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# Low-gain integral control for a class of discrete-time Lur'e systems with applications to sampled-data control

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**Abstract**

We study low-gain (P)roportional (I)ntegral control of multivariate discrete-time, forced Lur'e systems to solve the output-tracking problem for constant reference signals. We formulate an incremental sector condition which is sufficient for a usual linear low-gain PI controller to achieve exponential disturbance-to-state and disturbance-to-tracking-error stability in closed-loop, for all sufficiently small integrator gains. Output tracking is achieved in the absence of exogenous disturbance (noise) terms. Our line of argument invokes a recent circle criterion for exponential incremental input-to-state stability. The discrete-time theory facilitates a similar result for a continuous-time forced Lur'e system in feedback with sampled-data low-gain integral control. The theory is illustrated by two examples.

**KEYWORDS**

discrete-time, input-to-state stability, low-gain integral control, Lur'e systems, sampled-data control

## 1 | INTRODUCTION

We consider Proportional Integral (PI) control in the context that the plant is specified by a system of controlled nonlinear difference equations called a forced Lur'e (also Lurie or Lurye) system—a well-studied and ubiquitous class of nonlinear control systems—comprising the feedback connection of a linear control system and a static nonlinear output feedback.

Integral control is a classical control engineering technique for robustly tracking constant reference signals, and refers to the feedback connection of an integrator and a (stable) plant. Low-gain integral control is a special case wherein the integrator gain is sufficiently small, which is known to be sufficient for closed-loop stability under a sign condition on the steady-state gain. Early literature on integral control includes References 1-7. Low-gain integral control has been further generalized to, for example: discrete-time systems; sampled-data systems; input–output approaches; classes of distributed parameter systems; adaptively determined integrator gains, and; integral control in the presence of input and output nonlinearities. Literature across these areas is vast and includes References 8-17.

Given the importance of output regulation in applied settings, much attention has been devoted to the extension of integral control, and related notions, to nonlinear plants, with early contributions including References 18, and 19,20 are recent papers in the area. These latter works both contain a bibliographic overview of contributions to integral control in the nonlinear setting, see also Reference 21, the research monograph[22] and the references therein.

The academic study of stability and convergence properties of Lur'e systems is broadly termed *absolute stability theory*, and is also a much-researched area. Absolute stability theory seeks to conclude stability (referring to a number of possible notions) via the interplay of frequency-domain properties of the linear component and sector or boundedness properties of the nonlinearity. Relevant background on absolute stability theory includes the texts,[23,24] and the papers[25,26] specifically consider the discrete-time case. Recently, a line of enquiry has arisen investigating how classical absolute stability type results generalise to guarantee the so-called input-to-state stability (ISS) property; see, for instance References 27-29. As is well-known, ISS is a stability concept for nonlinear control systems which accommodates the contribution of exogenous control terms. For further background on ISS, we refer the reader to the survey.[30]

Here we exploit recent ISS theory for discrete-time Lur'e systems developed in Reference 31 to derive sufficient conditions for the feedback connection of a usual (linear) low-gain PI controller and a forced Lur'e system to admit an exponential disturbance-to-tracking-error estimate. This result is in the spirit of the well-known circle criterion from absolute stability theory, and is presented in Theorem 1. In the absence of forcing terms, exponential convergence of the output to the desired reference is guaranteed. The results in Reference 31 are applicable here as the studied feedback connection may itself be written as a Lur'e system with an augmented state. Our working assumption is that the nonlinear component in the Lur'e system is not known and, therefore, is not available for feedback purposes. The main technical challenge is to obtain a result which is in the spirit of low-gain integral control and absolute stability theory—namely ensuring that there is a sufficiently small positive integrator gain $\gamma_*$ such that the closed-loop feedback system has desired stability and convergence properties for *all* integrator gains $\gamma \in (0, \gamma_*)$ and *all* nonlinear terms satisfying, in this case, a suitable incremental sector condition.

As an application, we consider sampled-data low-gain integral control, now of a plant specified by a system of forced and controlled Lur'e differential equations. Sampled-data control broadly refers to controlling continuous-time plants by discrete-time controllers, via the use of sample- and hold-operations. For more background on sampled data control, we refer to the texts.[32,33] Proposition 1 is analogous to Theorem 1, and provides an exponential disturbance-to-tracking-error estimate for sufficiently small sampling times and integrator gains.

The manuscript is organised as follows. Section 2 contains the discrete-time theory, and also contains comparisons with known results from the literature. The results of Section 2 are used in the context of sampled-data control in Section 3. Two worked examples are presented in Section 4, and Section 5 is the conclusion. Proofs of our results appear in the Appendix.

## 1.1 Notation

Most notation we use is standard. As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote the positive integers (natural numbers), integers, real numbers, and complex numbers, respectively. Furthermore, let

$$\mathbb{Z}_+ := \{m \in \mathbb{Z} \ : \ m \geq 0\} \quad \text{and} \quad \mathbb{R}_+ := \{h \in \mathbb{R} \ : \ h \geq 0\}\,.$$

We let

$$\mathbb{C}_0 := \{s \in \mathbb{C} \ : \ \mathrm{Re}(s) > 0\} \quad \text{and} \quad \mathbb{E} := \{z \in \mathbb{C} \ : \ |z| > 1\}\,,$$

denote the (open) right-half complex plane and the exterior of the closed unit disc, respectively. For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we let $\mathbb{F}^n$ and $\mathbb{F}^{n\times m}$ denote $n$-dimensional (real or complex) Euclidean space, and the space of matrices with elements in $\mathbb{F}$ of format $n \times m$, respectively. We equip $\mathbb{F}^n$ with its usual 2-norm, denoted $\|\cdot\|$, which is induced by the usual inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{F}^n$. We use the same symbol $\|\cdot\|$ to denote the induced operator norm on $\mathbb{F}^{n\times m}$.

Given a square matrix $M$, the symbols $\sigma(M)$, $\rho(M)$, $M^*$ and $\mathrm{Re}\, M$ denote: the spectrum of $M$; the spectral radius of $M$; its Hermitian transpose; and real part of $M$, namely, $\mathrm{Re}\, M := (M + M^*)/2$, respectively.

For $\mathbb{F} = \mathbb{C}_0$ or $\mathbb{E}$, we let $H^\infty(\mathbb{F}, \mathbb{C}^{p\times m})$ denote the Hardy space of bounded, analytic, matrix-valued functions $\mathbb{F} \to \mathbb{C}^{p\times m}$, with respective norms

$$\|\mathbf{H}\|_{H^\infty(\mathbb{C}_0)} := \sup_{s\in\mathbb{C}_0} \|\mathbf{H}(s)\| \quad \text{and} \quad \|\mathbf{G}\|_{H^\infty} := \sup_{z\in\mathbb{E}} \|\mathbf{G}(z)\|.$$

As usual, such functions will play the role of transfer functions of stable continuous-time and discrete-time linear control systems, respectively.

Given a rational, matrix-valued function $\mathbf{H} : \mathbb{F} \to \mathbb{C}^{p\times m}$, we say that a matrix $K \in \mathbb{C}^{m\times p}$ is *feedback admissible* for $\mathbf{H}$ if $\det(I - K\mathbf{H}) \neq 0$ on $\mathbb{F}$. Furthermore, in the case that $m = p$, we call $\mathbf{H}$ *positive real* if $\operatorname{Re}\mathbf{H}(s)$ is positive semi-definite for all $s \in \mathbb{F}$ which are not poles of $\mathbf{H}$; see, for example, Reference 34. Here $\mathbb{F} = \mathbb{C}_0$ or $\mathbb{E}$ corresponds to positive realness in the continuous-time and discrete-time settings, respectively. It is well-known, for example from Reference 34, Proposition 3.3 or Reference 31, Lemma 3.5, that rational positive real functions have no poles in $\mathbb{C}_0$ or $\mathbb{E} \cup \{\infty\}$, respectively. Furthermore, such a function $\mathbf{H}$ is called *strongly positive real* or *strictly positive real* if $s \mapsto \mathbf{H}(s) - \varepsilon I$ or $s \mapsto \mathbf{H}(s - \varepsilon)$ is positive real, respectively, for some $\varepsilon > 0$.

Finally, $\mathcal{F}(\mathbb{Z}_+, V)$ are sequences $\mathbb{Z}_+ \to V$, for normed space $V$. We set

$$\|v\|_{\ell^\infty(t_1,t_2)} := \max\{\|v(\tau)\| \ : \ t_1 \leq \tau \leq t_2\} \quad \forall\, t_1, t_2 \in \mathbb{Z}_+,\ t_1 \leq t_2.$$

If $v \in \mathcal{F}(\mathbb{Z}_+, V)$ is bounded, then we set $\|v\|_{\ell^\infty} := \sup_{t\in\mathbb{Z}_+} \|v(t)\|$.

For presentational reasons, we write column vectors inline as row vectors.

## 2 | LOW-GAIN PI CONTROL FOR DISCRETE-TIME FORCED LUR'E SYSTEMS

### 2.1 | Preliminaries

Our focus is the following multivariate discrete-time controlled Lur'e system

$$\begin{cases} x^+ = Ax + BF(Gx + v_2) + Du + v_1, \quad x(0) = x^0, \\ y = Cx + v_3, \end{cases} \tag{1}$$

where $x^+(t) = x(t+1)$ for all $t \in \mathbb{Z}_+$. We denote the linear data in (1) by $\Sigma := (A, B, C, D, G)$, with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m_1}$, $C \in \mathbb{R}^{p_2\times n}$, $D \in \mathbb{R}^{n\times m_2}$, and $G \in \mathbb{R}^{p_1\times n}$. Here $m_1, m_2, n, p_1$, and $p_2$ are fixed positive integers. The term $F : \mathbb{R}^{p_1} \to \mathbb{R}^{m_1}$ is a (nonlinear) function which shall require the properties defined in Assumption (A3) and Theorem 2.2. Roughly speaking, $F$ is a function which will be required to satisfy a so-called incremental sector condition.

As usual, the variables $u, x$, and $y$ in (1) denote the input, state, and measured output, respectively, and they take values in $\mathbb{R}^n$, $\mathbb{R}^{m_2}$, and $\mathbb{R}^{p_2}$, respectively. The variables $v_i$ are exogenous input signals, which we call forcing terms. The terms $v_1$, $v_2$ and $v_3$ take values in $\mathbb{R}^n$, $\mathbb{R}^{p_1}$, and $\mathbb{R}^{p_2}$, respectively.

We assume throughout that we have access to only the (noisy) output for control purposes. Thus, for output regulation of constant references, we introduce the low-gain integrator

$$w^+ = w + \gamma L_{\mathrm{I}}(r - y), \quad w(0) = w^0, \tag{2}$$

where $r \in \mathbb{R}^{p_2}$ is the desired reference, $L_{\mathrm{I}} \in \mathbb{R}^{m_2\times p_2}$ is a (matrix) integrator gain, $\gamma > 0$ is a low-gain parameter, and $w^0 \in \mathbb{R}^{m_2}$ is the initial integrator state. In the case that $m_2 = p_2 = 1$, then we simply take $L_{\mathrm{I}} = 1$, leaving $\gamma$ as the sole integrator gain parameter.

We consider the feedback connection of (1), (2) and the PI controller $u := L_{\mathrm{P}}y + w$, where $L_{\mathrm{P}} \in \mathbb{R}^{m_2\times p_2}$ is the proportional feedback gain. Substituting the variables $u$ and $y$ into (1) and (2) yields the folllowing closed-loop feedback system

$$\begin{cases} x^+ = (A + DL_{\mathrm{P}}C)x + BF(Gx + v_2) + Dw + v_1 + DL_{\mathrm{P}}v_3, \quad x(0) = x^0, \\ w^+ = w + \gamma L_{\mathrm{I}}(r - Cx) - \gamma L_{\mathrm{I}}v_3, \quad w(0) = w^0. \end{cases} \tag{3}$$

It is clear that, for every $(x^0, w^0) \in \mathbb{R}^n \times \mathbb{R}^{m_2}$ and all $(v_1, v_2, v_3) \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^n \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_2})$, there is a unique solution of 3 which we denote by $(x, w)$. We comment that (3) is in fact a forced Lur'e system with an augmented state. A block diagram of the closed-loop feedback system (3) is contained in Figure 1.

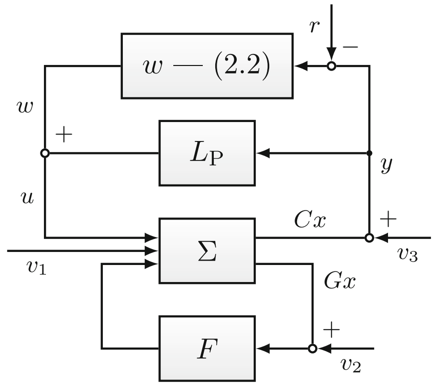


**FIGURE 1** Block diagram of the closed-loop feedback system (3). The bottom loop contains the nonlinear components in (1), and the top loops contain the controller components.

For $L_{\rm P} \in \mathbb{R}^{m_2 \times p_2}$, we set $A^{L_{\rm P}} := A + DL_{\rm P}C$, and let $\mathbf{G}_{CB}$ denote the transfer function

$$\mathbf{G}_{CB}(z) := C(zI - A^{L_{\rm P}})^{-1}B.$$

The functions $\mathbf{G}_{CD}$, $\mathbf{G}_{GB}$, $\mathbf{G}_{GD}$ are defined analogously, and they capture the various input-output relationships in (3).

We proceed to introduce assumptions used in our main result, Theorem 1 below, and convenient notation. To minimize disruption to the presentation, we provide commentary on the various assumptions after the statement of the theorem.

The first two assumptions pertain to the linear components of the model data $\Sigma$, $L_{\rm P}$ and $L_{\rm I}$.

(**A1**) $L_{\rm P} \in \mathbb{R}^{m_2 \times p_2}$ is such that $\rho(A + DL_{\rm P}C) < 1$.

(**A2**) $\mathbf{G}_{CD}(1)$ is invertible and $\sigma(\mathbf{G}_{CD}(1)L_{\rm I}) \subset \mathbb{C}_0$.

We introduce

$$v := (v_1, v_2, v_3) \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^n \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}) \quad \text{and} \quad v^\dagger := (\hat{v}_1, \hat{v}_2, \hat{v}_3) \in \mathbb{R}^n \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}, \tag{4}$$

as the collection of forcing terms in (3) and their respective reference values (which may simply be zero). We set

$$\|v(t)\| := \sum_{j=1}^{3} \|v_j(t)\| \quad \forall\, t \in \mathbb{Z}_+,$$

and define

$$\mathbf{P}(s) := \mathbf{G}_{GB}(1) - \mathbf{G}_{GD}(1)(sI + L_{\rm I}\mathbf{G}_{CD}(1))^{-1}L_{\rm I}\mathbf{G}_{CB}(1) \quad \forall\, s \in \mathbb{C}_0, \tag{5}$$

which shall play an important auxiliary and technical role in the current work. The third assumption connects the function $F : \mathbb{R}^{p_1} \to \mathbb{R}^{m_1}$ and $\mathbf{P}(0)$:

(**A3**) For all $z_1, z_2 \in \mathbb{R}^{p_1}$, there is a unique $q \in \mathbb{R}^{p_1}$ such that

$$\mathbf{P}(0)F(q + z_1) + z_2 = q. \tag{6}$$

Our first lemma introduces quantities which shall appear as state- and input-limits in the feedback system (3).

**Lemma 1.** *Assume that $\Sigma$, $L_{\rm P}$ and $L_{\rm I}$ satisfy Assumptions A1–A3, and let $\gamma > 0$, $r \in \mathbb{R}^{p_2}$ and $v^\dagger$ as in (4) be given. Define $\sigma_1 := r - \hat{v}_3$, $\sigma_2 := (I - A^{L_{\rm P}})^{-1}\hat{v}_1$ and*

$$\sigma_3 := \mathbf{G}_{GD}(1)\mathbf{G}_{CD}(1)^{-1}\sigma_1 + \left(G - \mathbf{G}_{GD}(1)\mathbf{G}_{CD}(1)^{-1}C\right)\sigma_2.$$

*Let* $q \in \mathbb{R}^{p_1}$ *be the unique solution of* $\mathbf{P}(0)F(q+\hat{v}_2)+\sigma_3 = q$. *Then,* $(x^\dagger, w^\dagger)$ *given by*

$$w^\dagger := \mathbf{G}_{CD}(1)^{-1}\left(\sigma_1 - \mathbf{G}_{CB}(1)F(q+\hat{v}_2) - C\sigma_2 - \mathbf{G}_{CD}(1)L_{\mathrm{P}}\hat{v}_3\right), \tag{7}$$

*and*

$$x^\dagger := (I - A^{L_{\mathrm{P}}})^{-1}\left(BF(q+\hat{v}_2) + \hat{v}_1 + DL_{\mathrm{P}}\hat{v}_3 + Dw^\dagger\right), \tag{8}$$

*is a constant solution of (3) with (constant)* $v = v^\dagger$, *and further satisfies* $Cx^\dagger = r - \hat{v}_3$.

## 2.2 | An exponential disturbance-to-state/tracking-error result

The following theorem is the main result of this section and, roughly, provides an exponential disturbance-to-state and disturbance-to-tracking-error estimate for (3) for all sufficiently small integrator gains and all nonlinear terms satisfying a given incremental sector condition. The result is in the spirit of the familiar circle criterion, but a recent version which is sufficient for exponential ISS. The derived estimates guarantee the complementary control objective of output-tracking, namely ensuring that $y(t) \to r$, if $v(t) \to \hat{v} = 0$ as $t \to \infty$.

**Theorem 1.** *Consider (3) and assume that* $\Sigma$, $L_{\mathrm{P}}$ *and* $L_{\mathrm{I}}$ *satisfy A1 and A2. Let* $\mathbf{P}$ *be as in (5). Given* $K_1, K_2 \in \mathbb{R}^{m_1 \times p_1}$, *assume that* $K_1$ *is feedback admissible for* $\mathbf{P}$ *and* $\mathbf{G}_{GB}$, *and further that*

*(i)* $(I - K_2\mathbf{P})(I - K_1\mathbf{P})^{-1} : \mathbb{C}_0 \to \mathbb{C}^{m_1 \times m_1}$ *is strictly positive real, and;*
*(ii)* $(I - K_2\mathbf{G}_{GB})(I - K_1\mathbf{G}_{GB})^{-1} : \mathbb{E}_0 \to \mathbb{C}^{m_1 \times m_1}$ *is strongly positive real.*

*Then there exists* $\gamma_* > 0$ *such that for every* $\gamma \in (0, \gamma_*)$, *there exist* $\Gamma > 0$ *and* $\theta \in (0,1)$ *such that, for every* $r \in \mathbb{R}^{p_2}$, *every* $v$ *and* $v^\dagger$ *as in (4), every* $F : \mathbb{R}^{p_1} \to \mathbb{R}^{m_1}$ *which satisfies A3 and*

$$\sup_{\substack{z_1, z_2 \in \mathbb{R}^{p_1} \\ z_1 \neq 0}} \frac{\langle F(z_1+z_2) - F(z_2) - K_1 z_1, F(z_1+z_2) - F(z_2) - K_2 z_1 \rangle}{\|z_1\|^2} < 0, \tag{9}$$

*and all* $(x^0, w^0) \in \mathbb{R}^n \times \mathbb{R}^{m_2}$, *the solution* $(x, w)$ *of (3) satisfies, for all* $\tau \in \mathbb{Z}_+$ *and all* $t \in \mathbb{N}$,

$$\left\| \begin{pmatrix} x(t+\tau) - x^\dagger \\ w(t+\tau) - w^\dagger \\ Cx(t+\tau) - r^\dagger \end{pmatrix} \right\| \leq \Gamma \left( \theta^t \left\| \begin{pmatrix} x(\tau) - x^\dagger \\ w(\tau) - w^\dagger \end{pmatrix} \right\| + \|v - v^\dagger\|_{\ell^\infty(\tau, t+\tau-1)} \right), \tag{10}$$

*where* $r^\dagger := r - \hat{v}_3$, *and* $w^\dagger, x^\dagger$ *are given by (7) and (8), respectively.*

*The constant* $\gamma_*$ *depends on* $\Sigma$, $L_{\mathrm{P}}$, $L_{\mathrm{I}}$, $K_1$, $K_2$, *and the left hand side of (9), but not on* $F$, $x^0$, $w^0$, *or* $r$. *The constants* $\Gamma$ *and* $\theta$ *depend on* $\gamma$, $\Sigma$, $L_{\mathrm{P}}$, $L_{\mathrm{I}}$, $K_1$, $K_2$, *and the left-hand side of (9), but not on* $F$, $x^0$, $w^0$, *or* $r$.

Recall from the notation section that the symbols $\langle \cdot, \cdot \rangle$ in (9) denote the usual inner-product on (in this case) $\mathbb{R}^{m_1}$. We provide commentary on the above theorem in terms of the result's hypotheses, conclusions and extensions.

## 2.3 | Hypotheses

The hypotheses of Theorem 1 are the Assumptions A1–A3, the positive real assumptions (i) and (ii), and the incremental sector condition (9) on $F$. In the context of low-gain PI control of discrete-time linear systems, Assumptions A1 and A2 are known together to be sufficient for low-gain output regulation; see, for example Reference 14, theorem 2.5, remark 2.7. Observe that the invertibility and spectrum requirement in Assumption A2 necessitates that $m_2 = p_2$ and that $L_{\mathrm{I}}$ is also invertible.

To discuss Assumption A3 requires more information on $\mathbf{P}$ in (5). In overview, $\mathbf{P}$ plays an important auxiliary and technical role in the current work by capturing essential input-output features of the linear components in the closed-loop

Lur'e system (3)—particularly for small $\gamma > 0$. For which purpose, for $\gamma > 0$ we introduce

$$\mathcal{A}_\gamma := \begin{pmatrix} A^{L_P} & D \\ -\gamma L_I C & I \end{pmatrix}, \quad \mathcal{B} := \begin{pmatrix} B \\ 0 \end{pmatrix} \quad \text{and} \quad \mathcal{G} := \begin{pmatrix} G & 0 \end{pmatrix}, \tag{11}$$

and the associated transfer function $\mathbf{K}_\gamma(z) = \mathcal{G}(I - \mathcal{A}_\gamma)^{-1}\mathcal{B}$. It is straightforward to see that $\mathcal{A}_\gamma$, $\mathcal{B}$ and $\mathcal{G}$ comprise the linear components of the closed-loop Lur'e system (3) with combined state $(x, w)$. A calculation shows that $\mathbf{P}(0) = \mathbf{K}_\gamma(1)$ for all $\gamma > 0$– the steady-state gain of $\mathbf{K}_\gamma$.

Assumption A3 plays a key role in the proof of Lemma 1 which, essentially, entails that, for all reference terms $r$ and persistent forcing terms $v^\dagger$, there is a unique constant solution $(x^\dagger, w^\dagger)$ of the feedback system (3) (with constant $v = v^\dagger$) which satisfies $Cx^\dagger = r - v_3^\dagger$. Assumptions of this type appear in other nonlinear integral control works, such as Reference 18, assumptions N.2 and N.3. That a steady-state gain $\mathbf{P}(0) = \mathbf{K}_\gamma(1)$ should appear in a condition for equilibria is natural.

Assumption A3 is always satisfied if $\mathbf{P}(0) = 0$. For nonzero $\mathbf{P}(0)$, a sufficient condition for A3 to hold is that $F$ is globally Lipschitz with Lipschitz constant less than $1/\|\mathbf{P}(0)\|$. In this case the continuous function

$$\mathbb{R}^{p_1} \to \mathbb{R}^{p_1}, \quad \zeta \mapsto \mathbf{P}(0)F(\zeta + z_1) + z_2 \quad \forall\, \zeta \in \mathbb{R}^{p_1},$$

is a contraction (for all $z_1, z_2 \in \mathbb{R}^{p_1}$), the upshot of which is that there is a unique solution of (6) by the Contraction Mapping Principle.

The assumptions (i) and (ii) are various (strengthened) positive-real hypotheses. Note that the former is in a "continuous-time" sense—meaning positive real on the open right-half complex plane $\mathbb{C}_0$, and the latter is in a "discrete-time" sense—meaning positive real on the exterior of the closed complex unit disc $\mathbb{E}_0$.

That a positive-real condition on $\mathbb{C}_0$ appears in a discrete-time result is as follows. We prove Theorem 1 by applying a recent circle criterion for exponential ISS (Reference 31, corollary 3.7) to the closed-loop Lur'e system (3) which, note, depends on $\gamma > 0$. Thus, we seek hypotheses on the model data that are both independent of $\gamma$ and guarantee that

$$(I - K_2\mathbf{K}_\gamma)(I - K_1\mathbf{K}_\gamma)^{-1} \quad \text{is positive real on } \mathbb{E}_0 \text{ for all sufficiently small } \gamma > 0. \tag{12}$$

Very roughly, a careful argument shows that $\mathbf{K}_\gamma(z)$ approaches $\mathbf{P}(s)$, for some $s \in i\mathbb{R} \cup \{\infty\}$, as $\gamma \to 0$ and $z \to 1$. In other words, there is no single limit of $\mathbf{K}_\gamma(z)$ as $(z, \gamma) \to (1, 0)$, rather the "limit" depends on the behavior of $(z-1)/\gamma$ as $z \to 1$ and $\gamma \to 0$. The upshot is that the conjunction of the positive-real assumptions in (i) and (ii) are sufficient for (12) (The precise result is statement (c) of Lemma 3).

The condition (9) is an incremental sector condition for $F$. To motivate this assumption note that, by Reference 31, corollary 3.7, positive-realness of $(I - K_2\mathbf{G}_{GB})(I - K_1\mathbf{G}_{GB})^{-1}$ and the (usual) sector condition

$$\sup_{\substack{z \in \mathbb{R}^{p_1} \\ z \neq 0}} \frac{\langle F(z) - K_1 z, F(z) - K_2 z\rangle}{\|z\|^2} < 0, \tag{13}$$

are together sufficient (up to some minor technical assumptions) for exponential ISS of $x$ given by the first equation in (3), with forcing term $Dw + v_1 + DL_P v_3$. The comparable hypotheses in Theorem 1 are the (stronger) positive-realness assumptions (i) and (ii), and the sector condition (9), the latter of which is simply an incremental version of (13). That an incremental condition should appear as a sufficient condition in Theorem 1 is unsurprising as, roughly, output regulation to a desired set point $r$ introduces a new equilibrium $(x^\dagger, w^\dagger)$ into (3) when unforced, (see Lemma 1), which varies as $r$ varies. Roughly, the conclusions of Theorem 1 follow once the shifted state $(x - x^\dagger, w - w^\dagger)$ is shown to be exponentially ISS.

This leads to the value of (and difficulty in establishing) Theorem 1. Chiefly, the hypotheses imposed are on the to-be-controlled Lur'e system (1) in terms of Assumptions A1–A3, which are primarily input–output/steady-state gain conditions, the positive-real conditions, and the already-mentioned incremental sector condition (9). Indeed, the small integrator parameter $\gamma > 0$ does not appear in the hypotheses of Theorem 1. However, in the spirit of low-gain integral control, the exponential disturbance-to-state and disturbance-to-tracking-error stability conclusions obtained are valid for *all* sufficiently small $\gamma$ and, in the spirit of absolute stability theory, for *all* nonlinear terms $F$ satisfying (9). The dependence of the constants on the various terms is carefully stated.

Put differently, a slight strengthening of sufficient conditions for exponential ISS of the nonlinear plant become, in conjunction with Assumptions A1–A3, sufficient conditions for the stability of the closed-loop feedback system (3), for all sufficiently small integrator gains.

**Conclusions** A consequence of the estimate (10) is that, if $v(t) \to v^\dagger$ as $t \to \infty$, then

$$x(t) \to x^\dagger, \quad w(t) \to w^\dagger \quad \text{and} \quad Cx(t) \to r - \hat{v}_3 \quad \text{as } t \to \infty.$$

In general, the rate of the above convergence will depend on the rate of convergence of $v(t)$ to $v^\dagger$. However, if $v = 0$ and $v^\dagger = 0$, then the estimate (10) ensures that the convergence is exponentially fast. Note that $Cx$ is the "true" output, whilst $y = Cx + v_3$ is the measured output, which is subject to the forcing (noise, measurement error) term $v_3$. Furthermore, we see that constant or convergent plant state forcing terms $v_1$ and $v_2$ are rejected, and constant or convergent output forcing terms $v_3$ lead to an asymptotic tracking-offset of $\hat{v}_3$ for the true output. These features are consistent with low-gain PI control of linear control systems.

## 2.4 | Extensions

It is straightforward to show that, under the hypotheses of Theorem 1, there exist $M_1 > 0$ and $\theta \in (0, 1)$ such that, for every $r \in \mathbb{R}^{p_2}$, and for all $x^0 \in \mathbb{R}^n$, the solution $x$ of (1) with $u = L_P y + w^\dagger$, with $y$ given by (1), satisfies, for all $t \in \mathbb{N}$ and $\tau \in \mathbb{Z}_+$,

$$\left\|x(t+\tau) - x^\dagger\right\| + \left\|Cx(t+\tau) - r^\dagger\right\| \le \Gamma_1 \left(\theta_1^t \left\|x(\tau) - x^\dagger\right\| + \|v - v^\dagger\|_{\ell^\infty(\tau, t+\tau-1)}\right).$$

In other words, a P-control plus a suitable constant provides a method for the output to track any prescribed reference.

Our proof of Theorem 1 shows that, if $v = 0$ and $v^\dagger = 0$, then the incremental sector condition (9) on $F$ in Theorem 1 can weakened to

$$\sup_{\substack{z \in \mathbb{R}^{p_1} \\ z \neq 0}} \frac{\langle F(z+q) - F(q) - K_1 z, F(z+q) - F(q) - K_2 z \rangle}{\|z\|^2} < 0, \tag{14}$$

for all $q = q(r)$ as in Lemma 1. However, verifying (14) requires additional knowledge of $F$ and the linear data to determine $q$, and may only be suitable if, in practice, output-tracking of only a few references $r$ is required.

Finally, by way of extensions and variations of Theorem 1, we comment that:

- Theorem 1 extends to the situation wherein the integrator state $w$ is subject to a forcing term $v_4$, that is, (2) is replaced by

  $$w^+ = w + \gamma L_I(r - y) + v_4, \quad w(0) = w^0,$$

  for some $v_4 \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^{m_2})$. Roughly, the hypotheses of of Lemma 1 and Theorem 1 do not change, and the stability conclusions of Theorem 1 remain valid, but the additional terms $v_4$ and $\hat{v}_4$ are introduced into $v$ and $v^\dagger$ in (4) and then the estimate (10), and the resulting equilibria $x^\dagger$, $w^\dagger$, and $r^\dagger$ change accordingly. For brevity, we do not give a formal statement.
- The conclusions of Theorem 1 remain true if the control variable $u$ is replaced by $u = L_P(r - y) + w$ (noting sign change on $L_P$ here), although the resulting equilibria $x^\dagger$, $w^\dagger$, and $r^\dagger$ change accordingly.
- Theorem 1 remains true if the state space for (1), which is currently $\mathbb{R}^n$, is replaced by a (possibly infinite-dimensional) Hilbert space $X$. We have presented the finite-dimensional case for simplicity, but the proof for more general $X$ basically remains unchanged. Indeed, the key results from the literature used in the proof are those in References 31 and 14, which both treat the infinite-dimensional case. The various transfer functions, such as $\mathbf{G}_{CB}$, are no longer necessarily rational, but essentially the same positive realness arguments apply as in the rational case.

In the single-input, single-output setting ($m_1 = m_2 = p_1 = p_2 = 1$), the positive real condition (i) in Theorem 1 essentially follows from the positive realness condition (ii) and an additional single point condition, detailed in the next lemma.

**Lemma 2.** *Consider (3) with $m_1 = m_2 = p_1 = p_2 = L_{\rm I} = 1$ and assume that $\Sigma$ and $L_{\rm P}$ satisfy Assumptions A1 and A2. Let $\mathbf{P}$ be as in (5). Assume that $K_1, K_2 \in \mathbb{R}$ are such that $1 - K_1\mathbf{G}_{GB} \neq 0$, and that $(1 - K_2\mathbf{G}_{GB})/(1 - K_1\mathbf{G}_{GB})$ is strongly positive real $\mathbb{E}_0 \to \mathbb{C}$. If*

$$\frac{1 - K_2\mathbf{P}(0)}{1 - K_1\mathbf{P}(0)} > 0,$$

*and $1 - K_1\mathbf{P}(0)$ and $1 - K_1\mathbf{G}_{GB}(1)$ have the same sign, then $(1 - K_2\mathbf{P})/(1 - K_1\mathbf{P})$ is strictly positive real $\mathbb{C}_0 \to \mathbb{C}$.*

## 2.5 | Estimating the maximal integrator gain and exponential decay rate

Theorem 1 guarantees the existence of two quantities which play an essential role—the so-called maximal integrator gain $\gamma_* > 0$ and closed-loop exponential decay rate $\theta \in (0, 1)$ in (10). Since the conclusions of Theorem 1 are only guaranteed to hold for integrator gains $\gamma \in (0, \gamma_*)$, determining suitable $\gamma_*$ is of great practical interest. Unfortunately, to the best of our knowledge, both $\gamma_*$ and $\theta$ are rather difficult to calculate exactly, and we proceed to discuss how they may be estimated, essentially by carefully inspecting the proof of Theorem 1.

In both cases, the key objects are the discrete-time triple $(\mathcal{A}_\gamma, \mathcal{B}, \mathcal{G})$ in (11), and the corresponding transfer function $\mathbf{K}_\gamma$. The proof of Theorem 1 relies on the properties that

$$\text{there exists } \gamma_0 > 0 \text{ such that} \quad \rho(\mathcal{A}_\gamma) < 1 \quad \forall\, \gamma \in (0, \gamma_0), \tag{15}$$

and that (12) holds, for all $\gamma \in (0, \gamma_*)$, for some $\gamma_* > 0$. These claims are established in the technical result Lemma 3. The term $\gamma_*$ in (12) is that which appears in Theorem 1.

Restricting attention to the single-input single-output case in the control loop (meaning $m_2 = p_2 = 1 = L_{\rm I}$), a consequence of the research of Coughlan[8] and his PhD thesis[35] is that (15) holds with $\gamma_0 := 1/|f(\mathbf{G}_{CD})|$ by Reference 35, theorem 6.2.4, where

$$f(\mathbf{G}_{CD}) := \sup_{q \geq 0} \left( \operatorname{essinf}_{\theta \in (0,2\pi)} \operatorname{Re} \left( \left( \frac{q}{e^{i\theta}} + \frac{1}{e^{i\theta} - 1} \right) \mathbf{G}_{CD}(e^{i\theta}) \right) \right).$$

It is shown in Reference 35, proposition 12.1.3, that

$$-\infty < f(\mathbf{G}_{CD}) < -\frac{\mathbf{G}_{CD}(1)}{2} < 0,$$

so that $1/|f(\mathbf{G}_{CD})| \in (0, \infty)$. The quantity $f(\mathbf{G}_{CD})$ can be difficult to compute, and a more conservative option for $\gamma_*$ is $1/|f_0(\mathbf{G}_{CD})|$ where

$$f_0(\mathbf{G}_{CD}) := \operatorname{essinf}_{\theta \in (0,2\pi)} \operatorname{Re} \left( \left( \frac{1}{e^{i\theta} - 1} \right) \mathbf{G}_{CD}(e^{i\theta}) \right) \leq f(\mathbf{G}_{CD}).$$

In particular, our line of argument gives that $\gamma_*$ is bounded from above by $\gamma_0$. The condition (12) for $\gamma_*$ does not appear to produce a constructive method for determining $\gamma_*$. Heuristically, the condition (12) can be tested graphically for candidate $\gamma_* > 0$.

We now turn attention to $\theta$. By way of context, for usual low-gain PI control of linear systems, the exponential decay rate equals $\rho(\mathcal{A}_\gamma) < 1$. Presently, the key stability result invoked in the proof of Theorem 1 is Reference 31, corollary 3.7, which itself draws upon Reference 31, theorem 3.2. An inspection of that proof (Reference 31, column 1, p. 3031) shows that any $\theta$ such that

$$\rho(\mathcal{A}_\gamma + \mathcal{B}M\mathcal{G}) < \theta < 1,$$

and

$$\|z \mapsto L\mathbf{K}_\gamma(\theta z)(I - M\mathbf{K}_\gamma(\theta z))^{-1}\|_{H^\infty} \leq 1,$$

will satisfy (10). Here $M := (K_1 + K_2)/2$ and $L := (K_1 - K_2)/2$. This latter condition does not seem to constructively determine $\theta$, but the above $H^\infty$-norm may be computed for candidate $\theta$. In particular, our argument gives that $\theta$ is bounded from below by $\rho(\mathcal{A}_\gamma + \mathcal{B}M\mathcal{G})$.

## 2.6 | Comparisons with existing literature

We provide some comparisons between our results and others available in the literature. Much attention has been devoted to output regulation of linear control systems subject to input saturation, including specifically in the discrete-time case the works References 36 and 37. Output regulation is a more general problem than integral control, in that the desired reference signal is typically generated by a so-called exosystem, such as periodic signals, and naturally need not be constant. At the heart of these works are stabilisation of discrete-time linear control systems by bounded (specifically saturated) feedback, including References 38 and 39.

The overlap between References 36,37 and the present work is minimal, however, owing essentially to the fact that the nonlinear term irreconcilably appears in different places in the models under consideration. We consider linear PI control applied to a nonlinear system, while References 36 and 37 consider nonlinear (saturated) controls of a linear system. Our work is closer in spirit to Reference 18, which considers the same problem we do, but for rather general continuous-time nonlinear control systems. The key assumptions in Reference 18 are that the nonlinear plant is globally uniformly exponentially stable, for all constant input signals, and that the steady-state gain map is monotone. The conclusions pertain to global asymptotic (exponential) stability, rather than input-to-state stability considered here. The approach taken in Reference 18 is somewhat different to that here, as there the low-gain integral parameter is viewed as "sufficiently small" so that the closed-loop feedback system may be rewritten as a standard singular perturbation model, and singular perturbation techniques applied. The recent paper[20] effectively generalises and strengthens,[18] by weakening various assumptions and by including an anti-windup component in the integrator. The conclusions of Reference 20 are local, but only local assumptions are imposed, and also does not consider ISS properties.

Finally, we comment that the paper[40] considers the (exponential) synchronization problem of two continuous time Lur'e systems using PID control, but that work does not consider output tracking and the overlap is otherwise minimal.

# 3 | SAMPLED-DATA LOW-GAIN INTEGRAL CONTROL OF LUR'E SYSTEMS

As an application we show that the theory developed in Section 2 facilitates the sampled-data low-gain integral control of the following multivariate continuous-time controlled Lur'e system

$$\dot{x} = Ax + BF(Gx) + Du + v_1, \quad x(0) = x^0, \quad y = Cx. \tag{16}$$

Here $u$ is a control input, and $v_1$ is an exogenous input, $x$ is the state variable and $y$ is the (true) output. We assume that $F$ is locally Lipschitz, and we let $\mathbf{H}_{CB}(s) := C(sI - A)^{-1}B$ denote the transfer function associated with the continuous-time triple $(A, B, C)$, and similarly for $\mathbf{H}_{CD}$ and so on. The linear data $(A, B, C, D, G)$ in (16) is as in Section 2.

For fixed sampling-period $\tau > 0$, we define the sampling operator $\mathcal{S}$ by

$$(\mathcal{S}y)(k) := y(k\tau) \quad \forall\, k \in \mathbb{Z}_+,$$

which is defined on all continuous functions $\mathbb{R}_+ \to \mathbb{R}^{p_2}$ and returns a sequence in $\mathcal{F}(\mathbb{Z}_+, \mathbb{R}^{p_2})$. The zero-order hold operator $\mathcal{H}$ is defined as

$$(\mathcal{H}w)(t) := w(k) \quad \forall\, u \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^{m_2}),\ \forall\, t \in [k\tau, (k+1)\tau),$$

which maps $\mathcal{F}(\mathbb{Z}_+, \mathbb{R}^{m_2})$ into the set of step-functions mapping $\mathbb{R}_+ \to \mathbb{R}^{m_2}$. We assume that the (noisy) sampled output $\mathcal{S}y + v_2$, where $v_2 \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^{p_2})$ is another forcing term, is available for feedback purposes. We consider the feedback connection of (16) and the discrete-time low-gain integrator

$$w^+ = w + \gamma L_{\mathrm{I}}(r - (Sy + v_2)), \quad w(0) = w^0, \tag{17}$$

where, as before, $r \in \mathbb{R}^{p_2}$ is the desired reference, $L_{\mathrm{I}} \in \mathbb{R}^{m_2 \times p_2}$ is a (matrix) integrator gain, $\gamma > 0$ is a low-gain parameter, and $w^0 \in \mathbb{R}^{m_2}$ is the initial integrator state. If $m_2 = p_2 = 1$, then we simply take $L_{\mathrm{I}} = 1$, so that $\gamma$ is the only integrator gain parameter.

The plant (16) and controller (17) are connected in feedback via the held integral control $u = \mathcal{H}(w)$, which yields the closed-loop feedback system

$$\dot{x} = Ax + BF(Gx) + D\mathcal{H}(w) + v_1, \quad x(0) = x^0, \tag{18a}$$

$$y = Cx, \tag{18b}$$

$$w^+ = w + \gamma L_{\mathrm{I}}\left(r - (Sy + v_2)\right), \quad w(0) = w^0. \tag{18c}$$

For given $v_1 \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, $v_2 \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^{p_2})$, and $(x^0, w^0) \in \mathbb{R}^n \times \mathbb{R}^{m_2}$, we say that $(x, w)$, where $x$ is an absolutely continuous function $\mathbb{R}_+ \to \mathbb{R}^n$ and $w \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^{m_2})$, is a solution of (18) if, for all $k \in \mathbb{Z}_+$ and all $t \in [0, k\tau]$,

$$x(t + k\tau) = e^{At}x(k\tau) + \int_{k\tau}^{t+k\tau} e^{A(t+k\tau-\sigma)}\left(BF(Gx(\sigma)) + Dw(k) + v_1(\sigma)\right)\,\mathrm{d}\sigma,$$

and (18b) and (18c) are satisfied. Existence of a unique solution of (18) is a consequence of the above equality and (18).

Our main result of this section is the following proposition which, roughly, shows that, for sufficiently small sampling-times and sufficiently small integrator gains, the feedback system (18) admits an exponential disturbance-to-state and disturbance-to-output estimate. Output tracking is guaranteed in the absence of forcing. For simplicity, we take the nominal forcing value $v^\dagger$ as zero.

**Proposition 1.** *Consider (18) with given $\Sigma$, $F : \mathbb{R}^{p_1} \to \mathbb{R}^{m_1}$ and $K_1, K_2 \in \mathbb{R}^{m_1 \times p_1}$, and assume that*

(**B1**) *$A$ is Hurwitz (all eigenvalues have negative real part)*
(**B2**) *$\mathbf{H}_{CD}(0)$ is invertible and $\sigma(L_{\mathrm{I}}\mathbf{H}_{CD}(0)) \subset \mathbb{C}_0$*
(**B3**) *With $\mathbf{Q}$ defined by*

$$\mathbf{Q}(s) := \mathbf{H}_{GB}(0) - \mathbf{H}_{GD}(0)(sI + L_{\mathrm{I}}\mathbf{H}_{CD}(0))^{-1}L_{\mathrm{I}}\mathbf{H}_{CB}(0) \quad \forall\, s \in \mathbb{C}_0, \tag{19}$$

*it follows that, for all $z_1, z_2 \in \mathbb{R}^{p_1}$, there is a unique solution $q \in \mathbb{R}^{p_1}$ to*

$$\mathbf{Q}(0)F(q + z_1) + z_2 = q. \tag{20}$$

(**B4**) *$K_1$ is feedback admissible for $\mathbf{H}_{GB}$ and $\mathbf{Q}$, and $(I - K_2\mathbf{H}_{GB})(I - K_1\mathbf{H}_{GB})^{-1}$ and $(I - K_2\mathbf{Q})(I - K_1\mathbf{Q})^{-1}$ are both strictly positive real.*

*Let $r \in \mathbb{R}^{p_2}$. The following statements hold.*

(**1**) *There exist $x^\dagger \in \mathbb{R}^n$ and $w^\dagger \in \mathbb{R}^{m_2}$ such that*

$$0 = Ax^\dagger + BF(Gx^\dagger) + Dw^\dagger \quad \text{and} \quad Cx^\dagger = r. \tag{21}$$

(**2**) *There exist $\tau > 0$ and $\gamma_* = \gamma_*(\tau) > 0$ such that for all $\gamma \in (0, \gamma_*)$, there exist $\theta_1 \in (0,1)$ and $\Gamma_1, \Gamma_2, \theta_2 > 0$ such that, for every $r \in \mathbb{R}^{p_2}$ and for all $v_1 \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, $v_2 \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^{p_2})$, all $F$ which satisfy (9), and all $(x^0, w^0) \in \mathbb{R}^n \times \mathbb{R}^{m_2}$, the solution $(x, w)$ of (18) satisfies*

$$\|w(k+m) - w^\dagger\| \le \Gamma_1\left(\theta_1^k \left\|\begin{pmatrix} x(m\tau) - x^\dagger \\ w(m) - w^\dagger \end{pmatrix}\right\| + \beta(m, k, 0)\right) \quad \forall\, k \in \mathbb{N},\ \forall\, m \in \mathbb{Z}_+, \tag{22}$$

*and*

$$\left\| \begin{pmatrix} x(t+(m+k)\tau) - x^\dagger \\ y(t+(m+k)\tau) - r \end{pmatrix} \right\| \leq \Gamma_2 \left( e^{-\theta_2(k\tau+t)} \left\| \begin{pmatrix} x(m\tau) - x^\dagger \\ w(m) - w^\dagger \end{pmatrix} \right\| + \beta(m,k,t) \right)$$
$$\forall\, m,k \in \mathbb{Z}_+, \forall\, t \in [0,\tau), \tag{23}$$

*where*

$$\beta(m,k,t) := \|v_1\|_{L^\infty(m\tau,(m+k)\tau+t)} + \|v_2\|_{\ell^\infty(m,\min\{k+m-1,m\})}. \tag{24}$$

*The suitable sampling period $\tau$ depends on the model data $\Sigma$, $L_I$, $K_1$, $K_2$ and the left hand side of (9), but not on $F$ itself, $(x^0, w^0)$ or $v$. The maximum integrator gain parameter $\gamma_*$ depends additionally on $\tau$, but again not on $F$ itself, $(x^0, w^0)$ or $v$.*

We provide commentary on the above result. First, the Assumptions B1–B3 are continuous-time analogues of Assumptions A1–A3. Here we are assuming that the to-be-controlled system (16) already has stable linear part $A$, and so no proportional feedback is included (i.e., there is no $L_P$ term in $u$). Including a sampled-and-hold proportional feedback term in (18) would complicate the analysis even further, and is beyond the scope of the present contribution.

Second, Assumption B2 requires that $p_2 = m_2$ and $L_I$ is invertible, and reduces to the usual low-gain integral control requirement that $\mathbf{H}_{CD}(0) > 0$ when $k_2 = p_2 = 1$.

Third, in light of combined assumptions on $\mathbf{H}_{GB}$ and $\mathbf{Q}$, it follows from Reference 34, corollary 4.5, that the hypothesis of strict positive realness of $(I - K_2\mathbf{H}_{GB})(I - K_1\mathbf{H}_{GB})^{-1}$ and $(I - K_2\mathbf{Q})(I - K_1\mathbf{Q})^{-1}$ is equivalent to that of strong positive realness of these functions.

Fourth, our proof shows that in fact, there is some $\tau_* > 0$, such that for each $\tau \in (0, \tau_*)$, there is some $\gamma_* > 0$ such that the conclusions of the above proposition hold. In other words, all sufficiently small sampling periods $\tau$ "work," but the permitted small integrator gains $\gamma$ will depend in general on the sampling period.

Unfortunately, estimating the maximal integrator gain $\gamma_*$ in the sampled-data setting seems even more challenging than in the wholly discrete-time situation considered in Section 2. Indeed, the proof of Proposition 1 extracts a discrete-time forced Lur'e system from (18) by considering the evolution of $x$ at the sampling points (and the discrete integrator state $w$), to which Theorem 1 applies. The key objects are now

$$\mathrm{A}_\gamma := \begin{pmatrix} A_\tau & D_\tau \\ -\gamma L_I C & I \end{pmatrix}, \quad \mathrm{B}_\tau := \begin{pmatrix} B_\tau \\ 0 \end{pmatrix} \quad \text{and} \quad \mathrm{G} := \begin{pmatrix} G & 0 \end{pmatrix}, \tag{25}$$

where

$$A_\tau := e^{A\tau}, \quad B_\tau := \int_0^\tau e^{As} B \, \mathrm{d}s \quad \text{and} \quad D_\tau := \int_0^\tau e^{As} D \, \mathrm{d}s. \tag{26}$$

The guidelines in Section 2.5 after Theorem 1 now apply for determining $\gamma_*$, replacing (11) by (25). For example, $\gamma_* \in (0, \gamma_0(\tau))$, where $\gamma_0$ is such that $\rho(\mathrm{A}_\gamma) < 1$.

The additional difficulty comes from the fact that $\tau$ in (25) must satisfy some additional "sufficiently small" properties, which are not constructive, although these may be artifacts of our proof.

## 4 | EXAMPLES

We illustrate our results through two examples.

**Example 1.** We consider the low-gain PI control of a scalar difference equation, a so-called Ricker model, namely

$$x^+ = e^{-(\mu+\eta)}x + \alpha x e^{-\beta x} \quad x(0) = x^0, \tag{27}$$

for the Gold-spotted grenadier anchovy (*Coilia dussumieri*), see Reference 41. Here the state $x(t)$ describes the biomass of mature individuals in a population at time-step $t \in \mathbb{Z}_+$, and $\mu$ and $\eta$ are positive parameters denoting the natural mortality and fishing mortality, respectively. The positive parameter $\alpha > 0$ is the maximum per-capita reproduction rate and $\beta > 0$ affects the density-dependent mortality near equilibrium abundance (Reference 41, supporting information). Although the model (27) is simple, its inclusion is intended to illustrate the key ideas behind our results, without being obscured by numerous technical details.

Note that (27) is a (scalar) unforced discrete-time Lur'e system with $A = a := e^{-(\mu+\eta)}$, $B = G = 1$ and $F(z) = f(z) := \alpha z e^{-\beta z}$. (Strictly speaking, $f$ is only defined for nonnegative arguments and, to fit the framework of the current paper, we define $f$ on all of $\mathbb{R}$ by extending by zero. This extension, although artifactual, is not seen in physically motivated examples.) We assume that $y = x + v_3$, where $v_3$ is a measurement error term, so that $C = 1$. For simplicity, we assume that the other exogenous forcing terms $v_1$ and $v_2$ in (1) are zero. The closed-loop feedback system (3) simplifies to

$$\begin{cases} x^+ = (a+k)x + f(x) + w, & x(0) = x^0, \\ w^+ = w + \gamma(r - x - v_3), & w(0) = w^0, \end{cases} \tag{28}$$

where $L_{\mathrm{P}} := k \in \mathbb{R}$ is a proportional feedback parameter, $r$ is the desired reference and $\gamma$ is the small integrator parameter. We set $L_{\mathrm{I}} = 1$. We shall assume that $k \in (-a, 0)$, so that $a + k \in (0, 1)$.

The zero equilibrium of the uncontrolled model (27) is globally asymptotically stable if $\alpha \leq 1 - a$, and if $\alpha > 1 - a$, then

$$x^\sharp := \frac{1}{\beta} \ln\left(\frac{\alpha}{1-a}\right) > 0,$$

is a nonzero equilibrium of the uncontrolled model (27), corresponding to a persistent population. We shall consider the latter situation and seek to apply Theorem 1 to (28) to raise $x$ to a limiting population $r > x^\sharp$. For this purpose, we verify the hypotheses of this result. Assumption A1 is satisfied as $\mu + \eta > 0$, and by our choice of $k$. Since $G = C = B = D$, the four transfer functions $\mathbf{G}_{CB}$, $\mathbf{G}_{CD}$ etc. are all equal, and are all equal to

$$\mathbf{G}(z) = \frac{1}{z - (a+k)} \quad \text{with} \quad \mathbf{G}(1) = \frac{1}{1-(a+k)} = \|\mathbf{G}\|_{H^\infty} > 0.$$

In light of the above, Assumption A2 is trivially satisfied. Note that $\mathbf{G}$ is a linear fractional transformation, and hence so is $\mathbf{J} := (1 - k_2\mathbf{G})/(1 - k_1\mathbf{G})$ for all real $k_1$ and $k_2$, as $\mathbf{J}$ is the composition of linear fractional transformations. Hence, the image of the complex unit circle under $\mathbf{J}$ is a circle whenever $1 - k_1\mathbf{G}(\pm 1) \neq 0$, which is symmetric with respect to the real axis and crosses the real axis at $\mathbf{J}(-1)$ and $\mathbf{J}(1)$. Thus, $\mathbf{J}$ is strongly positive real whenever $k_1$ and $k_2$ are such that

$$\min\{\mathbf{J}(-1), \mathbf{J}(1)\} = \min\left\{\frac{1 - k_2\mathbf{G}(-1)}{1 - k_1\mathbf{G}(-1)}, \frac{1 - k_2\mathbf{G}(1)}{1 - k_1\mathbf{G}(1)}\right\} > 0. \tag{29}$$

Here $\mathbf{P}$ in (5) is given by

$$\mathbf{P}(s) = \mathbf{G}(1) - \frac{\mathbf{G}(1)^2}{s + \mathbf{G}(1)} = \frac{s\mathbf{G}(1)}{s + \mathbf{G}(1)} \quad \forall\, s \in \mathbb{C}_0 \quad \text{with} \quad \mathbf{P}(0) = 0.$$

Therefore, Assumption A3 is also trivially satisfied, independently of $f$. Furthermore, $(1 - k_2\mathbf{P})/(1 - k_1\mathbf{P})$ is strictly positive real by Lemma 2 whenever (29) holds.

Since $v^\dagger = 0$, for each desired reference $r \geq 0$, the terms $\sigma_1, \sigma_2$ and $\sigma_3$ in Lemma 1 simplify somewhat, and the resulting unique $q \in \mathbb{R}$ in Lemma 1 is simply given by $q = \sigma_3 = r$.

To summarise, to track a particular (single) reference $r$, the hypotheses of Theorem 1 reduce to finding real $k_1$ and $k_2$ such that (29) holds and $f$ satisfies the incremental sector condition (14) with $q = r$. The former condition is algebraic and the latter may be verified graphically in this scalar case. Note that the these hypotheses are in fact independent of the functional form of $f$.

Common to all the following numerical simulations we take

$$x^0 = 1, \quad w^0 = 0, \quad a = 0.9, \quad k = -0.8, \quad \alpha = 0.8100, \quad \beta = 1.5, \quad r = 2, \tag{30}$$

which yield $x^\sharp = 1.3946 < r$. These parameter values have been chosen somewhat arbitrarily. We further take

$$k_1 := 0.0444 > 0.0403 = f(r)/r \quad \text{and} \quad k_2 := -0.125, \tag{31}$$

which have been chosen so that the strong positive-real condition (29) and the sector condition (14) hold — the latter is seen graphically in Figure 2A.

Numerical simulations of the closed-loop feedback system (28) with model data as in (30) and $v_3 = 0$ are contained in Figure 2B. The integrator gains are varied with values contained in Table 1. In each case, convergence $x(t) \to r$ as $t \to \infty$ is observed. The solution to the uncontrolled model (27) is also plotted for comparison.

We proceed to discuss various aspects of Theorem 1, starting with the choice of small integrator gain. Recall that estimating the maximal permitted integrator gain $\gamma_*$, the existence of which is guaranteed by Theorem 1, is difficult in general, but becomes more tractable in this simple example. We follow the approach discussed in Section 2.5. Here

$$\mathcal{A}_\gamma = \begin{pmatrix} a+k & 1 \\ -\gamma & 1 \end{pmatrix} \quad \text{with} \quad \det(zI - \mathcal{A}_\gamma) = z^2 - (a+k+1)z + (a+k+\gamma).$$

Routine stability analysis (such as the Jury criterion) gives $\rho(\mathcal{A}_\gamma) < 1$ whenever $0 < \gamma < \gamma_0 := 1 - (a+k)$ $(= 0.9$ with the numerical values in (30)). Further straightforward calculations give

$$\mathbf{K}_\gamma(z) = \frac{z-1}{(z-(a+k))(z-1)+\gamma} \quad \forall\, z \in \mathbb{E}.$$

The technical result, Lemma 3, ensures that $(1 - k_2\mathbf{K}_\gamma)/(1 - k_1\mathbf{K}_\gamma)$ is positive real as $\gamma \searrow 0$. Global exponential stability of the closed-loop feedback system

$$\begin{pmatrix} x-r \\ w-w_* \end{pmatrix}^+ = \begin{pmatrix} a+k & 1 \\ -\gamma & 1 \end{pmatrix} \begin{pmatrix} x-r \\ w-w_* \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} g\left( \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x-r \\ w-w_* \end{pmatrix} \right),$$

where

$$g(z) := f(z+r) - f(r) \quad \forall\, z \in \mathbb{R} \quad \text{and} \quad w_* := (1-(a+k))r - f(r),$$

is guaranteed for all $\gamma \in (0, \gamma_0)$ such that $(1 - k_2\mathbf{K}_\gamma)/(1 - k_1\mathbf{K}_\gamma)$ is positive real. Both conditions are satisfied by the gains used in Figure 2.

Next, regarding robustness with respect to the nonlinear term $f$, the maximal integrator gain $\gamma_*$ is independent of $f$. Figure 3B contains simulations of (28) with model data as in (30), $\gamma = 0.3$, $v_3 = 0$, and with nonlinear terms given by

$$f_1(z) := \min\{c_1 z, f(r)\}, \quad f_2(z) := \frac{c_2 z}{1+3z^2}, \quad f_3(z) := c_3(1 - 0.25\sin(3z))\ln(1+z), \tag{32}$$

for positive constants $c_1$, $c_2$, and $c_3$. The constants have been chosen so that $f_i(r) = f(r)$ for each $i$. The above functions have been chosen somewhat arbitrarily, and all satisfy the sector condition (14) with $q = r$. In each case, convergence of $x(t) \to r$ as $t \to \infty$ is observed. The difference in transient behavior appears negligible.

Finally, Figure 4 contains simulations of (28) with model data as in (30), $\gamma = 0.3$ and nonzero measurement forcing

$$v_3(t) = \varepsilon_i \sin(2\pi t/10) \quad \varepsilon_i = 0.25i, \quad i \in \{1, 2, 3, 4\}. \tag{33}$$

As expected by the estimate (10), we see that the deviation $|x(t) - r|$ is bounded, and grows as $\|v_3\|_{\ell^\infty} = \varepsilon_i$ grows.

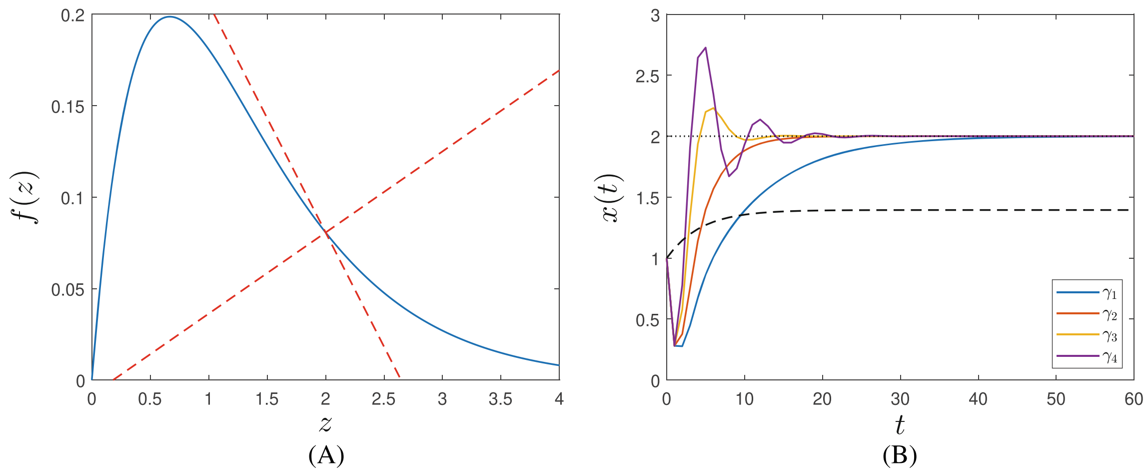


**FIGURE 2** Simulation results from Example 1. (A) Graph of function $f$ in blue curve. The red dashed straight lines have slopes $k_1$ and $k_2$, as in (31). (B) State $x(t)$ given by (28) plotted against $t$ with model data as in (30), varying $\gamma$ and $v_3 = 0$. The black dashed line is the solution of the uncontrolled model (27) from initial condition one.

**TABLE 1** Low-gain integrator parameters $\gamma_i$ used

| $i$ | $\gamma_i$ | $\rho(\mathcal{A}_{\gamma_i})$ | $(1 - k_2 \mathbf{K}_\gamma)/(1 - k_1 \mathbf{K}_\gamma)$ positive real |
|---|---|---|---|
| 1 | 0.1 | 0.8702 | ✓ |
| 2 | 0.2 | 0.6 | ✓ |
| 3 | 0.4 | 0.7071 | ✓ |
| 4 | 0.6 | 0.8367 | ✓ |

In closing we comment that, for the models (27) and (28) to be biologically meaningful, $x(t) \geq 0$ for every $t \in \mathbb{Z}_+$ is required, which is not guaranteed by Theorem 1. To ensure that $x(t) \geq 0$ is not violated, some constraint on $w(t)$ in the difference equation for $x$ in (28) is required, which is beyond the scope of the present contribution. Roughly, when trying to make $x$ larger than its initial value, as is the case here, our simulations show that $x$ does indeed remain nonnegative.

**Example 2.** We consider low-gain sampled-data integral control of the following mass-spring-damper system with forcing, namely

$$m_s \ddot{z} + d_s \dot{z} + k_s z + f(z) = u + v, \quad y = z. \tag{34}$$

Here $z(t)$ denotes the displacement of the mass from rest at time $t$, $u$ is a control signal and $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$ is a forcing term. The displacement is assumed to be measured, giving the observed variable $y = z$. Moreover, $m_s, k_s, d_s > 0$ are constants, and $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz with $f(0) = 0$. The above model has linear damping, and the restoring force depends nonlinearly on $z(t)$.

Writing (34) in first-order form, and connecting via sample-and-hold with a low-gain integrator gives (18) with

$$A := \begin{pmatrix} 0 & 1 \\ \frac{-k_s}{m_s} & \frac{-d_s}{m_s} \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ \frac{-1}{m_s} \end{pmatrix}, \quad D := -B, \quad C := G := \begin{pmatrix} 1 & 0 \end{pmatrix} \quad \text{and} \quad F := f.$$

Furthermore, $v_1 := Dv$ and, for simplicity, $v_2$ is assumed equal to zero.

We seek to apply Proposition 1. For this purpose, we proceed to verify the hypotheses. Assumption (B1) holds as $m_s, k_s, d_s$ are all positive. Furthermore, $\mathbf{H}_{CD}(0) = 1/k_m > 0$ and a straightforward calculation shows that $\mathbf{Q}(0)$, so that Assumptions (B2) and (B3) are satisfied.

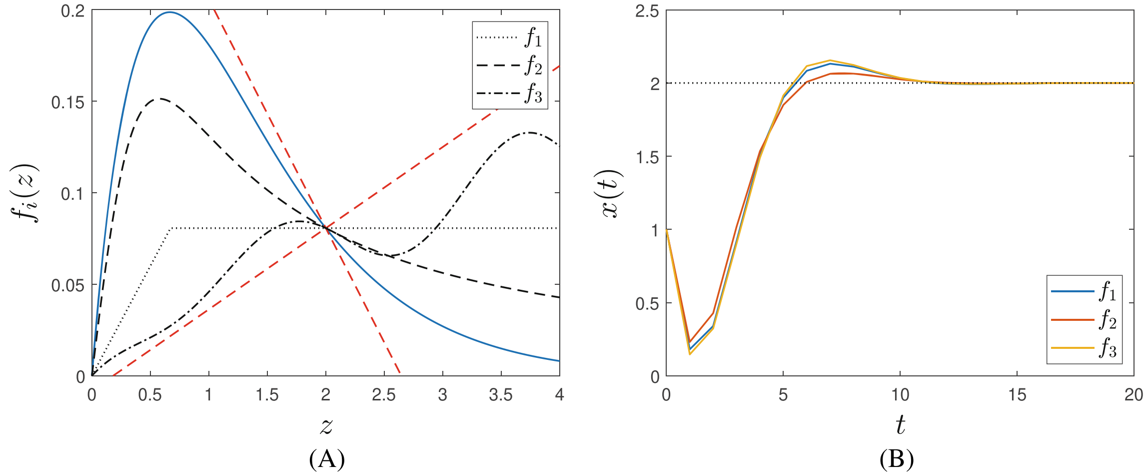


**FIGURE 3** Simulation results from Example 1. (A) Graphs of functions $f_i$ from (32). (B) State $x(t)$ given by (28) plotted against $t$ with model data as in (30), $v_3 = 0$ and functions $f_i$.

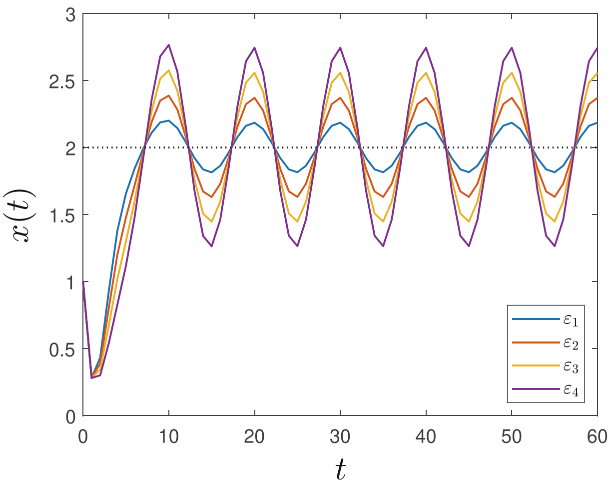


**FIGURE 4** Simulation results from Example 1. State $x(t)$ given by (28) plotted against $t$ with model data as in (30) and $v_3$ as in (33).

For illustrative numerical simulations of the sampled-data low-gain integral control feedback system (18), we take

$$m_s = 2, \quad k_s = 1, \quad d_s = 0.5, \quad r = 2, \quad w^0 = 0.$$

With these choices, it follows that $k_1 := -0.2$ and $k_2 := 0.5$ renders both $(I - K_2\mathbf{H}_{GB})(I - K_1\mathbf{H}_{GB})^{-1}$ and $(I - K_2\mathbf{Q})(I - K_1\mathbf{Q})^{-1}$ strictly positive real. Thus, the conclusions of Proposition 1 apply for every $r \in \mathbb{R}$ such that the function $f$ satisfies the incremental sector condition (14) with $q = r$ — in particular, for the present example we choose a saturated deadzone type function

$$f(z) := \operatorname{sign}(z) \min\{0.75, \max\{0, |z| - 0.3\}\} \quad \forall\, z \in \mathbb{R},$$

where $\operatorname{sign}(0) := 0$ and $\operatorname{sign}(z) := z/|z|$ otherwise. We explore varying $\tau, \gamma, x^0$ and $v$. We simulated (18) numerically in MATLAB R2020a. Specifically, the `ode113` command was used to solve the differential equation (18a) over the interval $[k\tau, (k+1)\tau]$, and then the difference equation (18c) was iterated once via (18a) and (18b). This provides the initial data to solve the differential equation (18a) on the next interval $[(k+1)\tau, (k+2)\tau]$, and the process repeats.

Figure 5A contains a contour plot of $\rho(\mathrm{A}_\gamma)$, where $\mathrm{A}_\gamma$ is as in (25), against varying sampling-period $\tau$ and integrator gain $\gamma$. Choosing $\tau, \gamma > 0$ such that $\rho(\mathrm{A}_\gamma) < 1$ is a requirement for the conclusions of Proposition 1 to hold, but is not

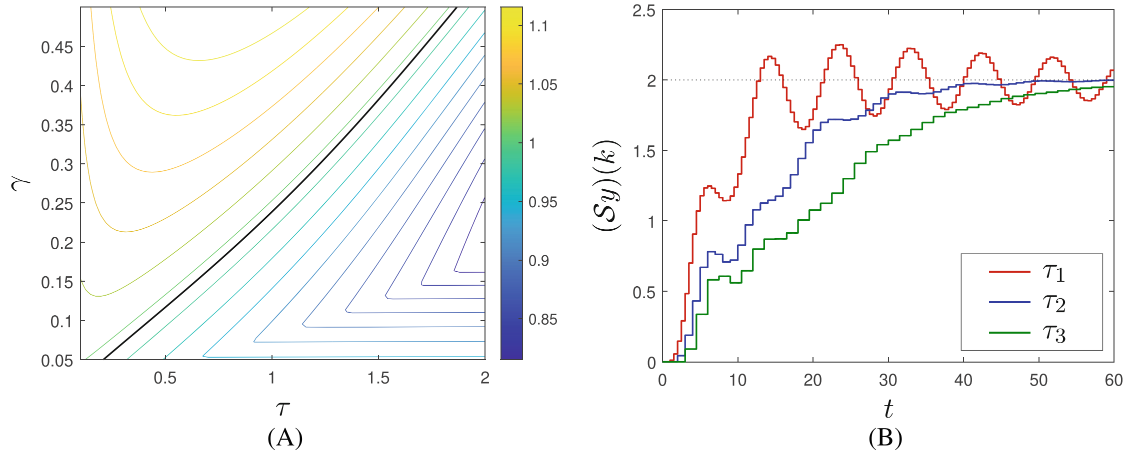


**FIGURE 5** Simulation results from Example 2. (A) Contour plot of $\rho(A_\gamma)$ against varying $\tau$ and $\gamma$. The black line is contour level one. (B) Sampled output-tracking of (18) for varying sampling time $\tau$ with model data as in (35).

sufficient by itself; see the commentary after the statement of the proposition. However, we use the contour plot to provide at least a guide for choosing $\tau$ and $\gamma$. Indeed, Figure 5B plots $(\mathcal{S}y)(k)$ against $k$ for varying $\tau$, with additional model data

$$\gamma = 0.1, \quad \nu = 0, \quad x^0 = 0, \quad \tau_i = 0.5i \quad i \in \{1, 2, 3\}. \tag{35}$$

As expected by the estimate (23), convergence over time of the sampled-output $\mathcal{S}y$ to $r$ is observed. The convergence appears slower as $\tau$ increases.

Figure 6A plots $(\mathcal{S}y)(k)$ against $k$ for varying initial states $x_i^0$, with additional model data

$$\tau = 0.75, \quad \gamma = 0.12, \quad \nu = 0, \quad x_1^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x_2^0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad x_3^0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \tag{36}$$

As expected by the estimate (23), convergence over time of the sampled-output $\mathcal{S}y$ to $r$ is observed. Finally, we illustrate the exponential disturbance-to-tracking error estimate by varying the forcing term $\nu = \nu^j$. Figure 6B contains three simulations with additional model data

$$\tau = 1.5, \quad \gamma = 0.25, \quad x^0 = 0, \quad \nu^j(t) = 2j\cos(2\pi \times 0.75t) \quad j \in \{1, 2, 3\}. \tag{37}$$

Again, as expected by the estimate (23), we see that the deviation $|y(k\tau) - r|$ is bounded, and grows as $\|\nu^j\|_{L^\infty} = 2j$ grows.

## 5 | SUMMARY

Low-gain PI control for a class of multivariate discrete-time Lur'e systems has been considered. Specifically, exponential disturbance-to-state and disturbance-to-tracking-error stability properties of the feedback connection of a Lur'e system and the usual linear low-gain integral controller have been investigated. Our main result is Theorem 1, which provides a sufficient condition in the spirit of an incremental circle criterion for these stability properties to hold for all sufficiently small integrator gains and, in particular, ensures that the usual control objective $y(t) \to r$ as $t \to \infty$ is achieved in the absence of persistent forcing or measurement error. The other key hypotheses for Theorem 1 are properties of the linear system, namely a stabilizability condition Assumption A1, a familiar sign condition on the steady-state gain Assumption A2, and a condition ensuring suitable state limits exist in closed loop Assumption A3.

The rationale for our choice of controller is that the nonlinear term $F$ in the to-be-controlled system (1) is not known and cannot be used in the design of the integrator. As such, we have sought to understand the performance (at least theoretically and qualitatively) of this controller when connected to Lur'e systems. A moral of our work is that, broadly, under certain assumptions, a linear PI controller qualitatively performs as expected when connected in feedback to a Lur'e system. As an application, in Section 3 we considered sampled-data low-gain integral control of

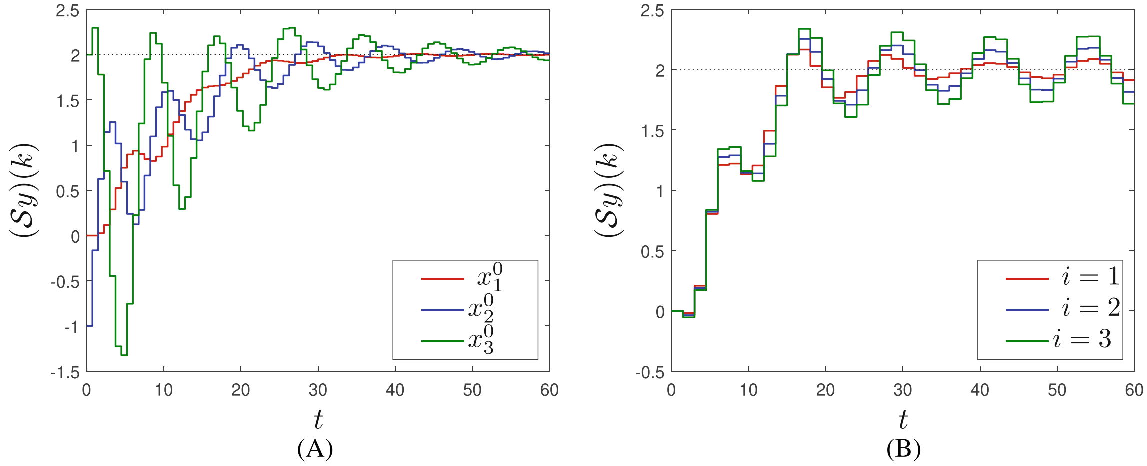


**FIGURE 6** Simulation results from Example 2. (A) Sampled output-tracking of (18) for varying $x_i^0$ with model data as in (36). (B) Forcing-to-tracking-error stability of (18) for varying $v^i$ with model data as in (37).

continuous-time controlled Lur'e systems, wherein a continuous-time Lur'e system is connected in feedback via sample-and hold-operations to the discrete-time low-gain integrator considered in Section 3. Proposition 1 is the main result of this section.

On the one hand, our results are in the spirit of low-gain integral control in that we conclude closed-loop stability for all sufficiently small integrator gains, based on assumptions which are independent of the integrator gain. On the other hand, our result is in the spirit of the circle criterion—a classical absolute stability result—both in terms of assumptions (briefly, positive realness on the linear data and an incremental sector condition on the nonlinear term) and conclusions which ensure stability for all such nonlinear terms.

In closing, we comment that the previous works[42,43] by the current authors have considered the utility of low-gain integral control for population management of linear models arising in theoretical ecology, as a potential tool for population conservation. However, linear models allow for unbounded, exponential growth which is not ecologically realistic and, in fact, nonlinear systems of Lur'e type are known to arise naturally in this setting; see, for example Reference 44. Therefore, the current paper in part paves the way for low-gain integral control in much more realistic ecological settings.

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## CONFLICT OF INTEREST

The authors have declared no conflict of interest.

## DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study. The MATLAB routines used to generate the numerical examples and figures in Section 4 are available from the corresponding author on request.

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## APPENDIX A. PRELIMINARIES

The Appendix is divided into three sections. The second and third contain the proofs of results in Sections 2 and 3, respectively. The first gathers some common preliminaries.

For $K_1, K_2$ as in the statements of Theorem 1 or Proposition 1, define

$$L := (K_1 - K_2)/2 \quad \text{and} \quad M := (K_1 + K_2)/2. \tag{A1}$$

Routine calculations give that the sector condition (9) can be rewritten as

$$\begin{aligned} &\|F(z_1 + z_2) - F(z_2) - Mz_1\|^2 - \|Lz_1\|^2 \\ &\quad = \langle F(z_1 + z_2) - F(z_2) - K_1 z_1, F(z_1 + z_2) - F(z_2) - K_2 z_1 \rangle \\ &\quad \leq -\varepsilon \|z_1\|^2 \quad \forall\, z_1, z_2 \in \mathbb{R}^{p_1}, \end{aligned} \tag{A2}$$

for some $\varepsilon > 0$ (independent of $z_1, z_2$).

In particular, the above inequality gives that $F$ is globally Lipschitz. Moreover, the inequality (A2) entails that $L$ is bounded from below, and so $L^*L$ is invertible. Therefore, $L^\sharp = (L^*L)^{-1}L^*$ is a left inverse for $L$. Further calculations from (A2) yield that

$$\|(F \circ L^\sharp)(z_1 + z_2) - (F \circ L^\sharp)(z_2) - ML^\sharp z_1\|^2 \leq (1 - \delta)\|z_1\|^2 \quad \forall\, z_1, z_2 \in \mathbb{R}^{p_1}, \tag{A3}$$

for some $\delta \in (0, 1)$ (e.g. Reference 31, proof of Corollary 3.7).

## APPENDIX B. PROOFS FOR SECTION 2

Recall the notation $A^{L_{\mathrm{P}}} := A + DL_{\mathrm{P}}C$, which satisfies $\rho(A^{L_{\mathrm{P}}}) < 1$ if 1 holds, and that $\mathbf{P}$ is defined in (5).

*Proof.* (Proof of Lemma 1) Let $r$, $\gamma$ and $v^\dagger$ be as described. We claim that

$$Gx^\sharp = q \quad \text{and} \quad Cx^\sharp = r - \hat{v}_3, \tag{B1}$$

where $q$ is as the statement of the result. The existence of such a $q$ is guaranteed by hypothesis 3. Indeed, we compute $Gx^\sharp$ using the definitions (7) and (8) to yield that

$$\begin{aligned} Gx^\sharp &= \mathbf{G}_{GB}(1)F(q + \hat{v}_2) + G\sigma_2 + \mathbf{G}_{GD}(1)L_{\mathrm{P}}\hat{v}_3 + \mathbf{G}_{GD}(1)w^\sharp \\ &= \mathbf{G}_{GB}(1)F(q + \hat{v}_2) + G\sigma_2 + \mathbf{G}_{GD}(1)L_{\mathrm{P}}\hat{v}_3 \end{aligned}$$

$$
\begin{aligned}
&+ \mathbf{G}_{GD}(1)(L_{\mathrm{I}}\mathbf{G}_{CD}(1))^{-1}L_{\mathrm{I}}\,(\sigma_1 - \mathbf{G}_{CB}(1)F(q+\hat{v}_2) - C\sigma_2 - \mathbf{G}_{CD}(1)L_{\mathrm{P}}\hat{v}_3)\\
&= \mathbf{P}(0)F(q+\hat{v}_2) + \mathbf{G}_{GD}(1)(L_{\mathrm{I}}\mathbf{G}_{CD}(1))^{-1}L_{\mathrm{I}}\sigma_1 + \left(G - \mathbf{G}_{GD}(1)(L_{\mathrm{I}}\mathbf{G}_{CD}(1))^{-1}L_{\mathrm{I}}C\right)\sigma_2\\
&= \mathbf{P}(0)F(q+\hat{v}_2) + \sigma_3 = q,
\end{aligned}
$$

where the final equality follows from (6). From the assumed invertibility of $\mathbf{G}_{CD}(1)$ (and consequently of $L_{\mathrm{I}}$), it follows that

$$
I - \mathbf{G}_{CD}(1)(L_{\mathrm{I}}\mathbf{G}_{CD}(1))^{-1}L_{\mathrm{I}} = 0.
$$

Therefore, again using (7) and (8), we compute that

$$
\begin{aligned}
Cx^{\sharp} &= \mathbf{G}_{CB}(1)F(q+\hat{v}_2) + C\sigma_2 + \mathbf{G}_{CD}(1)L_{\mathrm{P}}\hat{v}_3 + \mathbf{G}_{CD}(1)w^{\sharp}\\
&= \mathbf{G}_{CB}(1)F(q+\hat{v}_2) + C\sigma_2 + \mathbf{G}_{CD}(1)L_{\mathrm{P}}\hat{v}_3\\
&\quad + \mathbf{G}_{CD}(1)(L_{\mathrm{I}}\mathbf{G}_{CD}(1))^{-1}L_{\mathrm{I}}\,(\sigma_1 - \mathbf{G}_{CB}(1)F(q+\hat{v}_2) - C\sigma_2 - \mathbf{G}_{CD}(1)L_{\mathrm{P}}\hat{v}_3)\\
&= \sigma_1 = r - \hat{v}_3,
\end{aligned}
$$

establishing the second equality in (B1).

Next, rearranging (8) and invoking (B1) gives

$$
x^{\sharp} = A^{L_{\mathrm{P}}}x^{\sharp} + BF(Gx^{\sharp} + \hat{v}_2) + Dw^{\sharp} + \hat{v}_1 + DL_{\mathrm{P}}\hat{v}_3,
$$

and, evidently, (B1) also yields that

$$
w^{\sharp} = w^{\sharp} + \gamma L_{\mathrm{I}}\left(r - (Cx^{\sharp} + \hat{v}_3)\right) \quad \forall\, \gamma > 0,
$$

completing the proof. ■

*Proof.* (Proof of Lemma 2) For notational convenience, define $\mathbf{M} := (1 - K_2\mathbf{P})/(1 - K_1\mathbf{P})$. With a slight abuse of notation, we view $\mathbf{P}$ as a map from the extended complex plane to itself. Note that trivially every complex $s$ belongs to a set of the form $\rho + i\mathbb{R}$ with real $\rho$ — these are lines parallel to the imaginary axis.

If $\mathbf{G}_{CB}(1)\mathbf{G}_{GD}(1) = 0$, then $\mathbf{P}$ is obviously constant, and hence so is $\mathbf{M}$. Thus,

$$
\mathbf{M}(s) = \lim_{t\to\infty}\mathbf{M}(t) = \frac{1 - K_2\mathbf{G}_{GB}(1)}{1 - K_1\mathbf{G}_{GB}(1)} > 0 \quad \forall\, s \in \mathbb{C},
$$

by the strong positive realness hypothesis on $\mathbf{G}_{GB}$.

If $\mathbf{G}_{CB}(1)\mathbf{G}_{GD}(1) \neq 0$, then $\mathbf{P}$ is a linear fractional transformation. By the assumption of same signs of $1 - K_1\mathbf{G}_{GB}(1)$ and $1 - K_1\mathbf{P}(0)$, it follows by continuity that there exists sufficiently small $\rho_* \in (-\mathbf{G}_{CD}(1), 0)$ such that $1 - K_1\mathbf{P}(\rho_*)$ and $1 - \mathbf{K}_1\mathbf{G}_{GB}(1)$ have the same sign. Recall that $\mathbf{G}_{CD}(1) > 0$ by 2. Let $\rho \geq \rho_*$. By construction of $\rho_*$, it follows that $\mathbf{P}(\rho + i\mathbb{R})$ is bounded, and hence $\mathbf{P}(\rho + i\mathbb{R})$ is a circle. Moreover, $\mathbf{P}(\rho + i\mathbb{R})$ is symmetric with respect to the real axis as

$$
\overline{\mathbf{P}(\rho + \xi i)} = \mathbf{P}(\overline{\rho + \xi i}) \quad \forall\, \xi \in \mathbb{R}.
$$

The circle $\mathbf{P}(\rho + i\mathbb{R})$ crosses the real axis at $\mathbf{P}(\infty) = \mathbf{G}_{GB}(1)$ and $\mathbf{P}(\rho)$. The assumption on the signs of being equal implies that $1 - K_1\mathbf{P}(\rho) \neq 0$, as $t \mapsto \mathbf{P}(t)$ is monotone. In particular, $\mathbf{M}$ has no poles with real part greater than or equal to $\rho_*$.

Now $\mathbf{M}$ is the composition of linear fractional transformations, and so is itself a linear fractional transformation. In particular, $\mathbf{M}(\rho + i\mathbb{R})$ is a circle (as $\mathbf{M}(\rho + i\mathbb{R})$ is bounded), which is also symmetric with respect to the real axis as $\mathbf{P}$ is. Therefore, $\mathbf{M}(\rho + i\mathbb{R})$ crosses the real axis at $\mathbf{M}(\rho)$ and $\mathbf{M}(\infty)$. Since $t \mapsto \mathbf{M}(t)$ is a real-valued, continuous function, which is positive at zero and infinity by hypothesis, it follows that

$$
\operatorname{Re}\,\mathbf{M}(\rho + \xi i) \geq \varepsilon := \min_{t\geq\rho_*}\mathbf{M}(t) > 0 \quad \forall\, \xi \in \mathbb{R}.
$$

We conclude that $\mathbf{M}$ is strictly positive real, as required. ■

The next lemma collects properties of the linear components of the feedback Lur'e system (3), and is an essential ingredient in the proof of Theorem 1.

**Lemma 3.** *Assume that $\Sigma$, $L_{\mathrm{P}}$ and $L_{\mathrm{I}}$ satisfy Assumptions (A1) and (A2) and let $\gamma > 0$. Define $\mathcal{A}_\gamma$, $\mathcal{B}$ and $\mathcal{G}$ as in (11). The following statements hold.*

*(a) There exists $\gamma_0 > 0$ such that $\rho(\mathcal{A}_\gamma) < 1$ for all $\gamma \in (0, \gamma_0)$.*
*(b) For all $\gamma \in (0, \gamma_0)$, the transfer function of the triple $(\mathcal{A}_\gamma, \mathcal{B}, \mathcal{G})$ is given by*

$$\mathbf{K}_\gamma(z) = \mathbf{G}_{GB}(z) - \mathbf{G}_{GD}(z)((z-1)I + \gamma L_{\mathrm{I}} \mathbf{G}_{CD}(z))^{-1} \gamma L_{\mathrm{I}} \mathbf{G}_{CB}(z) \quad \forall\, z \in \mathbb{E}.$$

*(c) If the positive real conditions (i) and (ii) from Theorem 1 hold, then there exists $\gamma_* \in (0, \gamma_0)$ such that $(I - K_2\mathbf{K}_\gamma)(I - K_1\mathbf{K}_\gamma)^{-1}$ is positive real for all $\gamma \in (0, \gamma_*)$.*

*Proof.* (a) The claim is well-known and is a key ingredient for low-gain integral control of linear systems; see, for example, Reference 14, theorem 2.5, remark 2.7.

(b) A straightforward calculation using blockwise inversion gives, for $\gamma \in (0, \gamma_0)$ and $z \in \mathbb{E}$,

$$\begin{aligned}\mathbf{K}_\gamma(z) &= \begin{pmatrix} G & 0 \end{pmatrix} \begin{pmatrix} zI - A^{L_{\mathrm{P}}} & -D \\ \gamma L_{\mathrm{I}} C & (z-1)I \end{pmatrix}^{-1} \begin{pmatrix} B \\ 0 \end{pmatrix} \\ &= \mathbf{G}_{GB}(z) - \mathbf{G}_{GD}(z)((z-1)I + \gamma L_{\mathrm{I}} \mathbf{G}_{CD}(z))^{-1} \gamma L_{\mathrm{I}} \mathbf{G}_{CB}(z),\end{aligned}$$

as required.

(c) The proof is essentially a careful continuity argument. For symmetric matrices $Q_1$ and $Q_2$, the notation $Q_2 \preccurlyeq Q_1$ or $Q_1 \succcurlyeq Q_2$ means that $Q_1 - Q_2$ is positive semi-definite. We shall frequently use the routinely-established claim that

$$Q_1 - \|Q_2 - Q_1\| I \preccurlyeq Q_2 \preccurlyeq Q_1 + \|Q_2 - Q_1\| I,$$

and, as a corollary, if $Q_1$ is positive definite, and $\|Q_1 - Q_2\|$ is sufficiently small, then $Q_2$ is positive definite as well.

For notational convenience, define $\mathcal{M}$ by

$$\Gamma \mapsto \mathcal{M}(\Gamma) := (I - K_2\Gamma)(I - K_1\Gamma)^{-1}, \tag{B2}$$

which is a continuous function of a matrix variable $\Gamma$ (whenever $K_1$ is feedback admissible).

Observe that $K_1$ is feedback admissible for $\mathbf{K}_\gamma$, as $\mathbf{K}_\gamma$ is strictly proper for all $\gamma > 0$. In particular, $\mathcal{M}(\mathbf{K}_\gamma)$ is well defined for all $\gamma > 0$. Moreover, $\mathcal{M}(\mathbf{K}_\gamma)$ is rational, and so holomorphic on $\mathbb{E}$ with the possible exception of (necessarily isolated) poles. For $\gamma > 0$, let $\Lambda_\gamma \subseteq \mathbb{E}$ denote the (possibly empty) set of poles of $\mathcal{M}(\mathbf{K}_\gamma)$. We shall prove that there exists $\gamma_* > 0$ such that, for all $\gamma \in (0, \gamma_*)$,

$$\mathrm{Re}\ \mathcal{M}(\mathbf{K}_\gamma(z)) \geq 0 \quad \forall\, z \in \mathbb{E} \setminus \Lambda_\gamma, \tag{B3}$$

from which the claimed positive-realness follows Reference 31, lemma 3.5.

For this purpose, observe that $\mathcal{M}(\mathbf{P})$ is rational and strictly positive real by hypothesis, with $\mathbf{P} \in H^\infty(\mathbb{C}_0, \mathbb{C}^{p_1 \times m_1})$ by 2. Hence, $\mathcal{M}(\mathbf{P})$ does not have any poles in some open right-half complex plane containing $\mathbb{C}_0$. Since

$$\mathcal{M}(\mathbf{P}(s)) \to \mathcal{M}(\mathbf{G}_{GB}(1)) \quad \text{as } |s| \to \infty,$$

with positive-definite real part by hypothesis, the hypotheses of Reference 34, theorem 4.4, are satisfied and, by that result, there exist $\varepsilon_1, \lambda > 0$ such that

$$\mathrm{Re}\ \mathcal{M}(\mathbf{P}(s)) \succcurlyeq 2\varepsilon_1 I \quad \forall\, s \in \mathbb{C} \quad \text{such that} \quad \mathrm{Re}\,(s) \geq -\lambda. \tag{B4}$$

By continuity of $\mathcal{M}$, there exist $\delta_1, \varepsilon_1 > 0$ such that, for $\gamma > 0$,

$$\text{if} \quad z \in \mathbb{E} \setminus \Lambda_\gamma,\ \text{Re}\left(\frac{z-1}{\gamma}\right) \geq -\lambda \quad \text{and} \quad \|\mathbf{K}_\gamma(z) - \mathbf{P}((z-1)/\gamma)\| < \delta_1,$$
$$\text{then} \quad \text{Re}\ \mathcal{M}(\mathbf{K}_\gamma(z)) \succcurlyeq \text{Re}\ \mathcal{M}(\mathbf{P}((z-1)/\gamma)) - \varepsilon_1 I \succcurlyeq \varepsilon_1 I, \tag{B5}$$

where the final inequality above follows from the inequality (B4).

Similarly, there exist $\delta_2, \varepsilon_2 > 0$ such that,

$$\text{if} \quad z \in \mathbb{E} \quad \text{and} \quad \|\mathbf{K}_\gamma(z) - \mathbf{G}_{GB}(z)\| < \delta_2,$$
$$\text{then} \quad \text{Re}\ \mathcal{M}(\mathbf{K}_\gamma(z)) \succcurlyeq \text{Re}\ \mathcal{M}(\mathbf{G}_{GB}(z)) - \varepsilon_2 I \succcurlyeq 0, \tag{B6}$$

where the final inequality follows from the strong positive-realness of $\mathbf{G}_{GB}$.

Observe further that, for $\gamma > 0$, we may express $\mathbf{K}_\gamma(z)$ as

$$\mathbf{K}_\gamma(z) = \mathbf{G}_{GB}(z) - \mathbf{G}_{GD}(z)(((z-1)/\gamma)I + L_\text{I}\mathbf{G}_{CD}(z))^{-1} L_\text{I}\mathbf{G}_{CB}(z). \tag{B7}$$

Since $\mathbf{G}_{CB}, \mathbf{G}_{CD}, \mathbf{G}_{GD} \in H^\infty$, we may choose $R > 0$ sufficiently large so that, for $\gamma > 0$ and $z \in \mathbb{E}$,

$$\text{if} \quad \left|\frac{z-1}{\gamma}\right| > R, \quad \text{then} \quad \|\mathbf{K}_\gamma(z) - \mathbf{G}_{GB}(z)\| < \delta_2. \tag{B8}$$

Combining (B6) and (B8) yields that, for $\gamma > 0$,

$$\text{if } z \in \mathbb{E} \setminus \Lambda_\gamma \text{ is such that } \left|\frac{z-1}{\gamma}\right| > R, \quad \text{then} \quad \text{Re}\ \mathcal{M}(\mathbf{K}_\gamma(z)) \succcurlyeq \varepsilon_2 I. \tag{B9}$$

Moreover, in light of the expression (B7) for $\mathbf{K}_\gamma$ and the definition of $\mathbf{P}$, and since the functions $\mathbf{G}_{CB}, \mathbf{G}_{CD}, \mathbf{G}_{GB}$, and $\mathbf{G}_{GD}$ are all analytic on a neighborhood of one, it follows that there exists $\delta_3 > 0$ such that

$$\text{if} \quad z \in \mathbb{E} \text{ and } |z-1| < \delta_3, \quad \text{then} \quad \|\mathbf{K}_\gamma(z) - \mathbf{P}((z-1)/\gamma)\| < \delta_1. \tag{B10}$$

A careful estimation of the difference $\mathbf{K}_\gamma(z) - \mathbf{P}((z-1)/\gamma)$ shows that the above bound holds independently of $\gamma > 0$.

Simple geometric arguments show that there exists sufficiently small $\gamma_1 > 0$ so that, for all $\gamma \in (0, \gamma_1)$

$$\text{if } z \in \mathbb{E} \text{ and } |z-1|/\gamma \leq R, \quad \text{then} \quad -\lambda < \text{Re}\left(\frac{z-1}{\gamma}\right).$$

Set $\gamma_* := \min\left\{\frac{\delta_3}{R}, \gamma_0, \gamma_1\right\} > 0$. Therefore, for $\gamma \in (0, \gamma_*)$ and $z \in \mathbb{E}$ such that $\left|\frac{z-1}{\gamma}\right| \leq R$, we have that

$$\left|z-1\right| \leq \gamma R < \delta_3 \quad \text{and} \quad -\lambda < \text{Re}\left(\frac{z-1}{\gamma}\right) \leq 0.$$

The conjunction of (B5) and (B10) yields that

$$\text{if } z \in \mathbb{E} \setminus \Lambda_\gamma \text{ is such that } \left|\frac{z-1}{\gamma}\right| \leq R, \quad \text{then} \quad \text{Re}\ \mathcal{M}(\mathbf{K}_\gamma(z)) \succcurlyeq \varepsilon_1 I. \tag{B11}$$

Finally, combining (B9) and (B11) yields (B3), as required. ∎

We are now in position to prove Theorem 1.

*Proof.* (Proof of Theorem 1) Fix $v$ and $\hat{v}$ as in (4), $(x^0, w^0) \in \mathbb{R}^n \times \mathbb{R}^{m_2}$, and let $\gamma_0 > 0$ and $\gamma_* \in (0, \gamma_0)$ be as in statements (a) and (c) of Lemma 3, respectively.

Fix $\gamma \in (0, \gamma_*)$ and let $(x, w)$ denote the solution of (3). Letting $q$, $w^\dagger$ and $x^\dagger$ be as in (6), (7), and (8), respectively, we introduce the shifted variables $\tilde{x} := x - x^\dagger$, $\tilde{w} := w - w^\dagger$ and $\tilde{v} := v - v^\dagger$. It is routine to verify that

$$\tilde{x}^+ = A^{L_P}\tilde{x} + B\tilde{F}(G\tilde{x}) + D\tilde{w} + \xi_1, \quad \tilde{x}(0) = x^0 - x^\dagger, \tag{B12}$$

where $\tilde{F} : \mathbb{R}^{p_1} \to \mathbb{R}^{m_1}$ is given by

$$\tilde{F}(\zeta) := F(\zeta + q + \hat{v}_2) - F(q + \hat{v}_2) \quad \forall\, \zeta \in \mathbb{R}^{p_1}, \tag{B13}$$

and

$$\xi_1 := \tilde{v}_1 + DL_P\tilde{v}_3 + B(F(G\tilde{x} + q + v_2) - F(G\tilde{x} + q + \hat{v}_2)).$$

Similarly, since $Cx^\dagger = r - \hat{v}_3$, we see that

$$\tilde{w}^+ = \tilde{w} - \gamma L_I C\tilde{x} + \xi_2, \quad \tilde{w}(0) = w^0 - w^\dagger,$$

where $\xi_2 := -\gamma\tilde{v}_3$. Combining the equalities from (B12) to the definition of $\xi_2$ gives

$$\begin{cases} \begin{pmatrix} \tilde{x}^+ \\ \tilde{w}^+ \end{pmatrix} = \begin{pmatrix} A^{L_P} & D \\ -\gamma L_I C & I \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \tilde{F}\left( \begin{pmatrix} G & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} \right) + \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \\ \begin{pmatrix} \tilde{x}(0) \\ \tilde{w}(0) \end{pmatrix} = \begin{pmatrix} x^0 - x^\dagger \\ w^0 - w^\dagger \end{pmatrix}. \end{cases} \tag{B14}$$

Clearly, (B14) is a forced Lur'e system with nonlinear term $\tilde{F}$ and transfer function $\mathbf{K}_\gamma$. Since $F$ is globally Lipschitz (see Appendix A), estimating $\|\xi_1\|$ and $\|\xi_2\|$ yields a positive constant $c_1$ such that

$$\max\{\|\xi_1(t)\|, \|\xi_2(t)\|\} \le c_1\|\tilde{v}(t)\| \quad \forall\, t \in \mathbb{Z}_+. \tag{B15}$$

The incremental sector condition (9) entails that

$$\langle \tilde{F}(z) - K_1 z, \tilde{F}(z) - K_2 z\rangle \le -\varepsilon\|z\|^2 \quad \forall\, z \in \mathbb{R}^{p_1},$$

for some $\varepsilon > 0$. In light of the above inequality, the positive-realness of $\mathbf{K}_\gamma$ from statement (c) of Lemma 3, and the bounds (B15), an application of Circle Criterion for exponential ISS (Reference 31, corollary 3.7) to (B14) gives an estimate of the form (10) for the variables $\tilde{x}$ and $\tilde{w}$. The result of Reference 31, corollary 3.7 invokes theorem 3.2 of Reference 31. For ease of verification, Table B1 relates the relevant notation of these results to the notation used here.

We note that Reference,[31] corollary 3.7, requires that $\Sigma$ (notation there) is (exponentially) stabilizable and detectable—which is trivially satisfied in the present setting as $\rho(\mathcal{A}_\gamma) < 1$ by statement (a) of Lemma 3. Furthermore, $K_1$

**TABLE B1** Relationship between notation used in the derivation of (10)

| Notation in Reference 31 | Notation here |
|---|---|
| $\Sigma = (A, B, B_e, C, D, D_e)$ | $(\mathcal{A}_\gamma, \mathcal{B}, I, \mathcal{G}, 0, 0)$ — see (11) |
| $f$ | $\tilde{F}$ |
| $K$ | 0 |
| $r$ | $1/\|\mathbf{K}_\gamma\|_{H^\infty}$ |
| $(Y, S)$ | $(\mathbb{R}^{p_1}, 0)$ |
| $(v_1, w_1, x_1, y_1)$ | $(\xi, 0, (\tilde{x}, \tilde{w}), \mathcal{G}(\tilde{x}, \tilde{w}))$ |
| $(v_2, w_2, x_2, y_2)$ | 0 |

is required to be an admissible feedback operator, meaning $K_1 \in \mathbb{A}(D)$ in notation of Reference 31. However, this holds since $D = 0$ and so trivially $I - DK_1 = I$ is invertible.

The estimate for $Cx - r^\dagger$ follows from the estimate (10) and the fact that

$$Cx - r^\dagger = Cx - (r - \hat{v}_3) = C(x - x^\dagger) = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix}.$$

■

## APPENDIX C. PROOFS FOR SECTION 3

The proof of Proposition 1 draws upon the material in Section 2. It is reasonably lengthy, but none of the steps are too involved. Some notation and routine consequences are needed. The outline is as follows:

- By considering the evolution of $x$ at the sampling points $k\tau$ for $k \in \mathbb{Z}_+$, we extract a discrete-time Lur'e system with states $(x(k\tau), w(k))$, to which Theorem 1 applies.
- However, to apply Theorem 1 at the sampling points, an error term is introduced.
- A small-gain feedback connection argument then enables us to eliminate this error term and obtain an exponential ISS estimate for $x$ at the sampling points.
- As is standard for sampled-data arguments, it then remains to estimate $x(t)$ for $t$ between the sampling points.

*Proof.* (Proof of Proposition 1) The proof is divided into steps. ■

Step 1: Gathering notation and consequences

For $\tau > 0$, we define $A_\tau$, $B_\tau$, and $D_\tau$ as in (26). Since $A$ is assumed Hurwitz, it is clear that $\rho(A_\tau) < 1$ for all $\tau > 0$. We let $\mathbf{G}^\tau_{GB}$ denote the transfer function of the discrete-time triple $(A_\tau, B_\tau, G)$, and analogously for $\mathbf{G}^\tau_{CB}$, $\mathbf{G}^\tau_{CD}$, and $\mathbf{G}^\tau_{GD}$. It is straightforward to show that

$$\mathbf{G}^\tau_{GB}(1) = G(I - A_\tau)^{-1}B_\tau = G(-A)^{-1}B = \mathbf{H}_{GB}(0), \tag{C1}$$

and similarly for the other transfer functions. Note that $-A$ is invertible, as $A$ is assumed Hurwitz.

**Lemma 4.** *Given the notation and assumptions in Appendix C so far, it follows that there exists $\tau_0 > 0$ such that, for all $\tau \in (0, \tau_0)$, the function $(I - K_2\mathbf{G}^\tau_{GB})(I - K_1\mathbf{G}^\tau_{GB})^{-1}$ is strongly positive real.*

*Proof.* The proof is essentially another careful continuity argument, and is somewhat similar to the proof of statement (c) of Lemma 3, *mutatis mutandis*, with the role of small $\gamma > 0$ there played by small $\tau > 0$ here. We give a brief outline.

The function $\mathcal{M}(\mathbf{H}_{GB})$ satisfies the hypotheses of Reference 34, theorem 4.4 and, by that result, there exist $\varepsilon_1, \lambda > 0$ such that

$$\text{Re } \mathcal{M}(\mathbf{H}_{GB}(s)) \succcurlyeq 2\varepsilon_1 I \quad \forall\, s \in \mathbb{C} \quad \text{such that} \quad \text{Re }(s) \geq -\lambda, \tag{C2}$$

where, recall, $\mathcal{M}$ is defined in (B2).

Note, from their respective definitions in (26), that

$$\frac{B_\tau}{\tau} \to B, \quad \frac{D_\tau}{\tau} \to D \quad \text{and} \quad \frac{A_\tau - I}{\tau} \to A \quad \text{as} \quad \tau \searrow 0, \tag{C3}$$

and, trivially, for $\tau > 0$ that $\mathbf{G}^\tau_{GB}(z)$ may be expressed as

$$\mathbf{G}^\tau_{GB}(z) = G\left(\left(\frac{z-1}{\tau}\right)I - \frac{A_\tau - I}{\tau}\right)^{-1}\frac{B_\tau}{\tau}. \tag{C4}$$

In light of (C4), for fixed $\tau_1 > 0$, we can fix sufficiently large $R > 0$ such that, for all $\tau \in (0, \tau_1)$

$$\text{if} \quad z \in \mathbb{E} \quad \text{and} \quad \left|\frac{z-1}{\tau}\right| > R, \quad \text{then} \quad \|\mathbf{G}^{\tau}_{GB}(z)\| < \varepsilon_2,$$

for some small $\varepsilon_2 > 0$, and so, for these $z$ and $\tau$

$$\mathrm{Re}\ \mathcal{M}(\mathbf{G}^{\tau}_{GB}(z)) \succcurlyeq I - \varepsilon_3 I = (1-\varepsilon_3)I,$$

for some $\varepsilon_3 \in (0,1)$. Here we have used that $\mathcal{M}(0) = I$ is trivially symmetric positive definite. By choosing $\tau_2 \in (0, \tau_1)$ sufficiently small, it follows that

$$\text{if } z \in \mathbb{E} \text{ and } |z-1|/\tau \le R, \quad \text{then} \quad -\lambda < \mathrm{Re}\left(\frac{z-1}{\tau}\right).$$

In light of (C3) and (C4), we can choose $\tau_0 < \min\{\tau_1, \tau_2\}$ such that, for all $\tau \in (0, \tau_0)$,

$$\text{if } z \in \mathbb{E} \text{ and } |z-1|/\tau \le R, \quad \text{then} \quad \|\mathbf{G}^{\tau}_{GB}(z) - \mathbf{H}_{GB}((z-1)/\tau)\| < \varepsilon_4,$$

for some small $\varepsilon_4 > 0$ which is independent of $\tau \in (0, \tau_0)$. By the continuity of $\mathcal{M}$ and (C2), provided that $\varepsilon_4$ is sufficiently small, the above estimate entails

$$\text{if } z \in \mathbb{E} \text{ and } |z-1|/\tau \le R, \quad \text{then} \quad \mathrm{Re}\ \mathcal{M}(\mathbf{G}^{\tau}_{GB}(z)) \succcurlyeq \varepsilon_1 I,$$

completing the proof. ■

The conjunction of the hypotheses (B1)–(B3) and the equalities (C1) yields that the discrete-time model data $(A_\tau, B_\tau, C, D_\tau, G)$ satisfies Assumptions A1–A3 (there with $L_P = 0$), for all $\tau > 0$.

An application of Lemma 1, gives $w^\dagger$ and $x^\dagger$ as in (7) and (8), respectively, with the terms $\mathbf{G}_{CB}(1)$ replaced by $\mathbf{G}^{\tau}_{CB}(1)$, and likewise for the other steady-state gains. In light of (C1), it is clear that $w^\dagger$ and $x^\dagger$ are in fact independent of $\tau > 0$, and that $Cx^\dagger = r$, since all the $\hat{v}_i$ terms are here equal to zero.

By construction, it follows that

$$x^\dagger = A_t x^\dagger + B_t F(Gx^\dagger) + D_t w^\dagger \quad \forall\, t \ge 0. \tag{C5}$$

By rewriting the above as

$$0 = \frac{(A_t - I)}{t} x^\dagger + \frac{B_t}{t} F(Gx^\dagger) + \frac{D_t}{t} w^\dagger \quad \forall\, t > 0,$$

taking the limit as $t \searrow 0$, and invoking (C3), we conclude that (21) holds. This proves statement (1).

Step 2: Extracting a discrete-time Lur'e system at the sampling points

In what follows, to make the exposition clearer, and unless stated otherwise, $k \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ are arbitrary. For $\tau, \gamma > 0$, let $(x, w)$ denote the solution of (18). We define

$$\begin{cases} z(m) := x(m\tau) - x^\dagger, \quad \xi(m) := w(m) - w^\dagger, \\ v_1(m) := \int_0^\tau e^{As} v_1((m+1)\tau - s)\, \mathrm{d}s, \\ v_2(m) := \int_{m\tau}^{(m+1)\tau} e^{As} B\,(F(Gx(s)) - F(Gx(m\tau))\ \mathrm{d}s \\ v_3(m) := -\gamma v_2(m). \end{cases} \tag{C6}$$

Routine calculations give that

$$x((m+1)\tau) = A_\tau x(m\tau) + B_\tau F(Gx(m\tau)) + D_\tau w(m) + v_1(m) + v_2(m),$$

and so, in light of the definitions in (C5) and (C6), we have

$$\begin{cases} \begin{pmatrix} z^+ \\ \xi^+ \end{pmatrix} = \begin{pmatrix} A_\tau & D_\tau \\ -\gamma L_1 C & I \end{pmatrix} \begin{pmatrix} z \\ \xi \end{pmatrix} + \begin{pmatrix} B_\tau \\ 0 \end{pmatrix} \tilde{F}\left( \begin{pmatrix} G & 0 \end{pmatrix} \begin{pmatrix} z \\ \xi \end{pmatrix} \right) + \begin{pmatrix} v_1 \\ v_3 \end{pmatrix} + \begin{pmatrix} v_2 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} z(0) \\ \xi(0) \end{pmatrix} = \begin{pmatrix} x^0 - x^\dagger \\ w^0 - w^\dagger \end{pmatrix}, \end{cases} \tag{C7}$$

where $\tilde{F} : \mathbb{R}^{p_1} \to \mathbb{R}^{m_1}$ is defined as in (B13) with $\hat{v}_2 = 0$. We observe that (C7) is a special case of (B14), although the term $v_2$ depends on $x$, and so is not an exogenous forcing term.

Step 3: Applying Theorem 1 to (C7)

The remaining hypotheses of Theorem 1 are the positive real hypotheses, which we proceed to verify. In light of the equalities (C1), it follows that $(I - K_2\mathbf{P})(I - K_1\mathbf{P})^{-1}$ is strongly positive real, since $(I - K_2\mathbf{Q})(I - K_1\mathbf{Q})^{-1}$ is (see commentary after statement of the proposition). Recall that $\mathbf{P}$ is as in (5), but with linear data $(A_\tau, B_\tau, C, D_\tau, G)$. An application of Lemma 4 yields the existence of $\tau_0 > 0$ such that $(I - K_2\mathbf{G}^\tau_{GB})(I - K_1\mathbf{G}^\tau_{GB})^{-1}$ is strongly positive real for all $\tau \in (0, \tau_0)$.

All the hypotheses of Theorem 1 are satisfied for each $\tau \in (0, \tau_0)$, and an application of that result yields the existence of $\gamma_0 = \gamma_0(\tau) > 0$ such that, for all $\gamma \in (0, \gamma_0)$, there exist $c_1, c_2 > 0$ and $\mu_1 \in (0, 1)$ such that

$$\left\| \begin{pmatrix} z(k+m) \\ \xi(k+m) \end{pmatrix} \right\| \le c_1 \left( \mu_1^k \left\| \begin{pmatrix} z(m) \\ \xi(m) \end{pmatrix} \right\| + \|(v_1, v_3)\|_{\ell^\infty(m,m+k-1)} \right) + c_2 \left\| v_2 \right\|_{\ell^\infty(m,m+k-1)}. \tag{C8}$$

Here, and in what follows, $c_i > 0$ will be multiplicative constants which appear in estimates, $\mu_i \in (0, 1)$ shall be discrete-time exponential decay rates, and $\lambda_i > 0$ are continuous-time exponential decay rates, that is, they shall appear in terms of the form $e^{-\lambda_i t}$.

It is clear that there exist $c_3, c_4 > 0$ such that

$$\|v_1(m)\| \le c_3 \|v_1\|_{L^\infty(m\tau,(m+1)\tau)} \quad \text{and} \quad \|v_3(m)\| \le c_4 \|v_2(m)\|,$$

so that

$$\|(v_1, v_3)\|_{\ell^\infty(m,m+k-1)} \le \max\{c_3, c_4\}\, \beta(m, k, 0), \tag{C9}$$

where, recall, $\beta$ is as in (24).

The conjunction of (C8) and (C9) nearly yields the estimate (22) for $\|w - w^\dagger\|$ — but there is still an additive term $v_2$ involving the approximation error. We investigate this term next.

Step 4: A small-gain feedback connection argument

We consider the term $v_2$. The positive-real condition on $(I - K_2\mathbf{H}_{GB})(I - K_1\mathbf{H}_{GB})^{-1}$ entails that

$$\|L\mathbf{H}_{GB}(I - M\mathbf{H}_{GB})^{-1}\|_{H^\infty(\mathbb{C}_0)} = \|(L\mathbf{H}_{GB})^{ML^\sharp}\|_{H^\infty(\mathbb{C}_0)} \le 1, \tag{C10}$$

(see, e.g., Reference 31, equations (32) to (34), for a discrete-time argument—the continuous-time case is the same). Recall that $L$ and $M$ are defined in (A1), and $L^\sharp$ is a particular left inverse of $L$.

Next, observe that $x$ in (18a) may be expressed as

$$\begin{aligned} \dot{x} &= (A + B(ML^\sharp)LG)x + B\left(F(L^\sharp LGx) - (ML^\sharp)LGx\right) + D\mathcal{H}(w) + v_1 \\ &= A^{ML^\sharp}x + BF_\sharp(LGx) + D\mathcal{H}(w) + v_1, \end{aligned}$$

where $A^{ML^\sharp}$ and $F_\sharp$ are defined accordingly. The matrix $A^{ML^\sharp}$ is Hurwitz by the condition (C10), and as the pairs $(A, B)$ and $(LG, A)$ are stabilizable and detectable—trivially, as $A$ is Hurwitz). Note further that

$$0 = A^{ML^\sharp}x^\dagger + BF_\sharp(LGx^\dagger) + Dw^\dagger \quad \text{and} \quad x^\dagger = (A^{ML^\sharp})_t x^\dagger + B_t F_\sharp(LGx^\dagger) + D_t w^\dagger \quad \forall\, t \ge 0.$$

Routine calculations gives that

$$x(m\tau + t) - x(m\tau) = \int_{m\tau}^{m\tau+t} (A^{ML^\sharp})_{m\tau+t-s} \left(BF_\sharp(LGx(s)) + v_1(s)\right) \, \mathrm{d}s + D_t w(m) + \left((A^{ML^\sharp})_t - I\right) x(m\tau),$$

so that

$$\begin{aligned} x(m\tau + t) - x(m\tau) &= \int_0^t (A^{ML^\sharp})_{t-s} B \left(F_\sharp(LGx(m\tau + s)) - F_\sharp(LGx(m\tau))\right) \, \mathrm{d}s + D_t \xi(m) \\ &\quad + \left((A^{ML^\sharp})_t - I\right) z(m) + B_t(F_\sharp(LGx(m\tau)) - F_\sharp(LGx^\dagger)). \\ &\quad + \int_0^t (A^{ML^\sharp})_{t-s} v_1(s + m\tau) \, \mathrm{d}s. \end{aligned} \tag{C11}$$

Applying $LG$ to both sides of the above, usual estimates now give (with a slight abuse of notation)

$$\begin{aligned} \left\| LGx(m\tau + t) - LGx(m\tau) \right\|_{L^2(0,\tau)} &\leq \| F_\sharp(LGx(m\tau + t)) - F_\sharp(LGx(m\tau)) \|_{L^2(0,\tau)} \\ &\quad + c_6 \left\| \begin{pmatrix} z(m) \\ \xi(m) \end{pmatrix} \right\| + c_7 \| v_1 \|_{L^\infty(m\tau,(m+1)\tau)}, \end{aligned}$$

for some positive constants $c_6$ and $c_7$. Here we have used (C10) to majorise $\|(L\mathbf{H}_{GB})^{ML^\sharp}\|_{H^\infty(\mathbb{C}_0)}$, which appears as a multiplicative constant of the first term on the right-hand side of the above, by one. Setting $c_5 := 1 - \delta < 1$, and invoking the Lipschitz condition (A3), we rearrange the above to give

$$\left\| LGx(m\tau + t) - LGx(m\tau) \right\|_{L^2(0,\tau)} \leq \frac{c_6}{1 - c_5} \left\| \begin{pmatrix} z(m) \\ \xi(m) \end{pmatrix} \right\| + \frac{c_7}{1 - c_5} \| v_1 \|_{L^\infty(m\tau,(m+1)\tau)}. \tag{C12}$$

Again multiplying (C11) by $LG$, now taking $L^\infty$ norms and invoking (C12), gives

$$\left\| Gx(t + m\tau) - Gx(m\tau) \right\|_{L^\infty(0,\tau)} \leq c_8 \left\| \begin{pmatrix} z(m) \\ \xi(m) \end{pmatrix} \right\| + c_9 \| v_1 \|_{L^\infty(m\tau,(m+1)\tau)} \Bigg), \tag{C13}$$

for some $c_8, c_9 > 0$, where we have also used that $L$ is bounded from below.

Estimating $v_2$ in (C6) by Hölder's inequality and invoking (C13), it follows that

$$\| v_2(m) \| \leq \| A_t B \|_{L^1(0,\tau)} \left( c_8 \left\| \begin{pmatrix} z(m) \\ \xi(m) \end{pmatrix} \right\| + c_9 \| v_1 \|_{L^\infty(m\tau,(m+1)\tau)} \right), \tag{C14}$$

We view the conjunction of (C8) and (C14) as bounds for the output-feedback connection of two forced discrete-time systems. The first has state $(z, \xi)$, which is equal to its output, input $v_2$, and external input $(v_1, v_3)$. The second has zero state, output $v_2$, input $(z, \xi)$, and external input in terms of $v_1$.

We note that the constant $c_8 = c_8(\tau)$ is independent of $\gamma > 0$, and can be made arbitrarily small by choosing $\tau$ sufficiently small. A careful inspection of the proof of Reference 31, corollary 3.7, invoked in proving Theorem 1, shows that $c_2 \| A_t B \|_{L^1(0,\tau)}$ can be bounded independently of $\tau \in (0, \tau_0)$ and $\gamma \in (0, \gamma_0)$.

Therefore, there exist $\tau_* \in (0, \tau_0)$ and $\gamma_* = \gamma_*(\tau) \in (0, \gamma_0)$ such that

$$c_2 \| A_t B \|_{L^1(0,\tau_*)} c_8 < 1 \quad \forall \, \tau \in (0, \tau_*), \; \forall \, \gamma \in (0, \gamma_*).$$

We now fix $\tau \in (0, \tau_*)$ and $\gamma \in (0, \gamma_*)$. The small-gain theorem for exponential ISS/IOS feedback connections yields that

$$\left\| \begin{pmatrix} z(k+m) \\ \xi(k+m) \end{pmatrix} \right\| \leq c_{10} \left( \mu_2^k \left\| \begin{pmatrix} z(m) \\ \xi(m) \end{pmatrix} \right\| + \beta(m,k,0) \right), \tag{C15}$$

for some $c_{10} > 0$, $\mu_2 \in (0,1)$, both of which may depend on $\tau$ and $\gamma$. The small-gain theorem for ISS of discrete-time feedback connections can be found in Reference 45. It can be shown that if both systems are exponentially ISS and exponentially input-to-output stable (IOS), then so is the feedback connection. This is shown in the continuous-time case in the upcoming work.[46]

The estimate (22) now follows from (C15).

Step 6: Estimating $x$ between sampling times

We now estimate $\|x(k\tau + t) - x^\dagger\|$ for $t \in [0, k\tau]$. It follows from (18a) and (21) that

$$(x - x^\dagger)' = A(x - x^\dagger) + B\tilde{F}(G(x - x^\dagger)) + D(\mathcal{H}(w) - w^\dagger) + v_1.$$

An application of the circle criterion for exponential incremental ISS (Reference 47, corollary 4.5, special case 1) yields that there exist $c_{11}, \lambda_2 > 0$ such that

$$\|x(s+t) - x^\dagger\| \leq c_{11} \left( e^{-\lambda_2 t}\|x(s) - x^\dagger\| + \|\mathcal{H}(w) - w^\dagger\|_{L^\infty(s,s+t)} + \|v_1\|_{L^\infty(s,s+t)} \right) \quad \forall\, t, s \geq 0.$$

Taking $s = k\tau$ gives

$$\|x(k\tau + t) - x^\dagger\| \leq c_{11} \left( e^{-\lambda_2 t}\|x(k\tau) - x^\dagger\| + \|w(k) - w^\dagger\| + \|v_1\|_{L^\infty(k\tau,k\tau+t)} \right) \quad \forall\, t \in [0, \tau]. \tag{C16}$$

Substituting (C15) with $m = 0$ into (C16), we have that, for $k \in \mathbb{Z}_+$ and $t \in [0, \tau]$,

$$\begin{aligned} \|x(k\tau + t) - x^\dagger\| &\leq c_{12} \left( e^{-\lambda_2 t}\mu_2^k\|x(0) - x^\dagger\| + \mu_2^k\|w(0) - w^\dagger\| + \beta(0,k,t) \right) \\ &\leq c_{13} \left( e^{-\lambda_3(k\tau + t)} \left\| \begin{pmatrix} x(0) - x^\dagger \\ w(0) - w^\dagger \end{pmatrix} \right\| + \beta(0,k,t) \right), \end{aligned} \tag{C17}$$

for some $c_{12}, c_{13} > 0$ and $\lambda_3 > 0$ such that

$$c_{12}\mu_2^k(e^{-\lambda_2 t} + 1) \leq c_{13}e^{-\lambda_3(k+1)\tau} \leq c_{13}e^{-\lambda_3(k\tau+t)} \quad \forall\, t \in [0, \tau].$$

In light of (C17), and as $y - r = C(x - x^\dagger)$, a shift-invariance argument gives (23). Specifically, if $(v_1, v_2, x, w, y)$ satisfies (18), then

$$(\Lambda_{m\tau}v_1, \Lambda_m v_2, \Lambda_{m\tau}x, \Lambda_m w, \Lambda_{m\tau}y),$$

also satisfies (18), starting from $(v_1(m\tau), v_2(m), x(m\tau), w(m), y(m\tau))$, where $\Lambda_s$ is the left-shift operator $(\Lambda_s x)(t) := x(t + s)$, interpreted appropriately for both sequences and functions.