the condition of Theorem 4.13, the OQRW is reducible in the sense of [17]. Also, under the condition of Theorem 4.15, the OQRW is irreducible in the sense of [17].

5 Examples

5.1 OQRWs on the 1-dimensional integer lattice

In this subsection we give some examples of reducible and irreducible OQRWs on the 1-dimensional integer lattice. Of course the idea can be extended to multi-

dimensional models. First we consider reducible OQRWs.

Example 5.1 Let us consider a stationary OQRW on $\mathbb{Z}$ with nearest-neighbor jumps (see [14]). Let $\mathcal{H}$ be a Hilbert space and $B, C \in \mathcal{B}(\mathcal{H})$ such that $B^*B + C^*C = I$. We define the OQRW as follows:

$$B_i^{i-1} = B \text{ and } B_i^{i+1} = C$$

for all $i \in \mathbb{Z}$, and $B_j^i = 0$ in the other cases. Fix a density operator $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, of the form

$$\rho = \sum_i \rho_i \otimes |i\rangle\langle i|$$

with $\rho_i \neq 0$ for all i. We get

$$\mathcal{M}(\rho) = \sum_j (B\rho_{j+1}B^* + C\rho_{j-1}C^*) \otimes |j\rangle\langle j|. \tag{5.1}$$

In order to specify the model, let us consider the following matrices,

$$B = \begin{bmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad h = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

or

$$B = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad h = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

For both cases, B and C satisfy $B^*B + C^*C = I$ and $hB = B$, $hC = C$. By Theorem 4.13, the QMC corresponding to this OQRW is reducible.

The following is an example of reducible OQRW in 1 dimension with 3 states.

Example 5.2 Let us consider a stationary OQRW on $\mathbb{Z}$ with nearest-neighbor jumps. Let $\mathcal{H}$ be a Hilbert space and $L_i \in \mathcal{B}(\mathcal{H}), i = 1, 2, 3$, satisfy $\sum_{i=1}^3 L_i^* L_i = I$. We define the walk as follows:

$$B_i^{i-1} = L_1, \ B_i^i = L_2 \text{ and } B_i^{i+1} = L_3$$

for all $i \in \mathbb{Z}$, and $B_j^i = 0$ for the other cases. The evolution (5.1) becomes now

$$\mathcal{M}(\rho) = \sum_j (L_1\rho_{j+1}L_1^* + L_2\rho_j L_2^* + L_3\rho_{j-1}L_3^*) \otimes |j\rangle\langle j|.$$

If we take the matrices

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}, \ L_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}, \ L_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

it holds that $hL_i = L_i$, $i = 1, 2, 3$. Thus by Theorem 4.13 again, the QMC is reducible.

Next we consider irreducible OQRWs in 1-dimensional space.

Proposition 5.3 *In the 1-dimensional OQRW in* (5.1), *suppose that B and C satisfy the following condition:*

$$B^* x B = 0 \text{ and } C^* x C = 0 \text{ for nonnegative } x \in \mathcal{B}(\mathcal{H}) \text{ implies } x = 0. \tag{5.2}$$

Then $\mathcal{M}(\rho)$ is faithful whenever ρ is faithful. Therefore by Theorem 4.15, the QMC associated with the OQRW (5.1) *with a faithful initial state $\rho^{(0)}$ is irreducible.*

Proof: Let $\rho = \sum_{i \in \mathbb{Z}} \rho_i \otimes |i\rangle\langle i|$ be a faithful state. This means that ρ_i's are faithful for all $i \in \mathbb{Z}$. We have to show that

$$\mathcal{M}(\rho)_i = B\rho_{i+1}B^* + C\rho_{i-1}C^*$$

is faithful for each $i \in \mathbb{Z}$. So, let $x \in \mathcal{B}(\mathcal{H})$ be a nonnegative operator (matrix) and suppose that

$$\mathrm{Tr}(\mathcal{M}(\rho)_i x) = \mathrm{Tr}((B\rho_{i+1}B^* + C\rho_{i-1}C^*)x) = 0.$$

Since ρ_{i+1} as well as ρ_{i-1} are faithful, it implies that $B^* x B = 0$ and $C^* x C = 0$. By the condition (5.2) we get $x = 0$. The proof is completed. $\square$

The simplest example for which the condition (5.2) holds is the case where B or C is invertible. In the following example, the invertibility of B or C is not needed.

Example 5.4 Let $U = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$ be a 2×2 unitary matrix with column vectors $\mathbf{u}$ and $\mathbf{v}$. Let

$$B = \begin{bmatrix} \mathbf{u} & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & \mathbf{v} \end{bmatrix}.$$

Then we get

$$B^* x B = \begin{bmatrix} \langle \mathbf{u}, x\mathbf{u} \rangle & 0 \\ 0 & 0 \end{bmatrix} \text{ and } C^* x C = \begin{bmatrix} 0 & 0 \\ 0 & \langle \mathbf{v}, x\mathbf{v} \rangle \end{bmatrix}.$$

Thus the condition (5.2) is satisfied. By Proposition 5.3, if the OQRW has faithful initial state $\rho^{(0)}$, the associated QMC is irreducible.

5.2 Classical Markov chains

In this subsection we consider the classical Markov chains. The recovery of the classical Markov chains from the OQRWs was introduced in [14]. Let $\mathcal{H} = \mathbb{C}$ and $\mathcal{K} = l^2(\Lambda)$. Then $\mathcal{H} \otimes \mathcal{K} \approx l^2(\Lambda)$. Let $P = (P(i,j))_{i,j \in \Lambda}$ be a stochastic matrix, i.e., all the components are nonnegative and satisfy

$$\sum_{j \in \Lambda} P(i,j) = 1 \text{ for all } i \in \Lambda.$$

For each $i, j \in \Lambda$, let U_j^i be a unitary operator on $\mathcal{H} = \mathbb{C}$. (Thus U_j^i is a complex number with modulus 1, and in the sequel, it turns out that there is no difference with the choice $U_j^i \equiv 1$.) Define

$$B_j^i := \sqrt{P(j,i)}U_j^i, \quad i,j \in \Lambda.$$

We see that

$$\sum_i B_j^{i*} B_j^i = I, \quad j \in \Lambda.$$

We notice that since $\mathcal{H} = \mathbb{C}$ is a one-dimensional space, the algebra $\mathcal{B}$ consisting of the operators $x = \sum_j x_j \otimes |j\rangle\langle j|$, with (x_j) a bounded sequence in $\mathbb{C}$, is a commutative algebra. If $\rho = (\rho_i)_{i \in \Lambda}$ is a state, i.e., a probability measure on Λ, we denote by P_ρ the projection onto the support of ρ. Here the support of ρ is the set of $i \in \Lambda$ at which $\rho_i > 0$. By a direct computation from (4.4) we get

$$\mathcal{E}^{(n)}(x \otimes y) = P_{\rho^{(n)}} x Py, \tag{5.3}$$

where

$$(Py)_j = \sum_i P(j,i)y_i.$$

Notice that, in the classical Markov chain, if $\rho^{(0)}$ is the initial state (a probability measure) then $\rho^{(1)} = \rho^{(0)}P$, i.e.,

$$\rho_i^{(1)} = \sum_{j \in \Lambda} \rho_j^{(0)} P(j,i),$$

and

$$\rho^{(n)} = \rho^{(0)} P^n.$$

Proposition 5.5 *For any initial state $\rho^{(0)}$, the nth evolution of the open quantum random walk, $\mathcal{M}^n(\rho^{(0)})$, is $\rho^{(0)} P^n$. Therefore, the evolutions by classical Markov chain and by open quantum random walk are the same.*

Proof: By induction, it is enough to see $\mathcal{M}(\rho^{(0)})$.

$$
\begin{aligned}
(\mathcal{M}(\rho^{(0)}))_i &= \sum_j B_j^i \rho_j^{(0)} B_j^{i*} \\
&= \sum_j \rho_j^{(0)} P(j, i) \\
&= (\rho^{(0)} P)_i.
\end{aligned}
$$

The proof is complete. $\square$

Applying the formula (5.3) repeatedly we get

$$
\begin{aligned}
\rho(I \otimes \cdots \otimes I \otimes \overset{n\text{th}}{x} \otimes I \otimes \cdots) &= \rho^{(0)}(E_{0]}(I \otimes \cdots \otimes I \otimes x \otimes I \otimes \cdots)) \\
&= \rho^{(0)}(\mathcal{E}^{(0)}(I \otimes \mathcal{E}^{(1)}(I \otimes \cdots \mathcal{E}^{(n)}(x \otimes I)))) \\
&= \rho^{(0)}(P_{\rho^0} P P_{\rho^{(1)}} P \cdots P_{\rho^{(n-1)}} P P_{\rho^{(n)}} x) \\
&= \mathbb{E}_{\rho^{(0)}}[P^n x] = \mathbb{E}_{\rho^{(n)}}[x]. \qquad (5.4)
\end{aligned}
$$

The transition expectation thus recovers the classical Markov chain, which was observed in [6, 7].

Recall that in the classical Markov chain with transition matrix P, we say that a state j is accessible from i, written $i \to j$, if $P^n(i, j) > 0$ for some $n \in \mathbb{N}$. We say that i communicates with j, written $i \leftrightarrow j$, if $i \to j$ and $j \to i$. The relation "$\leftrightarrow$" is an equivalence relation. In the case when every states communicate with every other states, we say that the chain is irreducible. Otherwise, it is called reducible [28]. We want to see the reducibility or irreducibility of classical Markov chains also from the view point of quantum Markov chains. We emphasize here that, by definition, when we discuss the reducibility or irreducibility of QMCs, not only the transition expectations but also the initial states are concerned.

Proposition 5.6 *Suppose a classical Markov chain with transition matrix P is reducible. Then the QMC $(\rho^{(0)}, (\mathcal{E}^{(n)})_{n \geq 0})$ with a suitably chosen initial state (measure) $\rho^{(0)}$ and transition expectations $\mathcal{E}^{(n)}$ given by (5.3) is reducible.*

Proof: The state space Λ of the Markov chain is decomposed as $\Lambda = T \cup (\cup_k R_k)$, where T is the set of transient states and R_k's are closed, recurrent communicating classes. If there is a closed, recurrent communicating class, say R_1, by the hypothesis of the proposition, it holds that $R_1 \neq \Lambda$. Let $p := P_{R_1}$ be the projection onto the

set R_1, i.e., P_{R_1} is the indicator function 1_{R_1} looked as a multiplication operator on $l^2(\Lambda)$, and we consider $p_{[0} = p \otimes p \otimes \cdots$. Let $\rho^{(0)}$ be a state (measure) supported on R_1. Since R_1 is a closed communicating class, $\rho^{(n)}$ is also supported on R_1 for all $n \geq 1$. Now the condition $\rho_j^{(n)} p(j) = \rho_j^{(n)}$ is equivalent to saying that $p(j) = 1$ on the support of $\rho^{(n)}$, and this is the case by our construction. Therefore by Theorem 4.12 the QMC is reducible. If there is no closed, recurrent communicating class, then the set Λ consists only of transient states. Fix an $i_0 \in \Lambda$ and let C_0 be the communicating class containing i_0. By the assumption C_0 is not closed, i.e., there is a state $j \in \Lambda \setminus C_0$ such that $i_1 \to j$ for some $i_1 \in C_0$ and $j \not\to i$ for all $i \in C_0$. Let $C_1 := \{j' \in \Lambda : j \to j'\}$. Then $C_1 \cap C_0 = \emptyset$ and if the initial measure $\rho^{(0)}$ is supported on the set C_1, it follows that $\rho^{(n)}$ is also supported on the set C_1 for all $n \geq 1$. Defining now $p := P_{C_1}$, the projection onto the set C_1, we see as above that $p_{[0} = p \otimes p \otimes \cdots$ is a reducing projection for the QMC $(\rho^{(0)}, (\mathcal{E}^{(n)})_{n \geq 0})$. $\square$

Let us now consider the converse problem.

Proposition 5.7 *Suppose that the classical Markov chain with transition matrix P is irreducible. Then the QMC of transition expectation* (5.3) *with any faithful initial state is irreducible.*

Proof: Suppose that the transition expectation (5.3) is constructed from a faithful initial state $\rho^{(0)}$. From the assumption of irreducibility of the classical Markov chain, the distribution at any time has full support. This implies by Proposition 5.5 that the state $\rho^{(n)}$ is faithful for any $n \geq 0$. The result now follows from Theorem 4.15. $\square$

A   Equivalence of concepts of reducibility/irreducibility of OQRWs defined in [17] and in this paper

First of all we recall the definition of reducibility/irreducibility used in [17]. Let Φ be a positive map on the ideal $\mathcal{I}_1(\mathfrak{h})$ of trace class operators on a Hilbert space $\mathfrak{h}$. When we come to our model, $\mathfrak{h}$ is $\mathcal{H} \otimes \mathcal{K}$ and Φ is $\mathcal{M}$. Φ is said to be irreducible (see [17, Definition 3.1]) if the only orthogonal projections p reducing Φ, i.e. such that $\Phi(p\mathcal{I}_1(\mathfrak{h})p) \subset p\mathcal{I}_1(\mathfrak{h})p$, are $p = 0$ and I. Applying to OQRWs, Carbone and Pautrat have shown (terminology in our language):

Proposition A.1 *([17, Proposition 3.8]) The completely positive and trace preserving map $\mathcal{M}$ is irreducible if and only if for any $i, j \in \Lambda$ and any $\psi, \xi \in \mathcal{H} \setminus \{0\}$, there is a path $\pi \in \mathcal{P}(i, j)$ such that $\langle \xi, B_\pi \psi \rangle \neq 0$.*

Now we show the definitions of reducibility/irreducibility of OQRWs given in [17] and in the present paper are equivalent. First we remark that as given by [17, Proposition 6.1, item 3], once an OQRW is reducible (in the sense of [17]) one can always find a reducing projection p of the block-diagonal form: $p = \sum_j p(j) \otimes |j\rangle\langle j|$. Conversely speaking, if there is no nontrivial block-diagonal reducing projection the OQRW is irreducible. Suppose the OQRW is reducible in the sense of [17] with a reducing projection $p = \sum_j p(j) \otimes |j\rangle\langle j|$. By [17, Proposition 6.2], it holds that for any $i, j \in \Lambda$,

$$B_j^i p(j) = p(i) B_j^i p(j). \tag{A.1}$$

Take an initial state $\rho^{(0)} = \sum_j \rho_j^{(0)} \otimes |j\rangle\langle j|$ such that $p(j)\rho_j^{(0)} p(j) = \rho_j^{(0)}$ for all $j \in \Lambda$. We can show by induction that for all $n \geq 0$ and $j \in \Lambda$,

$$p(j)\rho_j^{(n)} p(j) = \rho_j^{(n)}. \tag{A.2}$$

In fact, suppose (A.2) holds for $n = 0, \cdots, k$. Then, by the assumption hypothesis and (A.1)

$$
\begin{aligned}
p(j)\rho_j^{(k+1)} p(j) &= \sum_i p(j) B_i^j \rho_i^{(k)} B_i^{j*} p(j) \\
&= \sum_i p(j) B_i^j p(i) \rho_i^{(k)} p(i) B_i^{j*} p(j) \\
&= \sum_i B_i^j p(i) \rho_i^{(k)} p(i) B_i^{j*} \\
&= \sum_i B_i^j \rho_i^{(k)} B_i^{j*} = \rho_j^{(k+1)}.
\end{aligned}
$$

Now (A.2) holds and by Theorem 4.12 the OQRW is reducible in the sense of this paper (recall (A.2) is equivalent to $\rho_j^{(n)} p(j) = \rho_j^{(n)}$).

Conversely, suppose that the OQRW is reducible in the sense of present paper. By Theorem 4.12, there is a nontrivial projection $p = \sum_j p(j) \otimes |j\rangle\langle j|$ such that (A.2) holds for $n \geq n_0$ for some n_0. Find a $j \in \Lambda$ such that $p(j) \neq I_\mathcal{H}$. By the assumption we have for any $k \geq 0$,

$$\mathrm{Tr}(\rho_j^{(n_0+k)} p(j)^\perp) = \mathrm{Tr}(\rho_j^{(n_0+k)} p(j) p(j)^\perp) = 0.$$

Take an $i \in \Lambda$ such that $\rho_i^{(n_0)} \neq 0$. From the above relation we have

$$
\begin{aligned}
0 &= \operatorname{Tr}(\rho_j^{(n_0+k)} p(j)^\perp) \\
&= \sum_{i_0,\cdots,i_{k-1}} \operatorname{Tr}\left(B_{\pi(i_0,\cdots,i_{k-1},j)} \rho_{i_0}^{(n_0)} B^*_{\pi(i_0,\cdots,i_{k-1},j)} p(j)^\perp \right) \\
&\geq \operatorname{Tr}\left(B_\pi \rho_i^{(n_0)} B^*_\pi p(j)^\perp \right) = \operatorname{Tr}\left(\rho_i^{(n_0)} B^*_\pi p(j)^\perp B_\pi \right) \geq 0,
\end{aligned}
$$

for any path $\pi \in \mathcal{P}(i,j)$ of length k. Thus for any $0 \neq \psi \in \mathcal{H}$ lying in the spectral projection of $\rho_i^{(n_0)}$ away from zero, e.g., any eigenvector of $\rho_i^{(n_0)}$ corresponding to nonzero eigenvalue,

$$
\langle \psi, B^*_\pi p(j)^\perp B_\pi \psi \rangle = 0.
$$

Therefore, for any such a vector $0 \neq \psi$ and $0 \neq \xi \in p(j)^\perp$, and for any path $\pi \in \mathcal{P}(i,j)$,

$$
\begin{aligned}
|\langle \xi, B_\pi \psi \rangle| &= |\langle \xi, p(j)^\perp B_\pi \psi \rangle| \\
&\leq \|\xi\| \langle p(j)^\perp B_\pi \psi, p(j)^\perp B_\pi \psi \rangle^{1/2} = 0.
\end{aligned}
$$

By Proposition A.1, it says that the OQRW is reducible in the sense of [17]. This completes the proof of equivalence.

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