

ON A CONJECTURE BY WILF ABOUT THE FROBENIUS NUMBER

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ABSTRACT. Given coprime positive integers $a_1 < \dots < a_d$, the Frobenius number F is the largest integer which is not representable as a non-negative integer combination of the a_i . Let g denote the number of all non-representable positive integers: Wilf conjectured that $d \geq \frac{F+1}{F+1-g}$. We prove that for every fixed value of $\lceil \frac{a_1}{d} \rceil$ the conjecture holds for all values of a_1 which are sufficiently large and are not divisible by a finite set of primes. We also propose a generalization in the context of one-dimensional local rings and a question on the equality $d = \frac{F+1}{F+1-g}$.

INTRODUCTION

The Diophantine Frobenius problem or coin problem consists of studying the greatest integer F , called the Frobenius number, that is not representable as a linear combination of given d coprime positive integers $a_1 < \dots < a_d$ with coefficients in \mathbb{N} . The problem has stimulated much research over the past decades, due to the applications to several areas of pure and applied mathematics including Coding Theory, Linear Algebra, Combinatorics and Commutative Algebra. The monograph [11] explores several viewpoints about the problem.

Let $\mathcal{S} = \{ \sum_{i=1}^d \lambda_i a_i : \lambda_i \in \mathbb{N} \}$ be the numerical semigroup generated by the a_i . The condition $\gcd(a_1, \dots, a_d) = 1$ implies that the set of gaps $\mathbb{N} \setminus \mathcal{S}$ is finite and its cardinality $g = |\mathbb{N} \setminus \mathcal{S}|$ is known as the genus of \mathcal{S} . In 1978 H. S. Wilf proposed a lower bound for the number of generators of \mathcal{S} in terms of the Frobenius number and the genus (cf. [14]):

$$(1) \quad d \geq \frac{F+1}{F+1-g}.$$

Although the problem has been considered by several authors (cf. [1], [3], [7], [8], [9], [13], [15]) only special cases have been solved and it remains wide open. The approach we follow in this paper is to study the poset structure of the so-called Apéry set once the value of the ratio $\rho = \lceil \frac{a_1}{d} \rceil$ is fixed. The cases $\rho = 1$ and $\rho = 2$ were solved in [7] and [13] respectively; in this work we focus on the general case. Our main result is the following:

Theorem 1. *For every value of $\rho = \lceil \frac{a_1}{d} \rceil$ Wilf's inequality (1) holds if a_1 is large enough and the prime factors of a_1 are greater than or equal to ρ .*

The explicit lower bound for a_1 in Theorem 1 is $a_1 \geq \frac{\rho(3\rho^2 - \rho - 4)(3\rho^2 - \rho - 2)}{8(\rho - 2)}$.

In the first section of the paper we recall some definitions and preliminary results, whereas the second section is devoted to proving Theorem 1. We conclude the paper with a word about the equality in (1) and a discussion of the conjecture in the context of Commutative Algebra.

2010 *Mathematics Subject Classification.* Primary: 11D07; Secondary: 11B75, 20M14.

Key words and phrases. Diophantine Frobenius problem; coin problem; Wilf's conjecture; numerical semigroup; Apéry set; length inequality; one-dimensional local ring.

1. PRELIMINARIES

We assume without loss of generality that a_1, \dots, a_d are the minimal generators of \mathcal{S} , i.e. no proper subset generates the semigroup; this implies that $d \leq a_1$. The Apéry set is defined as

$$\text{Ap}(\mathcal{S}) = \{\omega \in \mathcal{S} : \omega - a_1 \notin \mathcal{S}\}$$

and thus it consists of the smallest elements of \mathcal{S} in each residue class modulo a_1 . It follows that $0 \in \text{Ap}(\mathcal{S})$, $|\text{Ap}(\mathcal{S})| = a_1$ and that $F + a_1$ is the largest element in $\text{Ap}(\mathcal{S})$. We list the elements of $\text{Ap}(\mathcal{S})$ increasingly setting $\text{Ap}(\mathcal{S}) = \{\omega_0 < \omega_1 < \dots < \omega_{a_1-1}\}$, notice that we have $\omega_0 = 0, \omega_1 = a_2, \omega_{a_1-1} = F + a_1$. We establish a partial order on \mathbb{N} by setting $n_1 \preceq n_2$ if there exists $s \in \mathcal{S}$ such that $n_1 + s = n_2$ and we consider throughout the paper the poset $(\text{Ap}(\mathcal{S}) \setminus \{0\}, \preceq)$. The minimal elements in $(\text{Ap}(\mathcal{S}) \setminus \{0\}, \preceq)$ are exactly $\{a_2, \dots, a_d\}$ and if $\tau \in \mathcal{S}, \omega \in \text{Ap}(\mathcal{S}), \tau \preceq \omega$ then $\tau \in \text{Ap}(\mathcal{S})$. We refer to [12] for details on numerical semigroups.

Now we give further definitions in order to introduce a reformulation of Wilf's inequality. For each $k \in \mathbb{N}$ let $I_k = [ka_1, (k+1)a_1 - 1]$ and $n_k = |\mathcal{S} \cap [0, F] \cap I_k|$. We write

$$F + 1 = Qa_1 + R$$

with $Q, R \in \mathbb{N}$ and $2 \leq R \leq a_1$. Note that $R \neq 1$ as $F \notin \mathcal{S}$ and that I_Q is the interval containing the Frobenius number. We define the numbers

$$\eta_j = \left| \{k \in \mathbb{N} : |I_k \cap \mathcal{S}| = j\} \right| \quad \text{and} \quad \epsilon_j = \left| \{k \in \mathbb{N} : |I_k \cap \mathcal{S}| = j, 0 \leq k \leq Q-1\} \right|$$

for each $j \in \{1, 2, \dots, a_1 - 1\}$. In other words, η_j counts the intervals I_k containing exactly j elements of \mathcal{S} , while ϵ_j only counts such intervals among the first Q . The two definitions differ slightly, and the numbers η_j can be expressed in terms of $\text{Ap}(\mathcal{S})$:

Lemma 2 ([13], 12, 13). *The following properties hold:*

- (1) $\epsilon_j = \eta_j - 1$ if $j = |I_Q \cap \mathcal{S}|$ and $\epsilon_j = \eta_j$ otherwise;
- (2) $\eta_j = \lfloor \frac{\omega_j}{a_1} \rfloor - \lfloor \frac{\omega_{j-1}}{a_1} \rfloor$ for all $1 \leq j \leq a_1 - 1$.

The numbers ϵ_j give rise to an equivalent formulation of (1):

Proposition 3 ([13], 10, 11). *We have the equation*

$$d(F + 1 - g) - (F + 1) = \sum_{j=1}^{a_1-1} \epsilon_j(jd - a_1) + (n_Q d - R) =: \Delta.$$

2. PROOF OF THE MAIN THEOREM

In this section we prove Theorem 1 using Proposition 3. We need a series of lemmas first.

Lemma 4. *If $x \in \mathbb{N} \setminus \{0\}$ is smaller than every prime factor of a_1 then $\lfloor \frac{\omega_x}{a_1} \rfloor \leq x \lfloor \frac{\omega_1}{a_1} \rfloor + x - 1$.*

Proof. The assumption on x implies that $i\omega_1 \not\equiv j\omega_1 \pmod{a_1}$ for any $0 \leq i < j \leq x$: in fact $i\omega_1 \equiv j\omega_1 \pmod{a_1}$ yields the contradiction $(j-i)\omega_1 \equiv 0 \pmod{a_1}$ as $\gcd(j-i, a_1) = 1$ and $\omega_1 \not\equiv 0 \pmod{a_1}$. Thus the subset $\{\omega_1, 2\omega_1, \dots, x\omega_1\} \subseteq \mathcal{S}$ covers x different residue classes modulo a_1 , hence there are elements of at least x different classes less than or equal to $x\omega_1$ in \mathcal{S} . By definition of Apéry set we deduce that $\omega_x \leq x\omega_1$ and $\lfloor \frac{\omega_x}{a_1} \rfloor \leq \lfloor \frac{x\omega_1}{a_1} \rfloor \leq x \lfloor \frac{\omega_1}{a_1} \rfloor + x - 1$. \square

Lemma 5. *If $y \in \mathbb{N}$ satisfies $y \geq 2$ and $a_1 - d \geq \binom{y}{2} + 1$ then $\omega_{a_1-1} \geq \omega_y + \omega_1$. In particular, $\lfloor \frac{\omega_{a_1-1}}{a_1} \rfloor \geq \lfloor \frac{\omega_y}{a_1} \rfloor + \lfloor \frac{\omega_1}{a_1} \rfloor$ and $F > \omega_y$.*

Proof. Since the minimal elements in $(\text{Ap}(\mathcal{S}) \setminus \{0\}, \preceq)$ are exactly $\{a_2, \dots, a_d\}$ and $|\text{Ap}(\mathcal{S})| = a_1$, there are at least $\binom{y}{2} + 1$ non-minimal elements in $(\text{Ap}(\mathcal{S}) \setminus \{0\}, \preceq)$. The set $\mathcal{Y} = \{\omega_i + \omega_j \mid 1 \leq i \leq j \leq y-1\}$ contains at most $\binom{y}{2}$ distinct elements, so there exists $\tau \in \text{Ap}(\mathcal{S}) \setminus (\{0, a_2, \dots, a_d\} \cup \mathcal{Y})$. In particular τ is not minimal, thus $\tau = \omega_h + \omega_k$ for some $1 \leq h \leq k$, and $\tau \notin \mathcal{Y}$ yields $k \geq y$. Finally we have $\omega_{a_1-1} \geq \tau = \omega_h + \omega_k \geq \omega_1 + \omega_y$. The inequality with the floor function follows immediately, and from $\omega_1 > a_1$ we obtain $\omega_{a_1-1} = F + a_1 \geq \omega_y + \omega_1 > \omega_y + a_1$ and $F > \omega_y$. \square

Lemma 6. *If $y, z \in \mathbb{N}$ satisfy $y \geq 2$, $y \geq z$, $a_1 - d \geq \binom{y}{2} + 1$, and $\lfloor \frac{\omega_{a_1-1}}{a_1} \rfloor = \lfloor \frac{\omega_z}{a_1} \rfloor + \lfloor \frac{\omega_1}{a_1} \rfloor$ then $n_Q \geq y - z + 3$.*

Proof. Fix $z \leq i \leq y$, by Lemma 5 we have that $F + a_1 \geq \omega_1 + \omega_y \geq \omega_1 + \omega_i$ and

$$\left\lfloor \frac{\omega_{a_1-1}}{a_1} \right\rfloor \geq \left\lfloor \frac{\omega_y}{a_1} \right\rfloor + \left\lfloor \frac{\omega_1}{a_1} \right\rfloor \geq \left\lfloor \frac{\omega_i}{a_1} \right\rfloor + \left\lfloor \frac{\omega_1}{a_1} \right\rfloor \geq \left\lfloor \frac{\omega_z}{a_1} \right\rfloor + \left\lfloor \frac{\omega_1}{a_1} \right\rfloor = \left\lfloor \frac{\omega_{a_1-1}}{a_1} \right\rfloor$$

and this leads to $\lfloor \frac{\omega_y}{a_1} \rfloor = \lfloor \frac{\omega_i}{a_1} \rfloor = \lfloor \frac{\omega_z}{a_1} \rfloor$. From $F + 1 = Qa_1 + R$ we obtain $(Q+1)a_1 + (R-1) \geq \omega_1 + \omega_i$. Dividing by a_1 we get $\omega_1 = q_1 a_1 + r_1$ and $\omega_i = q_i a_1 + r_i$ with $0 \leq r_i \leq a_1 - 1$ and it follows that $Q+1 = \lfloor \frac{\omega_{a_1-1}}{a_1} \rfloor = \lfloor \frac{\omega_z}{a_1} \rfloor + \lfloor \frac{\omega_1}{a_1} \rfloor = \lfloor \frac{\omega_i}{a_1} \rfloor + \lfloor \frac{\omega_1}{a_1} \rfloor = q_1 + q_i$. From $\omega_{a_1-1} \geq \omega_i + \omega_1$ we obtain $(Q+1)a_1 + (R-1) \geq (q_i + q_1)a_1 + (r_i + r_1) = (Q+1)a_1 + (r_i + r_1)$, thus $R-1 \geq r_i + r_1$. In particular, $r_1, r_i < R$: we conclude that the elements $Qa_1, \omega_1 + l_1 a_1$ and $\omega_i + l_i a_1$ belong to $\mathcal{S} \cap I_Q \cap [0, F]$ for suitable $l_1, l_i \in \mathbb{N}$ for each $z \leq i \leq y$ so that $n_Q \geq 3 + y - z$. \square

Lemma 7. *If $y \in \mathbb{N}$ satisfies $y \geq 2$ and $a_1 - d \geq \binom{y}{2} + 1$ then $n_{Q-1} \geq y + 2 - n_Q$.*

Proof. By Lemma 5 we have $a_1 < \omega_y < F$, whence $Q > 0$ and $\{\omega_0, \dots, \omega_y\} \subseteq [0, F]$. At most $n_Q - 1$ of these elements belong to $\mathcal{S} \cap I_Q \cap [0, F]$, because $Qa_1 \notin \text{Ap}(\mathcal{S})$. Hence there are at least $y + 2 - n_Q$ elements ω_i smaller than Qa_1 , that is, $\omega_0 < \dots < \omega_{y-n_Q+1} < Qa_1$: we conclude that $\omega_i + l_i a_1 \in I_{Q-1}$ for suitable $l_i \in \mathbb{N}$ for each $0 \leq i \leq y - n_Q + 1$ so $n_{Q-1} \geq y + 2 - n_Q$. \square

We are now ready to prove the main result.

Proof of Theorem 1. Let $\rho = \lceil \frac{a_1}{d} \rceil$. Since the cases $\rho = 1, 2$ have been solved, we assume $\rho \geq 3$. Consider the integers

$$y = \frac{3\rho^2 - \rho - 4}{2} \quad \text{and} \quad z = \frac{\rho^2 + \rho - 2}{2}$$

notice that $y \geq z \geq \rho + 2$. We are going to make the assumption that $a_1 \geq \frac{\rho}{\rho-2} \binom{y}{2}$, so that the condition appearing in Lemmas 5, 6, 7 is satisfied:

$$a_1 - d \geq (\rho - 1)d + 1 - d = (\rho - 2)d + 1 \geq \frac{\rho - 2}{\rho} a_1 + 1 \geq \binom{y}{2} + 1.$$

Our task is to prove (1) by showing that the quantity Δ of Proposition 3 is non-negative; we will actually show that $\Delta > 0$. We are going to break the summation in two parts, at the index $j = \rho$. For the first part we use both properties of Lemma 2 and obtain:

$$\begin{aligned} \sum_{j=1}^{\rho} \epsilon_j (jd - a_1) &\geq \sum_{j=1}^{\rho} \eta_j (jd - a_1) - (\rho d - a_1) = \sum_{j=1}^{\rho} \left(\left\lfloor \frac{\omega_j}{a_1} \right\rfloor - \left\lfloor \frac{\omega_{j-1}}{a_1} \right\rfloor \right) (jd - a_1) - (\rho d - a_1) \\ &= \left\lfloor \frac{\omega_{\rho}}{a_1} \right\rfloor (\rho d - a_1) - d \sum_{j=1}^{\rho-1} \left\lfloor \frac{\omega_j}{a_1} \right\rfloor - (\rho d - a_1) \geq -d \sum_{j=1}^{\rho-1} \left\lfloor \frac{\omega_j}{a_1} \right\rfloor \end{aligned}$$

where we used that $\omega_0 = 0$ and $\omega_\rho > \omega_1 > a_1$. By the assumption on the factors of a_1 we can use Lemma 4 for each $j < \rho$, yielding

$$\sum_{j=1}^{\rho-1} \left\lfloor \frac{\omega_j}{a_1} \right\rfloor \leq \sum_{j=1}^{\rho-1} \left(j \left\lfloor \frac{\omega_1}{a_1} \right\rfloor + j - 1 \right) = \binom{\rho}{2} \left\lfloor \frac{\omega_1}{a_1} \right\rfloor + \binom{\rho-1}{2}$$

and therefore

$$\sum_{j=1}^{\rho} \epsilon_j(jd - a_1) \geq -\frac{d(\rho^2 - \rho)}{2} \left\lfloor \frac{\omega_1}{a_1} \right\rfloor - \frac{d(\rho^2 - 3\rho + 2)}{2}.$$

Moreover, since $\frac{1}{2}d(\rho^2 - \rho) \leq \frac{1}{2}d(\rho^2 - \rho) + \rho d - a_1 = \frac{1}{2}d(\rho^2 + \rho) - a_1 = (z+1)d - a_1$ then

$$\sum_{j=1}^{\rho} \epsilon_j(jd - a_1) \geq -((z+1)d - a_1) \left\lfloor \frac{\omega_1}{a_1} \right\rfloor - \frac{d(\rho^2 - 3\rho + 2)}{2}.$$

For the second part of the summation in Δ , as $z+1 \geq \rho+3$ we can write

$$\begin{aligned} \sum_{j=\rho+1}^{a_1-1} \epsilon_j(jd - a_1) &\geq \sum_{j=z+1}^{a_1-1} \epsilon_j(jd - a_1) \geq \sum_{j=z+1}^{a_1-1} \epsilon_j((z+1)d - a_1) = ((z+1)d - a_1) \sum_{j=z+1}^{a_1-1} \epsilon_j \geq \\ &\geq ((z+1)d - a_1) \left(\sum_{j=z+1}^{a_1-1} \eta_j - 1 \right) = ((z+1)d - a_1) \left(\left\lfloor \frac{\omega_{a_1-1}}{a_1} \right\rfloor - \left\lfloor \frac{\omega_z}{a_1} \right\rfloor - 1 \right) \end{aligned}$$

where we used again both properties of Lemma 2 in the last inequality. Combining the two parts gives

$$\begin{aligned} \Delta &= \sum_{j=1}^{a_1-1} \epsilon_j(jd - a_1) + (n_Q d - R) \geq \\ &\left(\left\lfloor \frac{\omega_{a_1-1}}{a_1} \right\rfloor - \left\lfloor \frac{\omega_z}{a_1} \right\rfloor - \left\lfloor \frac{\omega_1}{a_1} \right\rfloor - 1 \right) ((z+1)d - a_1) - \frac{d(\rho^2 - 3\rho + 2)}{2} + (n_Q d - R) =: \Pi. \end{aligned}$$

Since $z \leq y$, by Lemma 5 we have the inequality $\lfloor \frac{\omega_{a_1-1}}{a_1} \rfloor \geq \lfloor \frac{\omega_z}{a_1} \rfloor + \lfloor \frac{\omega_1}{a_1} \rfloor$. If the equality holds, we can use the bound for n_Q in Lemma 6 obtaining

$$\begin{aligned} \Delta \geq \Pi &\geq -((z+1)d - a_1) - \frac{d(\rho^2 - 3\rho + 2)}{2} + ((y - z + 3)d - R) = \\ &= d \left(y - 2z - \frac{\rho^2 - 3\rho + 2}{2} + 2 \right) + (a_1 - R) = d + (a_1 - R) > 0 \end{aligned}$$

by the definitions of y and z and the fact that $a_1 \geq R$. Suppose now we have the strict inequality $\lfloor \frac{\omega_{a_1-1}}{a_1} \rfloor > \lfloor \frac{\omega_z}{a_1} \rfloor + \lfloor \frac{\omega_1}{a_1} \rfloor$. In this case the first piece in Π is non-negative because $(z+1)d - a_1 \geq \rho d - a_1 \geq 0$, thus we can ignore it:

$$\Pi \geq -\frac{d(\rho^2 - 3\rho + 2)}{2} + (n_Q d - R).$$

If $n_Q \geq \frac{1}{2}(\rho^2 - \rho + 4)$, then $\Delta \geq \Pi \geq \rho d - R + d \geq \rho d - a_1 + d > 0$. Suppose finally that $n_Q \leq \frac{1}{2}(\rho^2 - \rho + 2)$. By Lemma 7 we know that $n_{Q-1} \geq y + 2 - n_Q$, i.e. $\mathcal{S} \cap I_{Q-1}$ contains at least $y + 2 - n_Q$ elements, and in this case from $\rho \geq 3$ it follows

$$y + 2 - n_Q \geq \frac{3\rho^2 - \rho - 4}{2} + 2 - \frac{\rho^2 - \rho + 2}{2} = \rho^2 - 1 \geq \frac{\rho^2}{2} + \frac{3\rho}{2} - 1 \geq \frac{\rho^2 + \rho - 2}{2} + \rho \geq z + 3.$$

Let $j_0 = n_{Q-1} \geq y + 2 - n_Q \geq z + 3 \geq \rho + 1$, we have $\epsilon_{j_0} \geq 1$ and therefore

$$\sum_{j=\rho+1}^{a_1-1} \epsilon_j(jd - a_1) \geq \sum_{j=z+1}^{a_1-1} \epsilon_j(jd - a_1) \geq (j_0d - a_1) - ((z+1)d - a_1) + \sum_{z+1}^{a_1-1} \epsilon_j((z+1)d - a_1)$$

it follows that

$$\begin{aligned} \Delta &= \sum_{j=1}^{a_1-1} \epsilon_j(jd - a_1) + (n_Qd - R) \geq \Pi + (y + 2 - n_Q - z - 1)d \geq \\ &= \frac{d(\rho^2 - 3\rho + 2)}{2} - R + (y - z + 1)d = \frac{d(\rho^2 + \rho - 2)}{2} - R \geq \frac{d(3\rho + 3 - 2)}{2} - R \geq (\rho d - R) + \frac{1}{2} > 0 \end{aligned}$$

where we used that $\rho \geq 3$ and $\rho d \geq a_1 \geq R$.

We have shown that $\Delta > 0$ in each case and the theorem is thus proved. \square

3. TWO FURTHER PROBLEMS

Whenever we have a bound, it is natural to ask ourselves whether the bound is sharp and, in case it is, to try to characterize the instances in which the bound is achieved. In his original paper [14], Wilf also asked whether the equality in (1) is attained if and only if $d = a_1$ and $a_i = a_1 + (i-1)$ for $i = 2, \dots, a_1$. This is not the case: a simple counterexample is $a_1 = 3, a_2 = 5$. However, we believe that equality can only occur in two cases.

Question 8. *Is it true that the equality $F + 1 = d(F + 1 - g)$ holds if and only if either $d = 2$ or $d = a_1$ and there exists $K \in \mathbb{N}$ such that $a_i = Ka_1 + (i - 1)$ for $i = 2, \dots, a_1$?*

Note that in the latter case the numerical semigroup has the form

$$\mathcal{S} = \{0, a_1, 2a_1, \dots, (K-1)a_1, Ka_1, Ka_1 + 1, Ka_1 + 2, \dots\}$$

and the equality follows from Proposition 3, while in the former case the equality was already known to Sylvester (cf. [12, 2.12]), so the interesting part is the ‘‘only if’’. We observe that, in order to show that these are the only two cases, it suffices to prove that either $d = 2$ or $d = a_1$. In fact, if $d = a_1$ and $F + 1 = d(F + 1 - g)$ then Proposition 3 implies the equation

$$\Delta = \sum_{j=1}^{a_1-1} \epsilon_j(j-1)a_1 + (n_Qa_1 - R) = 0$$

and since $n_Q \geq 1, a_1 \geq R$ we conclude that $\epsilon_j = 0$ for every $j \geq 2, n_Q = 1, R = a_1$; it follows that $a_i = (Q+1)a_1 + (i-1)$ for $i = 2, \dots, a_1$.

We have verified via GAP (cf. [6]) that Question 8 has an affirmative answer when $g \leq 35$. Further evidence in its favor is given by the proof of Theorem 1, as in all the cases investigated therein (where $\rho \geq 3$, hence $d < a_1$) the strict inequality was actually seen to hold: in other words, we give a positive answer under the assumptions of our theorem.

Finally, we remark that the conjecture has an interpretation in Commutative Algebra in terms of length inequalities. Let $\mathcal{R} = \mathbb{k}[[t^{a_1}, \dots, t^{a_d}]]$ be the local ring of a monomial curve, where \mathbb{k} is a field. Let $\overline{\mathcal{R}} = \mathbb{k}[[t]]$ be the integral closure of \mathcal{R} in its field of fractions $\mathcal{Q} = \mathbb{k}((t))$ and $\mathfrak{C} = (\mathcal{R} :_{\mathcal{Q}} \overline{\mathcal{R}})$ be the conductor of \mathcal{R} in $\overline{\mathcal{R}}$, that is the largest common ideal of \mathcal{R} and $\overline{\mathcal{R}}$. Denoting by $\ell(\cdot)$ the length of an \mathcal{R} -module, the values of $\ell(\overline{\mathcal{R}}/\mathcal{R})$ and $\ell(\mathcal{R}/\mathfrak{C})$ are both measures of the singularity of \mathcal{R} and they are related by $\ell(\mathcal{R}/\mathfrak{C}) \leq \ell(\overline{\mathcal{R}}/\mathcal{R})$, with equality holding if and only if \mathcal{R} is Gorenstein. Under the notation of this paper, the embedding

dimension of \mathcal{R} is $\text{edim}(\mathcal{R}) = d$, whereas $\ell(\overline{\mathcal{R}}/\mathfrak{C}) = F + 1$ and $\ell(\mathcal{R}/\mathfrak{C}) = F + 1 - g$ (cf. [2, II.1]). Therefore (1) is equivalent to

$$(2) \quad \ell(\overline{\mathcal{R}}/\mathfrak{C}) \leq \text{edim}(\mathcal{R})\ell(\mathcal{R}/\mathfrak{C}).$$

A similar well-known inequality holds in a more general context: if \mathcal{R} is a one-dimensional Cohen-Macaulay local ring with type $t(\mathcal{R})$ and such that the integral closure $\overline{\mathcal{R}}$ of \mathcal{R} in its total ring of fractions is a finite \mathcal{R} -module, then

$$(3) \quad \ell(\overline{\mathcal{R}}/\mathfrak{C}) \leq (t(\mathcal{R}) + 1)\ell(\mathcal{R}/\mathfrak{C})$$

see e.g. [4], [5], [10]. It follows that (2) is satisfied if $t(\mathcal{R}) < \text{edim}(\mathcal{R})$, but the two invariants are unrelated in general: J. Backelin exhibited a family of rings with $\text{edim}(\mathcal{R}) = 4$ and arbitrarily large type (cf. [8]). It would be natural to explore the form (2) of Wilf's inequality for more general classes of rings, therefore we conclude the paper with the following general question.

Question 9. *Let \mathcal{R} be a one-dimensional Cohen-Macaulay local ring such that the integral closure of \mathcal{R} in its total ring of fractions is a finite \mathcal{R} -module. Under what assumptions do we have $\ell(\overline{\mathcal{R}}/\mathfrak{C}) \leq \text{edim}(\mathcal{R})\ell(\mathcal{R}/\mathfrak{C})$?*

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