QUANTUM PROPERTIES OF MULTIPOLE RADIATION

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July 2002

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Abstract

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In this study, multipole expansion of quantum electromagnetic radiation is constructed and quantized by canonical transformation with increasing demand of some modern research areas of physics such as entanglement of the orbital angular momentum states, novel experiments with trapped atoms, and the atomic and molecular transitions with given angular momentum.

Also, the $SU(2)$ invariance of quantum field and the rotational symmetry of vacuum noise of polarization with respect to source location are proved.

It is shown that, at any point we can construct a proper frame in which the description of polarization is reduced to a conventional (2×2) polarization matrix. And peculiarities of electric and magnetic-type zero-point oscillations were examined, and as a result it is shown that the monochromatic zero-point oscillations of all types and modes, have constant level in the volume of quantization.

Finally, the complete local representation of photon operators, which correspond to the states of photons with given projection of angular momentum at any point, is constructed for the possible utility of near-field optics.

Keywords: Quantum multipole radiation, Polarization matrix, SU(2) invariance, Local polarization matrix, Zero point oscillations, Stokes parameters, Local photon operators.

Özet

ÇOK KUTUPLU IŞINIMIN KUVANTUM ÖZELLİKLERİ

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Temmuz 2002

Bu çalışmada, fiziğin, yörüngesel açısal momentum hallerinin dolaşıklığı, tuzağa düşürülmüş atomlarla yapılan son deneyler ve belirli açısal momentumlarda atomik, moleküler geçişler gibi bazı modern araştırma alanlarına paralel gelişim içinde, kuantum elektromanyetik ışınımın çok kutuplu formülasyonu oluşturulmuş ve kuvantize edilmiştir.

Ayrıca, kaynak konumuna bağlı olarak kuvantum alanının $SU(2)$ ve polarizasyon vakum gürültüsünün rotasyonal simetri değişmezlikleri gösterilmiştir.

Herhangi bir noktada polarizasyonun uygun bir referans sistemi seçimi ile geleneksel (2×2) polarizasyon matrisine dönüştüğü gösterilmiştir. Elektrik ve manyetik sıfır noktası osilasyonlarının tuhaflıkları incelenmiş ve sonuç olarak bütün modlar ve çeşitler için, kuvantize edilen hacim içinde, monokrom sıfır noktası osilasyonlarının sabit bir seviyesi olduğu gösterilmistir.

Son olarak da, yakın-alan optiğinin yararına olacağına inanılan, herhangi bir noktada verilen açısal momentuma sahip foton hallerine karşılık gelen, fotonların lokal betimlemesi oluşturulmuştur.

Anahtar sözcükler: Çok kutuplu kuvantum ışınımı, polarizasyon matrisi, SU(2) değişmezliği, lokal polarizasyon matrisi, sıfır noktası osilasyonları, Stokes parametreleri, lokal foton operatörleri.

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Chapter 1

Introduction

Conventional picture of quantum electromagnetic radiation is based on the use of expansion of the so-called free electromagnetic field over plane waves. At the same time, it is well known that this representation is not in a class by itself.[1] For example, the quantization in terms of spherical and cylindrical waves of photons can also be considered.[1–4]

There is a number of principle reasons to consider the representation of multipole photons based on the expansion of electromagnetic field over the spherical waves. First, the atomic and molecular transitions between the states with given angular momentum and projection of the angular momentum emit just the multipole photons.[5] In the second place, in many modern experiments with trapped atoms, the interatomic distances correspond to the intermediate or even near zone,[6,7] which requires the use of spherical rather than plane wave expansion for photons. Besides that, the problem of entanglement of the orbital angular momentum states[8] also implies the use of the representation of multipole photons.

There is no principle difference between the expansions over plane and spherical waves within the classical domain because both procedures are based on the use of the complete orthonormal set of functions of space and time.[9] However, according to one of the basic concepts of quantum mechanics has been introduced by Wigner,[11] the general properties of an arbitrary quantum mechanical system

are defined by the group of dynamical symmetry of the corresponding Hilbert space. From this point of view, the representations of plane and spherical photons are different in quantum domain. The former corresponds to the translational symmetry caused by the solutions of the homogeneous wave equation in a cube with the periodical boundary conditions, while the latter obeys the rotational symmetry with respect to the origin, which corresponds to the source location from the physical point of view. The translational symmetry corresponds to the states of photons with given linear momentum, while the rotational symmetry specifies the states with given angular momentum. Thus, the two representations correspond to the observables that cannot be measured at once, in spite of equivalence of classical expansions.

It is possible to say that the representation of multipole photons takes into account the real geometry of physical space provided by the presence of a local source at the origin. In turn, the representation of plane photons completely ignores the existence of a source.

In the case of plane waves of photons, an arbitrary state of the field is specified by a given energy (wave number k), direction of propagation ($\vec{k}/k$), and polarization μ, which can take only two values. In turn, the spherical waves of photons are described by the states with given energy, parity (the type of the multipole radiation $\lambda = E, M$ either electric or magnetic), angular momentum ($\ell \geq 1$), and projection of the angular momentum on the quantization axis ($m = -\ell, \cdots, \ell$).[1,2] Let us stress that, in contrast to the plane photons, the polarization of the multipole photons is not a global property of the field but changes from point to point.[12,13] The reason is that the polarization is defined to be a given spin state of photons,[3] while the quantum multipole field is specified by the total angular momentum which contains the orbital part in addition to the spin. Some properties of quantum multipole radiation have been considered recently in a number of papers.[14–16]

In chapter II, some basic properties of quantum multipole field will be discussed with brief summaries of classical and quantum electromagnetic field. In particular, the $SU(2)$ invariance of the operator vector potential will be proven.

In chapter III, the definition of polarization of quantum multipole radiation will be given by defining polarization matrix and corresponding Stokes parameters. Since the polarization is not a global characteristics of multipole field, the definition depends on the choice of the reference frame. In particular, in the "laboratory frame" connected with the source location, the three independent polarizations of multipole field should be considered. A more habitual picture of only two polarizations can also be constructed through the use of a certain local frame.

In chapter IV, the vacuum polarization matrix will be constructed by the aid of Weyl-Heisenberg algebra. Because of the invariance of operator vector potential under rotation proven in previous chapter, the local unitary transformation can be obtained. Also zero-point oscillations and quantum noise near source will be indicated.

Then, in chapter V, the problem of photon localization in terms of the local photon operators with given polarization will be examined and corresponding bare Stokes operators will be compared with the classical plane wave case.

Finally, in chapter VI, the obtained results will be discussed.

Chapter 2

Properties of Quantum Multipole Radiation

The central role of light in marking the frontiers of physics continues on into the twentieth century with the ultraviolet catastrophe associated with black-body radiation on the one hand and the photoelectric effect on the other. Indeed, it was here that the era of quantum mechanics was initiated Planck's introduction of the quantum of action that was necessary to explain the black-body radiation spectrum. However, it was left to Dirac to combine the wave and particle nature of light, so that the radiation field is capable of explaining all interference phenomena and shows the excitation of a specific atom located along a wave front absorbing one photon energy. Each mode of radiation field is associated with a quantized simple harmonic oscillator, this is the essence of quantum theory of radiation.

In this chapter, with the objective of quantizing the electromagnetic field in free space, it is convenient to begin with the classical description of the field based on Maxwell's equations in terms of vector potential. Then multipole expansion of electromagnetic field can be described in terms of spherical harmonics. Finally, one can use the quantum picture of electromagnetic field and second quantization becomes applicable as changing the complex field amplitudes with creation and annihilation operators by canonical transformation.

2.1 Classical Electromagnetic Field

An arbitrary classical free electromagnetic field is described by the vector potential $\mathcal{A}(\vec{r})$, which obeys the wave equation[9,10]

$$\nabla^2 \vec{\mathcal{A}} - \frac{1}{c^2}\frac{\partial^2 \vec{\mathcal{A}}}{\partial t^2} = 0 \tag{2.1}$$

and Coulomb gauge condition

$$\vec{\nabla} \cdot \vec{\mathcal{A}} = 0 \tag{2.2}$$

Then the field strengths are defined as follows:

$$\vec{\mathcal{E}} = -\frac{1}{c}\frac{\partial \vec{\mathcal{A}}}{\partial t}, \qquad \vec{\mathcal{B}} = \vec{\nabla} \times \mathcal{A} \tag{2.3}$$

Eq. 2.1 can be solved by seperation of variables[10]

$$\vec{\mathcal{A}}(\vec{r}, t) = \sum_{\ell} q_\ell(t)\vec{u}_\ell(\vec{r}) \tag{2.4}$$

Employing into Eq. 2.1 then yields the homogeneous Helmholtz wave equations of the form

$$\begin{aligned} \frac{d^2 q_\ell}{dt^2} + \omega_\ell^2 q_\ell &= 0 \\ \nabla^2 \vec{u}_\ell + \frac{\omega_\ell^2}{c^2}\vec{u}_\ell &= 0 \end{aligned} \tag{2.5}$$

where ω_ℓ are some constants. arising from the seperation of variables.[10] Solution of the first equation in 2.5 gives the harmonic time dependence $q_\ell = \exp(\pm i\omega_\ell t)$. Because of the harmonic time dependence in Eq. 2.4, it is customary to represent the vector potential in terms of the positive and negative frequency parts:

$$\vec{\mathcal{A}}(\vec{r}) = \vec{A}(\vec{r}) + \vec{A}^*(\vec{r}) \tag{2.6}$$

where $\vec{A} \sim \exp(-i\omega t)$. The energy density of the field is

$$W(\vec{r}) = \frac{1}{16\pi}[\vec{\mathcal{E}}^*(\vec{r}) \cdot \vec{\mathcal{E}}(\vec{r}) + \vec{\mathcal{B}}^*(\vec{r}) \cdot \vec{\mathcal{B}}(\vec{r})] \tag{2.7}$$

In turn, the flux of the energy is given by the real part of the complex Poynting vector

$$\vec{S}(\vec{r}) = \frac{1}{8\pi}\vec{E}(\vec{r}) \times \vec{B}^*(\vec{r}) \tag{2.8}$$

where, according to Eq. 2.3, we obtain

$$\vec{E}(\vec{r}) = -ik\vec{A}(\vec{r}), \qquad \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$$

The angular momentum density of the field has the form

$$\vec{M}(\vec{r}) = \frac{1}{4\pi c}\vec{r} \times [\vec{\mathcal{E}}(\vec{r}) \times \vec{\mathcal{B}}(\vec{r})] \tag{2.9}$$

One possible solution of the second equation in 2.5, corresponding to the plane waves, traveling along the z-axis and having the same amplitude and phase everywhere, has the form[9,10]

$$\vec{u}_\ell(\vec{r}) = \sum_\ell \sum_{\sigma=x,y} \vec{e}_{\ell\sigma} e^{i\vec{k}_\ell \cdot \vec{r}} a_{\ell\sigma} + c.c. \tag{2.10}$$

where c.c. denotes the complex conjugate of first part. Here $a_{\ell\sigma}$ are the complex field amplitudes, $\vec{e}_{x,y}$ are the unit vectors of polarization which, due to the Coulomb gauge condition 2.2, obey the relation

$$\forall \ell \qquad \vec{e}_{x,y} \cdot \vec{k}_\ell = 0 \tag{2.11}$$

and $k_\ell^2 = \nabla_\ell^2/c^2$. Employing Eqs. 2.3,2.6, and 2.10 then gives the following symmetry relations

$$\begin{aligned} E_x(\vec{r}) &= i\sum_k kA_{kx}(\vec{r}) = B_y(\vec{r}) \\ E_y(\vec{r}) &= i\sum_k kA_{ky}(\vec{r}) = B_x(\vec{r}) \end{aligned} \tag{2.12}$$

To simplify the notations, we omit the index ℓ here. According to Eq. 2.10, we have

$$A(\vec{r}) = \sum_k \gamma_k \sum_{\sigma=x,y} \vec{e}_{k\sigma} e^{i\vec{k}\cdot\vec{r}} a_{k\sigma} e^{-i\omega t} \tag{2.13}$$

where γ_k is the normalization factor. Another possible solution of the homogeneous Helmholtz wave equation 2.5 convenient for electromagnetic boundary-value problems possessing spherical properties is provided by the spherical waves.[10,23]

2.2 Multipole Expansion

For the time dependence $e^{-i\omega t}$ the Maxwell equations in a source-free region of empty space may be written

$$\begin{aligned} \vec{\nabla} \times \vec{E} = & \ ik\vec{H}, & \vec{\nabla} \times \vec{H} = & \ -ik\vec{E} \\ \vec{\nabla} \cdot \vec{E} = & \ 0, & \vec{\nabla} \cdot \vec{H} = & \ 0 \end{aligned} \tag{2.14}$$

where $k = \omega/c$. If $\vec{E}$ is eliminated by combining the two curl equations, we obtain for $\vec{H}$,

$$(\nabla^2 + k^2)\vec{H} = 0, \qquad \vec{\nabla} \cdot \vec{H} = 0 \tag{2.15}$$

with $\vec{E}$ given by

$$\vec{E} = \frac{i}{k}\vec{\nabla} \times \vec{H} \tag{2.16}$$

Alternatively, $\vec{H}$ can be eliminated to yield

$$(\nabla^2 + k^2)\vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{E} = 0 \tag{2.17}$$

with $\vec{H}$ given by

$$\vec{H} = -\frac{i}{k}\vec{\nabla} \times \vec{E} \tag{2.18}$$

Either Eqs. 2.15, 2.16 or 2.17, 2.18 is a set of three equations that is equivalent to the Maxwell equations 2.14.From these equations it is clear that each *Cartesian* component of $\vec{E}$ and $\vec{H}$ satisfies the Helmholtz wave equation

$$(\nabla^2 + k^2)\psi(\vec{x}, \omega) = 0. \tag{2.19}$$

Hence each such component can be written as an expansion of the general form in spherical coordinates as

$$\psi(\vec{x}) = \sum_{l,m} [A^{(1)}_{lm} h^{(1)}_{lm}(kr) + A^{(2)}_{lm} h^{(2)}_{lm}(kr)] Y_{lm}(\theta, \phi). \tag{2.20}$$

There remains the problem of orchestrating the different components in order to satisfy $\vec{\nabla} \cdot \vec{H} = 0$ and $\vec{\nabla} \cdot \vec{E} = 0$ and to give a pure multipole field of order (l, m).

However, a different and somewhat easier path was suggested by Bouwkamp and Casimir.[27] Consider the scalar quantity $\vec{r}\cdot\vec{A}$, where $\vec{A}$ is a well-behaved vector field. It is straightforward to verify that the Laplacian operator acting on this scalar gives

$$\nabla^2(\vec{r}\cdot\vec{A}) = \vec{r}\cdot(\nabla^2\vec{A}) + 2\vec{\nabla}\cdot\vec{A} \tag{2.21}$$

From Eqs. 2.15, 2.16 and Eqs. 2.17, 2.18, it is evident that the scalars, $\vec{r}\cdot\vec{E}$ and $\vec{r}\cdot\vec{H}$, both satisfy the Helmholtz wave equation:

$$(\nabla^2 + k^2)(\vec{r}\cdot\vec{E}) = 0, \qquad (\nabla^2 + k^2)(\vec{r}\cdot\vec{H}) = 0 \tag{2.22}$$

The general solution for $\vec{r}\cdot\vec{E}$ is given by Eq. 2.20, and similarly for $\vec{r}\cdot\vec{H}$.

Now *A magnetic multipole field of order* (l, m) can be defined by the conditions,

$$\begin{aligned} \vec{r}\cdot\vec{H}_{Mlm} &= \frac{l(l+1)}{k} g_l(kr) Y_{lm}(\theta,\phi) \\ \vec{r}\cdot\vec{E}_{Mlm} &= 0 \end{aligned} \tag{2.23}$$

where

$$g_l(kr) = A_l^{(1)} h_l^{(1)}(kr) + A_l^{(2)} h_l^{(2)}(kr) \tag{2.24}$$

The presence of the factor $l(l+1)/k$ is for normalization wonders. Using the curl equation 2.18 one can relate $\vec{r}\cdot\vec{H}$ to the electric field:

$$k\,\vec{r}\cdot\vec{H} = \frac{1}{i}\vec{r}\cdot(\vec{\nabla}\times\vec{E}) = \frac{1}{i}(\vec{r}\times\vec{\nabla})\cdot\vec{E} = \vec{L}\cdot\vec{E} \tag{2.25}$$

where $\vec{L}$ is given by $\vec{L} = -i(\vec{r}\times\vec{\nabla})$. With $\vec{r}\cdot\vec{H}$ given by Eq. 2.23, the electric field of the magnetic multipole must satisfy

$$\vec{L}\cdot\vec{E}_{Elm}(r,\theta,\phi) = l(l+1) g_l(kr) Y_{lm}(\theta,\phi) \tag{2.26}$$

and $\vec{r}\cdot\vec{E}_{Mlm} = 0$. To determine the purely transverse electric field from Eq. 2.26, note that the operator $\vec{L}$ acts only on angular variables (θ, phi), which means

that the radial dependence of E_{Mlm} must be given by $g_l(kr)$. On the other hand, the operator $\vec{L}$ acting on Y_{lm} transforms the m value according to

$$\begin{aligned} L_+ Y_{lm} &= \sqrt{(l-m)(l+m+1)}Y_{l,m+1} \\ L_- Y_{lm} &= \sqrt{(l+m)(l-m+1)}Y_{l,m-1} \\ L_z Y_{lm} &= mY_{lm} \end{aligned}$$

however, does not change the l value. Thus the components of E_{Mlm} can be at most linear combinations of Y_{lm}'s with different m values and a common l, equal to the l value on the right-hand side of Eq. 2.26. Therefore all shows that for $\vec{L} \cdot \vec{E}_{Mlm}$ to yield a *single* Y_{lm}, the components of E_{Mlm} must be prepared before-hand to compensate for whatever raising or lowering of m values is done by $\vec{L}$.Thus, in the term $L_- E_+$, for example, it must be that E_+ is proportional to $L_+ Y_{lm}$. What this amounts to is that the electric field should be

$$\begin{aligned} \vec{E}_{Mlm} &= g_l(kr)\vec{L}Y_{lm}(\theta,\phi) \\ \vec{H}_{Mlm} &= -\frac{i}{k}\vec{\nabla} \times \vec{E}_{Mlm} \end{aligned} \tag{2.27}$$

Equation 2.27 specifies the electromagnetic fields of a *magnetic* multipole of order (l, m). Because the electric field (Eq. 2.27) is transverse to the radius vector, these multipole fields are sometimes called *transverse electric(TE)* rather than magnetic.

The fields of an *electric* or *transverse magnetic(TM) multipole of order* (l, m) are specified similarly by the conditions,

$$\begin{aligned} \vec{r} \cdot \vec{E}_{Elm} &= -\frac{l(l+1)}{k} f_l(kr) Y_{lm}(\theta,\phi) \\ \vec{r} \cdot \vec{H}_{Elm} &= 0 \end{aligned} \tag{2.28}$$

Then the *electric* multipole fields are

$$\begin{aligned} \vec{H}_{Elm} &= f_l(kr)\vec{L}Y_{lm}(\theta,\phi) \\ \vec{E}_{Elm} &= \frac{i}{k}\vec{\nabla} \times \vec{H}_{Elm} \end{aligned} \tag{2.29}$$

The radial function $f_l(kr)$ is defined with an expression like (24).

The fields in Eq. 2.27 and Eq. 2.29 are in form of spherical wave expansion. These two sets of multipole fields can be shown to form a complete set of vector solutions to the Maxwell equations in a source-free region. Since the vector spherical harmonic, $\vec{L}Y_{lm}$, plays an important role, it is convenient to introduce the normalized form,

$$\vec{X}_{lm}(\theta,\phi) = \frac{1}{l(l+1)}\vec{L}Y_{lm}(\theta,\phi) \tag{2.30}$$

which is defined to be identically zero for $l = 0$. Spherically symmetric solutions to the source-free Maxwell's equations exist only in the static limit $k \to 0$. It has the following orthogonality properties,

$$\int \vec{X}^*_{l'm'} \cdot \vec{X}_{lm} d\Omega = \delta_{ll'}\delta_{mm'} \tag{2.31}$$

and

$$\int \vec{X}^*_{l'm'} \cdot (\vec{r} \times \vec{X}_{lm}) d\Omega = 0 \tag{2.32}$$

for all l, l', m, m'.

2.3 Quantum EM Field and Mode Functions

The canonical quantization of the field has introduced by Dirac[23,28,29] is provided by the substitution of the photon operators, forming a representation of the Weyl-Heisenberg algebra, into the expression for the vector potential instead of the complex field amplitudes. For positive-frequency part of the vector potential in the case of plane waves (Eq. 2.13), the following operator can be constructed

$$A(\vec{r}) = \sum_k \sum_{\sigma=x,y} \sqrt{\frac{2\pi\hbar c}{kV}} \vec{e}_{k\sigma} e^{i\vec{k}\cdot\vec{r}} a_{k\sigma} \tag{2.33}$$

where V is the volume of quantization, which is assumed to be a sufficiently large cubic box with periodical boundary conditions. The harmonic time dependence is included in the photon operators that obey the following commutation relations

$$[a_{k\sigma}, a^+_{k'\sigma'}] = \delta_{kk'}\delta_{\sigma\sigma'} \tag{2.34}$$

Because of the translational symmetry along the z-direction, the plane waves of photons, described by Eq. 2.33 and Eq. 2.34, correspond to the states of the radiation field with given linear momentum

$$\vec{P} = \sum_{k\sigma} \hbar \vec{k} a^{+}_{k\sigma} a_{k\sigma} \tag{2.35}$$

where $\vec{k} = k\vec{e}_z$.

The multipole electromagnetic field can be quantized in much the same way as plane waves.[1] Let us denote by $\vec{E}(\vec{r})$ the operator of the positive-frequency part of electric field strength defined at the point $\vec{r}$ and by $\vec{B}(\vec{r})$ that of magnetic induction. In the case of a monochromatic quantum multipole field, these operators are represented by the following expansions[1,2,4,9]

$$\begin{aligned} \vec{E}(\vec{r}) &= \sum_{\lambda,\ell,m} \vec{E}_{\lambda\ell m}(\vec{r}) a_{\lambda\ell m}, \\ \vec{B}(\vec{r}) &= \sum_{\lambda,\ell,m} \vec{B}_{\lambda\ell m}(\vec{r}) a_{\lambda\ell m}. \end{aligned} \tag{2.36}$$

Here $a_{\lambda\ell m}$ is the annihilation operator of multipole photon obeying the commutation relations

$$[a_{\lambda kjm}, a^{+}_{\lambda' k' j' m'}] = \delta_{\lambda\lambda'} \delta_{kk'} \delta_{jj'} \delta_{mm'} \tag{2.37}$$

As usually, the harmonic time dependence is included into the definition of the photon operator. The mode functions in Eq. 2.36 have the following form[1,9,10]

$$\begin{aligned} \vec{E}_{E\ell m}(\vec{r}) &= \frac{i}{k} \vec{\nabla} \times f_{E\ell}(kr) \vec{X}_{\ell m}(\theta, \phi), \\ \vec{E}_{M\ell m}(\vec{r}) &= f_{M\ell}(kr) \vec{X}_{\ell m}(\theta, \phi), \end{aligned} \tag{2.38}$$

where $\vec{X}_{\ell m}$ denotes the vector spherical harmonics

$$\vec{X}_{\ell m}(\theta, \phi) = \frac{-i}{\sqrt{\ell(\ell+1)}} (\vec{r} \times \vec{\nabla}) Y_{\ell m}(\theta, \phi) \tag{2.39}$$

while the radial dependence $f_{\lambda\ell}(r)$ is defined differently depending on the boundary conditions.[10] In the standard case of quantization of electromagnetic waves in a spherical cavity,[1] we have

$$f_{\lambda\ell}(x) = \gamma_{\lambda\ell} j_\ell(x) = \gamma_\lambda \sqrt{\frac{\pi}{2x}} J_{\ell+1/2}(x).$$

Here $j_\ell(x)$ is the spherical Bessel function and

$$\gamma_{E\ell} = \sqrt{\frac{2\pi\hbar c}{kV(2\ell+1)}}, \qquad \gamma_{M\ell} = \gamma_M = \sqrt{2\ell+1}\gamma_{E\ell}.$$

Here V is the volume of quantization. The mode functions specifying the magnetic induction in Eq. 2.36 can be determined through the use of the symmetry relations

$$\vec{B}_{E\ell m}(\vec{r}) = -\vec{E}_{M\ell m}(\vec{r}), \qquad \vec{B}_{M\ell m}(\vec{r}) = \vec{E}_{E\ell m}(\vec{r}).$$

The above expressions can be obtained from the following operator vector potential

$$\vec{A}_\lambda(\vec{r}) = \sum_{\mu=-1}^{1} (-1)^\mu \vec{\epsilon}_{-\mu} \sum_{\ell,m} V_{\lambda\ell m\mu}(\vec{r}) a_{\lambda\ell m} \tag{2.40}$$

through the use of relations

$$\vec{E}_\lambda(\vec{r}) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A}_\lambda(\vec{r}), \qquad \vec{B}_\lambda(\vec{r}) = \vec{\nabla} \times \vec{A}_\lambda(\vec{r}).$$

Here the unit vectors

$$\vec{\epsilon}_\pm = \mp\frac{\vec{\epsilon}_x \pm i\vec{\epsilon}_y}{\sqrt{2}}, \qquad \vec{\epsilon}_0 = \vec{\epsilon}_z$$

form the so-called helicity basis,[2]

$$\begin{aligned} V_{E\ell m\mu}(\vec{r}) &= \gamma_{E\ell}\{\sqrt{\ell} f_{\ell+1}(r)\langle 1, \ell+1, \mu, m-\mu|\ell m\rangle Y_{\ell+1,m-\mu}(\theta,\phi) \\ &- \sqrt{\ell+1} f_{\ell-1}(r)\langle 1, \ell-1, m-\mu|\ell m\rangle Y_{\ell-1.m-\mu}(\theta,\phi)\} \\ V_{M\ell m\mu}(\vec{r}) &= \gamma_{M\ell} f_\ell(r)\langle 1, \ell, \mu, m-\mu|\ell m\rangle Y_{\ell,m-\mu}(\theta,\phi), \end{aligned} \tag{2.41}$$

and $\langle\cdots|\cdots\rangle$ denotes the Clebsch-Gordon coefficients describing the addition of spin and orbital parts of the total angular momentum.

By definition, the mode functions in Eq. 2.41 obey the homogeneous Helmholtz equation

$$\nabla^2\vec{V}_{\lambda\ell m} + \omega^2\vec{V}_{\lambda\ell m} = 0,$$
$$\vec{V}_{\lambda\ell m}(\vec{r}) \equiv \sum_\mu (-1)^\mu \vec{\epsilon}_{-\mu} V_{\lambda\ell m\mu}(\vec{r}).$$

In fact, $\vec{V}_{\lambda\ell m}(\vec{r})$ can be considered as a function from the Euclidean three-dimensional space $\mathbf{E}$ into the Hilbert space $\mathbf{H}$ of complex linear functions on $\mathbf{E}$. The operator $\vec{A}_\lambda(\vec{r})$ (Eq. 2.40) satisfies the same wave equation but assumes values in the Hilbert space $\mathbf{H}\otimes\mathbf{H}$, where the second factor $\mathbf{H}$ comes from the spin states of a photon. In view of the Helmholtz equation, $\vec{V}_{\lambda\ell m}(\vec{r})$ can be treated as the wave function of a multipole photon.[10] For an arbitrary number n of the photons, the spin part of the Hilbert space is represented by the symmetric power $S^n\mathbf{H}$, corresponding to the Bose-Einstein statistics of photons, so that the wave function of n-photon state is the function of the type of $\vec{V}_{\lambda\ell m}(\vec{r})$ defined in the space $\mathbf{H}\otimes S^n\mathbf{H}$.

2.4 SU(2) Invariance

Let us now prove that the operator $\vec{A}_\lambda(\vec{r})$ (Eq. 2.40) is invariant with respect to the $SU(2)$ group.

Let $\varphi \in SU(2)$ be an arbitrary transformation belonging to the $SU(2)$ group. Taking into account the definition of the mode functions (Eq. 2.41), consider an auxiliary operator

$$\vec{\mathcal{A}}_{\lambda\ell}(\vec{r}) = \sum_{\mu,m} Y_{1\mu}(\vec{r})Y_{\ell m}(\vec{r})\vec{\epsilon}_\mu \otimes a_{\lambda\ell m}, \tag{2.42}$$

where the argument of spherical harmonics shows the direction in the three-dimensional space $\mathbf{E}$ and undoubtedly is independent of the distance r. It is easily seen that

$$\begin{aligned}
\vec{\mathcal{A}}_{\lambda\ell}(\varphi\vec{r}) &= \sum_{\mu,m} Y_{1\mu}(\varphi\vec{r})Y_{\ell m}(\varphi\vec{r})\vec{\epsilon}_\mu \otimes a_{\lambda\ell m} \\
&= \sum_{\mu,\mu'}\sum_{m,m'} Y_{1\mu'}(\vec{r})\varphi_{\mu\mu'}Y_{\ell m'}(\vec{r})\vec{\epsilon}_\mu \otimes \varphi_{mm'}a_{\lambda\ell m} \\
&= \sum_{\mu,\mu'}\sum_{m,m'} Y_{1\mu'}(\vec{r})Y_{\ell m'}(\vec{r})[\varphi_{\mu\mu'}\vec{\epsilon}_\mu] \otimes [\varphi a_{\lambda\ell m}] \\
&= \sum_{\mu\mu'} Y_{1\mu'}(\vec{r})Y_{\ell m'}(\vec{r})[\varphi\vec{\epsilon}_{\mu'}] \otimes [\varphi a_{\lambda\ell m}] \\
&= \varphi\vec{\mathcal{A}}_{\lambda\ell}(\vec{r}).
\end{aligned}$$

Thus, the auxiliary operator $\vec{\mathcal{A}}_{\lambda\ell}(\vec{r})$ (Eq. 2.42) is invariant with respect to the $SU(2)$ group.

Since the spherical harmonics form a basis of an irreducible representation $\mathbf{M}_\ell$ of the $SU(2)$ group, the product $Y_{1\mu}Y_{\ell m}$ in Eq. 2.42 form a basis of

$$\mathbf{M}_1 \otimes \mathbf{M}_\ell = \mathbf{M}_{\ell-1} \oplus \mathbf{M}_\ell \oplus \mathbf{M}_{\ell+1}.$$

In view of Eq. 2.41, the operator vector potential of Eq. 2.40 is defined just in $\mathbf{M}_{\ell-1} \oplus \mathbf{M}_\ell \oplus \mathbf{M}_{\ell+1}$. Let $(Y_{1\mu}Y_{\ell m})_s$, where $s = \ell, \ell \pm 1$, be the component (projection) of $Y_{1\mu}Y_{\ell m}$ in $\mathbf{M}_s$. Then

$$\vec{\mathcal{A}}_{\lambda\ell}(\vec{r}) = \sum_{\mu,m} (Y_{1\mu}Y_{\ell m})_s \vec{\epsilon} \otimes a_{\lambda\ell m}$$

is also invariant with respect to the $SU(2)$ group. Since the transformations $\varphi \in SU(2)$ does not affect the radial dependence in Eq. 2.40, this implies the $SU(2)$ invariance of the operator vector potential as well.

The property of the $SU(2)$ invariance of the operator vector potential $\vec{A}_\lambda(\vec{r})$ will be used in chapter IV in order to describe the vacuum properties of multipole field.

Chapter 3

Polarization of Multipole Photons

The states of polarization can be characterized by complex numbers, be it a trigonometric representation, Jones' representation or, a complex representation. Those representations only referred to the amplitudes of the components of the field. The problem is that only intensities are directly measurable in optics. However, in the Stokes parameters representation, the states of polarization resort to values stemmed from the intensities of the components of the field or from a combination of these intensities. Thus, this representation only yields to real numbers.

The polarization properties of electromagnetic waves, and of light in particular, are observed experimentally by passing the light to be investigated through various bodies and then observing the intensity of the transmitted light. From mathematical point of view this means that the polarization properties of light are obtained from the time averaged values of certain quadratic functions of its field.

3.1 Polarization Matrix

The polarization is known to be the measure of spatial anisotropy of the electromagnetic field oscillations.[17–19] The quantitative definition requires the measurement of all possible bilinear forms in complex field strengths, which form either the polarization matrix or the set of Stokes parameters.[18] Within the quantum picture, the complex field strengths are replaced by the corresponding operators.[20] These bilinear forms give the complete information about the contribution of spatial components of the field into the total intensity and about the phase differences between the components.

In general, the electromagnetic field is specified by the field-strength tensor[9]

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha, \tag{3.1}$$

where $\partial^\alpha = (\partial/\partial ct, -\vec{\nabla})$ and $A^\alpha = (\Phi, \vec{A})$ is the 4-vector potential. Assume that the elements of Eq. 3.1 are obtained from the positive-frequency part of the operator vector potential (2.40). Then, the general bilinear form in the field strengths is provided by (4×4) Hermitian matrix

$$R(\vec{r}) = F^+(\vec{r})F(\vec{r}), \tag{3.2}$$

which is similar, in a sense, to the Ricci tensor considered in the theory of relativity.[21]

Properties of electromagnetic field is completely described by Maxwell equations whether it is treated as a completely classical or quantum object. Since the field-strength tensor $F^{\mu\nu}$ which is a second rank antisymmetric tensor is constructed by the components of Electric and Magnetic field variables it includes all the physical information. Taking this into account one can define a construction similar to Ricci tensor directly from the field-strength tensor as

$$R = F^\dagger F = \begin{pmatrix} W & \vec{S} \\ \vec{S}^\star & 2P \end{pmatrix}$$

where

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

Although Maxwell Stress Tensor is a Lorentz invariant object, constructed by another combination of field strength tensor, the new Ricci Tensor is not. This is expected because the polarization matrix which is a sub-matrix of Ricci tensor is a local object. In other words polarization is measured at a definite point in space where the source of the field is located at another point.

It can be easily seen that the time-time component of Eq. 3.2 coincides with the energy density, while the time-space components give the linear momentum density (Poynting vector). In turn, the space-space (3×3) submatrix in Eq. 3.2 specifies the polarization properties of the field.[13] This submatrix is additive with respect to the contributions, coming from the electric and magnetic fields:

$$P(\vec{r}) = P_E(\vec{r}) + P_B(\vec{r}) = \begin{pmatrix} E_+^* E_+ & E_+^* E_0 & E_+^* E_- \\ E_0^* E_+ & E_0^* E_0 & E_0^* E_- \\ E_-^* E_+ & E_-^* E_0 & E_-^* E_- \end{pmatrix} + \begin{pmatrix} B_+^* B_+ & B_0^* B_+ & B_-^* B_+ \\ B_+^* B_0 & B_0^* B_0 & B_-^* B_0 \\ B_+^* B_- & B_0^* B_- & B_-^* B_- \end{pmatrix}. \tag{3.3}$$

This expression is written in the helicity basis. The first term here coincides with the polarization matrix of the electric-type multipole radiation has been defined in.[12] In turn, the second term in Eq. 3.3 represents a generalization of the polarization matrix of magnetic-type multipole radiation has been introduced by Wolf[22] (also see discussion in[23]). It is seen that the diagonal terms in P_λ determine the contributions of different spatial components into the intensity, while the off-diagonal elements specify the phase difference between the components. Since the base vectors $\vec{\epsilon}_\mu$ in Eq. 2.40 can be interpreted as the unit vectors of polarization, the components with $\mu = \pm 1$ correspond to the circular polarization with either

positive or negative helicity. In turn, the component $\mu = 0$ corresponds to the linear polarization along z-direction. Such a component always exists in the multipole radiation,[2,9] while vanishes in the case of plane waves.

The quantum counterpart of Eq. 3.3 is provided by the substitution of corresponding operators instead of the field strengths.[13]

Let us now stress that, in contrast to the stress tensor of electromagnetic field, the tensor Eq. 3.2 and the general polarization matrix Eq. 3.3 do not manifest the Lorentz invariance. This means that the polarization of multipole radiation depends on the choice of the reference frame, in which the polarization measurement is carried out. It should be emphasized that the plane waves are usually considered in a certain reference frame such that the z-axis coincides with the direction of propagation provided by the direction of Poynting vector. In this special case, $E_0(\vec{r}) = B_0(\vec{r}) = 0$ for all $\vec{r}$ and Eq. 3.3 is reduced to the conventional (2×2) polarization matrix of plane waves.

As a particular example of some considerable interest, we now investigate the polarization properties of electric-type multipole radiation. In this case, the first term in the general polarization matrix in Eq. 3.3 determined in the "laboratory frame" with the origin in the source location contains all nine elements, while the second term contains only four "transversal terms" because $B_{M\ell 0}(\vec{r}) = 0$ everywhere. Hence, the spatial anisotropy of the electric-type radiation is completely specified by the electric-field strength. Taking into account the relation $\vec{E}(\vec{r}) = ikA(\vec{r})$, we can represent the first term in Eq. 3.3 as follows

$$P_E(\vec{r}) = k^2 ||A^+_{E\mu}(\vec{r}) A_{E\mu'}(\vec{r})||, \tag{3.4}$$

where $A(\vec{r})$ is the operator vector potential of Eq. 2.40. This Eq. 3.4 defines the operator polarization matrix of the electric-type multipole radiation with the elements

$$P_{E\mu\mu'}(\vec{r}) = k^2 \sum_{\ell,\ell'} \sum_{m,m'} V^*_{E\ell m\mu}(\vec{r}) V_{E\ell' m'\mu'}(\vec{r}) a^+_{E\ell m} a_{E\ell' m'}. \tag{3.5}$$

Assume now that the source generates the photons with given ℓ and m by a given

atomic transition in a certain state, e.g. in the coherent state $|\alpha_{\ell m}\rangle$ such that

$$a_{E\ell' m'}|\alpha_{\ell m}\rangle = \begin{cases} \alpha_{\ell m}, & \text{if } \ell' = \ell \text{ and } m' = m \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

Then, the polarization matrix obtained from Eq. 3.5 by averaging over the coherent state under consideration takes the form

$$\langle P_{E\mu\mu'}(\vec{r})\rangle = k^2|\alpha_{\ell m}|^2 V^*_{E\ell m\mu}(\vec{r})V_{E\ell m\mu'}(\vec{r}). \tag{3.7}$$

Thus, the phase difference between the components with different polarization is completely defined by the geometrical phase

$$\phi_{\mu\mu'}(\vec{r}) = \arg[V_{E\ell m\mu'}(\vec{r})] - \arg[V_{E\ell m\mu}(\vec{r})], \tag{3.8}$$

that depends on the point of measurement $\vec{r}$.

Consider now the quantum uncertainty of the polarization measurement provided by the variance

$$\begin{aligned} \langle(\Delta P_{E\mu\mu'}(\vec{r}))^2\rangle &\equiv \langle(P_{E\mu\mu'}(\vec{r}))^2\rangle - \langle P_{E\mu\mu'}(\vec{r})\rangle^2 \\ &= \langle P_{E\mu\mu'}(\vec{r})\rangle \times \left(\sum_{\ell',m'} V^*_{E\ell' m'\mu}(\vec{r})V_{E\ell' m'\mu'}(\vec{r})\right). \end{aligned} \tag{3.9}$$

It is seen that the right-hand side of Eq. 3.9 consists of the product of the mean polarization (Eq. 3.7) and of a "geometrical" factor, which involves the contribution coming from all multipole terms although the "non-radiative" terms with $\ell' \neq \ell$, $m' \neq m$ do not contribute into the polarization in Eq. 3.7. It will be shown in the next section that this geometrical factor corresponds to the zero-point oscillations of multipole polarization. Thus, the quantum noise of a polarization measurements performed for a given mode of the radiation field depends on the vacuum noise of polarization of all possible modes.

3.2 Stokes Parameters

The polarization tensor $P_{\alpha\beta}$, as an every arbitrary tensor, can be split into two parts: a symmetric and an antisymmetric part. Of these, the first

$$S_{\alpha\beta} = \frac{1}{2}(P_{\alpha\beta} + P_{\beta\alpha})$$

is real because of the hermiticity of $P_{\alpha,\beta}$. The antisymmetric part is pure imaginary. Like any antisymmetric tensor of rank equal to the number of dimensions, it reduces to a pseudoscalar.

$$\frac{1}{2}(P_{\alpha\beta} - P_{\beta\alpha}) = -\frac{i}{2}e_{\alpha\beta}A$$

where A is a real pseudoscalar, $e_{\alpha\beta}$ is the unit antisymmetric tensor (with components $e_{12} = -e_{21} = 1$). Thus the polarization tensor has the form:

$$P_{\alpha\beta} = S_{\alpha\beta} - \frac{i}{2}e_{\alpha\beta}A, \qquad S_{\alpha\beta} = S_{\beta\alpha},$$

it reduces to one real symmetric tensor and one pseudoscalar.

For a circularly polarized wave, the vector $\vec{E}_0 = const$, where

$$E_{02} = \pm iE_{01}.$$

It is clear that while $A = \pm 1$, $S_{\alpha\beta} = \frac{1}{2}\delta_{\alpha\beta}$. On the other hand, for a linearly polarized wave the constant vector $\vec{E}_0$ can be chosen as real, so that $A = 0$. In the general case the quantity A is called the degree of circular polarization; it runs through values from +1 and -1, where the limiting values correspond to right- and left-circularly polarized wave, respectively.

The real symmetric tensor $S_{\alpha\beta}$, like any symmetric tensor, can be brought to principal axes, with different principal values which are denoted by λ_1 and λ_2. The directions of the principal axes are mutually perpendicular. Denoting the unit vectors along these directions by $\vec{n}^1$ and $\vec{n}^2$, $S_{\alpha\beta}$ can be written in the form

$$S_{\alpha\beta} = \lambda_1 \vec{n}^1_\alpha \vec{n}^1_\beta + \lambda_2 \vec{n}^2_\alpha \vec{n}^2_\beta, \qquad \lambda_1 + \lambda_2 = 1. \tag{3.10}$$

The quantities λ_1 and λ_2 are positive and have values from 0 to 1.

Suppose that $A = 0$, so that $P_{\alpha\beta} = S_{\alpha\beta}$. Each of the two terms in Eq. 3.10 has the form of a product of two components of a constant vector ($\sqrt{\lambda_1}\vec{n}^1 or \sqrt{\lambda_2}\vec{n}^2$). Physically each of the terms corresponds to linearly polarized light. Moreover, there is no term in Eq. 3.10 containing products of components of the two waves. This means that the two parts can be regarded as physically independent of one another, or, called *incoherent*.

Therefore, in the case of $A = 0$, the partially polarized light can be represented as a superposition of two incoherent waves (with intensities proportional λ_1 and λ_2), linearly polarized along mutually perpendicular directions. In the general case of a complex tensor $P_{\alpha\beta}$, the light can be represented as a superposition of two incoherent elliptically polarized waves, whose polarization ellipses are similar and mutually perpendicular.

Let ϕ be the angle between the y-axis(1) and the unit vector $\vec{n}^1$, then

$$\vec{n}^1 = (\cos\phi, \sin\phi), \qquad \vec{n}^2 = (-\sin\phi, \cos\phi).$$

Introducing the quantity $l = \lambda_1 - \lambda_2$, we write the components of the tensor (Eq. 3.10) in the following form:

$$S_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 1 + l\cos 2\phi & l\sin 2\phi \\ l\sin 2\phi & 1 - l\cos 2\phi \end{pmatrix}.$$

Thus, for an arbitrary choice of axes y and z, the polarization properties of the wave can be characterized by three real parameters: A (the degree of circular polarization), l (the degree of maximum linear polarization), ϕ (the angle between the direction $\vec{n}^1$ of maximum polarization and the y axis).

In place of these parameters another set of three parameters can be chosen

$$s_1 = l\sin 2\phi, \qquad s_2 = A, \qquad s_3 = l\cos 2\phi$$

which are called the *Stokes parameters*. The polarization tensor is expressed in terms of them as

$$P_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 1 + s_3 & s_1 - is_2 \\ s_1 + is_2 & 1 - s_3 \end{pmatrix}$$

All three parameters have values from -1 to +1. The parameter s_3 defines the linear polarization along the y and z axes; the value $s_3 = 1$ corresponds to complete linear polarization along the y axis, and $s_3 = -1$ to complete polarization along the z axis. Also s_1 characterizes the linear polarization along directions making an angle of $\pi/4$ with the y axis.And the fourth parameter is directly pointing itself

$$s_0 = \sqrt{s_1^2 + s_2^2 + s_3^2}$$

which is sometimes called as degree of polarization. Therefore, for a given degree of polarization, different types of polarization are possible.They form a sort of vector of fixed length. Note that the quantities $s_2 = A$ and $\sqrt{s_1^2 + s_3^2} - l$ are invariant under Lorentz transformations. This is almost obvious from the very meaning of these parameters as degrees of circular and linear polarization.

With the light of above definition, the polarization can also be described by the set of Stokes parameters or, in quantum case, by the Hermitian Stokes operators,[14,20] which can be directly measured.[17] In principle, they can be chosen differently. To establish contact with previous results,[14,16] the nine local Stokes operators of the electric-type multipole radiation can be chosen as follows:

$$\begin{aligned}
s_0(\vec{r}) &= \sum_{\mu} k^2 A^+_{E\mu} A_{E\mu}, \\
s_1(\vec{r}) &= -2k^2(A^+_{E+}A_{E0} + A^+_{E0}A_{E-} + A^+_{E-}A_{E+} + H.c.), \\
s_2(\vec{r}) &= \frac{2k^2}{i}(A^+_{E+}A_{E0} + A^+_{E0}A_{E-} + A^+_{E-}A_{E+} - H.c.), \\
s_3(\vec{r}) &= k^2(A^+_{E-}A_{E-} - A^+_{E+}A_{E+}), \\
s_4(\vec{r}) &= k^2(A^+_{E+}A_{E+} + A^+_{E-}A_{E-} - 2A^+_{E0}A_{E0}), \\
s_5(\vec{r}) &= -2k^2(A^+_{E+}A_{E-} + H.c.), \\
s_6(\vec{r}) &= \frac{2k^2}{i}(A^+_{E+}A_{E-} - H.c.), \\
s_7(\vec{r}) &= -2k^2(A^+_{E0}A_{E+} + H.c.), \\
s_8(\vec{r}) &= \frac{2k^2}{i}(A^+_{E0}A_{E+} - H.c.).
\end{aligned} \tag{3.11}$$

The bare operator form of the Stokes operators (Eq. 3.11), corresponding to the electric dipole radiation at $r \to 0$, was considered in.[14,16] It was shown that these bare operators are connected with the representation of the $SU(3)$ subalgebra in the Weyl-Heisenberg algebra of multipole photons.

Here the operator $s_0(\vec{r})$ defines the electric field contribution into the energy density in the point $\vec{r}$. The operators $s_{1,2,5,6,7,8}$ give the phase information concerning the phase differences between the components with different polarization measured in the laboratory frame. In turn, the operator $s_3(\vec{r})$ gives the preponderance of negative helicity over positive helicity, while

the operator $s_4(\vec{r})$ gives the preponderance of transversal polarization over linear polarization.

The measured Stokes parameters are obtained from Eq. 3.11 by averaging over a given state of the radiation field. Let us stress that the quantum noise of the measurement of Stokes operators in Eq. 3.11 again contains contribution of all multipoles even if the radiation field is in a state with given ℓ and m.

3.3 Local Polarization Matrix

We now show that the choice of a certain local frame enables us to fairly simplify the analysis of polarization of multipole radiation. We note that the field strengths are always orthogonal to the Poynting vector

$$\vec{S}(\vec{r}) = \frac{1}{8\pi}[\vec{E}^+(\vec{r}) + \vec{E}(\vec{r})] \times [\vec{B}^+(\vec{r}) + \vec{B}(\vec{r})].$$

Neglecting the fast oscillating terms, we can restrict our consideration by the Hermitian vector

$$\vec{S}(\vec{r}) = \frac{1}{8\pi}[\vec{E}^+ \times \vec{B} + \vec{E} \times \vec{B}^+]. \tag{3.12}$$

By construction, the above operator Poynting vector can have any direction. It is easily seen that, in the helicity basis defining the laboratory frame, the components of Eq. 3.12 has the following form

$$\begin{aligned}
S_+(\vec{r}) &= \frac{i}{8\pi}[-E_-^+(\vec{r})B_0(\vec{r}) + E_0^+(\vec{r})B_-(\vec{r}) - E_-(\vec{r})B_0^+(\vec{r}) + E_0(\vec{r})B_-^+(\vec{r})], \\
S_-(\vec{r}) &= \frac{i}{8\pi}[E_+^+(\vec{r})B_0(\vec{r}) - E_0^+(\vec{r})B_+(\vec{r}) + E_+(\vec{r})B_0^+(\vec{r}) - E_0(\vec{r})B_+^+(\vec{r})], \\
S_0(\vec{r}) &= \frac{i}{8\pi}[E_-^+(\vec{r})B_+(\vec{r}) - E_+^+(\vec{r})B_-(\vec{r}) + E_-(\vec{r})B_+^+(\vec{r}) - E_+(\vec{r})B_-^+(\vec{r})].
\end{aligned}$$

Let us now prepare the local frame by the shift of the origin to the point $\vec{r}$ and by successive rotation of the axes to put z'-direction along $\vec{S}(\vec{r})$. It is clear that only two polarizations would exist in this local frame.

The above local transformation of the reference frame can be expressed in terms of a proper rotation of the base vectors

$$U(\vec{r})[\{\vec{\epsilon}\}] = [\{\vec{\epsilon}'\}(\vec{r})], \tag{3.13}$$

where $[\{\vec{\epsilon}\}]$ denotes the column built from the base vectors forming the laboratory frame in the helicity basis and the matrix

$$U(\vec{r}) = \begin{pmatrix} \frac{1+\cos\theta_S}{2} e^{-i\phi_S} & \frac{1-\cos\theta_S}{2} e^{i\phi_S} & \frac{\sin\theta_S}{\sqrt{2}} \\ \frac{1-\cos\theta_S}{2} e^{-i\phi_S} & \frac{1+\cos\theta_S}{2} e^{i\phi_S} & -\frac{\sin\theta_S}{\sqrt{2}} \\ -\frac{\sin\theta_S}{\sqrt{2}} e^{-i\phi_S} & \frac{\sin\theta_S}{\sqrt{2}} e^{i\phi_S} & \cos\theta_S \end{pmatrix} \tag{3.14}$$

specifies the unitary transformation. Here the position-dependent angles $\theta_S(\vec{r})$ and $\phi_s(\vec{r})$ specify the direction of $\vec{S}(\vec{r})$ with respect to the laboratory frame. It is easy now to check that Eq. 3.14 transforms the polarization matrix (Eq. 3.4) of the electric-type radiation into the reduced (2×2) local polarization matrix

$$\tilde{P}_E(\vec{r}) = U(\vec{r}) P_E(\vec{r}) U^+(\vec{r}), \tag{3.15}$$

where

$$\tilde{P}_E(\vec{r}) = k^2 \begin{pmatrix} \tilde{A}^+_{E+}\tilde{A}_{E+} & 0 & \tilde{A}^+_{E+}\tilde{A}_{E-} \\ 0 & 0 & 0 \\ \tilde{A}^+_{E-}\tilde{A}_{E+} & 0 & \tilde{A}^+_{E-}\tilde{A}_{E-} \end{pmatrix} \tag{3.16}$$

and

$$\tilde{A}^+_{E\mu}(\vec{r}) = \sum_{\mu'} U_{\mu\mu'}(\vec{r}) A^+_{E\mu'}(\vec{r}).$$

This means that, if the polarization measurement is performed in the laboratory frame, the three polarizations described by the operator polarization matrix (Eq. 3.4) should be taken unto account. In other words, the polarization of multipole radiation in the laboratory frame is defined by the measurement of the nine Hermitian Stokes operators (Eq. 3.11). At the same time, it is always possible to chose a proper local frame in which the operator polarization matrix is reduced to Eq. 3.16 and the set of corresponding Stokes operators is also reduced to the set of only four operators as in conventional picture of plane waves.

Chapter 4

Vacuum Properties of Multipole Field

An interesting result of the quantization of radiation is the fluctuations associated with the zero-point energy or the so-called vacuum fluctuations. These fluctuations have no classical analog and are responsible for many interesting phenomena in quantum optics. In a semiclassical theory of atom-field interaction, only the atom is quantized and the field is defined classically. And it is capable to explain many of the phenomena which observed in modern optics. The quantization of the radiation field is needed to explain the effects such as spontaneous emission, the Lamb shift, the laser linewidth, the Casimir effect, and the full photon statistics of the laser. In fact, each of these effects can be understood from the point of view of vacuum fluctuations perturbing the atoms. However, beyond these reasons to quantize the radiation field, there are other strong reasons and logical arguments for quantizing the radiation field. For instance, the problem of quantum beat phenomena provides us with a simple example in which the results of self-consistent fully quantized calculation differ qualitatively from those obtained via a semiclassical theory with or without vacuum fluctuations.

4.1 Vacuum Polarization Matrix

Following,[15] consider now the zero-point oscillations of the multipole field. To avoid the infinite value of the zero-point energy density, consider the monochromatic field. Then

$$W(\vec{r}) = \frac{1}{8\pi}\langle 0|\vec{E}(\vec{r})\cdot\vec{E}^{+}(\vec{r}) + \vec{B}(\vec{r})\cdot\vec{B}^{+}(\vec{r})|0\rangle = \frac{k^2}{8\pi}\sum_{\lambda=E,M}\sum_{\mu}\sum_{\ell,m}|V_{\lambda\ell m\mu}(\vec{r})|^2. \quad (4.1)$$

Because of the definition of mode functions in Eq. 2.41, the zero-point energy density (Eq. 4.1) manifests a certain spatial inhomogeneity in contrast to the plane waves, that always have homogeneous zero-point energy described by the expression

$$\tilde{W} = \frac{\hbar k c}{V}, \quad (4.2)$$

where V is the volume of quantization.

In fact, the zero-point oscillations are concentrated in a certain vicinity of the origin (source location). To show this, consider the polarization matrix (Eq. 3.4), corresponding to the electric-type radiation. It is clear that the elements of Eq. 3.4 are the normal ordered quadratic forms in the photon operators. Besides Eq. 3.4, one can define the anti-normal ordered polarization matrix

$$P_E^{(an)}(\vec{r}) = k^2||A_{E\mu'}(\vec{r})A^{+}_{E\mu}(\vec{r})||.$$

In view of the commutation relations in Eq. 2.37, the difference

$$\begin{aligned} P_E^{(0)}(\vec{r}) \equiv P_E^{(an)}(\vec{r}) - P_E(\vec{r}) &= k^2||[A_{E\mu'}(\vec{r}), A^{+}_{E\mu}(\vec{r})]|| \\ &= k^2\sum_{\ell,m} V^{*}_{E\ell m\mu}(\vec{r})V_{E\ell m\mu'}(\vec{r}) \end{aligned} \quad (4.3)$$

is the (3×3) Hermitian matrix with c-number elements. By construction, this matrix defines the zero-point oscillations of polarization in an arbitrary point $\vec{r}$.[24] It is seen that the trace of Eq. 4.3 coincides, apart from an unimportant factor, with zero-point energy (Eq. 4.1).

Because of the rotational invariance of the multipole field has been proved in chapter II, the elements of Eq. 4.3 should be independent of the direction. In other words,

$$P_E^{(0)}(\vec{r}) = P_E^{(0)}(r).$$

The radial dependence here can be determined as follows. Consider the "north-pole" direction ($\theta = 0$). Then

$$\forall\phi \qquad Y_{\ell,m-\mu}(0,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\delta_{m\mu}.$$

Since $|\mu| \leq 1$, the multipole states with $|m| \geq 2$ does not contribute into the zero-point oscillations of polarization. By direct calculations, it is straightforward matter to arrive at conclusion that the vacuum polarization matrix (Eq. 4.3) in the "north-pole" direction is has the diagonal form:

$$P_E^{(0)}(r,0,0) = k^2 || \sum_{\ell\geq 1} |V_{E\ell\mu\mu}(r,0,0)|^2 \delta_{\mu\mu'} || = \begin{pmatrix} P_T(r) & 0 & 0 \\ 0 & P_L(r) & 0 \\ 0 & 0 & P_T(r) \end{pmatrix}. \quad (4.4)$$

Because of the invariance of $\vec{A}(\vec{r})$ under rotation proven in section II, there is a local unitary transformation

$$\mathcal{U}(\vec{r})P_E^{(0)}(r)\mathcal{U}^+(\vec{r}) = \mathcal{P}_E^{(0)}(r,0,0), \quad (4.5)$$

transforming Eq. 4.3 into Eq. 4.4.

4.2 Zero-Point Oscillations

The diagonal elements of Eq. 4.4 can be interpreted as follows. The elements

$$P_T(r) = k^2 \sum_{\ell\geq 1} |V_{E\ell\pm\pm}(r,0,0)|^2$$

describes the transversal with respect to $\vec{r}$ zero-point oscillations of the electric-field contribution into the energy density (to within an unimportant factor),

connected with the circular polarization of either helicity. In turn, the element

$$P_L(r) = k^2 \sum_{\ell \geq 1} |V_{E\ell 00}(r, 0, 0)|^2$$

gives contribution into the zero-point oscillations of the energy density coming from the longitudinal (radial) linear polarization. Because of invariance of trace, we get

$$W_E(\vec{r}) = W_E(r) \sim \frac{1}{8\pi}[2P_T(r) + P_L(r)]. \tag{4.6}$$

Taking into account the explicit form of the mode functions (Eq. 2.41), properties of the Clebsch-Gordon coefficients,[25] and spherical harmonics, it is straightforward to show that

$$\begin{aligned} P_T(r) &\sim \sum_{\ell \geq 1} \frac{1}{4(2\ell+1)} |\ell j_{\ell+1}(r) - (\ell+1) j_{\ell-1}(r)|^2, \\ P_L(r) &\sim \sum_{\ell \geq 1} \frac{\ell(\ell+1)}{2(2\ell+1)} |j_{\ell+1}(r) + j_{\ell-1}(r)|^2. \end{aligned}$$

Thus, the radial dependence of the zero-point oscillations of the energy density (Eq. 4.6) has the form

$$\begin{aligned} W_E(r) &\sim \frac{1}{32r} \sum_{\ell \geq 1} \left\{ \ell [J_{\ell+3/2}(kr)]^2 + (\ell+1)[J_{\ell-1/2}(kr)]^2 \right\} \\ &= \frac{1}{32r} \left\{ [J_{1/2}(kr)]^2 + \sum_{\ell \geq 1} (2\ell - 1)[J_{\ell-1/2}(kr)]^2 \right\}. \end{aligned}$$

For very large distances compared to the wavelength, $kr \gg 1$, employing the fact that

$$J_{1/2}(kr) = \sqrt{\frac{2}{\pi}} \frac{\sin(kr)}{\sqrt{kr}}$$

and Lommele's formula[26]

$$\sum_{\ell \geq 0} (\ell + 1/2)[J_{\ell+1/2}(kr)]^2 = \frac{kr}{\pi},$$

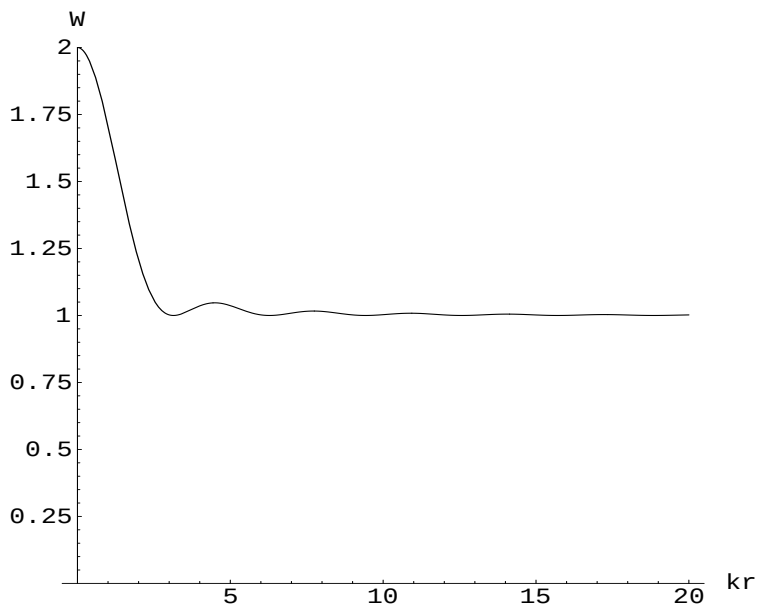

Figure 4.1: *Electric-type contributions to zero-point oscillations of energy density for all values of ℓ in the limit $kr \gg 1$, which means very large distances from source compared to wavelength.*

then gives

$$W_E(r) \sim \frac{k}{16\pi}\left[1 + \frac{\sin^2(kr)}{(kr)^2}\right]. \tag{4.7}$$

The radial behavior of Eq. 4.7, shown in Fig. 4.1, manifests the concentration of zero-point oscillations in a certain vicinity of the origin.

The same transformation is valid for the magnetic-type field as

$$\mathcal{U}(\vec{r})P_M^{(0)}(r)\mathcal{U}^+(\vec{r}) = \mathcal{P}_M^{(0)}(r,0,0) \tag{4.8}$$

Again the diagonal elements of Eq. 4.8 gives the same relations depends on mode functions, however now the definitions of mode functions for magnetic-type field is different in Eq. 2.41.

$$\begin{aligned} P_T(r) &= k^2 \sum_{\ell\geq 1} |V_{M\ell\pm\pm}(r,0,0)|^2, \\ P_L(r) &= k^2 \sum_{\ell\geq 1} |V_{M\ell 00}(r,0,0)|^2 \end{aligned}$$

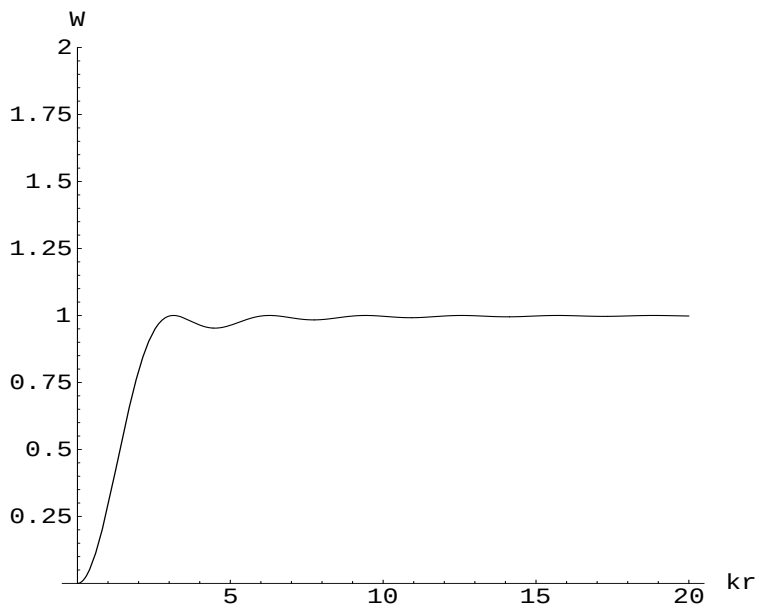

Figure 4.2: *Magnetic-type contributions to zero-point oscillations of energy density for all values of ℓ in the limit $kr \gg 1$, which means very large distances from source compared to wavelength.*

where P_T defines the transversal zero-point oscillations of the magnetic-field contribution into the energy density, corresponds to the circular polarization. Moreover, P_L describes the longitudinal linear polarization of zero-point oscillations of the energy density. Again by using the same invariance property of trace, W_M can be defined as follows in terms of P_T and P_L diagonal components.

$$W_M(\vec{r}) = W_M(r) \sim \frac{1}{8\pi}[2P_T(r) + P_L(r)] \tag{4.9}$$

Taking into account the explicit forms of mode functions of magnetic-type field in Eq. 4.9, P_L appears to be zero for all values of ℓ and P_T is found to be

$$P_T(r) \sim \sum_{\ell \geq 1} \frac{k(2\ell+1)}{4} j_\ell^2(kr). \tag{4.10}$$

Thus the radial dependence of zero-point oscillations of energy density for

magnetic-type field is

$$W_B(r) \quad \sim \quad \frac{k}{16\pi}\sum_{\ell\geq 1}(2\ell+1)j_\ell^2(kr). \tag{4.11}$$

For $kr \gg 1$, using the asymptotic form of Bessel function and Lommele's formula yields to

$$W_B(r) \sim \frac{k}{16\pi}\left[1-\frac{\sin kr^2}{kr^2}\right]. \tag{4.12}$$

The radial behavior is plotted in Fig. 4.2, indicating the zero-point energy oscillations in the vicinity of origin.

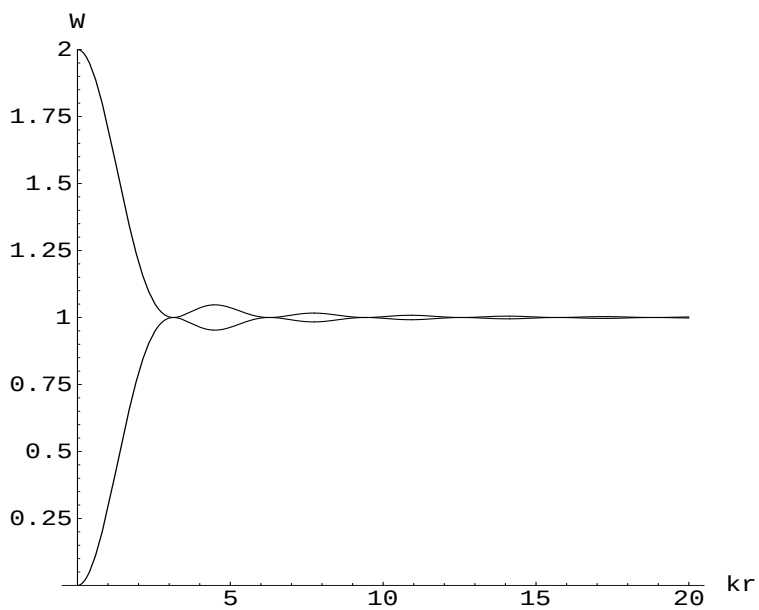

Figure 4.3: *Each type of zero-point oscillations of energy density for all values of ℓ in the limit $kr \gg 1$ shown separately, where their sum yields to a constant energy for all kr values.*

Because of the symmetry relations between the magnetic and electric field strengths, similar results were obtained in magnetic-type field as well. Moreover, total energy density is the sum of the both results in Eq. 4.7 and Eq. 4.12, which is giving a constant energy density for all values of kr(Fig. 4.3). This is not a

surprising result, in the case of large distances compared to wavelength, because it's just a limit case pointing to plane wave representation of radiation.

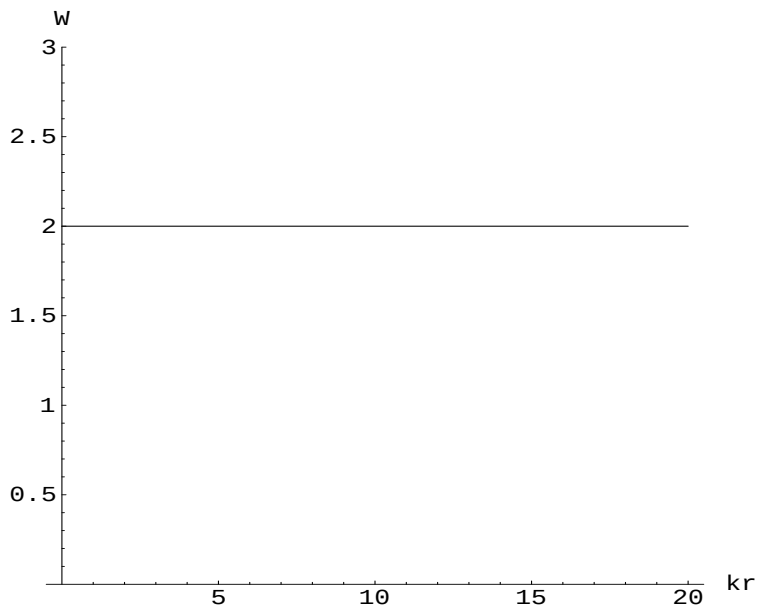

Figure 4.4: *Zero-point energy density, which is the sum of* W_E *and* W_B *for all values of* ℓ *in the limit* $kr \gg 1$.

Chapter 5

Local Representation of Photons

5.1 Electric Dipole Radiation

To clarify the structure of $P_{E\mu\mu'}$ and establish a contact with our results, consider the bare operator form of $P_{E\mu\mu'}$ in Eq. 3.5 in the case of the electric dipole radiation in a spherical cavity. The bare operator structure is provided by the limit $kr \to 0$ in the mode functions (Eq. 2.41) which corresponds to the consideration of the polarization directly near the source. Taking into account the explicit form of spherical Bessel functions

$$\begin{aligned} j_0(kr) &= \frac{\sin(kr)}{kr} \\ j_2(kr) &= \frac{3-(kr)^2}{\sin(kr)} - \frac{3\cos(kr)}{(kr)^2} \end{aligned}$$

and assuming atom the atom is the point-like object(in fact, very small with respect to the wavelength of radiationfield), we get

$$\lim_{kr\to 0} j_0(kr) = 1, \quad \lim_{kr\to 0} j_2(kr) = 0$$

Using the properties of Glebsch-Gordon coefficients[25] and spherical harmonics, the mode functions (Eq. 2.41) in this limit takes the form $V_{E\ell\mu}(0) \sim -\delta_{m\mu}$, which are inserted in Eq. 2.40. Then operator vector potential becomes

$$\vec{A}_{E\ell}(0) = -\sum_{m=-1}^{1} (-1)^{\mu} \vec{\chi}_{-\mu} a_{E1m} \delta_{m\mu} \tag{5.1}$$

This means that the electric dipole transition $|\,1,m\rangle \rightarrow |\,0,0\rangle$ creates a photon with spin state (polarization) $\mu = m$. Then the bare operator structure is

$$P_{Emm'} = \gamma_{E1} a^{+}_{E1m} a_{E1m'} \tag{5.2}$$

where γ_{E1} is an unimportant normalization factor.

The set of Stokes operators can be obtained by canonical quantization of Eq. 3.11. On the other hand, the Stokes operators should, by definition, represent the complete set of independent Hermitian bilinear forms in the photon operators of creation and annihilation. It is clear that such a set is represented by the generators of the $SU(3)$ subalgebra in the Weyl-Heisenberg algebra of electric dipole radiation. The nine generators have the form[14]

$$\begin{array}{ccc} (a^{+}_{+}a_{+} - a^{+}_{0}a_{0}) & (a^{+}_{0}a_{0} - a^{+}_{-}a_{-}) & (a^{+}_{-}a_{-} - a^{+}_{+}a_{+}) \\ \frac{1}{2}(a^{+}_{+}a_{0} + a^{+}_{0}a_{+}) & \frac{1}{2}(a^{+}_{0}a_{-} + a^{+}_{-}a_{0}) & \frac{1}{2}(a^{+}_{-}a_{+} + a^{+}_{+}a_{-}) \\ \frac{1}{2i}(a^{+}_{+}a_{0} - a^{+}_{0}a_{-}) & \frac{1}{2i}(a^{+}_{0}a_{-} - a^{+}_{-}a_{0}) & \frac{1}{2i}(a^{+}_{-}a_{+} - a^{+}_{+}a_{-}) \end{array} \tag{5.3}$$

and only eight of them are independent. To get the set of the Stokes operators, we have to use the generators in Eq. 5.3 or independent linear combinations of this generators together with the operator

$$S_0 = \sum_{m=-1}^{1} a^{+}_{m} a_{m} \tag{5.4}$$

describing the total number of multipole photons. Therefore the rest of the set of Stokes operators as follows[14]

$$\begin{aligned} S_1 &= (\varepsilon_{rad} + \varepsilon^{+}_{rad}) \\ S_2 &= i(\varepsilon^{+}_{rad} - \varepsilon_{rad}) \\ S_3 &= a^{+}_{+}a_{+} - a^{+}_{-}a_{-} \\ S_4 &= a^{+}_{+}a_{+} + a^{+}_{-}a_{-} - 2a^{+}_{0}a_{0} \\ S_5 &= (a^{+}_{+}a_{0} + a^{+}_{0}a_{+}) \\ S_6 &= a(a^{+}_{0}a_{+} - a^{+}_{+}a_{0}) \\ S_7 &= (a^{+}_{0}a_{-} + a^{+}_{-}a_{0}) \\ S_8 &= i(a^{+}a_{0} - a^{+}_{0}a_{0}) \end{aligned} \tag{5.5}$$

where $\varepsilon_{rad} = a^{+}_{+}a_{0} + a^{+}_{0}a_{-} + a^{+}_{-}a_{+}$.

5.2 Local Photon Operator

Appropriate choice of the reference frame leads to the reduced (2×2) local polarization matrix (Eq. 3.16). The use of the unitary transformation allows the operator polarization matrix to be cast into the form of Eq. 4.5

$$\mathcal{P}_E(r,0,0) = \mathcal{U}(\vec{r})\tilde{P}_E(r)\mathcal{U}^+(\vec{r}) \tag{5.6}$$

where

$$\mathcal{P}_{E\mu\mu'}(\vec{r}) = k^2\mathcal{A}^+_{E\mu}(\vec{r})\mathcal{A}_{E\mu'}(\vec{r}) \tag{5.7}$$

and if $\tilde{\mathcal{U}}(\vec{r})$ is defined by $\mathcal{U}(\vec{r})U(\vec{r})$ then gives

$$\mathcal{A}_{E\ell\mu}(\vec{r}) = \sum_{\mu'=-1}^{1} \tilde{\mathcal{U}}^*_{\mu\mu'}(\vec{r}) \sum_{m=-\ell}^{\ell} V_{E\ell m\mu'}(\vec{r})a_{E\ell m} \tag{5.8}$$

In view of Weyl-Heisenberg commutation relations (Eq. 2.37), the operators Eq. 5.8 obey the following relation

$$[\mathcal{A}_{E\ell\mu}(\vec{r}), \mathcal{A}^+_{E\ell'\mu'}(\vec{r})] = \delta_{EE'}\delta_{\ell\ell'}\delta_{\mu\mu'} \times P_T(r), \qquad \mu = \pm 1 \tag{5.9}$$

$P_{\mu=\pm} \equiv P_T$ is the transversal element of the diagonal vacuum polarization matrix (Eq. 4.4). As mentioned in section II, it is not surprising that $P_L(\vec{r})$ does not reveal, because of the local transformation of the reference frame (Eq. 3.16).

The only difference between the Eq. 5.9 and commutation relations in Eq. 2.37 is the presence of position-dependent factor in the $P_T(\vec{r})$. It seems to be quite tempting to introduce the normalized local operators

$$b_{E\ell\mu}(\vec{r}) = \frac{\mathcal{A}_{E\ell\mu}(\vec{r})}{\sqrt{P_\mu(\vec{r})}} \tag{5.10}$$

where $\mu = \pm 1$, hence instead of three, only two normalized local operators obtained which obey the standard Weyl-Heisenberg commutation relations

$$[b_{\lambda\ell\mu}(\vec{r}), b^+_{\lambda'\ell'\mu'}(\vec{r})] = \delta_{\lambda\lambda'}\delta_{\ell\ell'}\delta\mu\mu' \tag{5.11}$$

at any point $\vec{r}$, where λ is the type of radiation either electric or magnetic.

Due to the form of the operator polarization matrix in Eq. 3.5 and corresponding Stokes operators, the polarization, defined to be the spin state of photons,[2,3] is not a global property of the quantum multipole radiation. Any atomic transition emitting photons with given quantum number m, which yields the polarization of all two types depending on the distance from the atom. The forms of $\mathcal{A}_{E\ell\mu}(\vec{r})$ in Eq. 5.8 and $b_{\lambda\ell\mu}(\vec{r})$ in Eq. 5.10 just point how the photons with different m contribute into the polarization at an arbitrary point $\vec{r}$. Using the operators in Eq. 5.8, the local bare operator representation of the polarization matrix can be constructed as follows

$$P_{E\mu\mu'}(\vec{r}) = b^{+}_{E\ell\mu}(\vec{r})b_{E\ell\mu'}(\vec{r}) \tag{5.12}$$

as well as Stokes operators

$$\begin{aligned} S_0(\vec{r}) &= b^{+}_{-}(\vec{r})b_{-}(\vec{r}) + b^{+}_{+}(\vec{r})b_{+}(\vec{r}) \\ S_1(\vec{r}) &= b^{+}_{-}(\vec{r})b_{+}(\vec{r}) + b^{+}_{+}(\vec{r})b_{-}(\vec{r}) \\ S_2(\vec{r}) &= -i(b^{+}_{-}(\vec{r})b_{+}(\vec{r}) - b^{+}_{+}(\vec{r})b_{-}(\vec{r})) \\ S_3(\vec{r}) &= b^{+}_{+}(\vec{r})b_{+}(\vec{r}) - b^{+}_{-}(\vec{r})b_{-}(\vec{r}) \end{aligned} \tag{5.13}$$

By choosing a proper local frame, the set of corresponding Stokes operators is also reduced to the set of only four operators as in the case of plane waves, where only two spin states are allowed.

There is a very important difference between the representations of Stokes operators in Eq. 5.5 and Eq. 5.13. If the former is valid only for the electric dipole radiation (at $r \to 0$), then the latter describes an arbitrary multipole radiation with any λ and ℓ . The similarity in the operator structure is caused by the same number of degrees of freedom defining the representation of the $SU(2)$ subalgebra in the Weyl-Heisenberg algebra.

Chapter 6

Conclusion

The quantum multipole radiation is emitted from the atomic transitions between the states with given angular momenta. The states of spherical (multipole) photons are specified by given angular momentum and its projection as the spin state is changed in space-time. However, the plane wave representation of photons is specified by a given linear momentum and polarization everywhere. This difference reflects the boundary conditions used in the canonical quantization of the free electromagnetic radiation.

Results in all chapters can be summarized as follows:

• It is proved that the quantum field obeys the $SU(2)$ invariance. As a consequence, the vacuum noise of polarization has rotation symmetry with respect to the origin (source location).[Chap. II]

• It is known that, the polarization of multipole radiation in the laboratory reference frame is described by the (3×3) Hermitian polarization matrix, or by corresponding set of nine Stokes parameters. It is shown that, at any point we can construct a proper frame in which the description of polarization is reduced to a conventional (2×2) polarization matrix. This proper frame moves together with the photon, therefore is not convenient for the interpretation of standard measurement.[Chap. III]

• The polarization of multipole radiation manifests the zero-point oscillations, reflecting the vacuum properties of the space. The peculiarities of electric and

magnetic-type zero-point oscillations of energy density were examined.

In particular, it is shown that the monochromatic zero-point oscillations at all types and modes, have a constant level in the volume of quantization.[Chap. IV]

• The special properties of polarization of multipole radiation can be used to construct the local representation of photon operators. These operators correspond to the states of photons with given projection of angular momentum of any spin at any given point. This representation can be useful in the quantum near-field optics. The local representation of multipole photons is compatible with the Mandel operational definition of photon localization.[30][Chap. V]

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