CUTOFF RATE FOR FIXED-COMPOSITION ON-OFF KEYING OVER DIRECT DETECTION PHOTON CHANNELS

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
M. Şenol TOYCAN
June 1990

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I certify that I have read this thesis and that, in my opinion, it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Associate Prof. Dr. Erdal Arıkan (Principal Advisor)

I certify that I have read this thesis and that, in my opinion, it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Associate Prof. Dr. Melek Yücel

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to my family

Contents

ABSTRACT

CUTOFF RATE FOR FIXED-COMPOSITION ON-OFF KEYING
OVER DIRECT DETECTION PHOTON CHANNELS

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M.S. in Electrical and Electronics Engineering
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June, 1990

In this thesis, we consider direct detection photon channel with peak and average power constraints. This channel is modelled as a binary input discrete memoryless channel. We study the cutoff rate for different modulation formats on this channel since it is a measure of decoding complexity when sequential decoding is used and also, it gives an upper bound for the probability of error which decreases exponentially with the constraint length of convolutional code.

Cutoff rates for the ensembles of fixed-composition and independent-letters codes along with ON-OFF keying are computed numerically and also some bounds are given. Curoff rates versus signal-to-noise ratio or peak power are plotted for blocklengths of $N = 40, 100$ and for both ensembles.

Comparison of cutoff rates for these two ensembles shows that for the direct detection photon channel the cutoff rate of fixed-composition ensemble is significantly greater than that of independent-letters ensemble for small values of signal-to-noise ratio and when the average power is a small fraction of peak power, say, 5-30%.

In an uncoded system, for achieving a probability of error $P(E) = 10^{-9}$, we should send 10 photons/slot with rate $R = 1$ bit/slot, resulting in an efficiency of 0.1 bits/photon. However, using coding we can make probability of error arbitrarily small achieving an efficiency of 1 bit/photon.

Also, some remarks on the implementation of fixed-composition trellis codes and on multi-level signalling instead of ON-OFF keying are given in conclusions.

Key words: Cutoff rate, fixed-composition codes, photon channel, ON-OFF keying

ÖZET

DOĞRUDAN SAPTAMALI FOTON KANALLARINDA SABİT BİLEŞİMLİ AÇ-KAPA ANAHTARLAMA İÇİN KESİLİM HIZI

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Bu tez çalışmasında, doğrudan saptamalı foton kanalı tepe ve ortalama güç kısıtlamaları altında incelenmiştir. Bu kanal iki seviyeli bir girişe sahip ayırtık, belleksiz kanal olarak modellenmiştir. Bundan sonra, dizinsel çözümleme teknikleri kullanıldığında çözümleme güçlüğünün bir ölçütü olması ve evrimsel kodun kısıtlama uzunluğu ile üssel olarak azalan hata olasılığı üst sınırını belirlediğinden, kesilme hızı parametresi bu kanal modeli üzerinde değişik modülasyon formatları için incelenmiştir.

Kesilim hızları, sabit bileşimli ve bağımsız harfli kod toplulukları için AÇ-KAPA anahtarlama tekniği göz önünde tutularak sayısal olarak hesaplanmış ve aynı zamanda bazı alt ve üst sınır ifadeleri verilmiştir. Her iki kod topluluğu ve iki farklı blok uzunluğu ($N = 40, 100$) için sinyal gürültü oranı veya tepe güç değerine karşı kesilim hızı çizilmiştir.

Bu iki kod topluluğu için, kesilim hızlarının karşılaştırılması göstermiştir ki, sinyal gürültü oranının düşük değerlerinde ve ortalama güç değerinin tepe güç değerinin küçük bir kesri olduğu zaman (%5-30), sabit bileşimli kod topluluğunun kesilim hızı bağımsız harfli kod topluluğunun kesilim hıindan oldukça büyüktür.

Kodsuz bir sistemde, 10^{-9}'luk bir hata olasılığına ulaşabilmek için, her aralıkta 1 bit için 10 foton yollamamız gerekir; buda foton başına 0.1 bit/foton'luk verimlilik sağlamaktadır. Halbuki, kod kullanarak istenildiği kadar küçük hata olasılığına ulaşabiliriz ve hala hazırda 1 bit/foton gibi yüksek bir verimlilik elde edebiliriz.

Ayrıca, sonuç bölümünde sabit bileşimli kafes kodlarının gerçekleştirilmesi ve AÇ-KAPA anahtarlama yerine çok seviyeli sinyalleme tekniğinin kullanılması ile ilgili bazı açıklamalara yer verilmiştir.

Anahtar sözcükler: Kesilim hızı, sabit bileşimli kodlar, foton kanalı, AÇ-KAPA anahtarlama

ACKNOWLEDGEMENT

I am grateful to Assoc. Prof. Dr. Erdal Arıkan for his invaluable guidance and suggestions during the development of this study. My special thanks are due to Prof. Dr. Abdullah Atalar who helped me very much by his encouragement. It is also my pleasure to express my thanks to my friends who assisted in the editing or made life easier with their suggestions.

Nomenclature

γ	Threshold for ML receiver
ϵ, δ	Cross-over probabilities for binary DMC
R_{0i}	Cutoff rate for independent-letters ensemble
R_{0fc}	Cutoff rate for fixed-composition ensemble
$R_0^{(N)}(Q_N), R_0$	Cutoff rate
α	Improvement factor in cutoff rate
z	Channel parameter which is $\sqrt{\epsilon(1-\delta)} + \sqrt{\delta(1-\epsilon)}$
λ_0	Dark current level
Q_N	Input probability distribution
$P_N(\mathbf{y}\vert\mathbf{x})$	Channel transition probabilities
Δ	Slot length
A	Peak power level
SNR	peak signal-to-noise ratio
UB	Upper bound
LB	Lower bound

Chapter 1

INTRODUCTION

In this chapter, direct detection photon channel is described. Capacity, cutoff rate, R_0, and sequential decoding concepts with a brief review of previous work are given.

1.1 Direct Detection Photon Channel

The channel input is a waveform $\lambda(t)$, $0 \leq t < \infty$, which satisfies

$$0 \leq \lambda(t) \leq A$$

where the parameter A is called the peak power. The waveform $\lambda(\cdot)$ defines a poisson counting process $\nu(t)$ with rate $\lambda(t) + \lambda_0$, where $\lambda_0 \geq 0$ is the dark current level (background noise). Thus, the statistics for $\nu(t)$ can be written as

$$\nu(0) = 0$$

$$P(\nu(t+\tau) - \nu(t) = j) = \frac{e^{-\Lambda}\Lambda^j}{j!} \quad , j = 0, 1, 2, \ldots \quad 0 \leq \tau, t < \infty$$

where

$$\Lambda = \int_t^{t+\tau} (\lambda(s) + \lambda_0)\, ds$$

Jumps in $\nu(\cdot)$ correspond to photon arrivals at the receiver and they can be determined by using a photon detector.

For a set of M messages, $\lambda_m(t)$ is sent for message m, $1 \leq m \leq M$, where $\lambda_m(t)$ is nonzero only for $0 \leq t \leq T$ where T is the signalling interval. In addition to the peak power constraint, these waveforms also satisfy the average power constraint

$$\frac{1}{T}\int_0^T \lambda_m(t)\, dt \leq pA \qquad 0 < p \leq \frac{1}{2}$$

(Actually, for the codes we consider this condition will always hold with equality.)

Shannon [1] proved that there is a parameter C, called channel capacity which is the maximum of achievable rates allowing reliable communication. This capacity incorporates the effects of noise, constrained bandwidth and power limitations related to any physical channel. The significance of channel capacity can be stated as follows If there are $M = [e^{RT}]$[1] messages for some fixed R, called the rate of the code, then:

- arbitrarily small error probability can be obtained for T large enough if $R < C$.
- the probability of error must go to 1 as T increases if $R > C$.

Shannon did not, however, tell how to find suitable codes to construct a reliable system which works at rates close to channel capacity; his achievement was to prove only the existence of such codes. Since then, major part of the communication research has been devoted to extend these results and achieve rates closer to the channel capacity.

1.2 R_0 and Sequential Decoding

The aim of this thesis is to study the R_0 parameter for various modulation formats on the direct detection photon channel. The motivation for using the R_0 parameter as a criterion for comparing different coding and modulation schemes arises from using trellis coding along with sequential decoding. Wozencraft and Kennedy [2] argued in favor of the cutoff rate as a criterion because it is the upper limit of code rates R for which the average decoding computation per source digit is finite when sequential decoding is used. Viterbi [3] showed, for convolutional coding and maximum likelihood decoding on the discrete memoryless channel (DMC), that the error probability is upper bounded by

$$P(E) \leq C_R L e^{-NR_0} \quad \text{if} \quad R < R_0$$

where N is the constraint length of the convolutional code, R is the code rate, L is the total number of source letters encoded, and C_R is a weakly dependent function of R and not a function of L or N. Thus, the single parameter R_0 provides a measure of both reliable rates and code complexity.

So, if the communication rate is less than R_0, it is possible to construct sequential decoders that have error probability approaching zero exponentially by increasing the constraint length N of the trellis code.

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[1] Note that $M = e^{RT}$ and without loss of generality blocklength $N = \frac{T}{\Delta}$ can be replaced by T, where Δ is bit interval.

1.3 Brief Review of Previous Work

Snyder et. al. [5] examined cutoff rate as a performance measure in the design of encoder, optical modulator and demodulator of the direct detection photon channel. Channel is modelled as a memoryless channel with continuous output alphabet that corresponds to the limiting case of infinitely fine quantization. Davis [6] computed the capacity of a Poisson-type channel subject to peak amplitude and average energy constraints. In [11] capacity and error exponent of the direct detection photon channel is calculated, and an explicit construction for an exponentially optimum family of codes for this channel is given. In [7] and [8], assuming a noiseless photon channel, capacity and cutoff rate are calculated. While capacity can be made arbitrarily large, cutoff rate is bounded, $R_0 \leq 1$. Also, some codes are discussed in these papers. Pulse position modulation (PPM) for noiseless photon channel is examined by Zwillinger [10] and Bar-David et. al. [9] considering capacity and cutoff rate. Georghiades [12] showed how trellis coded modulation can be used to improve the performance of the direct detection photon channel. Forestieri et. al. [13] studied the performance of convolutional codes in this channel.

1.4 Summary of Results

The work in this thesis differs from the previous work in that here an ensemble of fixed-composition codes along with ON-OFF keying is considered. The cutoff rate parameter for the resulting channel is computed numerically and asymptotic bounds are given. The results demonstrate that significant coding gains are achievable by using fixed-composition ensembles of codes (rather than the more commonly used independent-letters ensemble).

Chapter 2

R_0 ANALYSIS FOR FIXED-COMPOSITION AND INDEPENDENT-LETTERS ENSEMBLES

2.1 Discrete-time Photon Channel

Consider the direct detection photon channel described in Section 1.1. Let signalling interval $[0,T]$ be divided into N slots. Let Δ be the slot length

$$\Delta = \frac{T}{N}.$$

Consider ON-OFF signalling on this slotted channel. That is, denote message m by a binary vector $\mathbf{x_m} = (x_{m1}, x_{m2}, \cdots, x_{mN})$, $\quad x_{mi} = 0, 1$ and let the corresponding signal waveform be given by

$$\lambda_m(t) = x_{mn}A \quad \text{for} \quad (n-1)\Delta < t \leq n\Delta \quad n = 1, 2, ..., N.$$

Thus, $\lambda_m(t)$ takes only the values A or 0 in the bit interval (slot) $((n-1)\Delta, n\Delta]$ according as x_{mn} is 1 or 0, respectively. We assume that the receiver is an ML decoder which bases its decisions on the increments

$$y_n = \nu(n\Delta) - \nu((n-1)\Delta) \quad , \text{where } \nu(0) = 0.$$

That is, the receiver decides that the bit value in the n'th time slot was a

$$1 \text{ if } P(y_n|x_{mn} = 1) > P(y_n|x_{mn} = 0)$$

$$0 \text{ if } P(y_n|x_{mn} = 0) \geq P(y_n|x_{mn} = 1).$$

We have

$$P(j|i) \triangleq P(y_n = j|x_{mn} = i) = \frac{e^{-\Lambda_i}\Lambda_i^j}{j!}, \quad i = 0, 1$$

where

$$\Lambda_0 = \int_{(n-1)\Delta}^{n\Delta} (0 + \lambda_0)\, ds = \lambda_0 \Delta$$

$$\Lambda_1 = \int_{(n-1)\Delta}^{n\Delta} (A + \lambda_0)\, ds = (A + \lambda_0)\Delta.$$

Hence,

$$\frac{P(y|1)}{P(y|0)} = e^{\Lambda_0 - \Lambda_1} \left(\frac{\Lambda_1}{\Lambda_0}\right)^j.$$

So, the receiver decides that a 1 was sent if and only if the number of received photons, j, exceeds the threshold γ, where

$$\gamma = \frac{\Lambda_1 - \Lambda_0}{\ln(\frac{\Lambda_1}{\Lambda_0})} = \frac{A\Delta}{\ln(1 + SNR)}, \quad SNR = \frac{A}{\lambda_0}.^{1} \tag{2.1}$$

As a result, the slotted direct detection photon channel with ON-OFF keying can be modelled as a DMC with the following cross-over probabilities

$$\begin{aligned} \epsilon &= P(0|1) = \sum_{0 \le j \le \gamma} \frac{e^{-(A+\lambda_0)\Delta}[(A + \lambda_0)\Delta]^j}{j!} \\ \delta &= P(1|0) = \sum_{j > \gamma} \frac{e^{-\lambda_0 \Delta}(\lambda_0 \Delta)^j}{j!} \end{aligned} \tag{2.2}$$

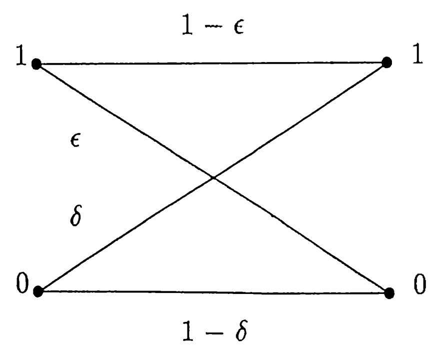

Figure 2.1: Binary DMC model for the direct detection photon channel

In the case of $\lambda_0 = 0$ (no dark current), the above DMC becomes a Z-channel with $\delta = 0$ (Figure 2.2)

[1] Actually this should be defined as peak signal-to-noise ratio, but for simplicity it will be refered as SNR througout the text.

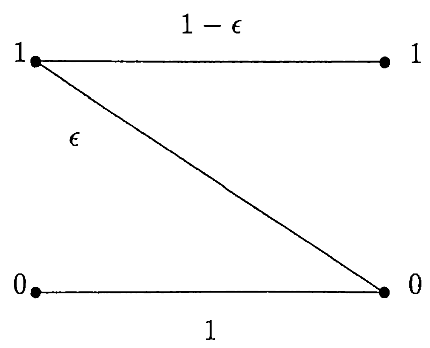

Figure 2.2: Z-channel model for the direct detection photon channel in the case of $\lambda_0 = 0$

2.2 Calculation of Cutoff Rates

Consider a code ensemble of blocklength N over an alphabet $\mathcal{A}$. Thus, each word in the ensemble belongs to $\mathcal{A}^N$ and there is a probability distribution Q_N defined on $\mathcal{A}^N$. $Q_N(\mathbf{x})$ is then the probability of $\mathbf{x}$ being chosen as a codeword. We let $P_N(\mathbf{y}|\mathbf{x})$ denote the probability that the channel output block $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ is received when the channel input block $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ is transmitted. For a DMC, $P_N(\mathbf{y}|\mathbf{x}) = \prod_{i=1}^{N} P(y_i|x_i)$

The cutoff rate parameter for the above ensemble and associated channel is defined by [4, p.135]

$$R_0^{(N)}(Q_N) = -\frac{1}{N} \ln \left[\sum_{\mathbf{y}} \left(\sum_{\mathbf{x}} Q_N(\mathbf{x}) \sqrt{P_N(\mathbf{y}|\mathbf{x})} \right)^2 \right] \tag{2.3}$$

The goal is to maximize this parameter by variation of Q_N and N subject to certain constraints on the code ensemble such as peak or average power constraints. This leads to the definition

$$R_0 = \sup_{N \geq 1} \max_{Q_N} R_0^{(N)}(Q_N) \tag{2.4}$$

where Q_N, for each N, varies over probability distributions satisfying the constraints.

We now consider two specific ensembles for the channel of Section 2.1.

2.2.1 Independent-Letters Ensemble

For an independent-letters ensemble over a binary alphabet $\mathcal{A} = \{0, 1\}$, the probability of choosing $\mathbf{x}$ as a codeword has a product form:

$$Q_N(\mathbf{x}) = p^{n_1(\mathbf{x})}(1-p)^{n_0(\mathbf{x})} = \prod_{i=1}^{N} Q(x_i)$$

where $n_1(\mathbf{x})$ and $n_0(\mathbf{x})$ are the number of 1's and 0's respectively in $\mathbf{x} \in \mathcal{A}^N$. That is, each letter of each codeword is chosen independently. Using Equations 2.4 and 2.3, R_0 for independent-letters ensemble can be calculated as follows:

$$\begin{aligned}
\sum_{\mathbf{y}} \left(\sum_{\mathbf{x}} Q_N(\mathbf{x}) \sqrt{P_N(\mathbf{y}|\mathbf{x})} \right)^2 &= \sum_{\mathbf{y}} \sum_{\mathbf{x}} \sum_{\mathbf{x}'} Q_N(\mathbf{x}) Q_N(\mathbf{x}') \sqrt{P_N(\mathbf{y}|\mathbf{x}) P_N(\mathbf{y}|\mathbf{x}')} \\
&= \prod_{i=1}^{N} \sum_{y_i} \sum_{x_i} \sum_{x_i'} Q(x_i) Q(x_i') \sqrt{P(y_i|x_i) P(y_i|x_i')} \\
&= \left[\sum_{y_1=0}^{1} \sum_{x_1=0}^{1} \sum_{x_1'=0}^{1} Q(x_1) Q(x_1') \sqrt{P(y_1|x_1) P(y_1|x_1')} \right]^N \\
&= \left[p^2 + (1-p)^2 + 2p(1-p)\left(\sqrt{\epsilon(1-\delta)} + \sqrt{\delta(1-\epsilon)}\right) \right]^N
\end{aligned}$$

Thus, the supremum over N in Equation 2.4 is achieved at $N = 1$ for this ensemble, and cutoff rate for independent-letters ensemble, R_{0i} is given by

$$R_{0i} = \sup_{N \geq 1} \max_{Q_N} R_0^{(N)}(Q_N) = \max_{p} - \ln[p^2 + (1-p)^2 + 2p(1-p)z] \tag{2.5}$$

where

$$z = \sqrt{\epsilon(1-\delta)} + \sqrt{\delta(1-\epsilon)} \tag{2.6}$$

and the maximum over p must be carried out subject to power constraints, if there are any. As Equation 2.5 implies, we gain nothing by increasing the blocklength, N, for independent-letters ensembles.

2.2.2 Fixed-Composition Ensemble

Each codeword from the ensemble of fixed-composition codes has the same fixed number K of 1's. Thus, if the blocklength is N then we have

$$\binom{N}{K}$$

words in the ensemble. We choose codewords from this ensemble with probability distribution

$$Q_N(\mathbf{x}) = \frac{1}{\binom{N}{K}}$$

Cutoff rate for the fixed-composition ensemble, R_{0fc}, can be calculated as follows.

$$\begin{aligned}
e^{-NR_{0fc}^{(N)}(Q_N)} &= \sum_{\mathbf{y}} \left(\sum_{\mathbf{x}} Q_N(\mathbf{x}) \sqrt{P_N(\mathbf{y}|\mathbf{x})} \right)^2 \\
&= \sum_{\mathbf{x}} \sum_{\mathbf{x}'} Q_N(\mathbf{x}) Q_N(\mathbf{x}') \sum_{y_1} \sum_{y_2} \cdots \sum_{y_N} \prod_{i=1}^{N} \sqrt{P(y_i|x_i)P(y_i|x_i')} \\
&= \sum_{\mathbf{x}} \sum_{\mathbf{x}'} Q_N(\mathbf{x}) Q_N(\mathbf{x}') \prod_{i=1}^{N} \sum_{y_i} \sqrt{P(y_i|x_i)P(y_i|x_i')}
\end{aligned}$$

$$\text{if } x_i = x_i' \text{ then } \sum_{y_i} \sqrt{P(y_i|x_i)P(y_i|x_i')} = \sum_{y_i} P(y_i|x_i) = 1$$

$$\text{if } x_i \neq x_i' \text{ then } \sum_{y_i} \sqrt{P(y_i|x_i)P(y_i|x_i')} = \sqrt{\epsilon(1-\delta)} + \sqrt{\delta(1-\epsilon)}$$

Therefore,

$$e^{-NR_{0fc}^{(N)}(Q_N)} = \sum_{\mathbf{x}} \sum_{\mathbf{x}'} Q_N(\mathbf{x}) Q_N(\mathbf{x}') z^{d(\mathbf{x},\mathbf{x}')}$$

where $d(\mathbf{x},\mathbf{x}')$ is the distance between the codewords $\mathbf{x}$ and $\mathbf{x}'$ which is defined as the number of bits where one codeword differs from the other one, and z is given by Equation 2.6.

$$e^{-NR_{0fc}^{(N)}(Q_N)} = \sum_{\mathbf{x}'} Q_N(\mathbf{x}') \sum_{\mathbf{x}} Q_N(\mathbf{x}) z^{d(\mathbf{x},\mathbf{x}')}$$

for any $\mathbf{x}'$ inner summation will be the same, so we can write

$$\begin{aligned}
e^{-NR_{0fc}^{(N)}(Q_N)} &= \sum_{\mathbf{x}} Q_N(\mathbf{x}) z^{d(\mathbf{x},\mathbf{x}')} \\
&= \frac{1}{\binom{N}{K}} \sum_{d=0}^{N} N_d(x_0) z^d
\end{aligned} \tag{2.7}$$

where M is the total number of the codewords for the fixed-composition ensemble and $N_d(x_0)$ is the number of the codewords that are at distance d from the codeword x_0. Thus,

$$R_{0fc}^{(N)} = -\frac{1}{N} \ln \left(\frac{1}{\binom{N}{K}} \sum_{d=0}^{N} N_d(x_0) z^d \right) \tag{2.8}$$

$$N_d(x_0) = \begin{cases} 0 & d \text{ is odd} \\ 0 & d > 2K \\ \binom{K}{d/2}\binom{N-K}{d/2} & d \text{ is even and } d \leq 2K \end{cases}$$

where it is assumed that $K \leq \frac{N}{2}$. Therefore, $N - K \geq K$. Unlike to the independent-letters ensemble, here we have the possibility to increase R_0 by increasing blocklength, N. This result is demonstrated in Section 2.3.

2.3 Comparison of Cutoff Rates

For a fair comparison of the two ensembles, we take $K = pN$, and compute the cutoff rates as a function of p and N. Thus, for the independent-letters ensemble an average number of $NA\Delta p$ photons are sent per codeword; for the fixed-composition ensemble exactly $NA\Delta p$ photons are sent per codeword.

For the independent-letters ensemble R_0 does not depend on N, but for the fixed-composition ensemble it improves as N is increased. The main point we wish to demonstrate is that the fixed-composition ensemble has a significantly larger R_0 than the independent-letters ensemble.

Cross over probabilities (ϵ and δ) for the DMC in Figures 2.1 and 2.2 depend on the signal-to-noise ratio ($SNR = \frac{A}{\lambda_0}$), and also on $A\Delta$ (photons/slot) which is the number of photons per bit interval. For $A\Delta = 2, 5, 10, 50$ various plots of R_0 versus SNR are given with the parameter p changing from 0.05 to 0.50 by increments of 0.05 in Figures 2.3 to 2.42. In each case R_{0i} and R_{0fc} with $N = 40, 100$ are plotted. In the case of no dark current ($\lambda_0 = 0$) similarly R_0 versus $A\Delta$ curves for the ensembles of independent-letters and fixed-composition codes are given in Figures 2.43 to 2.52.

Based on these figures our first observation is that one can get considerable improvements in the cutoff rate with fixed-composition codes. These improvements are listed in Tables 2.1 to 2.5. The improvement percentage is calculated by using R_{0fc} and R_{0i} values (from the saturated region) for a fixed N and p as follows

$$\alpha = \frac{R_{0fc} - R_{0i}}{R_{0i}} \times 100$$

We can achieve more than 80% improvement in cutoff rate for small values of p if fixed-composition codes are used. This improvement decreases in low noise case as p increases, and for values of p close to 0.50, R_{0i} becomes greater than R_{0fc}. The reason is that the independent-letters ensemble has more codewords than the fixed-composition ensemble, and in the low noise case (ideally in noiseless case) this is more dominating for obtaining higher cutoff rates.

Another observation is the increase in cutoff rate when $A\Delta$ is increased at a fixed value the signal-to-noise ratio. This is because of the dependence of the cross-over probabilities of the DMC on $A\Delta$ as well as on the signal-to-noise ratio.

For given $A\Delta$ and SNR, increasing p results in an increase in the cutoff rate. This is an expected result, because by increasing p we increase the power for the corresponding message signal of the codeword.

The slope discontinuities in Figures 2.13 to 2.22 are due to the fact that as γ in Equation 2.1 moves over integer values, the number of terms under summation in Equation 2.2 changes in a discrete manner.

In $\lambda_0 = 0$ case, there is only quantum noise and R_0 increases obviously with $A\Delta$, since we send more photons for one bit of information. For a specific value of $A\Delta$, R_0 increases with p similar to the case of $\lambda_0 > 0$.

After some value of N we expect that there will not be an improvement in cutoff rate by increasing N. For $A\Delta = 2$ case, cutoff rates for $N = 40$ and $N = 100$ are same because saturation value of N is possibly achieved for smaller values than 40.

Consider an uncoded system in which there is no dark current, $\lambda_0 = 0$. Then, due to Equation 2.2 (Z-channel of Figure 2.2) probability of error will be

$$P(E) = \epsilon = e^{-A\Delta}$$

where on the average we send $\frac{1}{2}A\Delta + \frac{1}{2}0 = \frac{A\Delta}{2}$ photons/slot assuming 0 and 1 are equally likely. In order to achieve a probability of error, say, $P(E) = 10^{-9}$ using this uncoded system one should send approximately 10 photons/slot with rate, $R = 1$ bit/slot. Then, efficiency will be 0.1 bits/photon. However, using coding, we can make $P(E)$ arbitrarily small by increasing constraint length and still send 10 photons/slot with a higher efficiency. For instance, using Figure 2.44 take $A\Delta = 3$ photons/slot, since $p = 0.10$ we send on the average $pA\Delta = 0.3$ photons/slot with rate $R_0 = 0.21$ nats/slot $= 0.3$ bits/slot. This results in an efficiency of 1 bit/photon which is ten times greater than that of uncoded system.

We can increase efficiency by increasing A and decreasing Δ where $A\Delta$ is held constant. Note that the rate of increase of R_0 decreases as $A\Delta$ increases and at some point R_0 saturates. But this causes an increase in the bandwidth. Large bandwidth is not a problem for optical systems but some practical problems may arise due to the hardware that should work with this system.

p	N	R_{0fc}	R_{0i}	α
0.05	100	0.17918	0.09749	83.78
0.05	40	0.16556	0.09749	69.81
0.10	100	0.30085	0.19360	55.40
0.10	40	0.28384	0.19360	46.62
0.15	100	0.37810	0.28681	31.83
0.15	40	0.33283	0.28681	16.04
0.20	100	0.47066	0.37528	25.42
0.20	40	0.45091	0.37528	20.15
0.25	100	0.53083	0.45679	16.21
0.25	40	0.51041	0.45679	11.74
0.30	100	0.57803	0.52880	9.31
0.30	40	0.55714	0.52880	5.36
0.35	100	0.61363	0.58863	4.25
0.35	40	0.59242	0.58863	0.65
0.40	100	0.63852	0.63366	0.77
0.40	40	0.61711	0.63366	-2.61
0.45	100	0.65325	0.66169	-1.28
0.45	40	0.63172	0.66169	-4.53
0.50	100	0.65813	0.67121	-1.95
0.50	40	0.63656	0.67121	-5.16

Table 2.1: Improvement in the cutoff rate for the case $A\Delta = 50$.

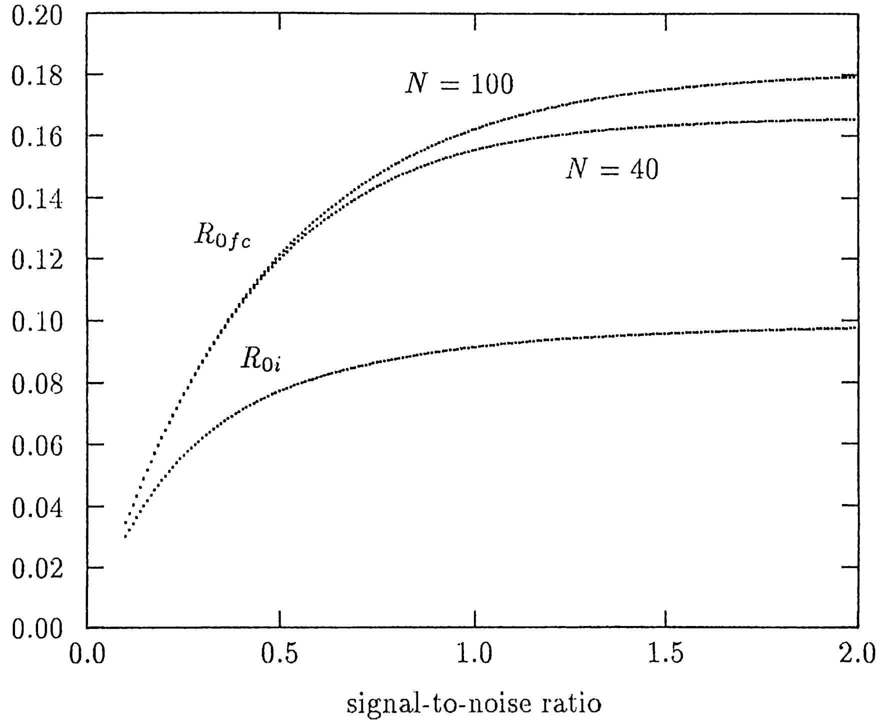

Figure 2.3: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 50$, $p = 0.05$

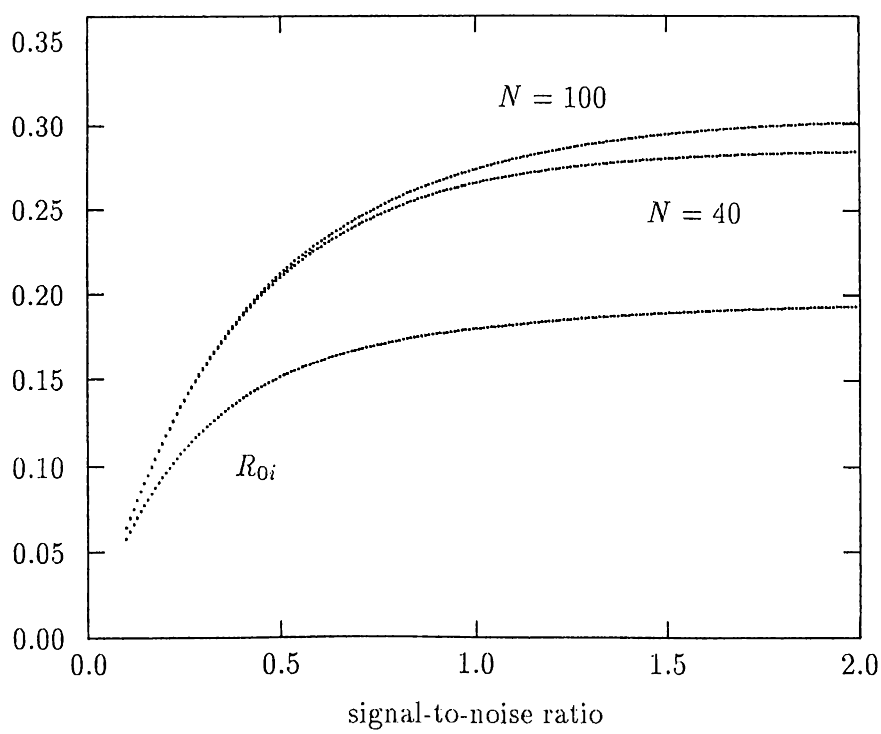

Figure 2.4: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 50$, $p = 0.10$

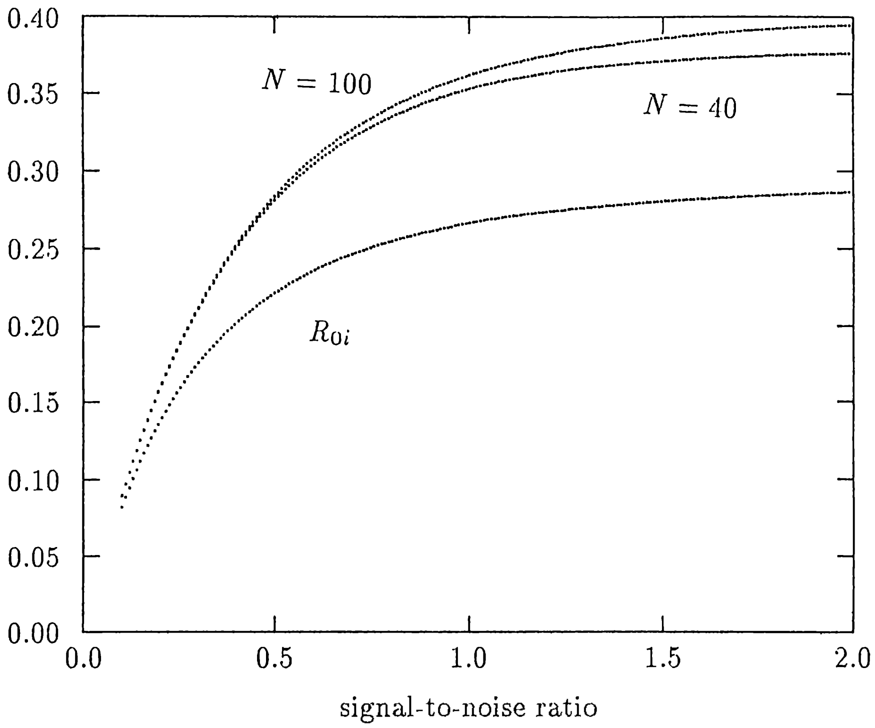

Figure 2.5: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 50$, $p = 0.15$

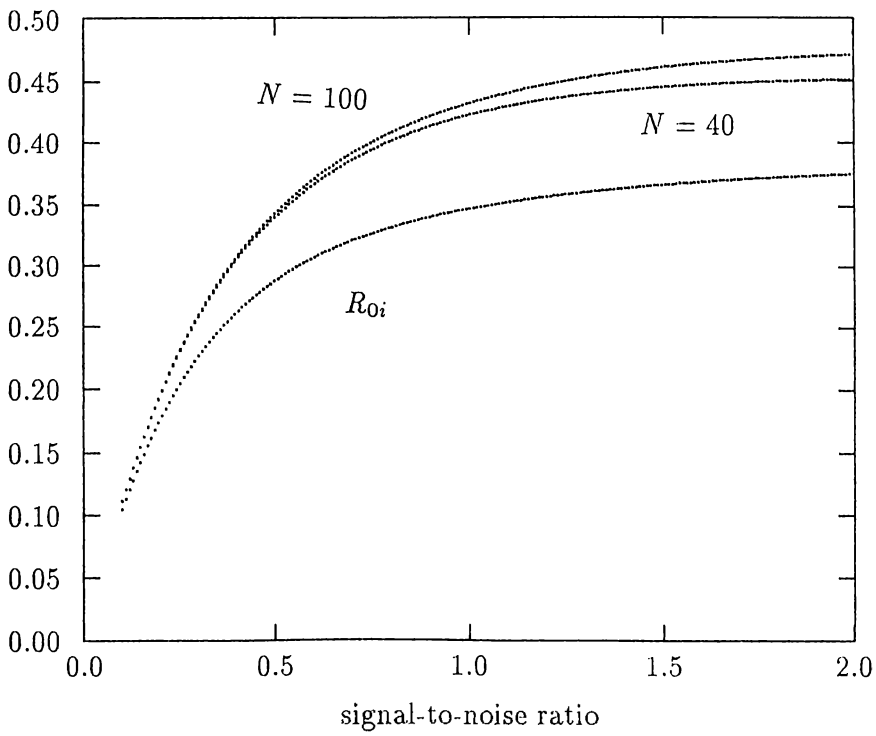

Figure 2.6: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 50$, $p = 0.20$

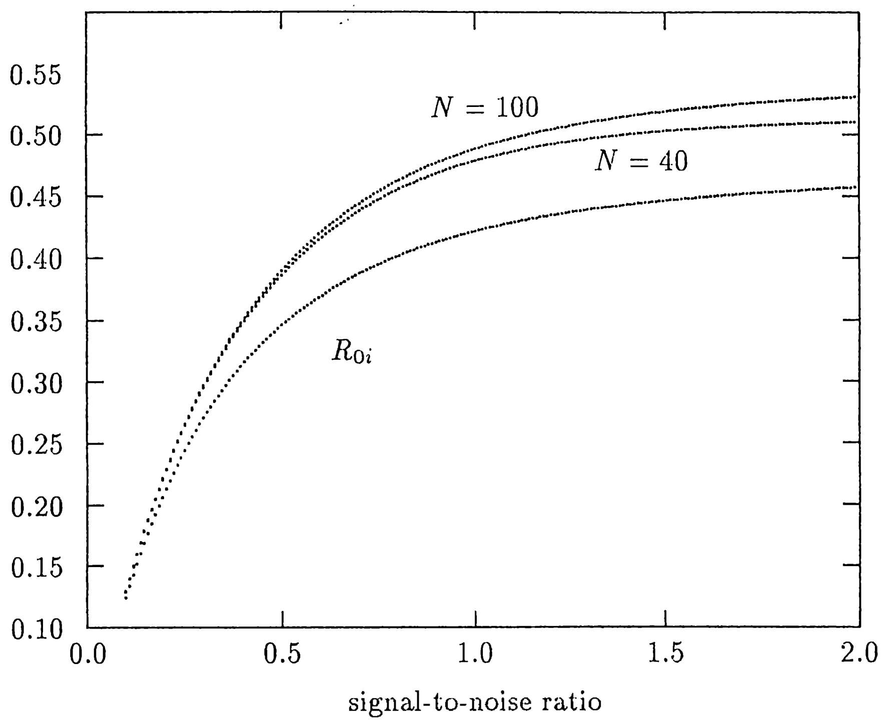

Figure 2.7: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 50$, $p = 0.25$

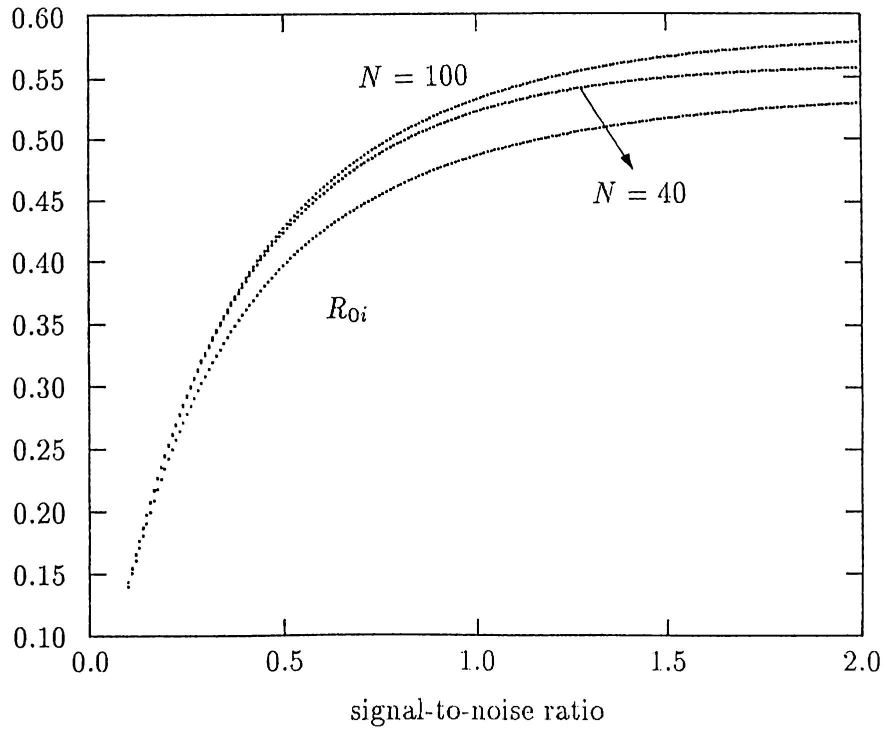

Figure 2.8: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 50$, $p = 0.30$

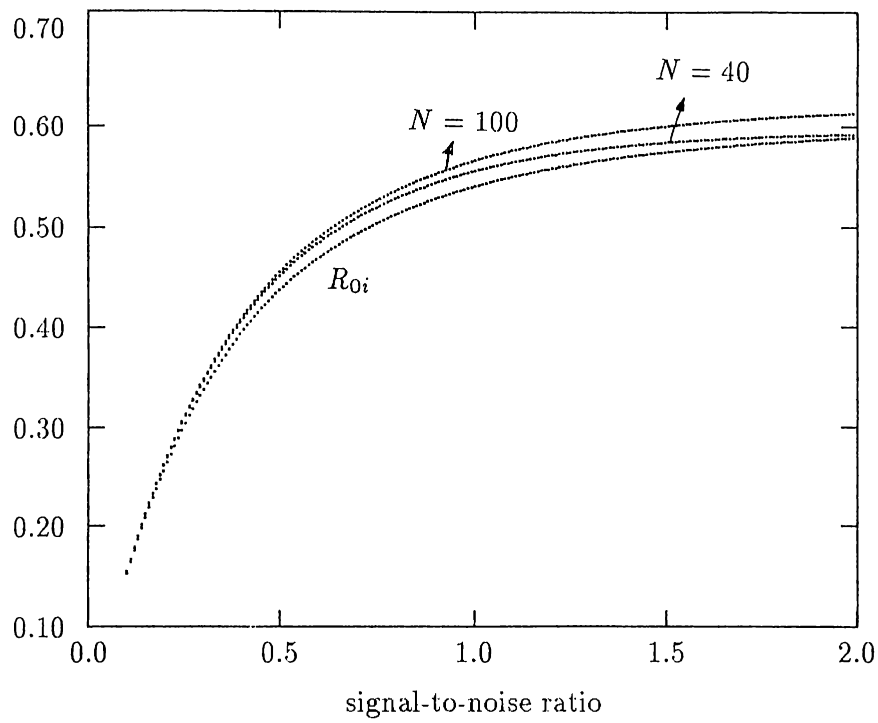

Figure 2.9: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 50$, $p = 0.35$

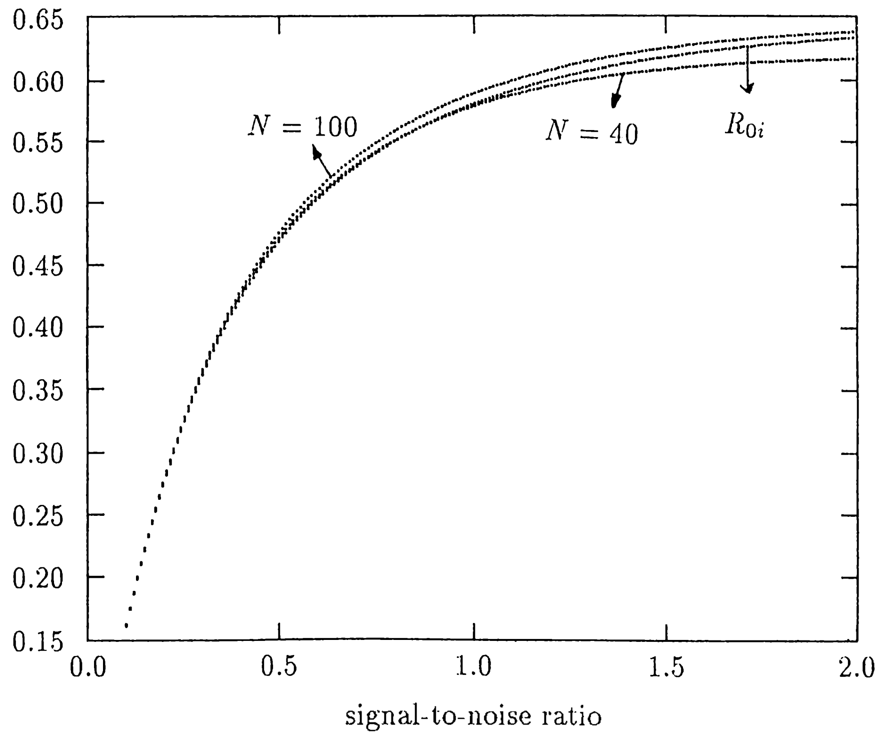

Figure 2.10: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 50$, $p = 0.40$

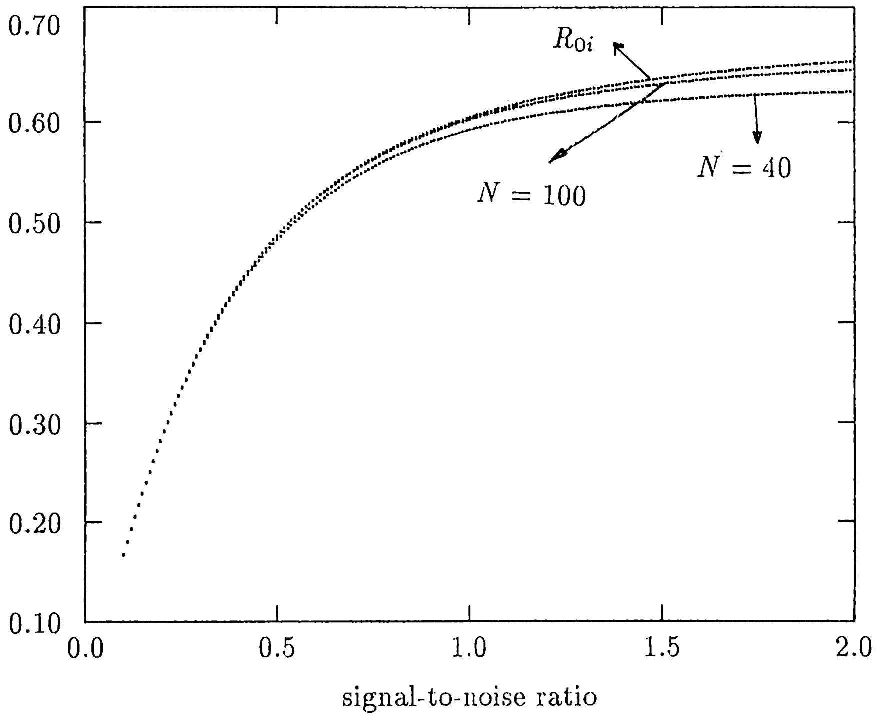

Figure 2.11: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 50$, $p = 0.45$

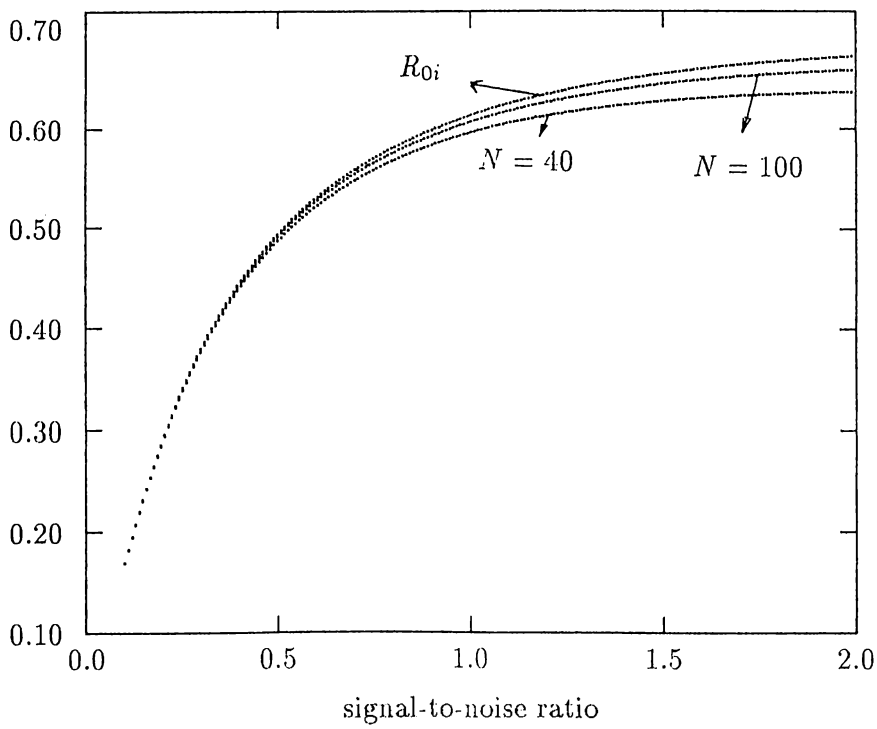

Figure 2.12: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 50$, $p = 0.50$

p	N	R_{0fc}	R_{0i}	α
0.05	100	0.16915	0.09350	80.90
0.05	40	0.16019	0.09350	71.32
0.10	100	0.28473	0.18529	53.67
0.10	40	0.27420	0.18529	47.98
0.15	100	0.37482	0.27392	36.83
0.15	40	0.36349	0.27392	32.70
0.20	100	0.44688	0.35765	24.95
0.20	40	0.43504	0.35765	21.64
0.25	100	0.50451	0.43443	16.13
0.25	40	0.49230	0.43443	13.32
0.30	100	0.54976	0.50195	9.53
0.30	40	0.53728	0.50195	7.04
0.35	100	0.58392	0.55781	4.68
0.35	40	0.57125	0.55781	2.41
0.40	100	0.60781	0.59970	1.35
0.40	40	0.59501	0.59970	-0.78
0.45	100	0.62196	0.62572	-0.60
0.45	40	0.60908	0.62572	-2.66
0.50	100	0.62664	0.63454	-1.24
0.50	40	0.61374	0.63454	-3.28

Table 2.2: Improvement in the cutoff rate for the case $A\Delta = 10$.

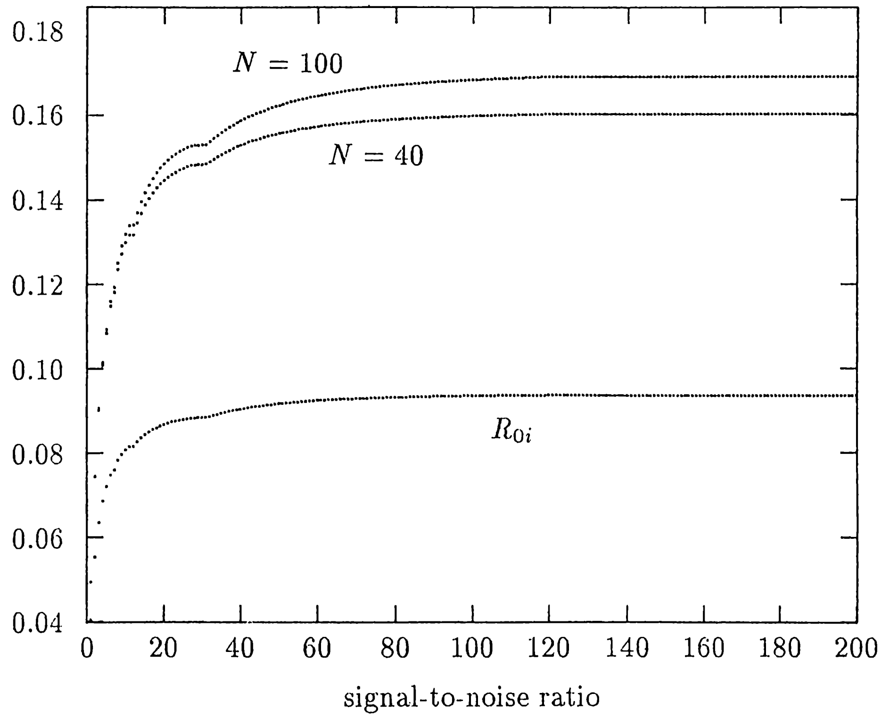

Figure 2.13: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 10$, $p = 0.05$

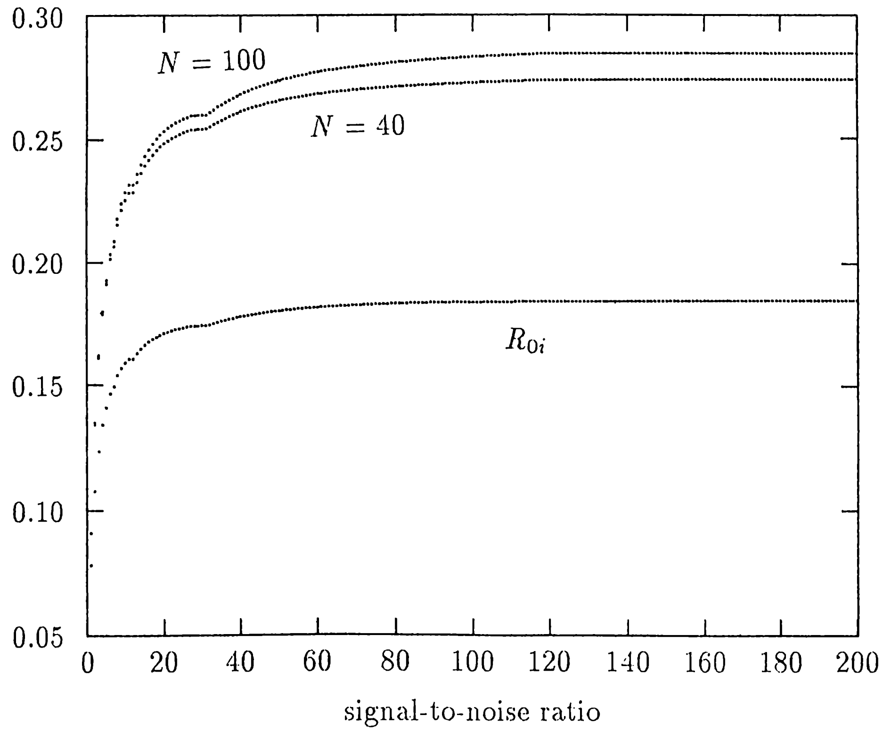

Figure 2.14: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 10$, $p = 0.10$

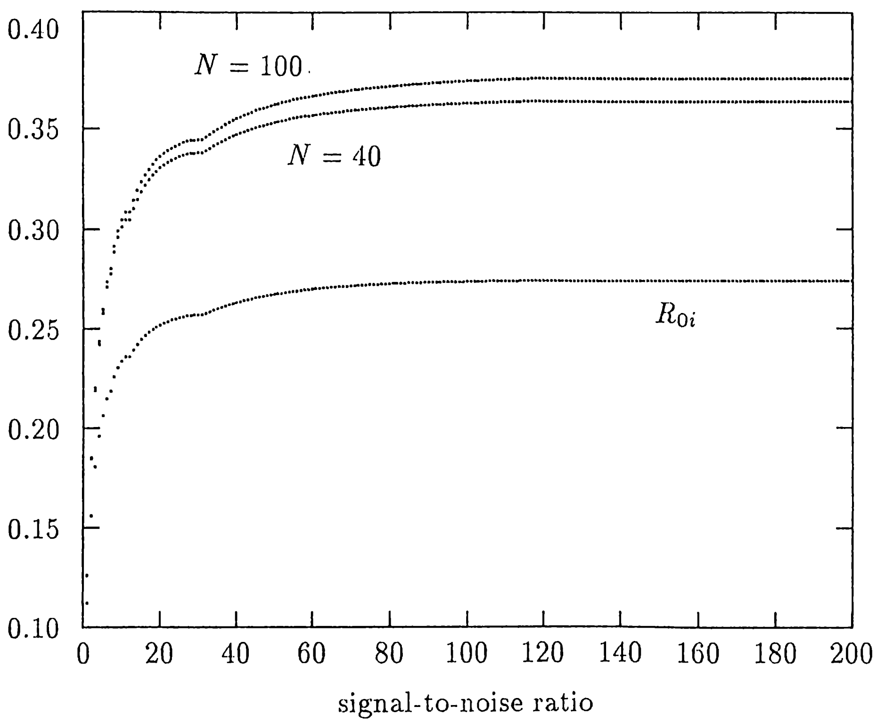

Figure 2.15: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 10$, $p = 0.15$

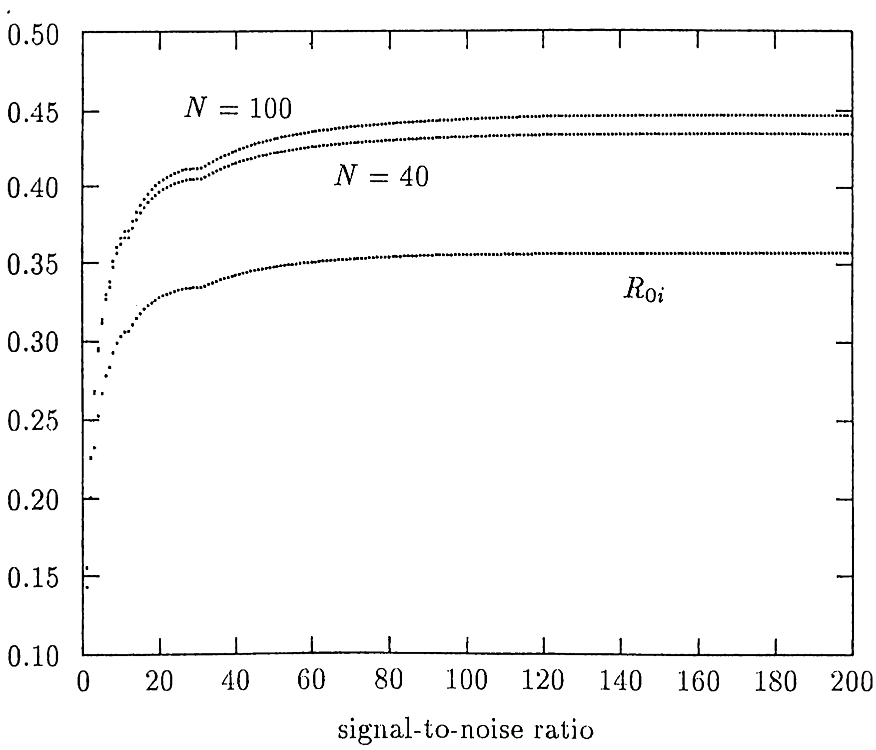

Figure 2.16: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 10$, $p = 0.20$

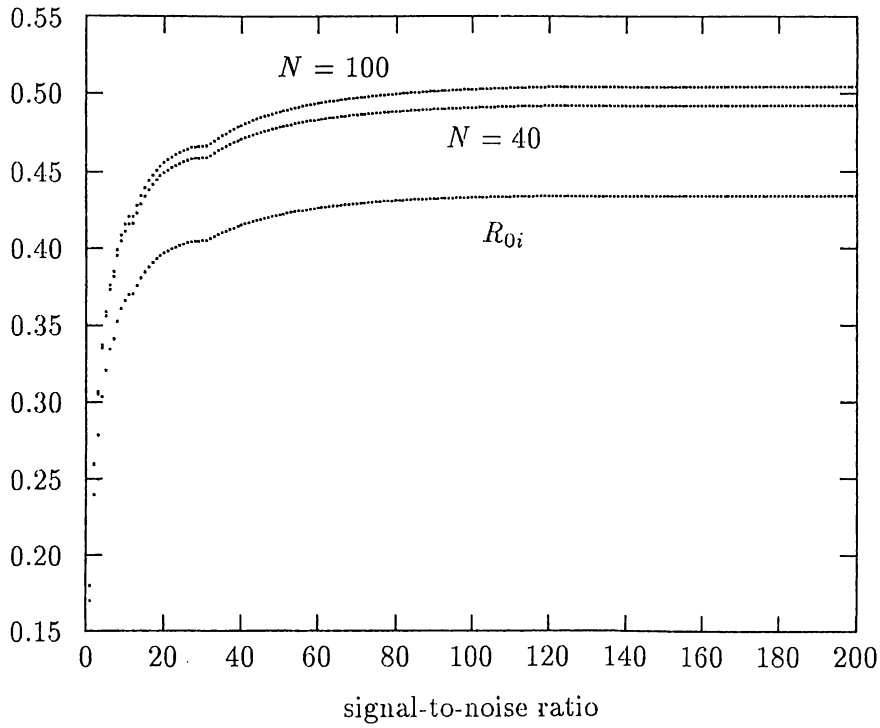

Figure 2.17: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 10$, $p = 0.25$

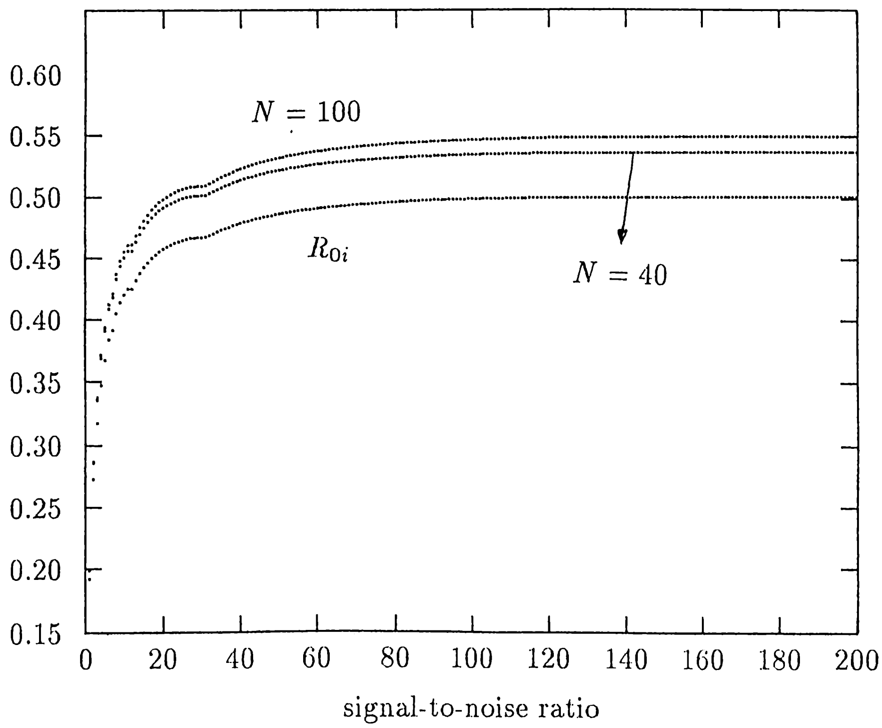

Figure 2.18: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 10$, $p = 0.30$

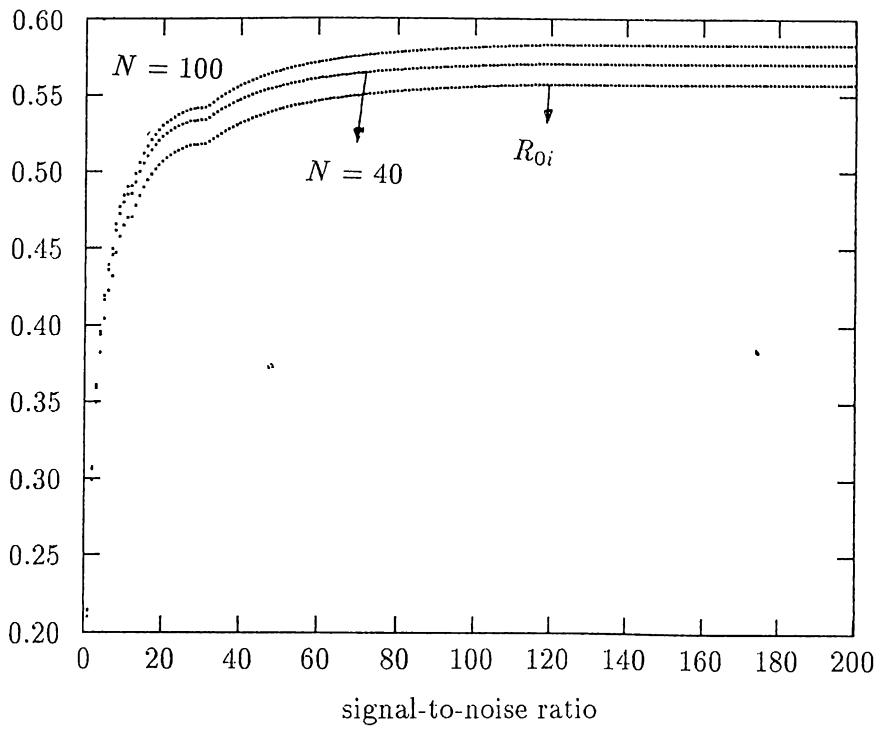

Figure 2.19: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 10$, $p = 0.35$

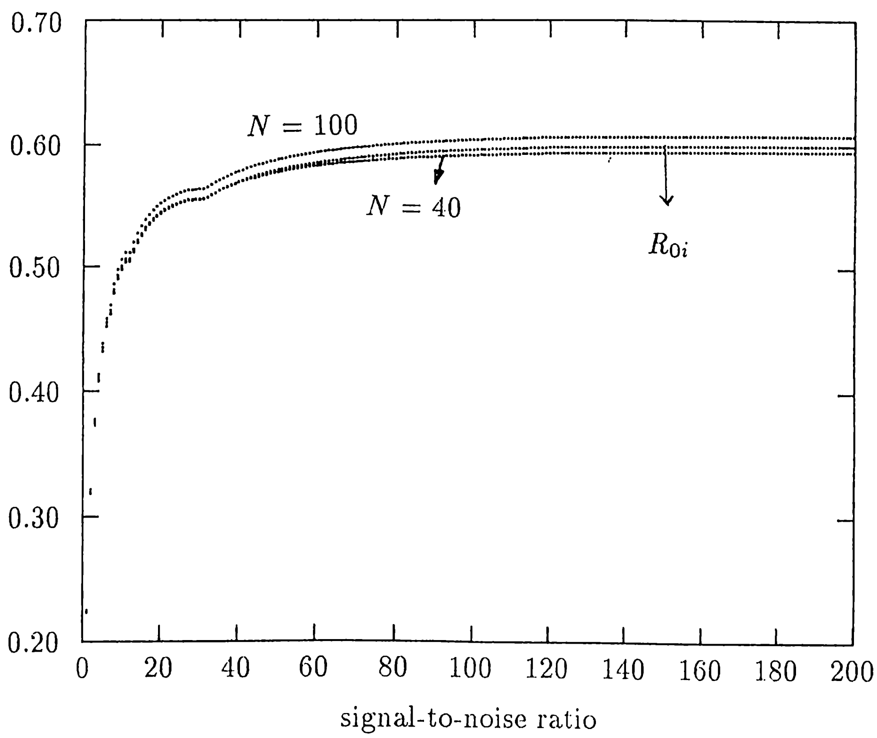

Figure 2.20: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 10$, $p = 0.40$

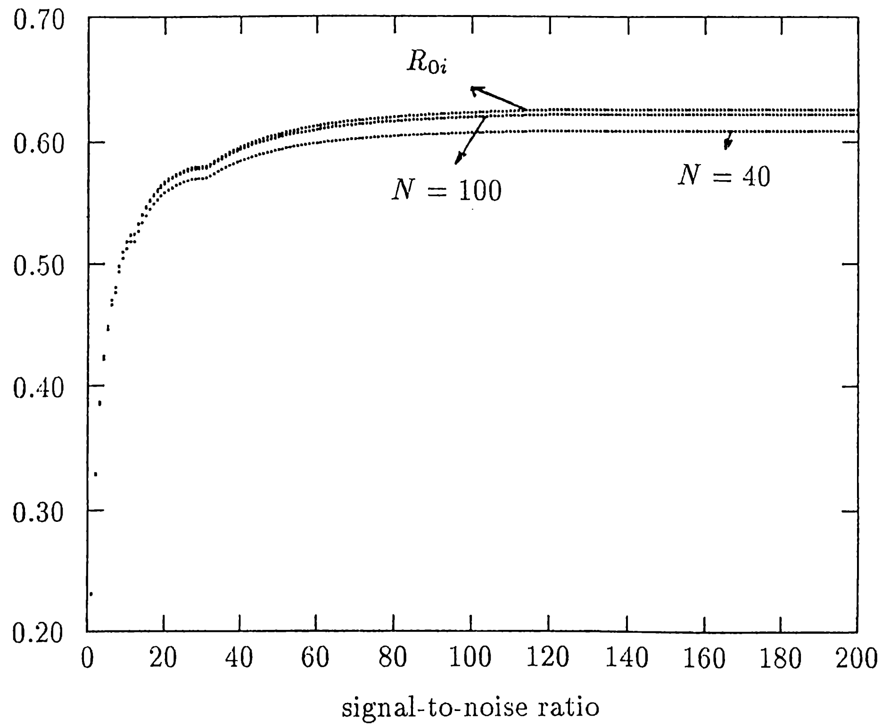

Figure 2.21: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 10$, $p = 0.45$

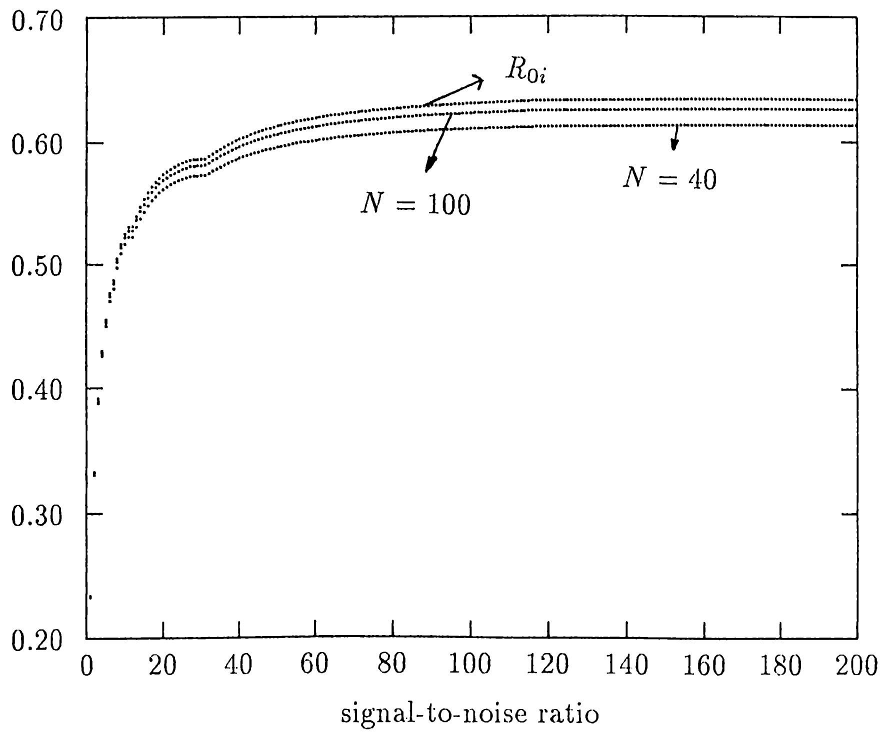

Figure 2.22: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 10$, $p = 0.50$

p	N	R_{0fc}	R_{0i}	α
0.05	100	0.12109	0.07673	57.81
0.05	40	0.11975	0.07673	56.05
0.10	100	0.21138	0.15077	40.21
0.10	40	0.20921	0.15077	38.76
0.15	100	0.28350	0.22098	28.29
0.15	40	0.28075	0.22098	27.05
0.20	100	0.34192	0.28609	19.52
0.20	40	0.33876	0.28609	18.41
0.25	100	0.38900	0.34470	12.85
0.25	40	0.38553	0.34470	11.85
0.30	100	0.42616	0.39535	7.79
0.30	40	0.42246	0.39535	6.86
0.35	100	0.45431	0.43659	4.06
0.35	40	0.45044	0.43659	3.17
0.40	100	0.47405	0.46713	1.48
0.40	40	0.47006	0.46713	0.63
0.45	100	0.48575	0.48591	-0.03
0.45	40	0.48169	0.48591	-0.87
0.50	100	0.48963	0.49225	-0.53
0.50	40	0.48555	0.49225	-1.36

Table 2.3: Improvement in the cutoff rate for the case $A\Delta = 5$.

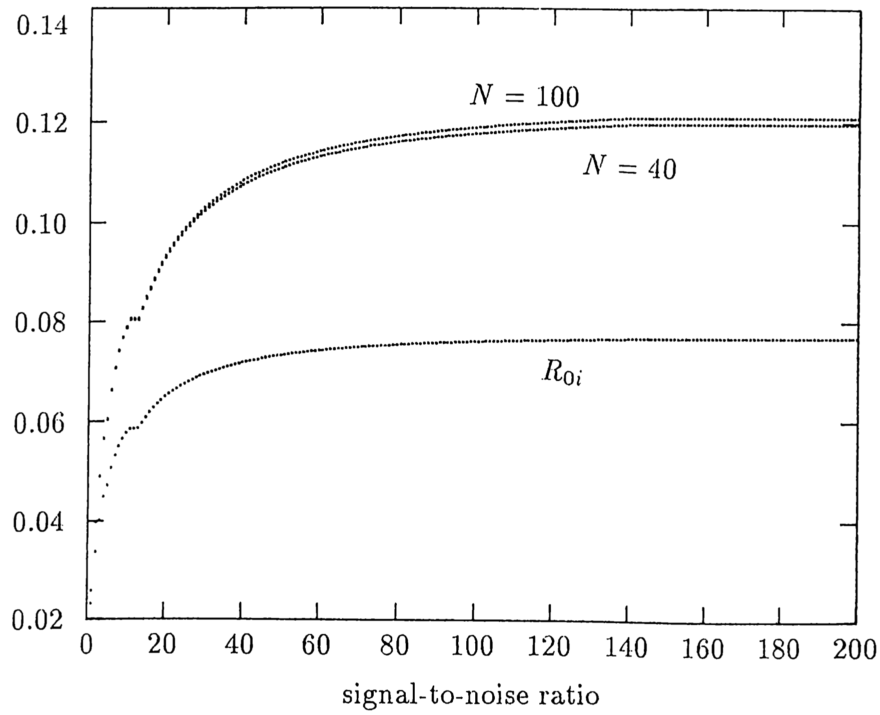

Figure 2.23: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 5$, $p = 0.05$

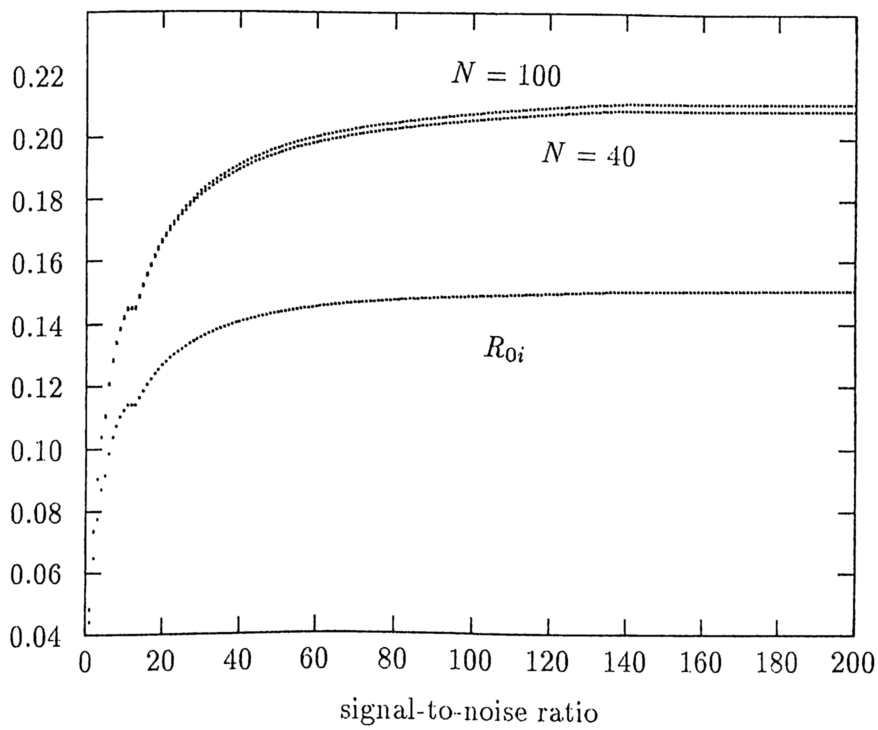

Figure 2.24: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 5$, $p = 0.10$

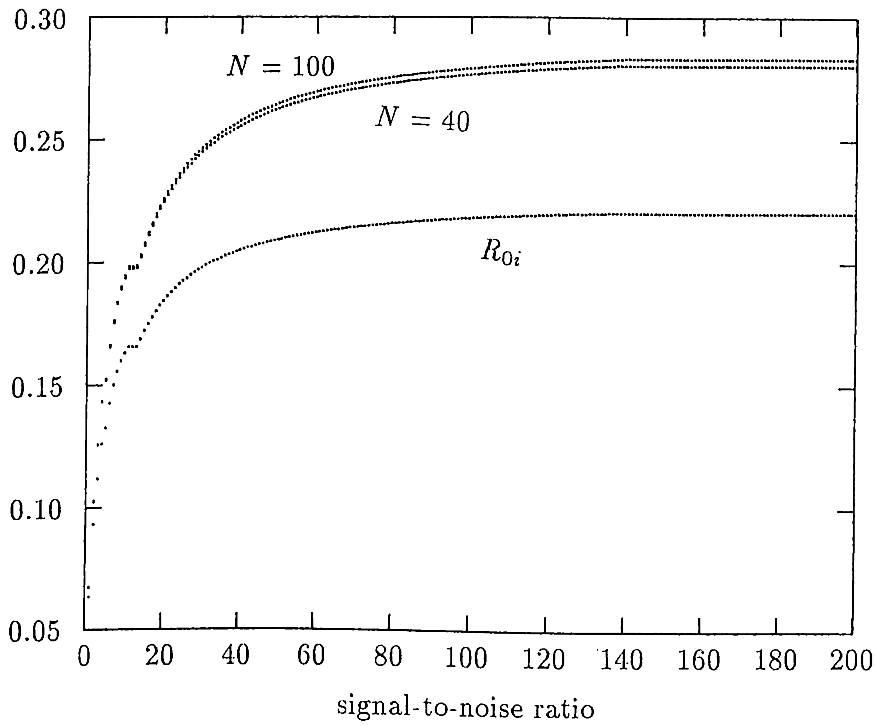

Figure 2.25: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 5$, $p = 0.15$

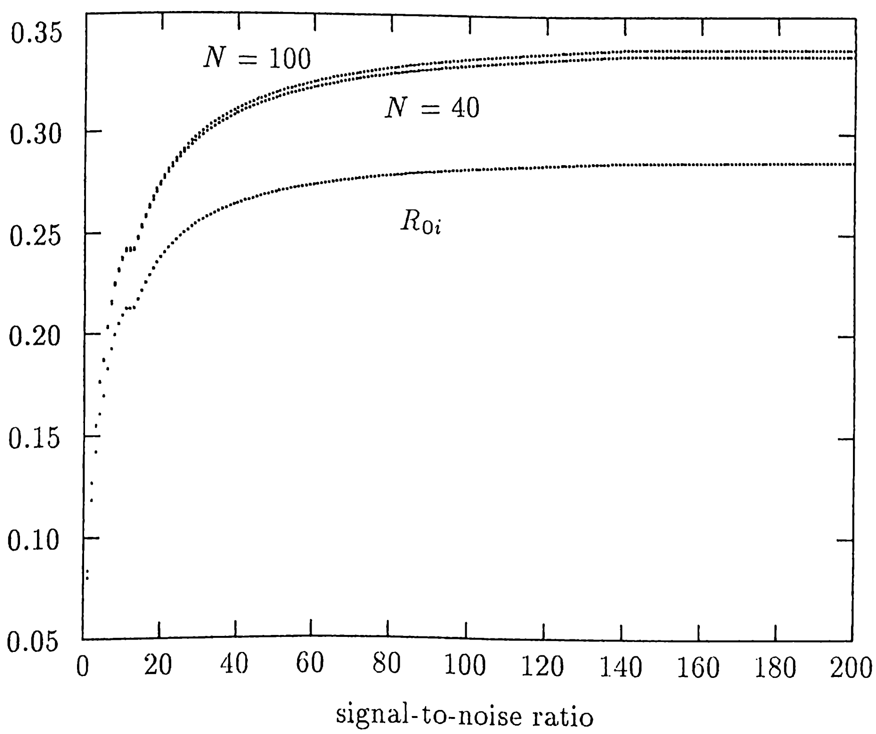

Figure 2.26: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 5$, $p = 0.20$

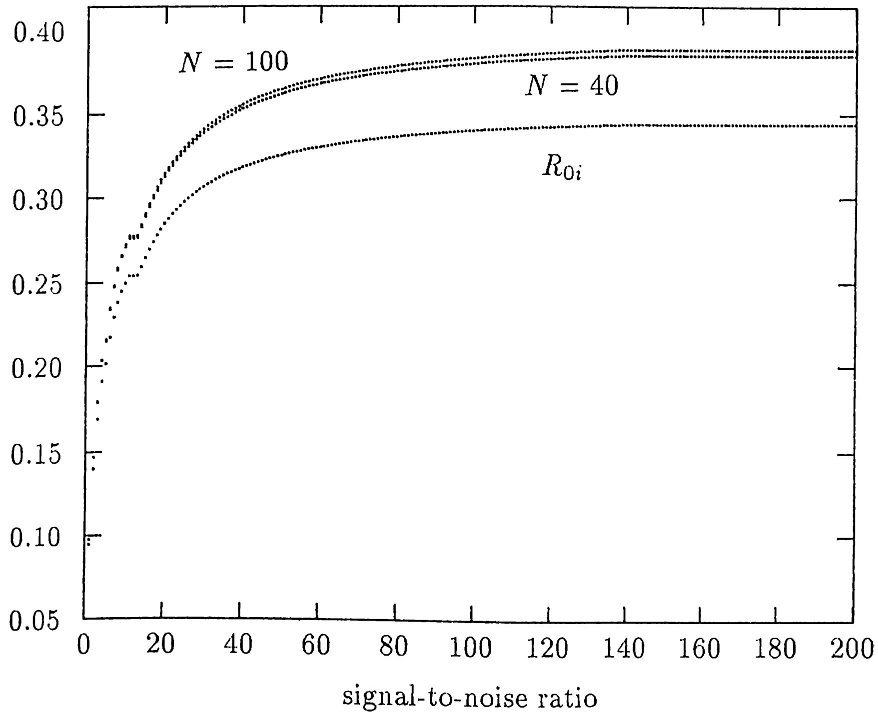

Figure 2.27: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 5$, $p = 0.25$

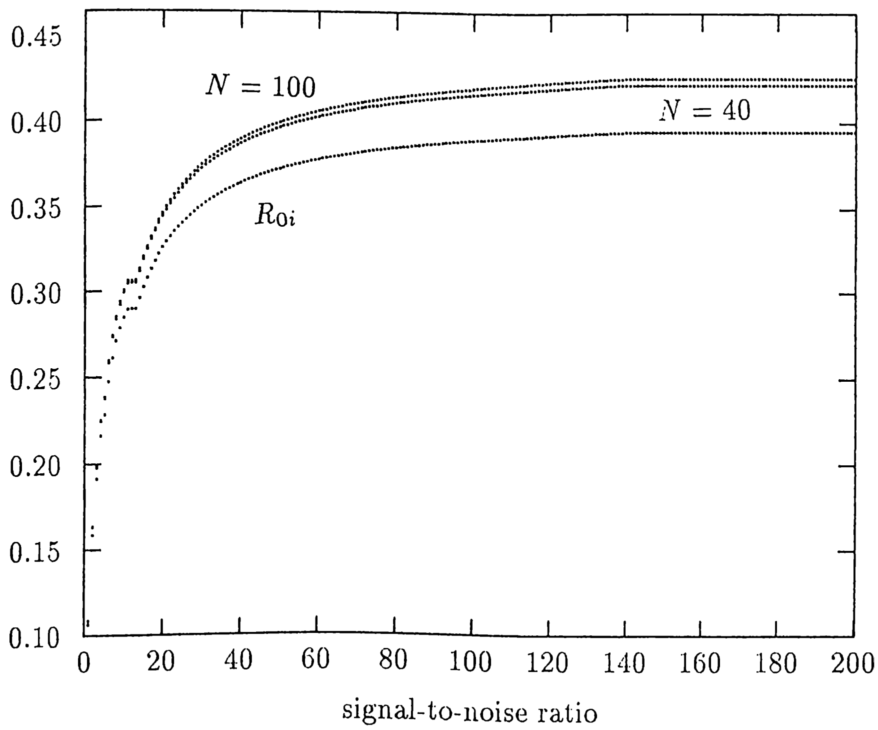

Figure 2.28: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 5$, $p = 0.30$

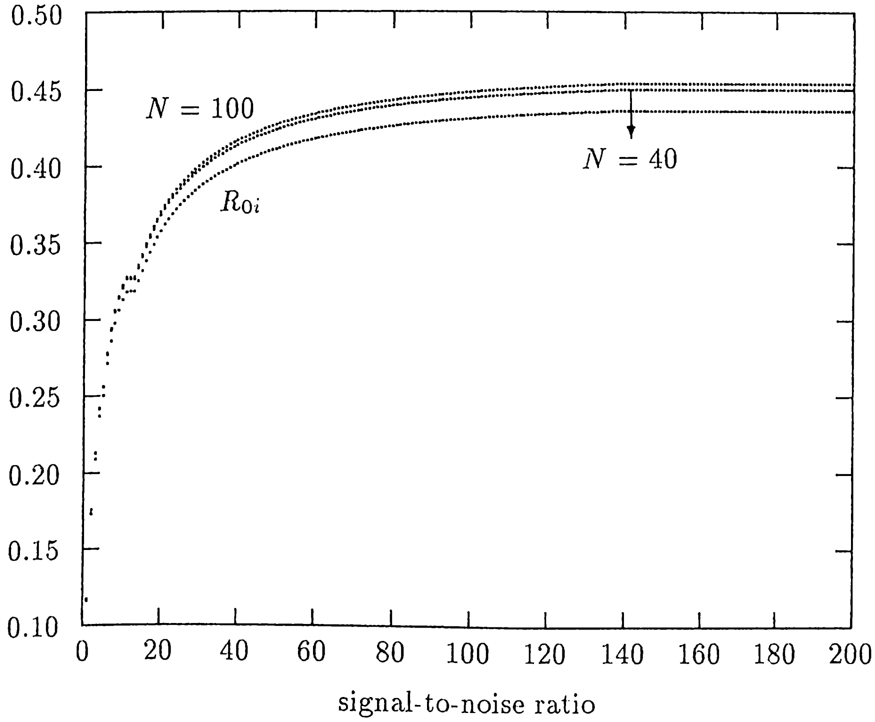

Figure 2.29: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 5$, $p = 0.35$

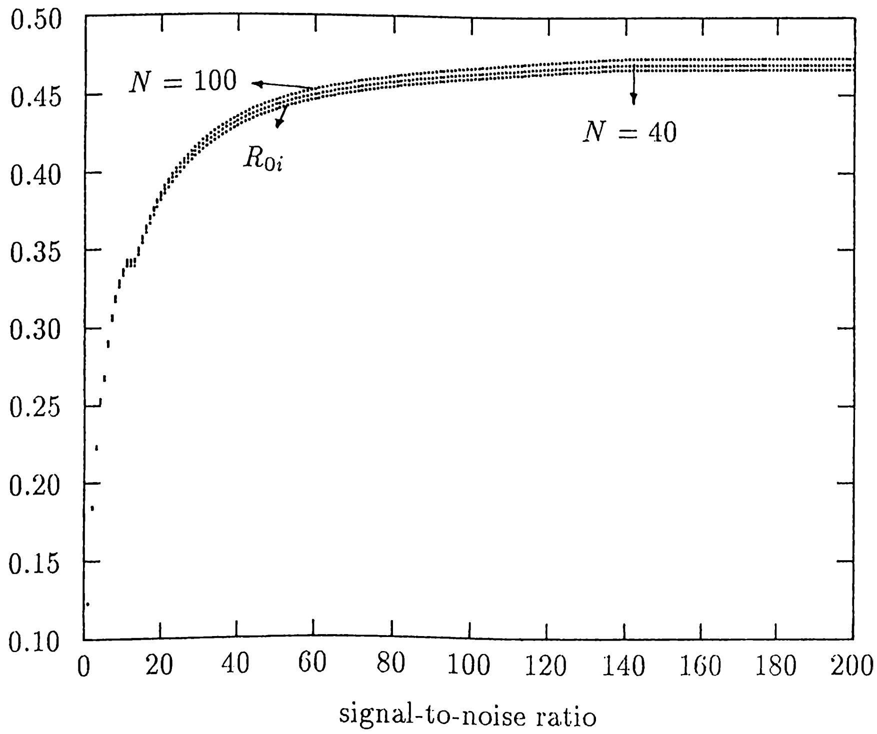

Figure 2.30: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 5$, $p = 0.40$

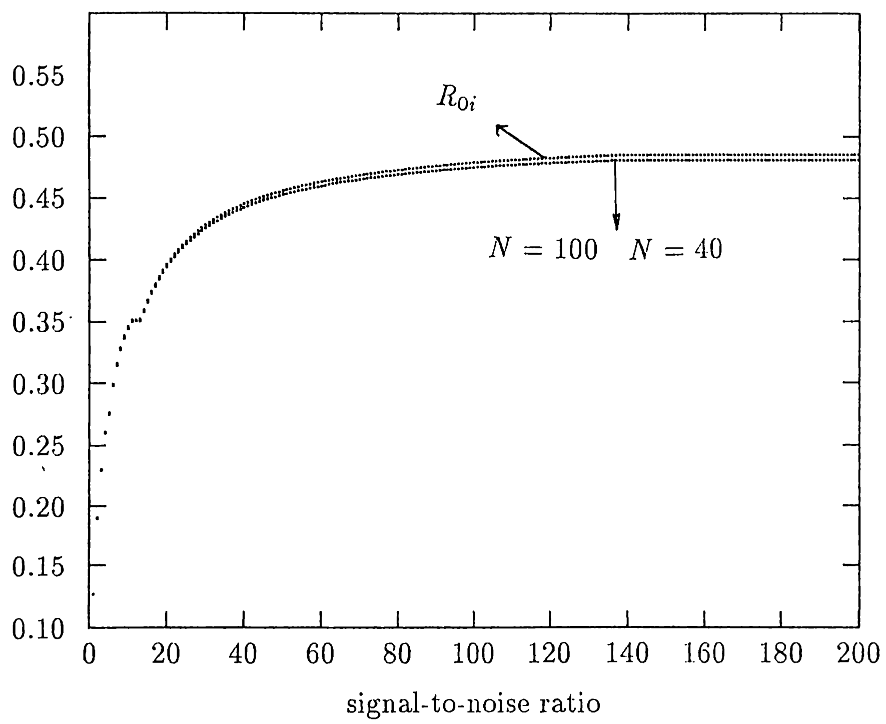

Figure 2.31: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 5$, $p = 0.45$

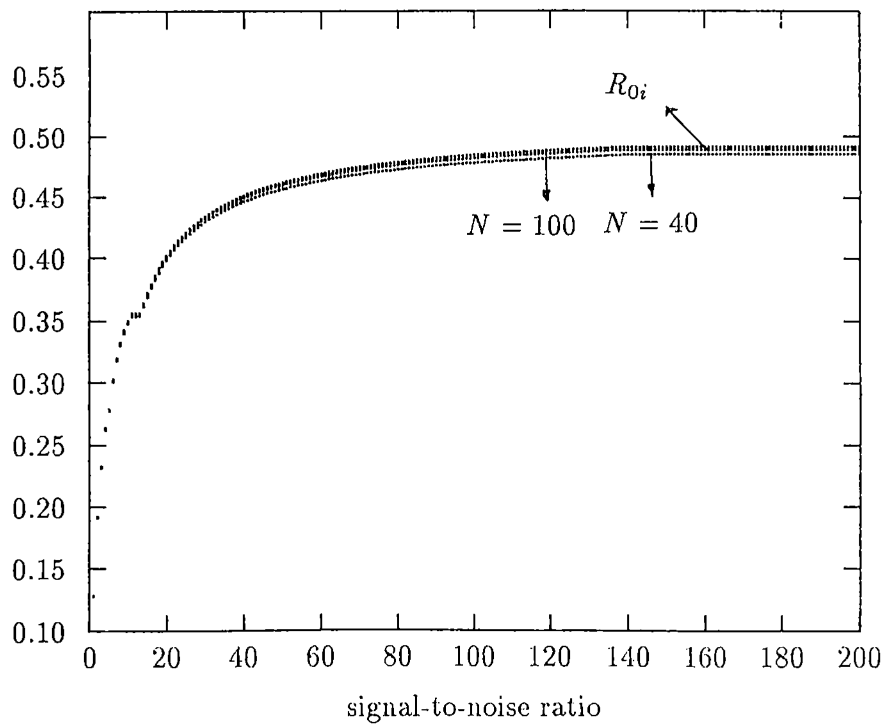

Figure 2.32: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 5$, $p = 0.50$

p	N	R_{0fc}	R_{0i}	α
0.05	100	0.07002	0.05291	32.32
0.05	40	0.06986	0.05291	32.02
0.10	100	0.12724	0.10276	23.82
0.10	40	0.12687	0.10276	23.46
0.15	100	0.17468	0.14890	17.31
0.15	40	0.17411	0.14890	16.93
0.20	100	0.21395	0.19069	12.20
0.20	40	0.21321	0.19069	11.81
0.25	100	0.24603	0.22746	8.16
0.25	40	0.24516	0.22746	7.78
0.30	100	0.27160	0.25859	5.03
0.30	40	0.27062	0.25859	4.65
0.35	100	0.29109	0.28349	2.68
0.35	40	0.29003	0.28349	2.31
0.40	100	0.30482	0.30166	1.05
0.40	40	0.30370	0.30166	0.68
0.45	100	0.31298	0.31273	0.08
0.45	40	0.31183	0.31273	-0.29
0.50	100	0.31569	0.31644	-0.24
0.50	40	0.31453	0.31644	-0.61

Table 2.4: Improvement in the cutoff rate for the case $A\Delta = 2$.

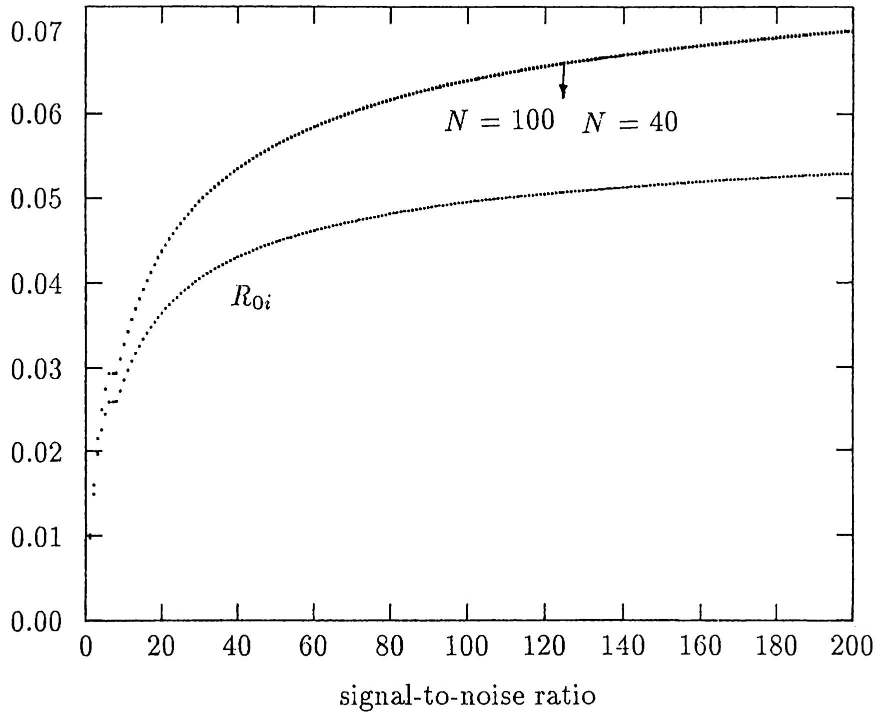

Figure 2.33: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 2$, $p = 0.05$

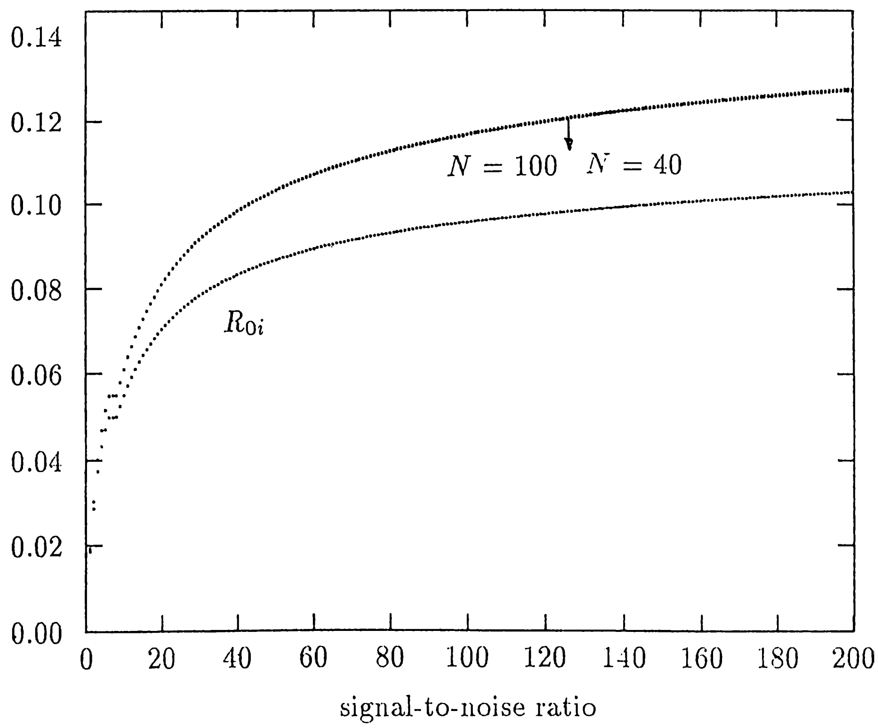

Figure 2.34: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 2$, $p = 0.10$

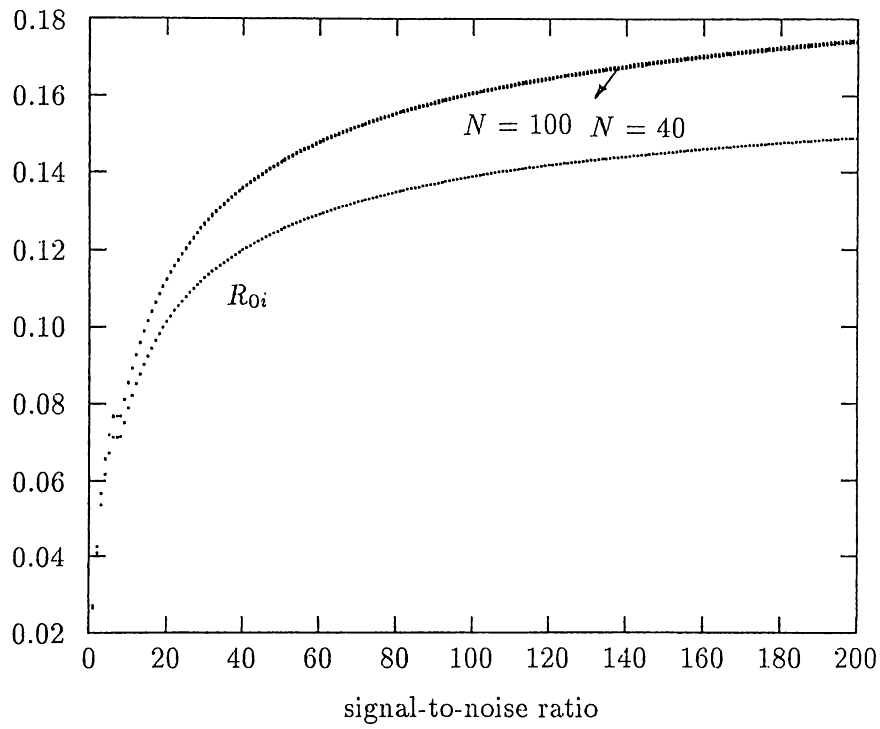

Figure 2.35: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 2$, $p = 0.15$

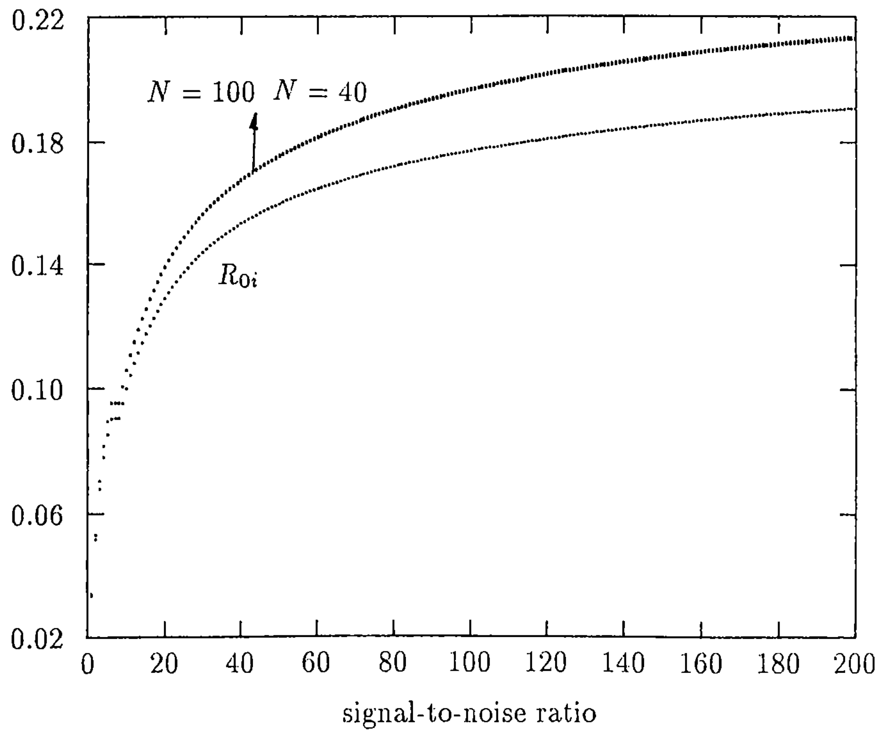

Figure 2.36: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 2$, $p = 0.20$

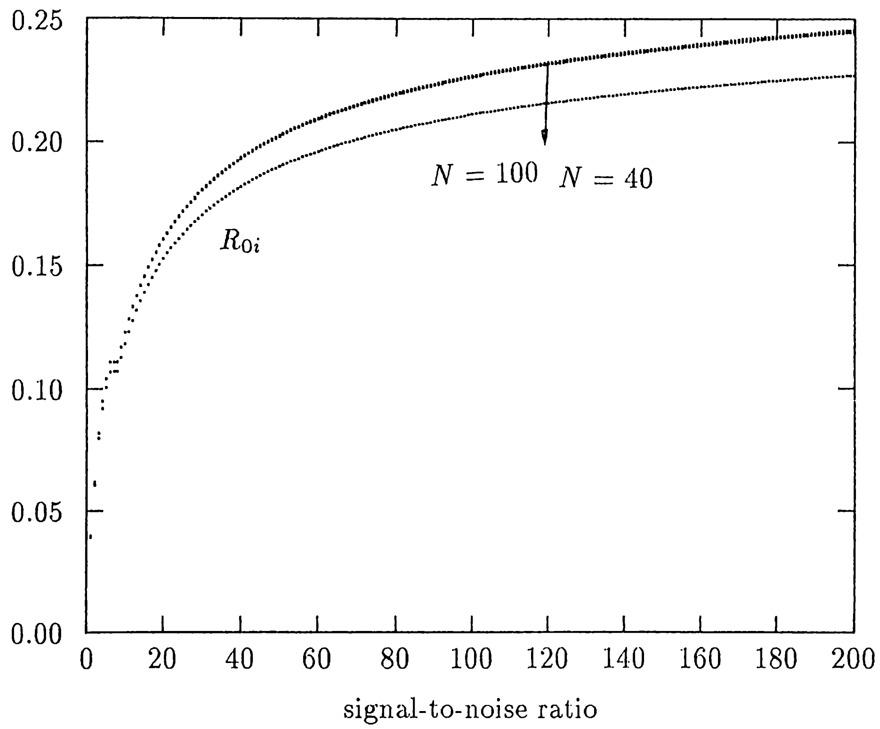

Figure 2.37: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 2$, $p = 0.25$

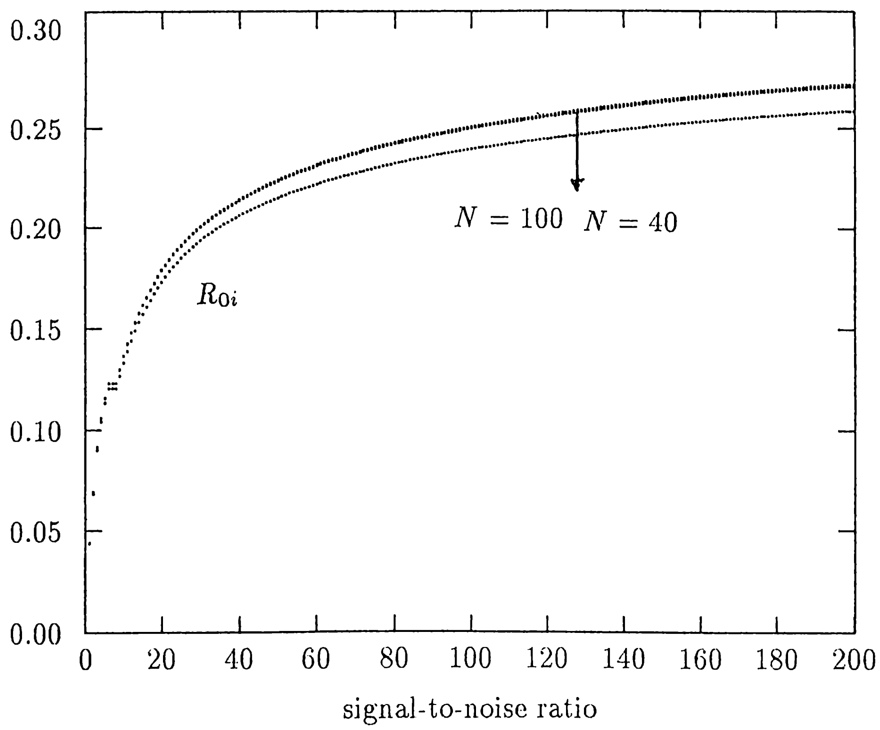

Figure 2.38: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 2$, $p = 0.30$

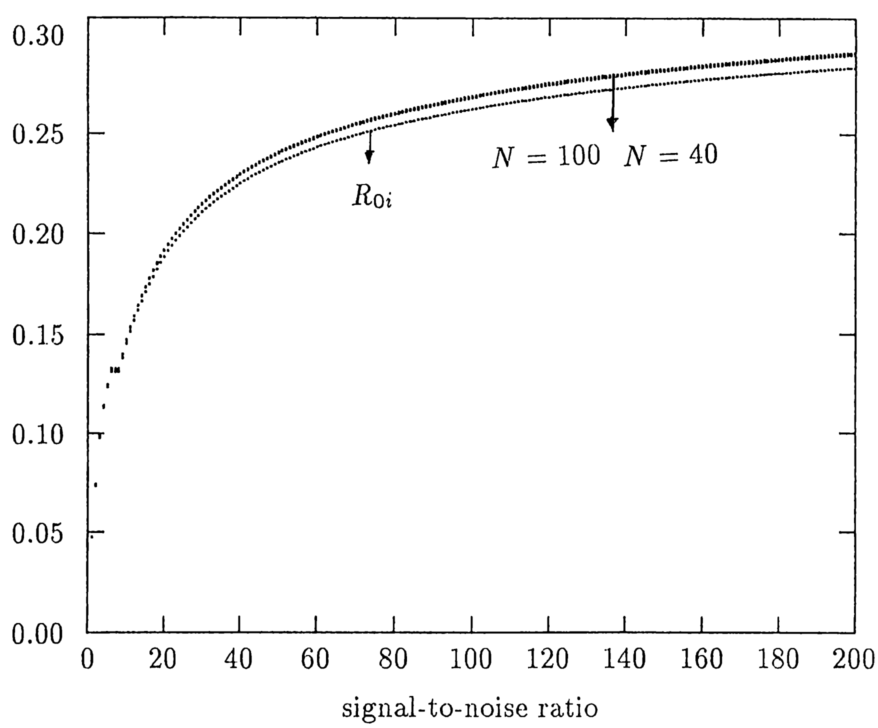

Figure 2.39: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 2$, $p = 0.35$

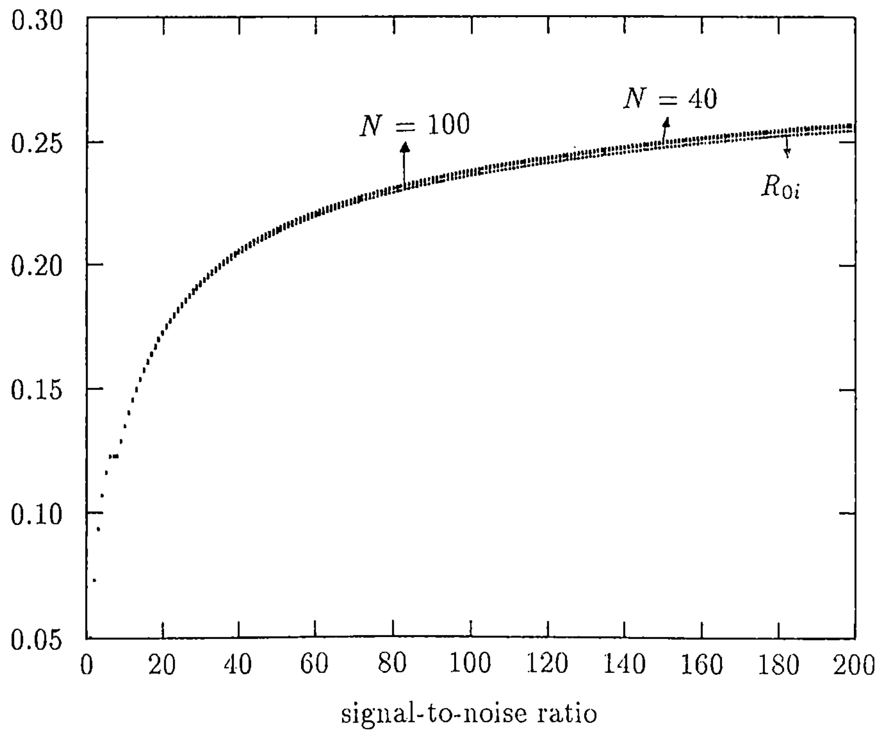

Figure 2.40: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 2$, $p = 0.40$

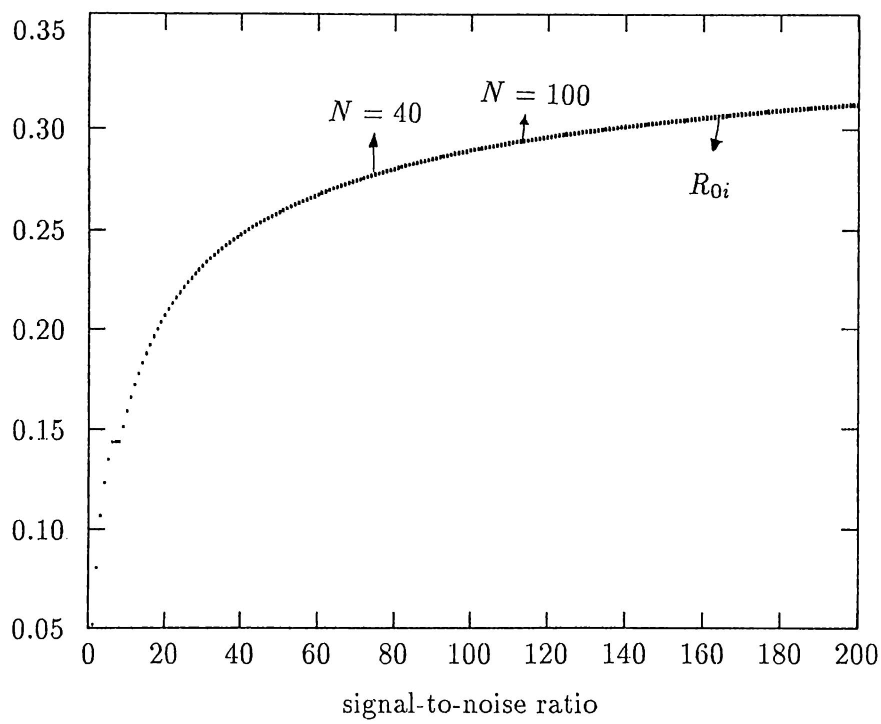

Figure 2.41: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 2$, $p = 0.45$

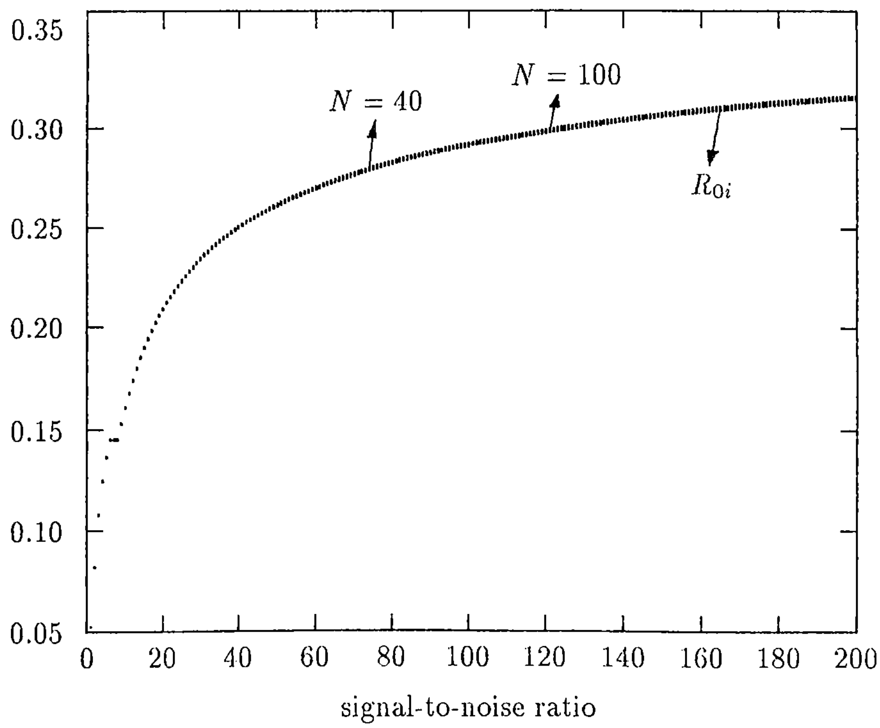

Figure 2.42: R_{0fc} for $N = 100, 40$ and R_{0i}, $A\Delta = 2$, $p = 0.50$

p	N	R_{0fc}	R_{0i}	α
0.05	100	0.18137	0.09976	81.80
0.05	40	0.16648	0.09976	66.88
0.10	100	0.30482	0.19832	53.70
0.10	40	0.28557	0.19832	43.99
0.15	100	0.40073	0.29417	36.22
0.15	40	0.37901	0.29417	28.84
0.20	100	0.47730	0.38539	23.85
0.20	40	0.45395	0.38539	17.79
0.25	100	0.53845	0.46966	14.65
0.25	40	0.51395	0.46966	9.43
0.30	100	0.58641	0.54431	7.73
0.30	40	0.56109	0.54431	3.08
0.35	100	0.62260	0.60649	2.66
0.35	40	0.59669	0.60649	-1.62
0.40	100	0.64790	0.65340	-0.84
0.40	40	0.62160	0.65340	-4.87
0.45	100	0.66287	0.68264	-2.90
0.45	40	0.63635	0.68264	-6.78
0.50	100	0.66783	0.69257	-3.57
0.50	40	0.64123	0.69257	-7.41

Table 2.5: Improvement in the cutoff rate for the case $\lambda_0 = 0, A\Delta = 15$.

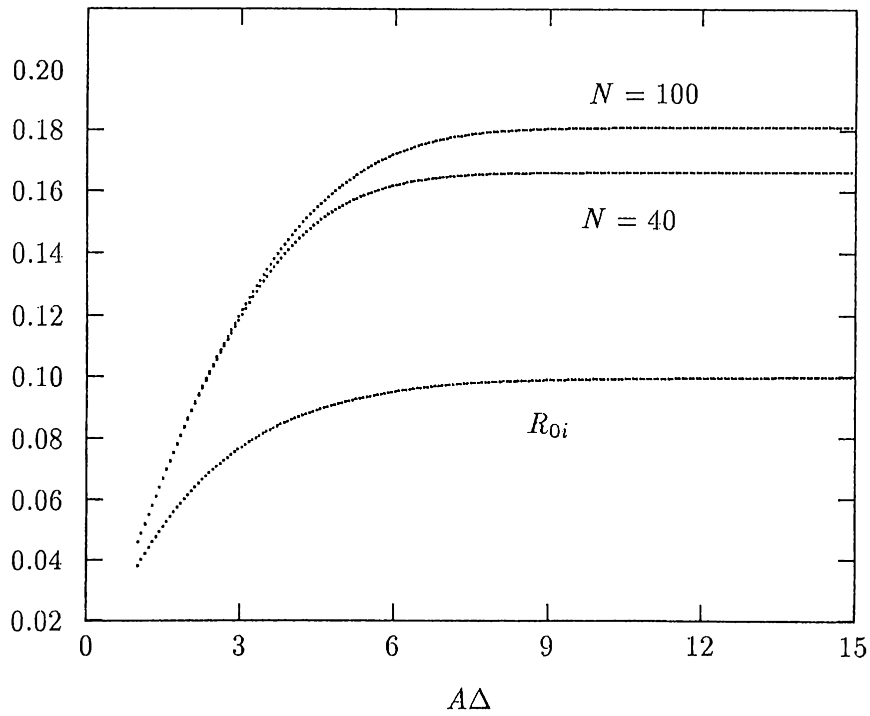

Figure 2.43: R_{0fc} for $N = 100, 40$ and R_{0i}, $\lambda_0 = 0$, $p = 0.05$

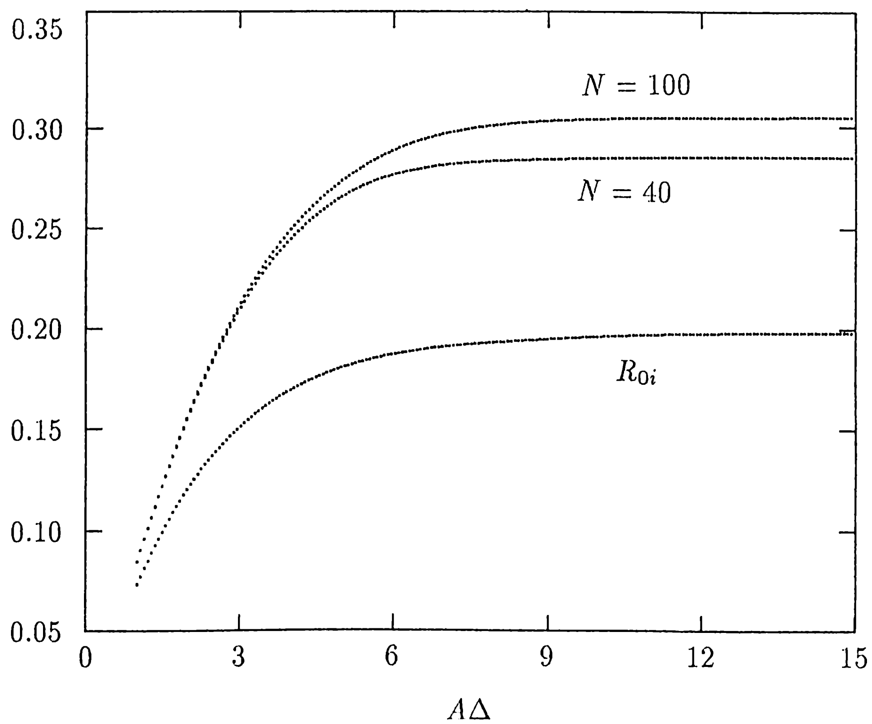

Figure 2.44: R_{0fc} for $N = 100, 40$ and R_{0i}, $\lambda_0 = 0$, $p = 0.10$

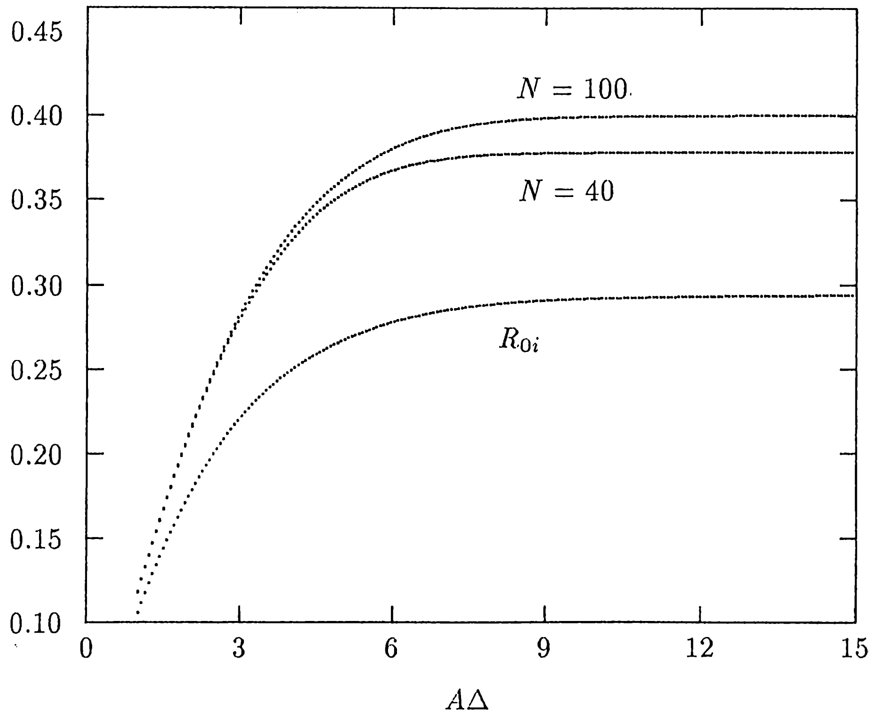

Figure 2.45: R_{0fc} for $N = 100, 40$ and R_{0i}, $\lambda_0 = 0$, $p = 0.15$

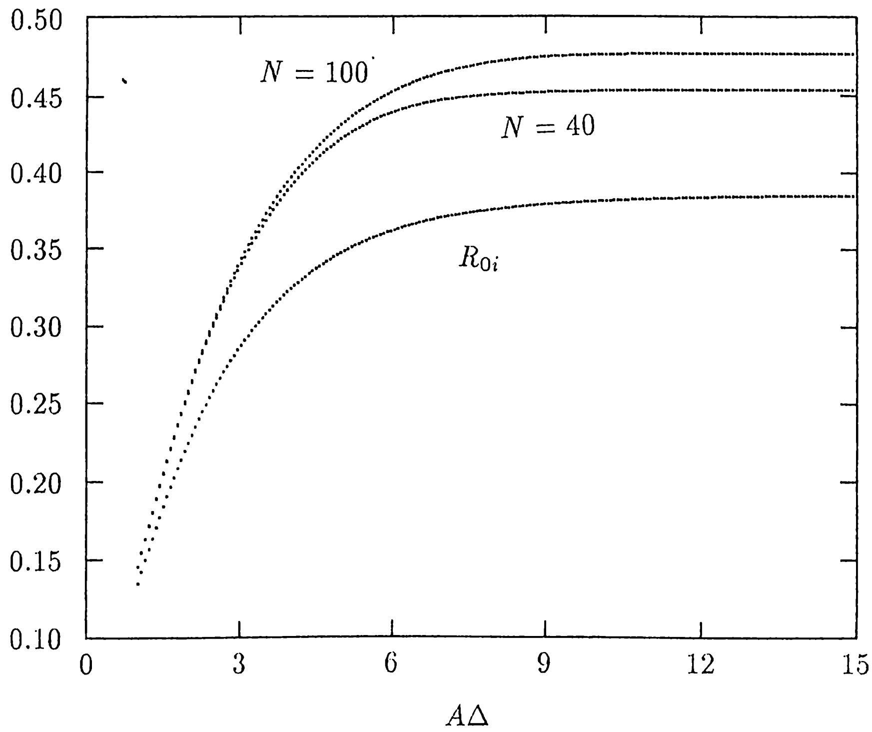

Figure 2.46: R_{0fc} for $N = 100, 40$ and R_{0i}, $\lambda_0 = 0$, $p = 0.20$

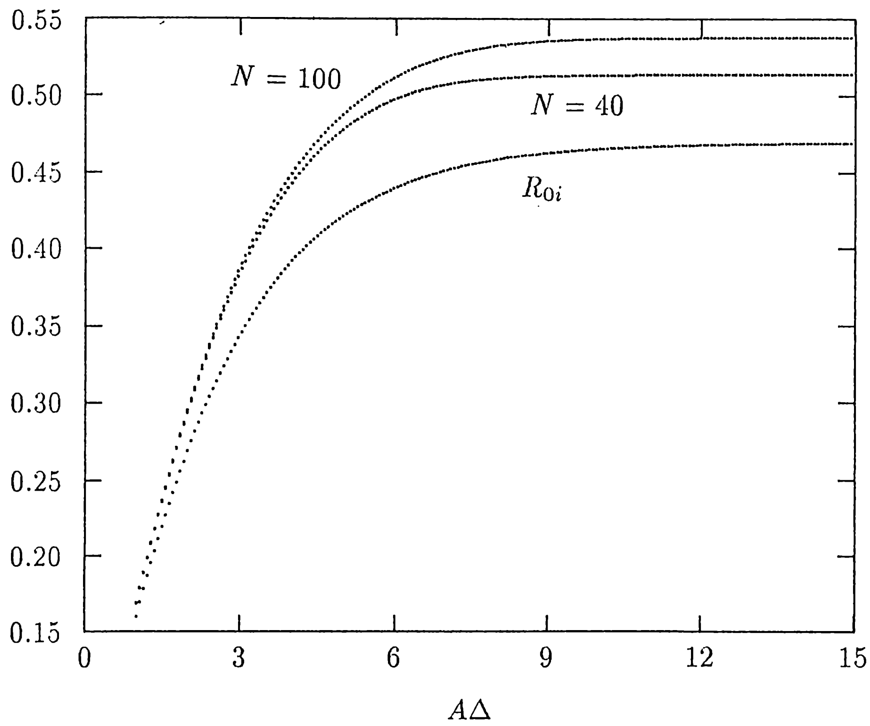

Figure 2.47: R_{0fc} for $N = 100, 40$ and R_{0i}, $\lambda_0 = 0$, $p = 0.25$

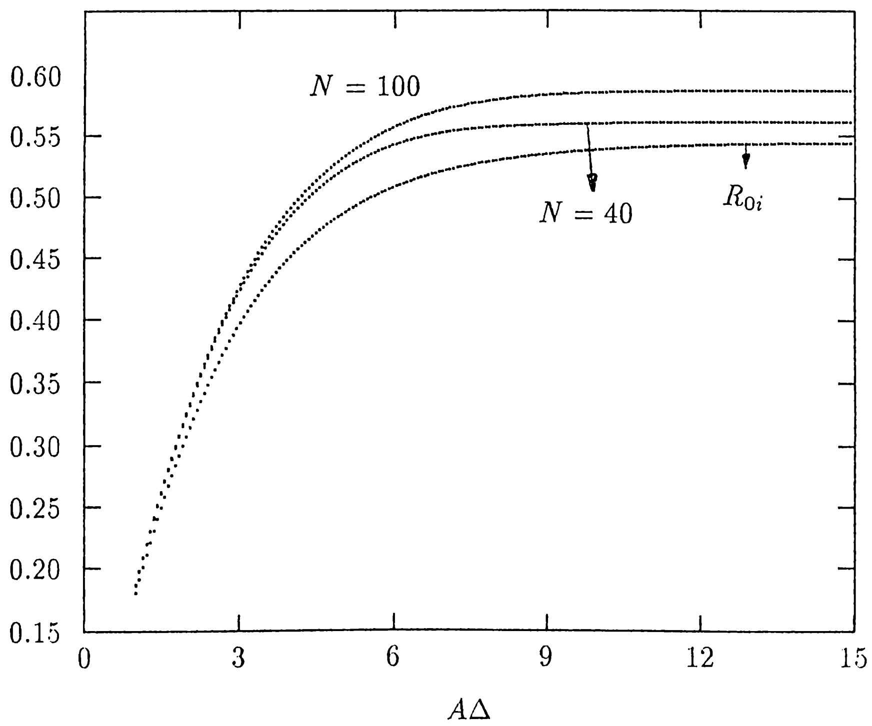

Figure 2.48: R_{0fc} for $N = 100, 40$ and R_{0i}, $\lambda_0 = 0$, $p = 0.30$

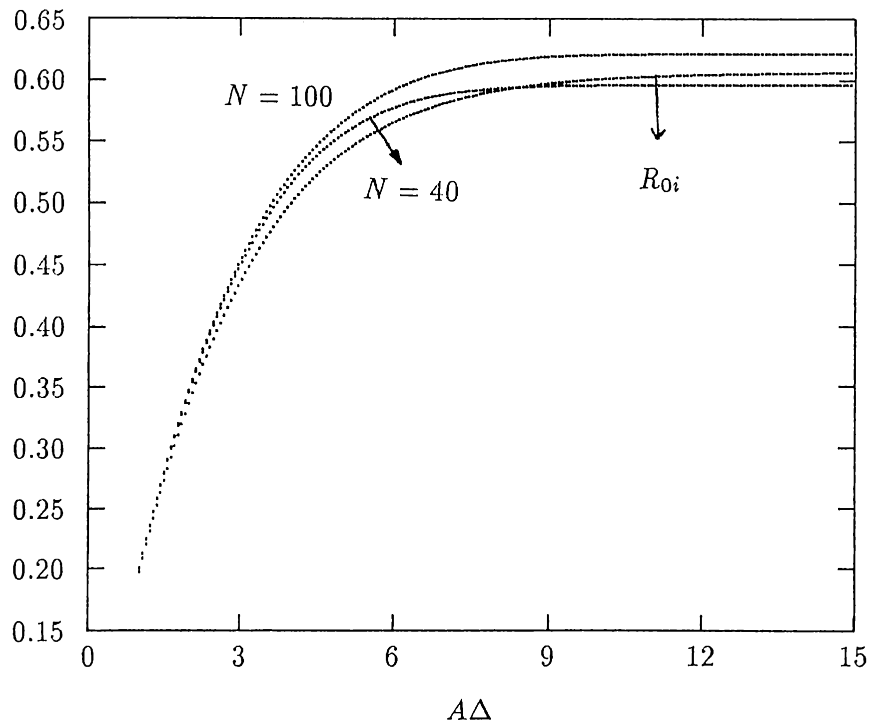

Figure 2.49: R_{0fc} for $N = 100, 40$ and R_{0i}, $\lambda_0 = 0$, $p = 0.35$

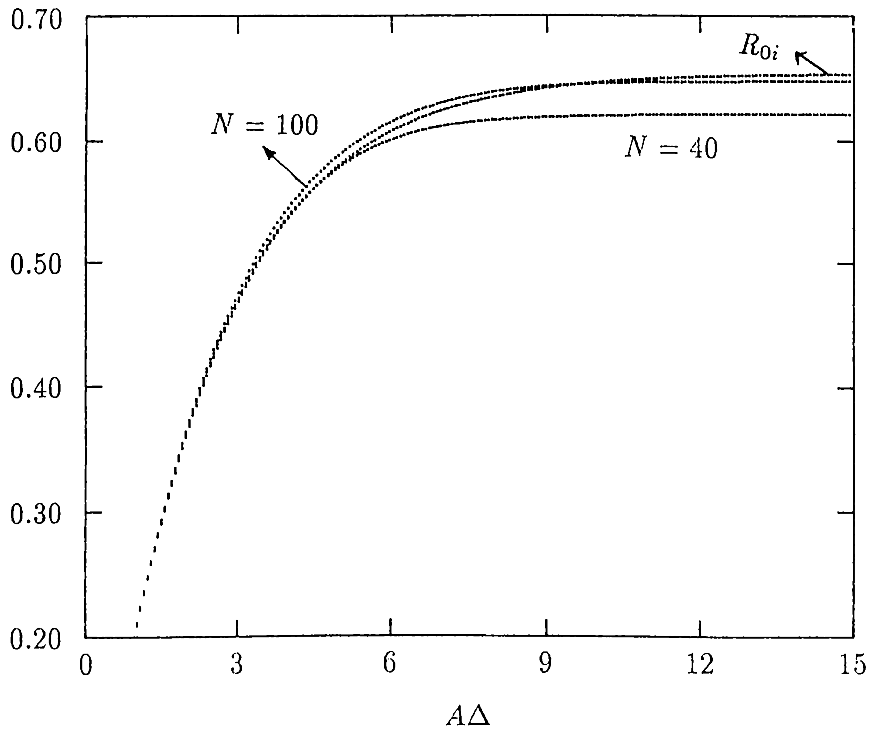

Figure 2.50: R_{0fc} for $N = 100, 40$ and R_{0i}, $\lambda_0 = 0$, $p = 0.40$

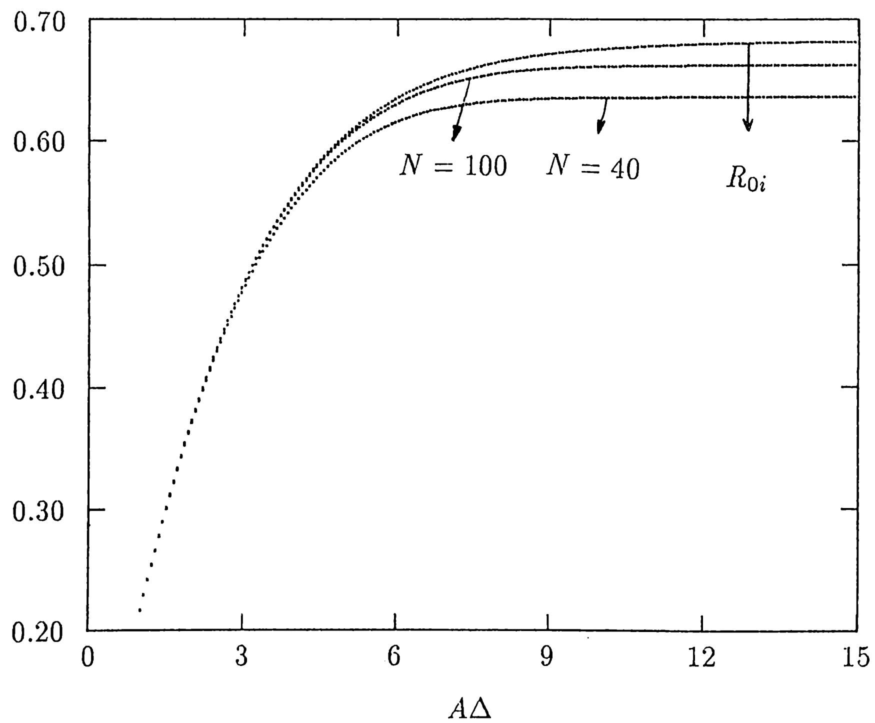

Figure 2.51: R_{0fc} for $N = 100, 40$ and R_{0i}, $\lambda_0 = 0$, $p = 0.45$

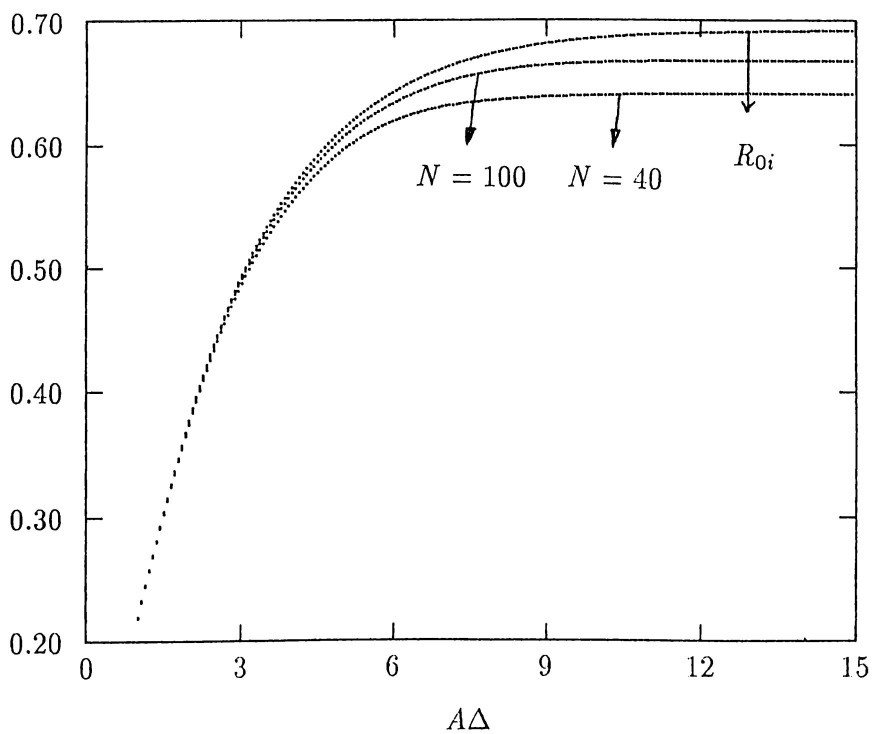

Figure 2.52: R_{0fc} for $N = 100, 40$ and R_{0i}, $\lambda_0 = 0$, $p = 0.50$

2.4 Bounds for Cutoff Rates

An upper bound for R_{0i} given by Equation 2.5 is

$$R_{0i} = -\ln(p^2 + (1-p)^2 + 2p(1-p)z) \leq \mathcal{H}(p) \quad \text{for} \quad 0 \leq z \leq 1 \tag{2.9}$$

where

$$\mathcal{H}(p) = -p\ln p - (1-p)\ln(1-p) \tag{2.10}$$

and z is given in Equation 2.6.
This bounding inequality is illustrated in Figure 2.53

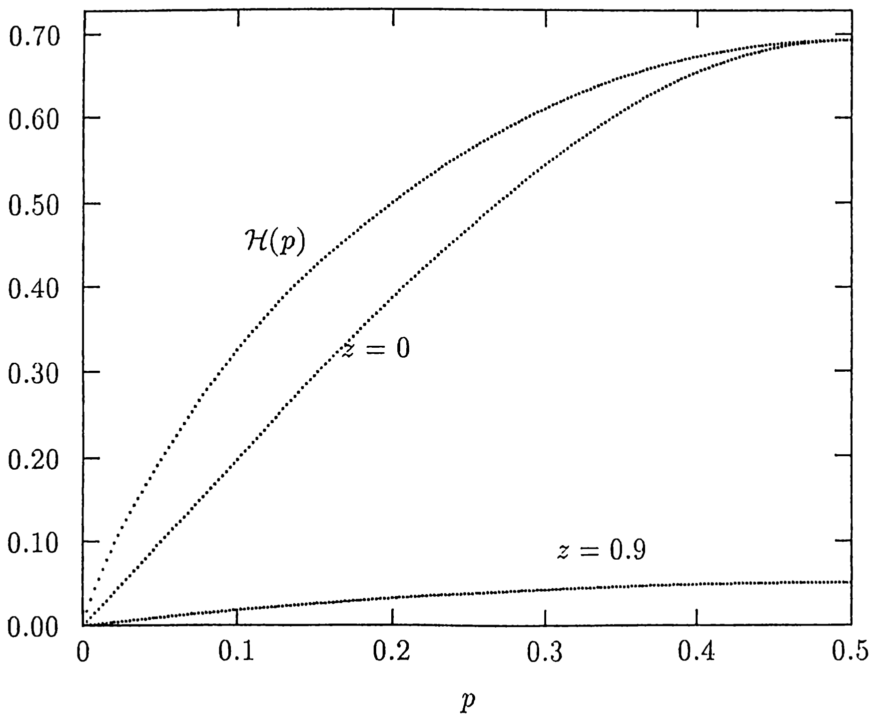

Figure 2.53: $\mathcal{H}(p)$ and R_{0i}, for $z = 0$ $z = 0.9$

We can also prove this fact as follows:
It is shown in Appendix that for any $a, b > 0$ and $0 \leq \alpha \leq 1$

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b \tag{2.11}$$

For $p = 0$ it is obvious that[2] $p^p(1-p)^{1-p} = p^2 + (1-p)^2$. For $0 < p \leq \frac{1}{2}$ substituting the values $\alpha = a = p$, $b = 1 - \alpha = 1 - p$ into Equation 2.11, we obtain

$$p^p(1-p)^{1-p} \leq p^2 + (1-p)^2$$

[2]Note that $\lim_{p\to 0} p^p = 1$

Taking the natural logarithm of both sides of this final inequality we get

$$\ln(p^p(1-p)^{1-p}) \le \ln(p^2+(1-p)^2) \le \ln(p^2+(1-p)^2+2p(1-p)z) \quad 0 \le z \le 1$$

which is nothing but Equation 2.9.

For $R_{0fc}^{(N)}$ lower and upper bounds can be derived. In order to find an upper bound for $R_{0fc}^{(N)}$ we can proceed by rewriting Equation 2.7

$$\begin{aligned}
e^{-R_{0fc}^{(N)}} &= \left(\frac{1}{\binom{N}{K}} \sum_{i=0}^{K} \binom{K}{i} \binom{N-K}{i} z^{2i} \right)^{\frac{1}{N}} \qquad K = pN \\
&= \left(\frac{1}{\binom{N}{pN}} \sum_{i=0}^{pN} \binom{pN}{i} \binom{(1-p)N}{i} z^{2i} \right)^{\frac{1}{N}}
\end{aligned}$$

$$\begin{aligned}
\binom{(1-p)N}{i} &\ge \frac{((1-p)N-i)^i}{i!} \\
&\ge \left(\frac{(1-2p)N}{pN}\right)^i = \left(\frac{1-2p}{p}\right)^i \quad \text{for } i \le pN \\
e^{-R_{0fc}^{(N)}} &\ge \left(\frac{1}{\binom{N}{pN}} \sum_{i=0}^{pN} \binom{pN}{i} \left(\frac{1-2p}{p} z^2\right)^i \right)^{\frac{1}{N}}
\end{aligned}$$

For $\binom{N}{pN}$ following inequalities hold [4, p.530]

$$\frac{1}{\sqrt{2N}} e^{N\mathcal{H}(p)} \le \binom{N}{pN} \le e^{N\mathcal{H}(p)}$$

where $\mathcal{H}(p)$ is defined by Equation 2.10. Since

$$\left[\binom{N}{pN}\right]^{-\frac{1}{N}} \ge e^{-\mathcal{H}(p)}$$

we can write

$$e^{-R_{0fc}^{(N)}} \ge \left(\sum_{i=0}^{pN} \binom{pN}{i} \left(\frac{1-2p}{p} z^2\right)^i \right)^{\frac{1}{N}} e^{-\mathcal{H}(p)}$$

Using the well known binomial identity

$$\sum_{i=0}^{pN} \binom{pN}{i} \left(\frac{1-2p}{p} z^2\right)^i = \left(1 + \frac{1-2p}{p} z^2\right)^{pN}$$

we obtain

$$e^{-R_{0fc}^{(N)}} \geq \left(1 + \frac{1-2p}{p} z^2\right)^{pN/N} e^{-\mathcal{H}(p)}$$

Hence, an upper bound for $R_{0fc}^{(N)}$ is found as follows:

$$R_{0fc}^{(N)} \leq \mathcal{H}(p) - p \ln\left(1 + \frac{1-2p}{p} z^2\right) \tag{2.12}$$

For obtaining lower bounds to $R_{0fc}^{(N)}$ we can proceed as follows: Using the inequality

$$\binom{(1-p)N}{i} \leq ((1-p)N)^i$$

we can write

$$\begin{aligned} e^{-R_{0fc}^{(N)}} &\leq \left(\sum_{i=0}^{pN} \binom{pN}{i} ((1-p)Nz^2)^i\right)^{\frac{1}{N}} (2N)^{1/2N} e^{-N\mathcal{H}(p)} \\ &\leq 1.06(1 + (1-p)Nz^2)^p e^{-N\mathcal{H}(p)} \quad N \geq 40 \end{aligned}$$

This gives us the following lower bound on $R_{0fc}^{(N)}$

$$R_{0fc}^{(N)} \geq \mathcal{H}(p) - p \ln(1 + (1-p)Nz^2) - \ln(1.06) \tag{2.13}$$

Another lower bound can be found [4, p.530] by noting that

$$\begin{aligned} \binom{(1-p)N}{i} &\leq \sqrt{\frac{(1-p)N}{2\pi((1-p)N - i)}} e^{(1-p)N\mathcal{H}\left(\frac{i}{(1-p)N}\right)} \\ &\leq \sqrt{\frac{(1-p)N}{2\pi(1-2p)N}} e^{(1-p)N\mathcal{H}\left(\frac{p}{1-p}\right)} \end{aligned}$$

Observing that $\frac{1-p}{2\pi(1-2p)} < 1$ if $p < 0.45$ and for large N $\sqrt{\frac{1-p}{2\pi(1-2p)}}^{1/N} \to 1$ we obtain

$$e^{-R_{0fc}^{(N)}} \leq \frac{e^{(1-p)\mathcal{H}\left(\frac{p}{1-p}\right)}(1+z^2)^p}{e^{\mathcal{H}(p)}}$$

So, an alternate lower bound for $R_{0fc}^{(N)}$ is

$$\mathcal{H}(p) - (1-p)\mathcal{H}\left(\frac{p}{1-p}\right) - p\ln(1+z^2) \leq R_{0fc}^{(N)} \tag{2.14}$$

Letting

$$B1 = \mathcal{H}(p) - p\ln(1+(1-p)Nz^2) - \ln(1.06))$$

and

$$B2 = \mathcal{H}(p) - (1-p)\mathcal{H}\left(\frac{p}{1-p}\right) - p\ln(1+z^2)$$

as a result it can be written that

$$\max(B1, B2) \leq R_{0fc}^{(N)} \leq \mathcal{H}(p) - p\ln\left(1 + \frac{1-2p}{p}z^2\right).$$

For some cases the lower and upper bounds to R_{0fc}, and R_{0i} versus p are given in Figures 2.54 to 2.59.

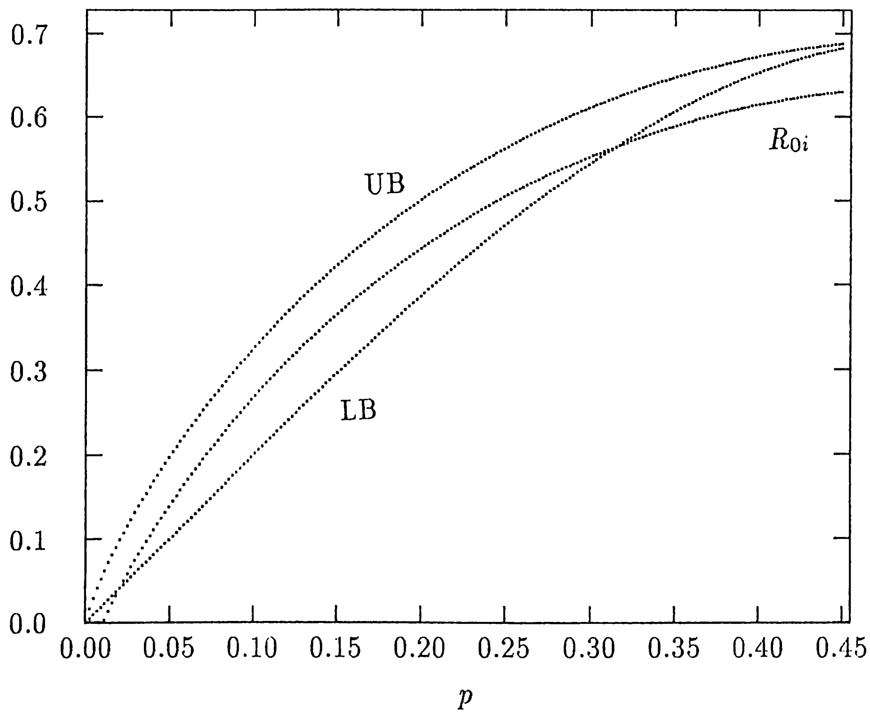

Figure 2.54: Upper and Lower Bounds for R_{0fc} and R_{0i}, $A\Delta = 15$, $\lambda_0 = 0$

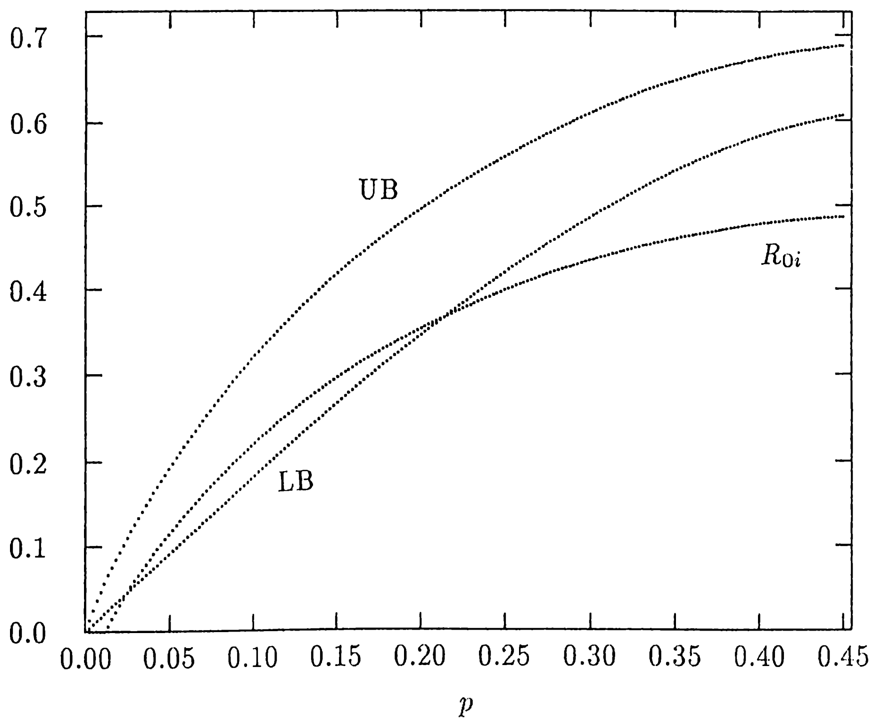

Figure 2.55: Upper and Lower Bounds for R_{0fc} and R_{0i}, $A\Delta = 5$, $\lambda_0 = 0$

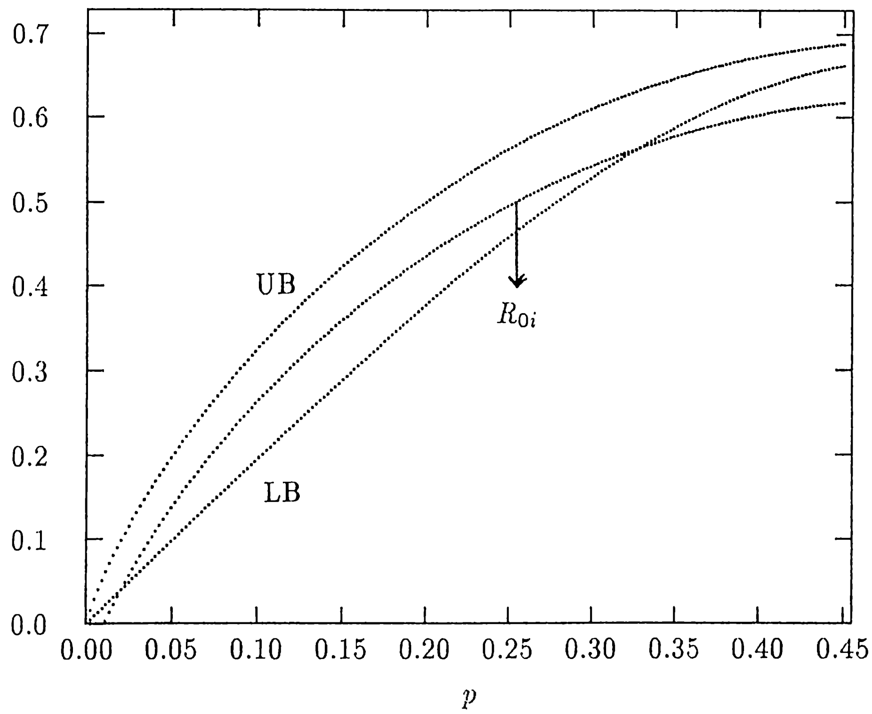

Figure 2.56: Upper and Lower Bounds for R_{0fc} and R_{0i}, $A\Delta = 50$, $SNR = 2$

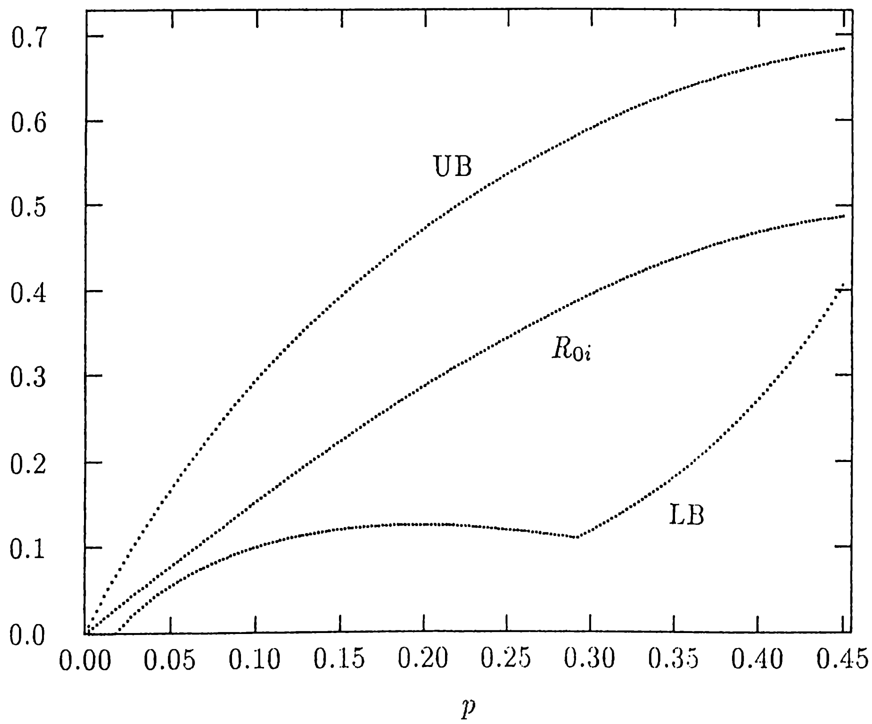

Figure 2.57: Upper and Lower Bounds for R_{0fc} and R_{0i}, $A\Delta = 50$, $SNR = 0.5$

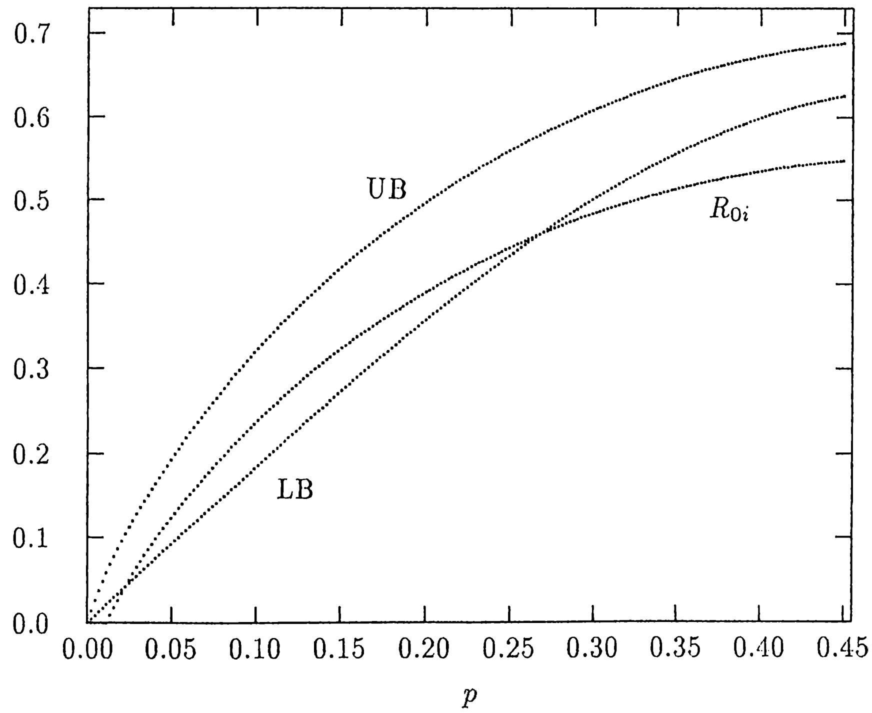

Figure 2.58: Upper and Lower Bounds for R_{0fc} and R_{0i}, $A\Delta = 10$, $SNR = 200$

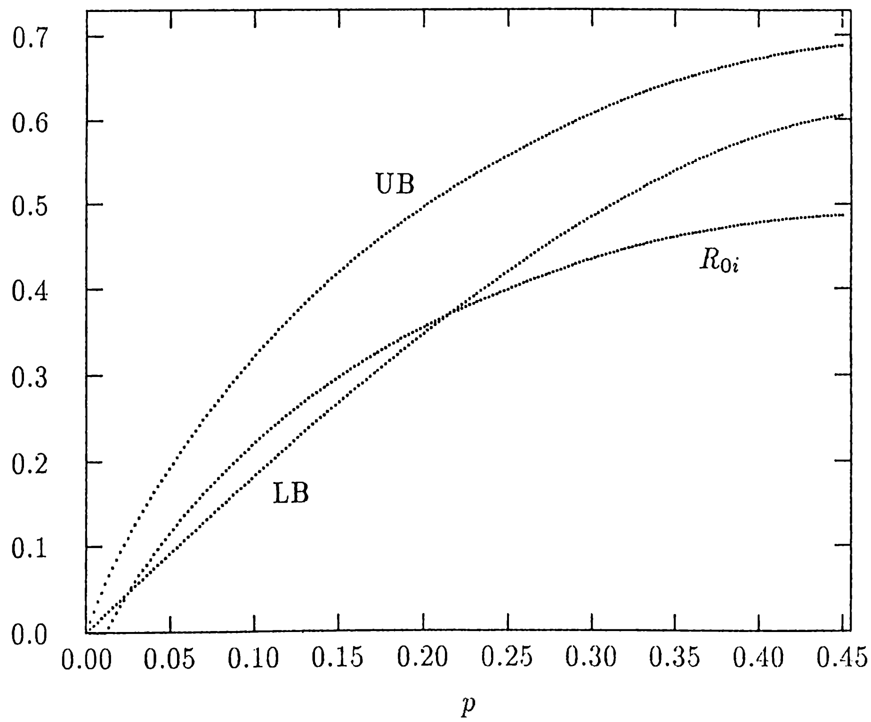

Figure 2.59: Upper and Lower Bounds for R_{0fc} and R_{0i}, $A\Delta = 10$, $SNR = 50$

Chapter 3

CONCLUSION

As stated in Chapter 2, for the direct detection photon channel, fixed-composition codes exhibit better performance than independent-letters codes and we obtain a considerable improvement in cutoff rate. Also, the practical implementation is possible since Arıkan [14] has recently proposed a method for constructing fixed-composition trellis codes with smallest possible degree which is independent of the blocklength.

In this work, we considered only ON-OFF keying in which letters are chosen from a binary alphabet. It may be of interest to study multi-level signalling. If $\mathcal{A} = \{A_1, A_2, ..., A_L\}$ is an alphabet of size L with probability distribution $\{p_1, p_2, ..., p_L\}$ then one should solve the problem of the optimization of R_0 under the constraints:

$$\sum p_i = 1 \ , \quad \sum p_i A_i = \text{constant}$$

which is a formidable problem.

Also, the sequential decoding of fixed-composition codes needs to be investigated further as pointed out in [14]. Namely, the problem stems from the memory introduced by the fixed-composition constraint; hence, optimum metrics for sequential decoding require excessive computation. However, trellis coding and sequential decoding part of the problem are left beyond the scope of this thesis work.

Appendix

Proposition For any $a, b > 0$ and $0 \leq \alpha \leq 1$

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \tag{3.1}$$

Proof

For $\alpha = 1$ and $\alpha = 0$ the above inequality holds with equality, so assume that $0 < \alpha < 1$. For $t \geq 0$ define $\Psi(t) = 1 - \alpha + \alpha t - t^{\alpha}$ then,

$$\Psi'(t) = \alpha - \alpha t^{\alpha-1} = \alpha \left(1 - \frac{1}{t^{1-\alpha}}\right) = 0 \;\; \Rightarrow \;\; t = 1$$

For $0 \leq t < 1$ $\Psi'(t) < 0$ and for $t > 1$ $\Psi'(t) > 0$. Therefore $\Psi(t)$ has its minimum at $t = 1$, hence

$$\forall t \geq 0 \;\; \Psi(t) \geq \Psi(1) \;\; \Rightarrow \;\; \forall t \geq 0 \;\; 1 - \alpha + \alpha t - t^{\alpha} \geq 0$$

Substituting $t = \frac{a}{b}$ we obtain

$$\frac{a^{\alpha}}{b^{\alpha}} \leq 1 - \alpha + \alpha \frac{a}{b} \text{ multiplying by } b > 0 \quad a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b$$

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