OPTIMAL EXERCISE OF COLLAR TYPE AND MULTIPLE TYPE PERPETUAL AMERICAN STOCK OPTIONS IN DISCRETE TIME WITH LINEAR PROGRAMMING

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By

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June, 2014

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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June, 2014

An American option is an option that entitles the holder to buy or sell an asset at a pre-determined price at any time within the period of the option contract. A perpetual American option does not have an expiration date. In this study, we solve the optimal stopping problem of a perpetual American stock option from optimization point of view using linear programming duality under the assumption that underlying's price follows a discrete time and discrete state Markov process. We formulate the problem with an infinite dimensional linear program and obtain an optimal stopping strategy showing the set of stock-prices for which the option should be exercised. We show that the optimal strategy is to exercise the option when the stock price hits a special critical value. We consider the problem under the following stock price movement scenario: We use a Markov chain model with absorption at zero, where at each step the stock price moves up by Δx with probability p, and moves down by Δx with probability q and does not change with probability $1-(p+q)$. We examine two special type of exotic options. In the first case, we propose a closed form formula when the option is collar type. In the second case we study multiple type options, that are written on multiple assets, and explore the exercise region for different multiple type options.

Keywords: Perpetual American options, Collar type options, Multiple type options, Triple random walk, Difference equations.

ÖZET

YAKA TİPİ VE ÇOKLU VARLIKLI VADESİZ AMERİKAN HİSSE SENEDİ OPSİYONLARININ KESİKLİ ZAMANDA DOĞRUSAL PROGRAMLAMA İLE EN İYİ KULLANIM DEĞERLERİNİN BELİRLENMESİ

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Amerikan opsiyonu, sahibine bir varlığı herhangi bir zamanda önceden belirlenmiş bir fiyata alma ya da satma hakkını verir. Vadesiz Amerikan opsiyonunun bitiş zamanı yoktur. Bu çalışmada, opsiyonun yazıldığı hisse senedinin kesikli zamanda kesikli değerler aldığı Markov süreçlerini izlediği varsayımıyla doğrusal programlama yöntemleri kullanılarak vadesiz Amerikan opsiyonunun en iyi karı verecek şekilde ne zaman kullanılması gerektiği problemi ele alınmıştır. Problem, sonsuz değişkenli doğrusal programlama ile modellenmiş ve en iyi opsiyon kullanım stratejisini veren hisse senedi değerleri belirlenmiştir. En iyi kullanım stratejisinin aslında hisse senedinin belli bir değere ulaşır ulaşmaz opsiyonun kullanılması olduğu gösterilmiştir. Problem, hisse senedinin şu şekilde bir hareket izlediği varsayımıyla modellenmiştir: Hisse senedi, 0 değeri aldığında artık işlem göremeyeceği varsayımını yapan Markov zinciri modelini izler, hisse senedinin değeri her bir adımda ya p olasılıkla Δx kadar artar, ya q olasılıkla Δx kadar azalır, ya da $1-(p+q)$ olasılıkla sabit kalır. İki farklı egzotik opsiyon tipi incelenmiştir. Birincisinde, yaka tipi opsiyonlar için kapalı çözüm formülü elde edilmiştir. İkincisinde ise birden fazla hisse senedi üzerine yazılmış birkaç tip çoklu varlıklı opsiyon için opsiyonu kullanma alanları belirlenmiştir.

Anahtar sözcükler: Vadesiz Amerikan opsiyonları, Yaka tipi opsiyonlar, Çoklu varlıklı opsiyonlar, Üçlü rassal yürüyüş, Fark denklemleri.

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Chapter 1

Introduction

Mathematical finance is one of the fields of applied mathematics, mainly concerned with financial markets. This field derives and extends mathematical models by assuming market prices given while financial economics might investigate the structural reasons behind it. There are two main research branches in mathematical finance: One is derivative pricing which we will be dealing with in this thesis, and the other one is risk and portfolio management.

A *derivative* can be defined as financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables [1]. For example, a stock option is a derivative whose value is dependent on the price of the underlying stock. The most common underlying assets are commodities, stocks, bonds, interest rates and currencies. In this thesis, we will focus on stocks as an underlying asset.

In the last 30 years, derivatives have become increasingly important in finance. They can be used for either risk management by hedging, meaning that providing offsetting compensation in case of an undesired event, or speculation which means a financial bet to make money.

Derivatives are traded on financial markets like stocks, currencies, and commodities, as well. Most traded derivatives are futures and options. A *future* *contract* is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price. An *option contract*, on the other hand, is an agreement between two parties to have the right but not the obligation to buy or sell an asset at a certain time in the future for a certain price. The main difference between these two derivatives is that for futures, there is an obligation to honor the contract at the expiration date, but not for options. That is why there is no cost to enter a future contract but there is for options, called *premium*.

Determining the premium is the fundamental question for both buyer and seller of an option contract in the financial market. It is called the *fair price* of an option. Valuation of options have some complexities compared to valuation of traditional equities. It is even more complicated when the option holder has right to use the contract at any time within the contract period which is the case in so-called American type options. In this study, we will construct an optimal trading strategy for an American option holder under the assumption that stock price follows discrete time discrete state triple random walk process. Optimal trading strategy will show to the holder at which states (s)he will use the option contract to get the best expected future pay-off.

1.1 Basic Terminology on Options

A *call option* gives the owner the right, but not the obligation, to buy an asset at a specified price within a specified time. A *put option* gives the owner the right, but not the obligation, to sell an asset in a similar fashion. *Strike price* of an option is the fixed price at which the owner of the option can buy or sell the underlying asset. The date on which the option expires is called *maturity date*. When the option holder uses the contract, meaning that (s)he buys or sells the underlying asset at strike price, we say that the holder has *exercised* the option. In a given state of the world, the amount that the option holder gains or loses as a function of underlying's payoff is called *option payoff*. In this work, we will use S for strike price, T for maturity date, the real-valued function $f : E \to \mathbb{R}$ for the option payoff where E is the set of all possible states of the world.

A *European option* can be exercised only at the expiration date, i.e., single pre-defined point in time. On the other hand, an *American option* can be exercised at any time before the expiration date. If there is no expiration date, it is called *perpetual option*. In this thesis, we work on perpetual American options. American type options require dynamic valuation process since the option holder must observe the underlying's price through time and decide on a time to exercise to maximize his/her earnings. This type of analysis is not required in European type options since exercise date is fixed during the agreement.

American and European types of options are called *plain vanilla options* and these types of options are used in financial markets most frequently. Options which do not fall this category is called *exotic options*. Exotic options are rarely used compared to plain vanilla options but they are more complex derivatives and constitute an interesting background in the derivative pricing literature.

Assume that for some state of the world $x \in E$, the payoff of the underlying is represented by $X(x)$. For the call option holder, if $X(x)$ is greater than the strike price S at the maturity date, it is meaningful for the holder to exercise the option for an immediate gain of $X(x) - S$, since the contract gives her the right to buy a unit of the underlying at the price S. Then, by selling this unit in the original market for its real market value $X(x)$, the owner can have the specified gain. If the price of the underlying, however, is lower than S, it will not be profitable to exercise the option because the same asset is already available cheaper in the exchange market. For a call option, the payoff function corresponds to:

$$f(x) = max\{X(x) - S, 0\} = (X(x) - S)^+$$

In the case of a put option, the condition on trade is reversed, and the owner has the right to sell the option at the maturity date. Note that this strategy is only profitable when $X(x) < S$, hence, the payoff of a put option is:

$$f(x) = max\{S - X(x), 0\} = (S - X(x))^+$$

Note that, in this study, an option is assumed to be call option unless otherwise stated.

1.2 Motivation

Let us consider a perpetual American option holder. Since there is no maturity date for the contract, the holder will decide when to exercise the option only based on the observations on the underlying stock price movement. The objective of the trader is to find the states where (s)he should exercise the option to obtain the best expected payoff. The purpose of this study is based on this important need of the option holder. In this study, we will characterize the optimal exercising strategy so that the option holder will know at which states to exercise the option and at which states to wait further in order to obtain more with respect to the expected pay-off.

Suppose that underlying stock follows a stochastic process X_t on the state space E. For any initial state x and any time t, expected payoff at time t can be denoted as $E[f(X_t)|X_0 = x]$. Let us introduce a *discount factor* α (a number slightly less than one) to account for the fact that future dollars are worth less than present dollars. Value function v for any option is represented by the maximum of all such functions over any future time t, that is

$$v(x) = \max_{t \in T} \mathbb{E}_x \left[\alpha^t f(X_t)\right].$$

This value function v tells both to the seller and buyer of the contract that fair price of the option is $v(x)$ if the current stock price is x. In each period, the option holder observes current stock price x, and decides either to exercise the option or keep it for one more period. This decision is made by looking at the gap between value function v and option payoff f. Note that it is not possible to have $v(x) < f(x)$ since the time-index set involves time zero. Another important note is that delaying the decision to exercise when v cannot beat $f(x)$ will clearly be suboptimal, due to the time value of money.

Exercise region can be characterized by a subset OPT of state space E such that for all $x \in OPT$, $v(x) = f(x)$. For all $x \notin OPT$, $v(x)$ must be greater than $f(x)$, meaning that expected payoff of future gains is greater than current option payoff, therefore, option holder should wait further to gain more. This region is called *continuation region.*

Optimal trading strategy is to exercise the option immediately once we have $v(x) = f(x)$, and to keep option as long as $v(x) > f(x)$. In this thesis, our aim is to find the correct value function and set OPT to tell the trader optimal exercising strategy.

In this work, we assume that (1) stock price follows a discrete time discrete state triple random walk, (2) discount factor $\alpha \in (0, 1)$ is used for the time value of money, and (3) option is perpetual American option with no expiration date.

Basically, we will deal with two different special cases of this problem. First, we will solve the problem when the option is *collar type*. Second, we will explore the exercise region when option is *multiple stock options* which is written on more than one stock (two different stocks in our case). These two types of options are exotic options and details will be given in corresponding chapters.

Chapter 2

Literature Review

Financial market instruments can be divided into two distinct species. There are the underlying stocks, shares, bonds, currencies; and their derivatives, claims that promise some payment or delivery in the future contingent on an underlying's behavior. Pricing of this second type, derivative pricing is well established and documented research area in mathematical finance. Since options are widely used derivatives in the financial market, they have taken researcher's attention more than other derivative instruments. This field of research has again two distinctive branches, based on being European or American type option. We are focusing on optimal strategy of American type perpetual stock options written on stocks which follows discrete time discrete state triple random walks.

Option pricing has well-known history and there are many books and papers written on this field of study. Comprehensive treatments of option pricing can be found in Hobson's survey [2], if reader is interested in specifically of American options, (s)he can refer to Myneni's [3] surveys. Hull's text [1] is a good reference for fundamental models in derivative pricing such as Black-Scholes option pricing model.

Option pricing literature started by Bachelier [4], he studied European type options. Later, Samuelson [5] provided a comprehensive treatment on the theory of option pricing. With the contribution of Black and Scholes [6] and Merton

[7], option pricing literature has become a well-established research field. In their work, Black and Scholes show that in a frictionless and arbitrage-free market, the price of an option solves a special differential equation, a variant of the heat equation arising in physical problems. Their assumption that the stock price follows a geometric Brownian motion has been a very key and much cited assumption.

The problem of determining correct prices for American type contingent claims was first handled by McKean upon a question posed by Samuelson (see appendix of [8]). In his response, McKean transformed the problem of pricing American options into a free boundary problem. The formal treatment of the problem from an optimal stopping perspective was later done by Moerbeke [9] and Karatzas [10], who used hedging arguments for financial justification. Wong, in a recent study, has collected the optimal stopping problems arising in the financial markets [11].

In this thesis, we want to provide a different approach to solve American option pricing problem in the perspective of linear programming when stock price follows discrete time discrete state triple random walk. Our aim is to characterize the exercise region to show the option holder when to exercise the option. It is well known that the value function of an optimal stopping problem for a Markov process is the minimal excessive function majorizing the pay-off of the reward process (see [12] and [13]). With this property of the value function, the problem can be structured as an infinite dimensional linear model and solved by linear programming duality concepts. This approach is first used in [14] to treat singular stochastic control problems. Then, Vanderbei and Pınar [15] use this approach to propose an alternative method for the pricing of American perpetual warrants. In their work, they solve the problem when stock price follows simple random walk and underlying option is plain vanilla option. In this thesis, we will solve the problem when stock price follows triple random walk, meaning that stock price can remain at the same price with some probability, and underlying option is collar type option which is one of the favorite exotic options used for hedging. Next, we will model the problem when option is multiple type, i.e., option is written on more than one stock, and explore the exercise region.

Chapter 3

Background

This chapter contains mathematical definitions and main concepts that will be used in the remaining part of the thesis. This develops a base to approach this problem from the linear programming perspective and details can be found in [13].

3.1 Markov Processes

Let the triplet $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 3.1.1. A stochastic process $\boldsymbol{X} = \{X_t, t \in T\}$ with the state space E is a collection of E-valued random variables indexed by a set T, often interpreted as the *time*. $\boldsymbol{X}$ is a discrete-time stochastic process if T is countable and a continuous-time stochastic process if T is uncountable. Likewise, $\boldsymbol{X}$ is called a discrete-state stochastic process if E is countable and called a continuous-state stochastic process if E is an uncountable set.

We model the stock price movement as a triple random walk which is a classical example of stochastic processes. We use memoryless property for stock price movement model, i.e., future of the process only depends on the current state of the process. This property is called Markov property.

Definition 3.1.2. A stochastic process $\boldsymbol{X}$ having the property,

$$\mathbb{P}\{\boldsymbol{X}(t+h) = y \mid \boldsymbol{X}(s), \forall s \leq t\} = \mathbb{P}\{\boldsymbol{X}(t+h) = y \mid \boldsymbol{X}(t)\}$$

for all $h > 0$ is said to have the Markov property. A stochastic process with the Markov property is called a Markov process.

In our study, stock price follows discrete time discrete state Markov process and by the Markov property, next state of stock price is only dependent to current state.

3.2 Potentials and Excessive Functions

Potentials and excessive functions are the fundamental concepts in optimal stopping literature. They allow us to link stochastic processes and outcomes. Here are some important definitions from Chapter 7 of [13].

Definition 3.2.1. A real-valued function $g : E \to \mathbb{R}$ is called the *reward* function of a stochastic process $\boldsymbol{X}$.

Definition 3.2.2. Let $g : E \to \mathbb{R}$ be a reward function defined on E. The function $Rg : E \times T \to \mathbb{R}$ defined as

$$Rg_t(i) = \mathbb{E}_i \sum_{h=0}^{\infty} g(X_{t+h})$$

is called the potential of g.

Definition 3.2.3. Let $g : E \to \mathbb{R}$ be a reward function defined on E and $\alpha \in (0, 1]$. The function $R^\alpha g : E \times T \to \mathbb{R}$ defined as

$$R^\alpha g_t(i) = \mathbb{E}_i \sum_{h=0}^{\infty} \alpha^n g(X_{t+h})$$

is called the α-potential of g.

Payoff of an option at any arbitrary state is defined by a reward function. By using this function, we can model the payoff of an option contract with respect to the stock price at any given state.

Given a state, an option holder should decide between exercising the option in that state or keeping it one more period. The decision can be made by comparing expected value of the option in that state and pay-off when option is exercised. The expected value of the option is modeled by potentials as defined above. We use α-potentials to include time value of the money in the model.

Definition 3.2.4. Let f be a finite-valued function defined on E and P be a transition matrix. The function f is said to be α-*excessive* provided that $f \geq 0$ and $f \geq \alpha P f$. If f is 1-*excessive*, it is simply called *excessive*.

This property of value function is very crucial in our work. This tells us that at any given state, the associated reward at time 0 is always greater than or equal to the discounted expected value of any future reward. This will constitute a lower bound for our value function in the linear programming model.

3.3 Optimal Stopping on Markov Processes

Definition 3.3.1. The real valued function v on E given by

$$v(i) = \sup_{\tau} \mathbb{E}_i \left[\alpha^{\tau} f(X_{\tau}) \right]$$

is said to be the value function of a game associated with the Markov process X and the reward function f.

This measure basically gives the supremum of the discounted expected future rewards over all stopping times τ when the initial state is i. In order to make a stopping decision, we need to determine the set of states, which is $OPT \subset E$, such that $v(j) = f(j)$, $\forall j \in OPT$.

3.4 The Fundamental Theorem

Theorem 3.4.1. *Let f be a bounded function on E. The value function v is the minimal α-excessive function greater than or equal to the pay-off function f.*

Proof. Both given in [12], p.105 and [13], p.221. □

This theorem is a key theorem that we will base our model on. By using excessive and majorant properties of value function, we can construct the following LP to model this optimal stopping problem.

$$\begin{aligned} \min \quad & \sum_{i \in E} v(i) & & \\ \text{s.t.} \quad & v(i) \geq f(i) & , \quad & i \in E \\ & v(i) \geq \alpha P v & , \quad & i \in E \\ & v(i) \geq 0 & , \quad & i \in E \end{aligned}$$

One important note is that the theorem states that payoff function f must be bounded. In our case, f is bounded above by the second strike price S_2, therefore the theorem can be applied.

3.5 Triple Random Walk

As we discussed in Chapter 1, we will model stock price movement as triple random walk.

Let Δx be a fixed positive integer in the interval $(0, 1]$. We define the set $E_1 = \{j \cdot \Delta x,\ j \in \mathbb{N}\}$ to be the state space for the stock price process. Let t_0 denote the beginning period of analysis and define the collection $\{X_t,\ t \in \mathbb{N}\}$ of random variables for each $X_t \in E_1$ to be the stock price process. Then, triple random walk on E_1 can be defined by letting:

$$X_{t+1} = \begin{cases} X_t + \Delta x & \text{w.p.} \quad p \\ X_t - \Delta x & \text{w.p.} \quad q \\ X_t & \text{w.p.} \quad 1 - p - q \end{cases}$$

for each period $t \in \mathbb{N}$ and $X_t \in E_1 - \{0\}$.

State 0 is an absorbing state; as stock price hits zero, it means associated company goes bankrupt and stock price will never be able to go back to any positive value.

3.6 Collar Type Options

Collar type options are used basically for hedging purposes if holder already has a stock in hand. This type of options allow the holder to gain some amount if stock prices goes up but it puts a cap on the gain in return for putting a floor on the downside in order to prevent risk of big stock price decline.

A collar type option is created by an investor being (1) long the underlying stock, (2) long a put option at strike price S_1, and (3) short a call option at strike price S_2.

The payoff of collar type options are as follows [16]:

$$f(x) = min\{max\{X(x), S_1\}, S_2\}$$

When option is exercised, if stock price is lower than first strike price, then holder will get first strike even if actual stock price is lower. If stock price is between two strikes, holder will get actual amount of stock. If stock price is greater than second strike price, then holder will get second strike even if stock has greater value than second strike. With this strategy, holder guarantees a payoff between two levels and limits herself for downside risk in return for sacrificing upside gain.

3.7 Multiple Type Options

In Chapter 6 of [17], Detemple considered American options written on multiple underlying assets. In his book, he studied different cases such as options written on minimum, average, and maximum of two assets in continuous time finance point of view.

Among those three multiple type options, we will consider and model *max-call options.* These type of options are exotic options and rarely used in trade market. However, it is an attractive and challenging research area since value and payoff function is in third dimension in this case which makes it complicated to find exercise region.

Given two stocks X and W which follows discrete time discrete state random walk, payoff function of max-call option written on these two stocks is as follows:

$$f(x,w) = max\{max\{X(x), W(w)\} - S, 0\} = (max\{X(x), W(w)\} - S)^+$$

At any given state, option holder observes stock price level of both stocks and decides based on the comparison of maximum of both stock prices and strike price.

Chapter 4

Optimal Stopping for Collar Type Options

4.1 Primal Problem

Under the triple random walk assumption, the problem is modeled by the following infinite dimensional LP:

$$\begin{array}{lll} minimize & \sum_{j=0}^{\infty} v_j & \\ s.t. & v_j \geq f_j, & j \geq 0, \\ & v_j \geq \alpha(pv_{j+1} + qv_{j-1} + (1-p-q)v_j), & j \geq 1. \end{array}$$

where $x_j = j\Delta x$, $v_j = v(x_j)$, $f_j = f(x_j)$, and $f(x_j) = min\{max\{x_j, S_1\}, S_2\}$.

This problem stands for the optimal stopping problem of a perpetual collar type American option written on a stock following triple random walk. If there exists an optimal solution for this problem, based on the gap between v and f, the option holder will find out on which states to exercise the contract, and on which states to wait further.

Therefore, our aim is to solve this problem optimally, and characterize the exercise region.

4.2 Dual Problem

The solution procedure is based on linear programming duality. Therefore, next step is to find the dual problem to the primal problem above. The associated dual problem is

$$\begin{array}{lll} maximize & \sum_{j=0}^{\infty} f_j y_j & \\ s.t. & y_0 - \alpha q z_1 = 1, & \\ & y_1 + (1-\alpha(1-p-q))z_1 - \alpha q z_2 = 1, & \\ & y_j - \alpha p z_{j-1} + (1-\alpha(1-p-q))z_j - \alpha q z_{j+1} = 1, & j \geq 2, \\ & y_j \geq 0, & j \geq 0, \\ & z_j \geq 0, & j \geq 1. \end{array}$$

Here, y_j's and z_j's are dual variables correspond to first and second set of constraints of the primal problem, respectively.

4.3 Complementary Slackness Conditions

Based on the primal and dual problem, let us write the complementary slackness conditions:

$$\begin{array}{rcll} (f_j - v_j)y_j & = & 0, & j \geq 0, \\ (\alpha(pv_{j+1} + qv_{j-1} + (1-p-q)v_j) - v_j)z_j & = & 0, & j \geq 1, \\ v_0(1 - y_0 + \alpha q z_1) & = & 0, & \\ v_1(1 - y_1 - (1-\alpha(1-p-q)z_1) + \alpha q z_2) & = & 0, & \\ v_j(1 - y_j + \alpha p z_{j-1} - (1-\alpha(1-p-q)z_j) + \alpha q z_{j+1}) & = & 0, & j \geq 2, \end{array}$$

4.4 Solution Procedure for Primal and Dual Problem

Let v_j denote the optimal primal solution and y_j and z_j the optimal dual solution (i.e., we are dropping the usual "stars" that denote optimality). Finite approximations of the primal problem suggests that there exists a j^* such that

$$\begin{aligned} v_0 &= f_0, \\ v_j &= \alpha(pv_{j+1} + qv_{j-1} + (1-p-q)v_j) > f_j, \quad \text{for } 0 < j < j^*, \\ v_j &= f_j > \alpha(pv_{j+1} + qv_{j-1} + (1-p-q)v_j), \quad \text{for } j^* \leq j. \end{aligned}$$

Due to our assumption, we already know the values of v^* for $j \geq j^*$. To determine v_j^* for $0 < j < j^*$, we need to solve the second order homogeneous difference equation:

$$v_j - \alpha(pv_{j+1} + qv_{j-1} + (1-p-q)v_j) = 0, \quad 0 < j < j^*$$

with the boundary conditions:

$$v_0 = S_1,$$

$$v_{j^*} = f_{j^*}.$$

From the numerical analysis, we know that value function behaves as an exponential function. Therefore, to this end, suppose that

$$v_j = \xi^j$$

for some positive real number ξ. If we substitute it into the difference equation, we get

$$\xi^j - \alpha(p\xi^{j+1} + q\xi^{j-1} + (1-p-q)\xi^j) = 0.$$

Dividing by ξ^{j-1}, we get a quadratic equation

$$-\alpha p\xi^2 + (1 - \alpha(1-p-q))\xi - \alpha q = 0.$$

The two roots of this equation are

$$\xi_- = \frac{\alpha(1-p-q) - 1 - \sqrt{(1-\alpha(1-p-q))^2 - 4\alpha^2 pq}}{-2\alpha p},$$

$$\xi_+ = \frac{\alpha(1-p-q) - 1 + \sqrt{(1-\alpha(1-p-q))^2 - 4\alpha^2 pq}}{-2\alpha p}.$$

where we used ξ_- to denote the larger root and ξ_+ for the smaller root. The general solution to the difference equation is therefore

$$v_j = c_+ \xi_+^j + c_- \xi_-^j.$$

From the first boundary condition $v_0 = S_1$, we get that $c_+ + c_- = S_1$. This relation together with the second boundary condition $v_{j^*} = f_{j^*}$ gives

$$c_+ = \frac{f_{j^*} - S_1 \xi_-^{j^*}}{\xi_+^{j^*} - \xi_-^{j^*}}$$

Hence,

$$v_j = \left(\frac{f_{j^*} - S_1 \xi_-^{j^*}}{\xi_+^{j^*} - \xi_-^{j^*}} \right) (\xi_+^j - \xi_-^j) + S_1 \xi_-^j, \qquad 0 < j < j^*.$$

Now, let us solve for z_j. Since v_{j^*} is optimal for primal problem, it has to satisfy CS conditions. We know that $v_j^* \neq f_j$ for $0 < j < j^*$, thus, $y_j^* = 0$ when $0 < j < j^*$ and last CS condition reduces to:

$$(1 - \alpha(1-p-q)) z_j - \alpha(p z_{j-1} + q z_{j+1}) = 1, \qquad 0 < j < j^*.$$

with the boundary conditions:

$$z_0 = 0,$$

$$z_{j^*} = 0.$$

First boundary condition is just an additional variable which does not affect the problem formulation. Second boundary condition is obtained from the second CS condition; since for $j \geq j^*$, first part of the multiplication is nonzero which forces z_j to be 0.

We need a particular solution to the equation and the general solution to the associated homogeneous equation. For a particular solution, we try a constant value

$$z_j \equiv c.$$

Substituting into the difference equation, we get $c = 1/(1-\alpha)$.

Let us write the equation for z_j in different form:

$$z_j - \alpha(qz_{j+1} + pz_{j-1} + (1-p-q)z_j) = 1, \qquad 0 < j < j^*.$$

This homogeneous equation is exactly the same as the equation for v_j except with p and q interchanged. Hence the general solution, which is the sum of the particular and the homogeneous, is given by

$$z_j = \frac{1}{1-\alpha} + c_+\zeta_+^j + c_-\zeta_-^j$$

where

$$\begin{aligned}
\zeta_+ &= 1/\xi_- = \frac{\alpha(1-p-q) - 1 - \sqrt{(1-\alpha(1-p-q))^2 - 4\alpha^2 pq}}{-2\alpha q}, \\
\zeta_- &= 1/\xi_+ = \frac{\alpha(1-p-q) - 1 + \sqrt{(1-\alpha(1-p-q))^2 - 4\alpha^2 pq}}{-2\alpha q}.
\end{aligned}$$

Using the boundary conditions to eliminate the two undetermined constants, we get

$$z_j = \left(1 - \frac{\zeta_-^{j^*} - 1}{\zeta_-^{j^*} - \zeta_+^{j^*}}\zeta_+^j - \frac{\zeta_+^{j^*} - 1}{\zeta_+^{j^*} - \zeta_-^{j^*}}\zeta_-^j\right) \Big/ (1-\alpha), \qquad 0 < j < j^*.$$

It only remains to show the values of y_j. The dual constraint ensure that $y_j = 1$ for $j > j^*$ since $z_j = 0$ in this region. Then,

$$y_j = \begin{cases} 1 + \alpha q z_1 & j = 0 \\ 0 & 0 < j < j^* \\ 1 + \alpha p z_{j^*-1} & j = j^* \\ 1 & j > j^* \end{cases}$$

To summarize, we have

$$v_j = \begin{cases} S_1 & j = 0 \\ (f_{j^*} - S_1 \xi_-^{j^*}) \left(\frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}} \right) + S_1 \xi_-^j, & 0 < j < j^* \\ f_j & j^* \leq j \end{cases}$$

$$z_j = \begin{cases} \left(1 - \frac{\zeta_-^{j^*} - 1}{\zeta_-^{j^*} - \zeta_+^{j^*}} \zeta_+^j - \frac{\zeta_+^{j^*} - 1}{\zeta_+^{j^*} - \zeta_-^{j^*}} \zeta_-^j \right) \Big/ (1 - \alpha) & 0 < j < j^* \\ 0 & j^* \leq j \end{cases}$$

$$y_j = \begin{cases} 1 + \alpha q z_1 & j = 0 \\ 0 & 0 < j < j^* \\ 1 + \alpha p z_{j^*-1} & j = j^* \\ 1 & j^* < j \end{cases}$$

Now, we have candidate optimal solutions v_j, y_j, and z_j to the primal and dual problems. The only thing left we have to show is that these candidate solutions satisfy all constraints in both primal and dual problems.

4.5 Check the Inequalities

Inequalities we need to check are as follows:

$$y_j \geq 0, \qquad j \geq 0, \tag{4.1}$$

$$z_j \geq 0, \qquad j \geq 1, \tag{4.2}$$

$$v_j \geq f_j, \qquad j \geq 0, \tag{4.3}$$

$$v_j \geq \alpha(p v_{j+1} + q v_{j-1} + (1 - p - q) v_j), \qquad j \geq 1. \tag{4.4}$$

In [15], a methodology to check these inequalities for a plain vanilla option under simple random walk assumption is proposed. We will follow the same methodology with some modifications for collar type options under triple random walk assumption.

4.5.1 Inequalities 4.2

For $j \geq j^*$, it directly follows from the formula of z_j. We are done when we show that it holds for $0 < j < j^*$.

By a contrary, let us assume that $z_j < 0$ for some $0 < j < j^*$. Then, there must be a k at which z_k is negative and a local minimum:

$$z_k \leq z_{k-1} \quad \text{and} \quad z_k \leq z_{k+1}.$$

If j' is the index of the local minimum, that is $j' = k$, from the last CS condition and $y_j = 0$ for $0 < j < j^*$, we have:

$$\begin{aligned}
(1-\alpha(1-p-q))z_{j'} &= 1+\alpha(pz_{j'-1}+qz_{j'+1}) \\
&\geq 1+\alpha(pz_{j'}+qz_{j'}) \\
z_{j'}-\alpha z_{j'}+\alpha p z_{j'}+\alpha q z_{j'} &\geq 1+\alpha p z_{j'}+\alpha q z_{j'} \\
z_{j'}(1-\alpha) &\geq 1.
\end{aligned}$$

which implies that $z_{j'} \geq 1/(1-\alpha)$. This contradicts with $z_{j'}$ being negative. If j' is not the index of the local minimum then we either have $z_{j'-1} > z_{j'} > z_{j'+1}$ or $z_{j'-1} < z_{j'} < z_{j'+1}$. Note that due to the boundary conditions, that are $z_0 = 0, z_{j^*}$=0, we must have at least one local minimum as we change j' towards 0 or j^* which will again lead to a contradiction. Hence, $z_j \geq 0$ for all $0 < j < j^*$

4.5.2 Inequalities 4.1

We proved that $z_j \geq 0$ for $j \geq 1$. p, q, and α are nonnegative. Combining these facts, from the formula of y_j, it is easy to see that $y_j \geq 0$ for $j \geq 0$.

4.5.3 Inequalities 4.4

These hold trivially for $j < j^*$. For $j \geq j^*$, We will split it by $j > j^*$ and $j = j^*$. Let us start with the former case. To show that inequality holds, we need make an assumption. We will assume that $\alpha p \leq 1/3$, $\alpha q \leq 1/3$, and $\alpha(1-p-q) \leq 1/3$.

Let $j = j^* + k$, $k = 1, 2, \ldots$ and assume $\alpha p \leq 1/3$, $\alpha q \leq 1/3$, and $\alpha(1-p-q) \leq 1/3$. Then,

$$\begin{aligned}
& \alpha p v_{j^*+k+1} + \alpha q v_{j^*+k-1} + \alpha(1-p-q) v_{j^*+k} \\
= \; & \alpha p f_{j^*+k+1} + \alpha q f_{j^*+k-1} + \alpha(1-p-q) f_{j^*+k} \\
\leq \; & \frac{1}{3} f_{j^*+k+1} + \frac{1}{3} f_{j^*+k-1} + \frac{1}{3} f_{j^*+k} \\
= \; & \frac{1}{3}(f_{j^*} + (k+1)\Delta x) + \frac{1}{3}(f_{j^*} + (k-1)\Delta x) + \frac{1}{3}(f_{j^*} + k\Delta x) \\
= \; & f_{j^*} + k\Delta x \\
= \; & f_{j^*+k} \\
= \; & v_{j^*+k}.
\end{aligned}$$

The one just given suffices but is not necessary. A necessary and sufficient condition is obtained by recalling that $v_j = f_j$ for $j > j^*$. It is easy to see after some simple algebraic manipulation that for $j = j^* + k$, and $k = 1, 2, \ldots$ we have

$$\alpha p v_{j^*+k+1} + \alpha q v_{j^*+k-1} + \alpha(1-p-q) v_{j^*+k} \leq v_{j^*+k}$$
$$\alpha p f_{j^*+k+1} + \alpha q f_{j^*+k-1} + \alpha(1-p-q) f_{j^*+k} \leq f_{j^*+k}$$

Since

$$f_{j^*+k} = f_{j^*} + k\Delta x$$

We have

$$\begin{aligned}
f_{j^*} + k\Delta x \; & \geq \; \alpha p f_{j^*} + \alpha p(k+1)\Delta x + \alpha q f_{j^*} + \alpha q(k-1)\Delta x + \alpha(1-p-q) f_{j^*} \\
& \quad + \alpha(1-p-q) k\Delta x \\
f_{j^*} + k\Delta x \; & \geq \; \alpha p\Delta x - \alpha q\Delta x + \alpha f_{j^*} + \alpha k\Delta x \\
f_{j^*}(1-\alpha) \; & \geq \; \alpha\Delta x(p-q) + k(\alpha-1)\Delta x
\end{aligned}$$

Since $k(\alpha-1)\Delta x$ is negative ($\alpha < 1$), the left hand side is maximized at $k = 1$. Hence, (4.4) holds with $v_j = f_j$ if and only if

$$(1-\alpha) f_{j^*} \geq \alpha\Delta x(p-q) + (\alpha-1)\Delta x.$$

We'll come back to the inequality (4.4) for $j = j^*$ after we consider inequalities (4.3).

4.5.4 Inequalities 4.3

For $j \geq j^*$, $v_j = f_j$ from the formula for v_j. We need to show that $v_j \geq f_j$ for $j \leq j^*$. In order to have these inequalities hold for $j < j^*$, we need to pick

$$j^* \in K := \left\{ k : (f_k - S_1 \xi_-^k) \left(\frac{\xi_+^{k-1} - \xi_-^{k-1}}{\xi_+^k - \xi_-^k} \right) + S_1 \xi_-^{k-1} > f_{k-1} \right\}.$$

We need to assume that K is nonempty. We need to show that K has finite number of elements.In order to show it, let us observe the following inequality for $k \in K$:

$$(f_k - S_1 \xi_-^k) \left(\frac{\xi_+^{k-1} - \xi_-^{k-1}}{\xi_+^k - \xi_-^k} \right) + S_1 \xi_-^{k-1} \quad > \quad f_{k-1}$$

It can be written as:

$$\frac{\xi_+^{k-1} - \xi_-^{k-1}}{\xi_+^k - \xi_-^{k-1} \xi_-} \quad > \quad \frac{f_{k-1} - S_1 \xi_-^{k-1}}{f_k - S_1 \xi_-^k}$$

The LHS behaves as $1/\xi_-$ and the right hand as 1 for large k, which implies that k cannot grow without bound since $\xi_- > 1$.

For an h_j and the assumption that $j^* \in K$, we have that $v_{j^*} = h_{j^*}$ and $v_{j^*-1} > h_{j^*-1}$. Suppose that $v_{j'} < h_{j'}$ for some $j' < j^*$. Then the sequence $u_j := v_j - h_j$ must have a local maximum at some point, say k, strictly between j' and j^*. That is, $u_k > u_{k-1}$ and $u_k > u_{k+1}$. But, we also have

$$\begin{aligned}
u_k &= v_k - h_k \\
&= \alpha(p v_{k+1} + q v_{k-1} + (1-p-q) v_k) - \frac{1}{3}(h_{k+1} + h_{k-1} + h_k) \\
&\leq \frac{1}{3}(v_{k+1} + v_{k-1} + v_k) - \frac{1}{3}(h_{k+1} + h_{k-1} + h_k) \\
&= \frac{1}{3}(u_{k+1} + u_{k-1} + u_k) \\
&< u_k.
\end{aligned}$$

It is a contradiction. Therefore, u_j can't have a local maximum and therefore v_j cannot dip below h_j and this ends the proof.

4.5.5 Inequality 4.4 with $j = j^*$

Finally, to get inequality (4.4) for $j = j^*$, we need to assume that $j^* + 1 \notin K$, meaning that following inequality must hold:

$$(f_{j^*+1} - S_1\xi_-^{j^*+1})\left(\frac{\xi_+^{j^*} - \xi_-^{j^*}}{\xi_+^{j^*+1} - \xi_-^{j^*+1}}\right) + S_1\xi_-^{j^*} \leq f_{j^*}$$

Let w_j denote the solution to the difference equation

$$\begin{aligned} w_j - \alpha(pw_{j+1} + qw_{j-1} + (1-p-q)w_j) &= 0, \qquad 0 < j, \\ w_0 &= 0, \\ w_{j^*} &= f_{j^*}. \end{aligned}$$

This is the same as the difference equation we used to solve for v_j but extended to all j. Clearly we have $v_{j^*} = w_{j^*}$ and $v_{j^*-1} = w_{j^*-1}$. Hence, (4.4) at j^* will hold if and only if $v_{j^*+1} \leq w_{j^*+1}$:

$$f_{j^*+1} = v_{j^*+1} \leq w_{j^*+1} = (f_{j^*} - S_1\xi_-^{j^*})\left(\frac{\xi_+^{j^*+1} - \xi_-^{j^*+1}}{\xi_+^{j^*} - \xi_-^{j^*}}\right) + S_1\xi_-^{j^*+1}$$

The resulting inequality is clearly equivalent to the one above we are trying to show after some algebraic manipulation

A natural consequence of our analysis above is that j^* should be chosen as:

$$j^* := max\left\{k : (f_k - S_1\xi_-^k)\left(\frac{\xi_+^{k-1} - \xi_-^{k-1}}{\xi_+^k - \xi_-^k}\right) + S_1\xi_-^{k-1} > f_{k-1}\right\}.$$

Note that there is no S_2 in the closed form formula to calculate j^*. However, it externally bounds the j^* as it is not meaningful to wait further once stock price hits second strike price.

4.6 Numerical Results

Theoretically, we have shown that there exists a j^* such that after that critical point, the option holder should exercise the option in order to collect maximum gain among all other expected payoffs and we give a closed form formulation to calculate j^*.

If we solve the finite approximation of infinite dimensional LP for 20 states, for chosen parameters p, q, α, S_1, S_2 (shown in the definition of figure), the obtained results are presented in figure 4.1.

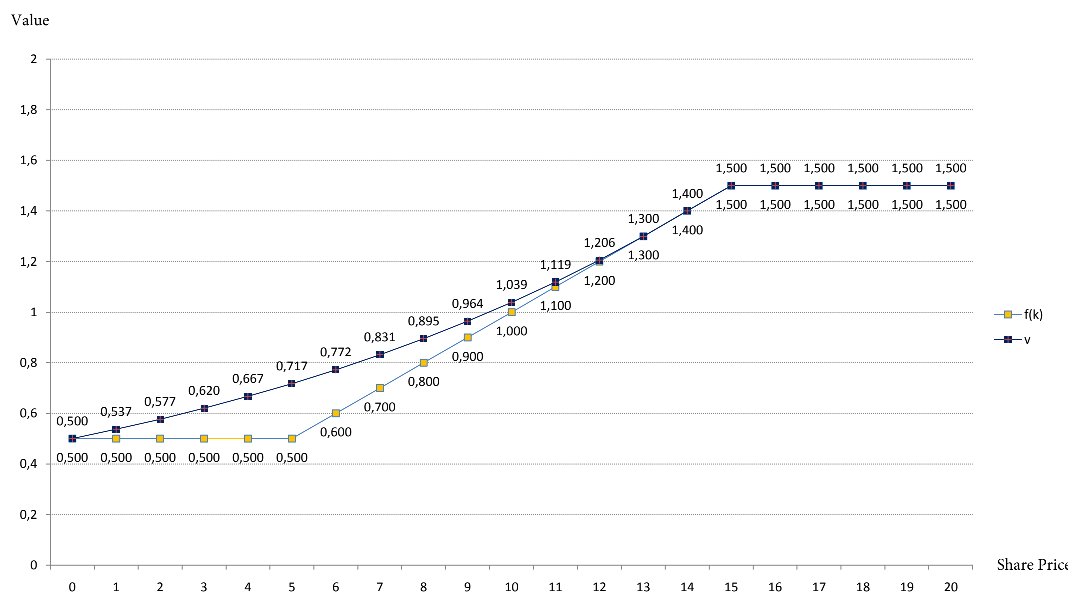

Figure 4.1: Value and pay-off function ($p = 0.35$, $q = 0.35$, $\alpha = 0.998$, $S_1 = 0.5$, $S_2 = 1.5$, $\Delta x = 0.1$).

Optimal v values and payoff function f are graphed. For chosen parameters, finite approximation of our LP model says that j^* is obtained at state 13, i.e., when stock price hits 1.3 (note that $\Delta x = 0.1$), option should be exercised.

One immediate observation from the figure is that, as claimed, until critical value, value function stands above the payoff function. It means that keeping option for one more period is more beneficial than exercising the option. In those states, $v_j > f_j$. After the critical point $j \geq j^*$, value function sticks to the payoff function ($v_j = f_j$) and continues to grow together.

For given parameters, proposed formula suggests to pick maximum index j

k	f(k)	ξ+(k-1)	ξ-(k-1)	ξ+(k)	ξ-(k)	result	f(k-1)	v
0	0,5							0,500
1	0,5	1,000	1,000	0,927	1,079	0,500	0,5	0,537
2	0,5	0,927	1,079	0,860	1,163	0,499	0,5	0,577
3	0,5	0,860	1,163	0,797	1,255	0,497	0,5	0,620
4	0,5	0,797	1,255	0,739	1,353	0,496	0,5	0,667
5	0,5	0,739	1,353	0,685	1,460	0,494	0,5	0,717
6	0,6	0,685	1,460	0,635	1,574	**0,575**	0,5	0,772
7	0,7	0,635	1,574	0,589	1,698	**0,661**	0,6	0,831
8	0,8	0,589	1,698	0,546	1,832	**0,749**	0,7	0,895
9	0,9	0,546	1,832	0,506	1,976	**0,839**	0,8	0,964
10	1	0,506	1,976	0,469	2,131	**0,930**	0,9	1,039
11	1,1	0,469	2,131	0,435	2,298	**1,022**	1	1,119
12	1,2	0,435	2,298	0,403	2,479	**1,114**	1,1	1,206
13	1,3	0,403	2,479	0,374	2,674	**1,206**	1,2	1,300
14	1,4	0,374	2,674	0,347	2,884	1,299	1,3	1,400
15	1,5	0,347	2,884	0,321	3,110	1,392	1,4	1,500
16	1,5	0,321	3,110	0,298	3,355	1,393	1,5	1,500
17	1,5	0,298	3,355	0,276	3,618	1,395	1,5	1,500
18	1,5	0,276	3,618	0,256	3,903	1,396	1,5	1,500
19	1,5	0,256	3,903	0,238	4,210	1,396	1,5	1,500
20	1,5	0,238	4,210	0,220	4,540	1,397	1,5	1,500

Figure 4.2: Calculation of j^* with proposed closed form formula.

as j^* which makes the LHS of the expression greater than f_{k-1}. In this example, state 13 is the last state that makes LHS greater than RHS, therefore 13 is chosen as j^*. (See figure 4.2)

Thus, results from the solution of LP model and output of closed form formula coincides. Hence, obtaining exercising strategy using proposed formula provides computational efficiency.

Note that the given closed form formula works even if the assumption $\alpha p \leq 1/3$ is violated. Low interest rates (α very close to one) and up probability greater than 1/3 are very likely in any stock market. Therefore, it is enough to check necessary and sufficient condition obtained which is as follows:

$$(1-\alpha)f_{j^*} \geq \alpha\Delta x(p-q)+(\alpha-1)\Delta x.$$

4.7 Sensitivity Analysis

In this section, we will investigate how exercising strategy can be affected by the changes in parameters.

4.7.1 Sensitivity on p and q

Probabilities p, q and $1-p-q$ represent the market projection for the future of stock price. If up probability p is higher than q, it means that in the long run, it is more probable that stock price increases.

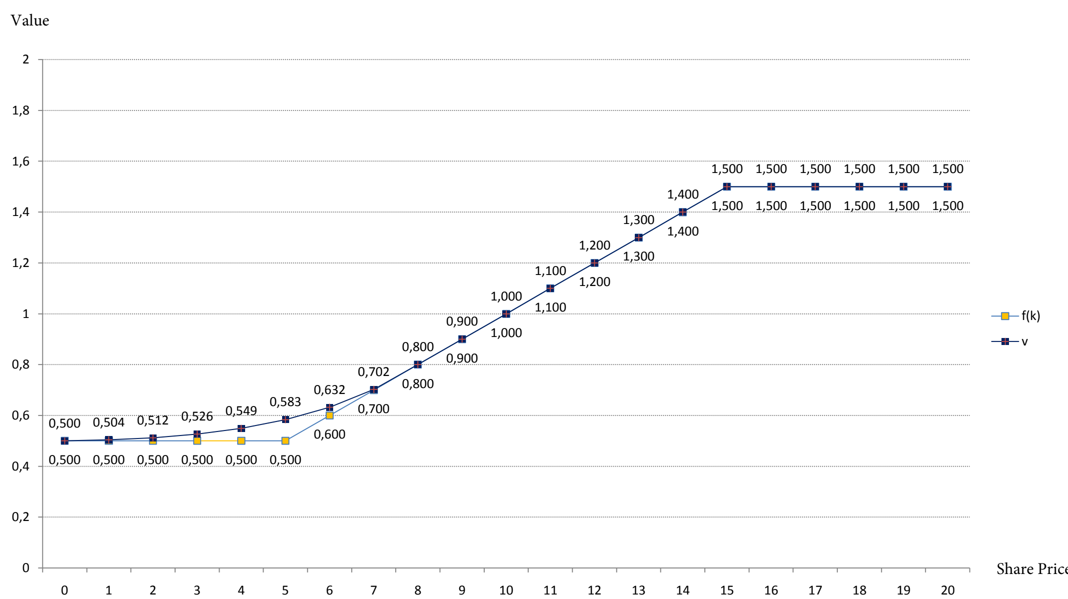

Figure 4.3: Value and pay-off function ($p = 0.3$, $q = 0.4$, $\alpha = 0.998$, $S_1 = 0.5$, $S_2 = 1.5$, $\Delta x = 0.1$).

By keeping all other parameters same as in the example above, we just change up and down probabilities to see their impacts on j^*.

If we decrease up probability to 0.3 and increase down probability to 0.4, j^* is chosen as 8 instead of 13. (See figure 4.3) It means that, since the probability that stock prices go up is relatively lower, it is more beneficial to exercise the option at an earlier state.

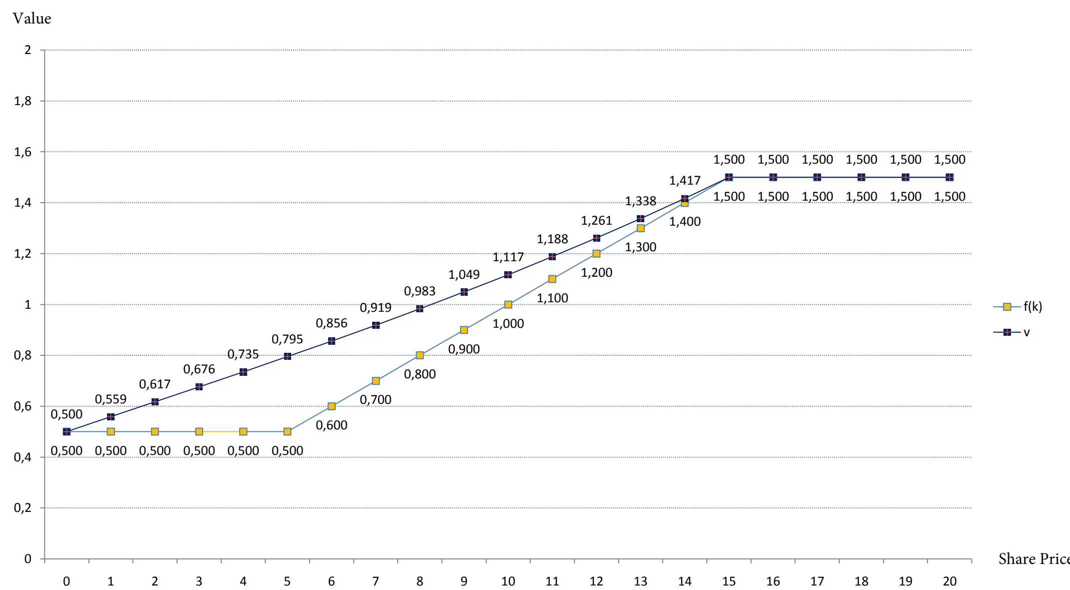

Figure 4.4: Value and pay-off function ($p = 0.36$, $q = 0.34$, $\alpha = 0.998$, $S_1 = 0.5$, $S_2 = 1.5$, $\Delta x = 0.1$).

In figure 4.4, we graphed the opposite case, i.e., probability that stock price goes up is relatively higher than the one in main example. As expected, optimal strategy is to wait more until $j^* = 15$ as market is relatively more favorable for this stock.

4.7.2 Sensitivity on α

The parameter α stands for time value of money in the model. Our model is very much dependent on the choice of α. For small and high (very close to 1) α values, j^* will position at the extremes, at S_1, or S_2.

We investigate the cases where $\alpha = 0.99$, and $\alpha = 0.999$. Former is lower than the one in main example and latter is higher. Resulting graphs can be seen in figure 4.5, and figure 4.6, respectively.

In the first case, money deflates at a higher rate compared to the main example, therefore keeping option to obtain higher return is not a good strategy. Therefore, model suggests a critical point which is earlier. (exercise when $j^* = 8$)

In the second case, larger α value favors the strategy to wait for obtaining higher stock prices to gain more. It is optimal to exercise the option when stock price hits 1.5.

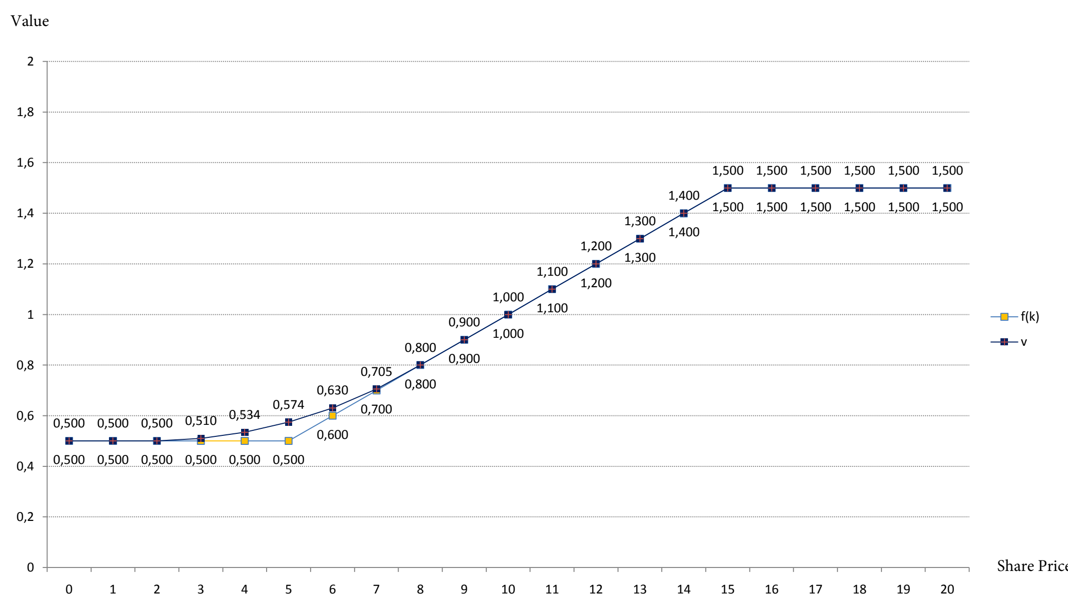

Figure 4.5: Value and pay-off function ($p = 0.35$, $q = 0.35$, $\alpha = 0.99$, $S_1 = 0.5$, $S_2 = 1.5$, $\Delta x = 0.1$).

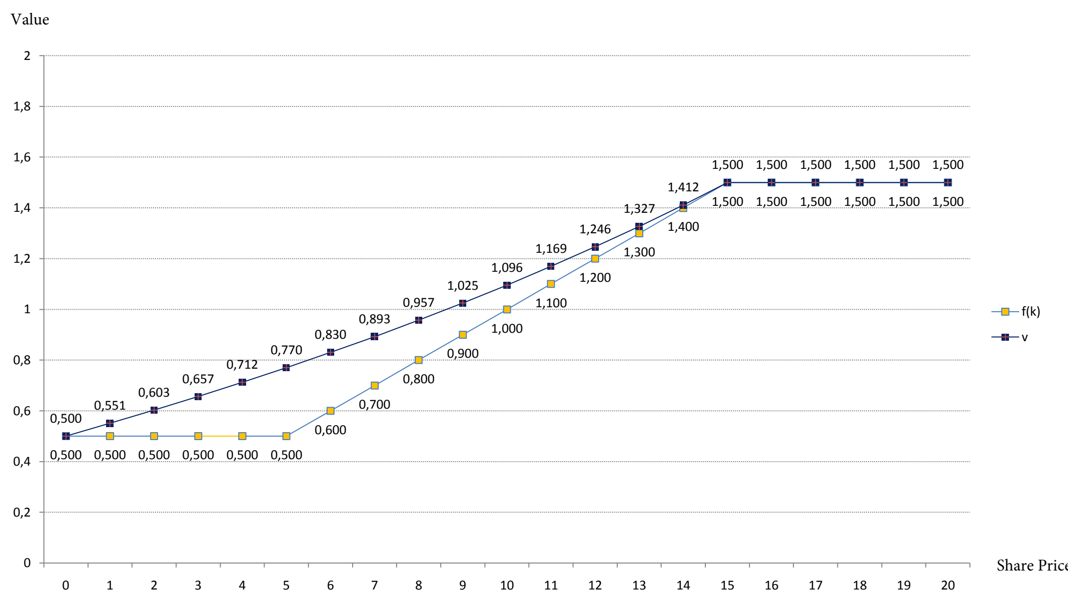

Figure 4.6: Value and pay-off function ($p = 0.35$, $q = 0.35$, $\alpha = 0.999$, $S_1 = 0.5$, $S_2 = 1.5$, $\Delta x = 0.1$).

4.7.3 Sensitivity on S_1

S_1 is the first strike price of collar type option. It is the minimum amount that the option holder can get in the worst case.

Figure 4.7 and 4.8 represents two cases when $S_1 = 0.3$, and $S_1 = 0.8$. For lower S_1 values, value function sticks to the payoff earlier, therefore option holder should exercise the option at earlier states.

For higher S_1's, the holder starts to gain at relatively higher stock prices which leads holder to wait more to obtain the optimal gain.

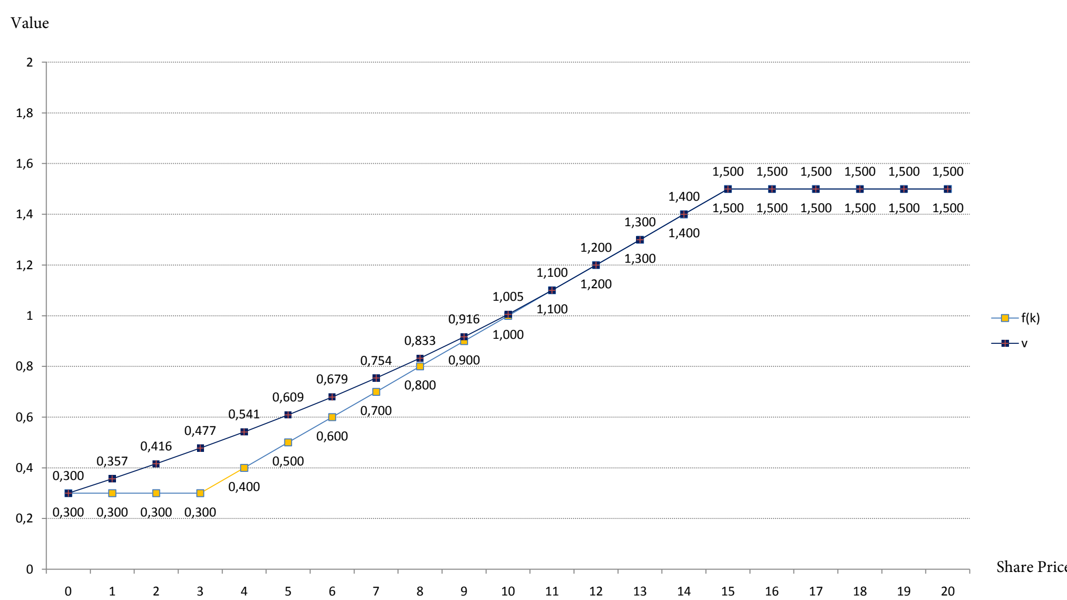

Figure 4.7: Value and pay-off function ($p = 0.35$, $q = 0.35$, $\alpha = 0.998$, $S_1 = 0.3$, $S_2 = 1.5$, $\Delta x = 0.1$).

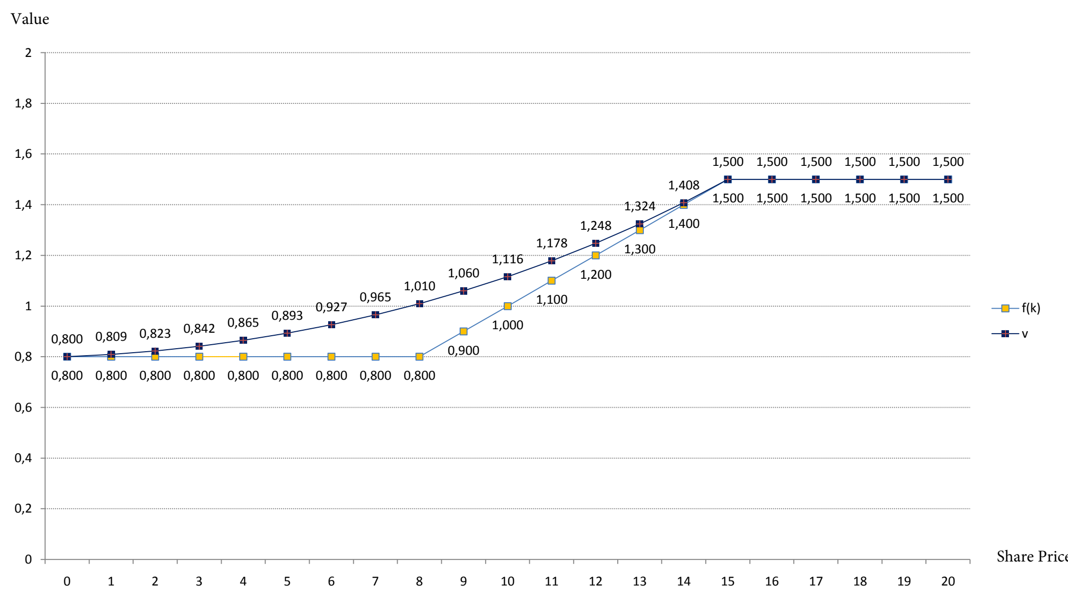

Figure 4.8: Value and pay-off function ($p = 0.35$, $q = 0.35$, $\alpha = 0.998$, $S_1 = 0.8$, $S_2 = 1.5$, $\Delta x = 0.1$).

4.7.4 Sensitivity on S_2

S_2 is the second boundary on the option payoff function. It is not present in the closed form formula to calculate j^*. However, it externally bounds the j^* as it is not meaningful to wait further once stock price hits second strike price. It is the ultimate amount that can be obtained by the option holder and if the holder does not exercise the option at that time, (s)he will lose money when stock price reaches the same level again because of time value of money.

In figure 4.9, when $S_2 = 1.0$, value function sticks to the pay-off at state 10. Recall that in the main example, it was state 13. Here, S_2 bounds the value function and forces it to equalize with option payoff at an earlier state.

As in figure 4.10, increasing S_2 from 1.5 to 1.8 does not affect the result. It behaves like a redundant constraint since j^* was already picked as 13 when $S_2 = 1.5$ in the main example, so no change in other parameters results with the same j^* even if S_2 is increased.

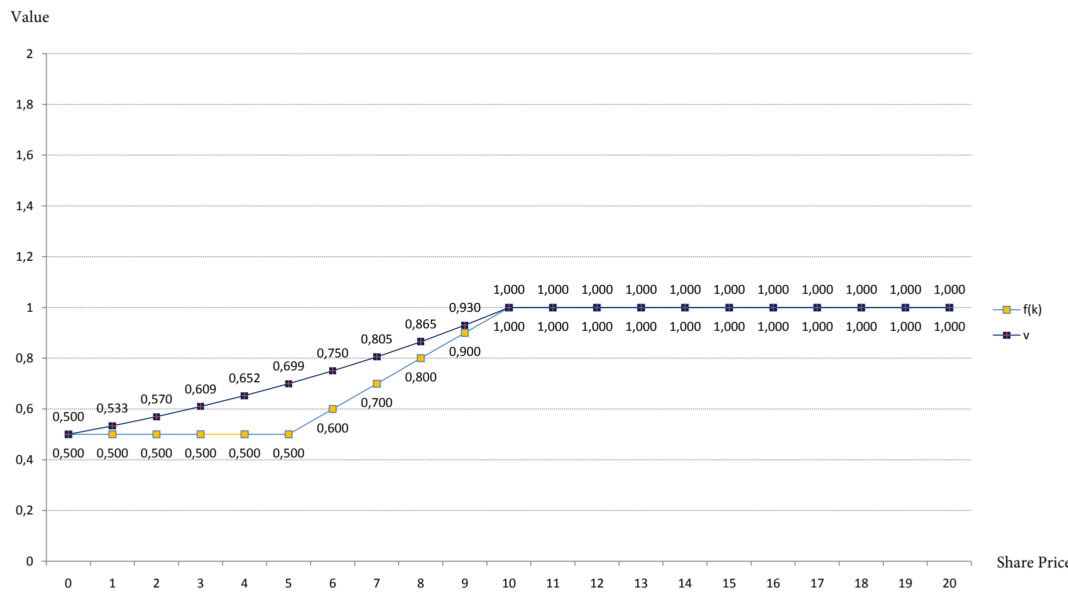

Figure 4.9: Value and pay-off function ($p = 0.35$, $q = 0.35$, $\alpha = 0.998$, $S_1 = 0.5$, $S_2 = 1.0$, $\Delta x = 0.1$).

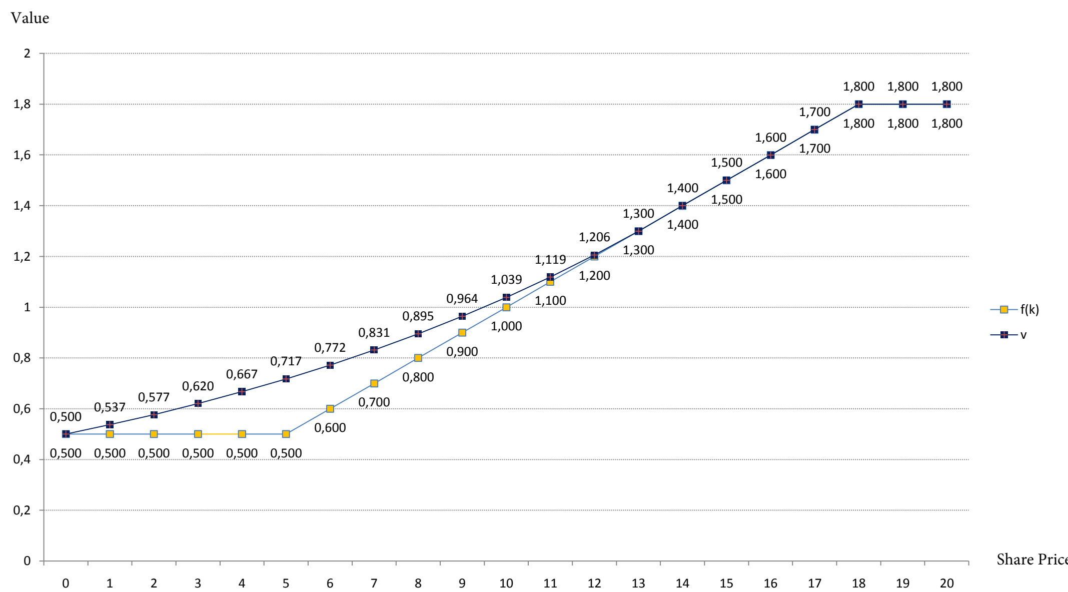

Figure 4.10: Value and pay-off function ($p = 0.35$, $q = 0.35$, $\alpha = 0.998$, $S_1 = 0.5$, $S_2 = 1.8$, $\Delta x = 0.1$).

Chapter 5

Optimal Stopping for Multiple Type Options

Multiple type options are options which are written on more than one asset. A standard example is the case of an index option that is based on the value of a portfolio of assets. Options on the S&P 100, which have traded on the Chicago Board of Options Exchange (CBOE) are American options on a value weighted index of 100 stocks. Options on the maximum of two or more assets can be found in firms choosing among mutually exclusive investment alternatives, or in employment switching decisions by agents. Gasoline crack spread options traded in New York Mercantile Exchange (NYMEX) are examples for multiple type spread options.

In this thesis, we will study multiple type options written on two assets. We will consider 4 different types of multiple options. These are maximum, minimum, average, and spread options. In the main part, we will work on maximum call options which have payoff structures as in the following:

$$f(x,w) = max\{max\{X(x), W(w)\} - S, 0\} = (max\{X(x), W(w)\} - S)^+$$

There are two underlying stocks, denoted as X, and W, and the option holder gain the difference between maximum of two stock prices and strike price if the maximum of those stock prices is greater than strike price; (s)he does not exercise the option if it is less than strike, because in that case, there is no gain.

In this chapter, we will model the problem again with an infinite dimensional LP as in Chapter 4. Note that we need two indices to include both stocks in our model and it makes the problem complicated to solve by following the same methodology. Instead of coming up with a closed form formula to calculate critical optimal stopping point here, we will explore the exercise region for different kinds of multiple type options.

We assume that both stocks follow different triple random walk process. We will use p_x, and p_w for up probabilities for stock X, and W, respectively. In a same manner, we will use q_x, and q_w for down probabilities. And we will define r_x, and r_w for $1-(p_x+q_x)$, and $1-(p_w+q_w)$ to simplify the model.

Based on those assumptions, by the fundamental theorem, we model the primal problem as follows:

5.1 Primal Problem

The primal problem is

$$\begin{array}{lll} \textit{minimize} & \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} v_{i,j} & \\ \textit{s.t.} & v_{i,j} \geq f_{i,j}, & i,j \geq 0, \\ & v_{i,j} \geq \alpha(p_x p_w v_{i+1,j+1} + r_x p_w v_{i,j+1} + q_x p_w v_{i-1,j+1} & \\ & +p_x r_w v_{i+1,j} + r_x r_w v_{i,j} + q_x r_w v_{i-1,j} & \\ & +p_x q_w v_{i+1,j-1} + r_x q_w v_{i,j-1} + q_x q_w v_{i-1,j-1}), & i,j \geq 1. \end{array}$$

where $x_i = i\Delta x$, $w_j = j\Delta x$, $v_{i,j} = v(x_i, w_j)$, $f_{i,j} = f(x_i, w_j)$, and $f(x_i, w_j) = max\{max\{x_i, w_j\} - S, 0\}$.

5.2 Dual Problem

The associated dual problem is

$$
\begin{array}{lll}
\textit{maximize} & \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{i,j}y_{i,j} & \\
\textit{s.t.} & y_{0,0} - \alpha q_x q_w z_{1,1} = 1, & \\
& y_{1,1} + (1-\alpha r_x r_w) z_{1,1} - \alpha q_x q_w z_{2,2} - \alpha r_x q_w z_{1,2} - \alpha q_x r_w z_{2,1} = 1, & \\
& y_{0,1} - \alpha q_x r_w z_{1,1} - \alpha q_x q_w z_{1,2} = 1, & \\
& y_{1,0} - \alpha r_x q_w z_{1,1} - \alpha q_x q_w z_{2,1} = 1, & \\
& y_{0,j} - \alpha q_x p_w z_{1,j-1} - \alpha q_x r_w z_{1,j} - \alpha q_x q_w z_{1,j+1} = 1, & j \geq 2, \\
& y_{i,0} - \alpha p_x q_w z_{i-1,1} - \alpha r_x q_w z_{i,1} - \alpha q_x q_w z_{i+1,1} = 1, & i \geq 2, \\
& y_{1,j} + (1-\alpha r_x r_w) z_{1,j} - \alpha r_x q_w z_{1,j+1} - \alpha q_x q_w z_{2,j+1} - \alpha q_x r_w z_{2,j} & \\
& -\alpha q_x p_w z_{2,j-1} - \alpha r_x p_w z_{1,j-1} = 1, & j \geq 2, \\
& y_{i,1} + (1-\alpha r_x r_w) z_{i,1} - \alpha q_x r_w z_{i+1,1} - \alpha q_x q_w z_{i+1,2} - \alpha r_x q_w z_{i,2} & \\
& -\alpha p_x q_w z_{i-1,2} - \alpha p_x r_w z_{i-1,1} = 1, & i \geq 2, \\
& y_{i,j} + (1-\alpha r_x r_w) z_{i,j} - \alpha q_x q_w z_{i+1,j+1} - \alpha r_x q_w z_{i,j+1} - \alpha p_x q_w z_{i-1,j+1} & \\
& -\alpha q_x r_w z_{i+1,j} - \alpha p_x r_w z_{i-1,j} - \alpha q_x p_w z_{i+1,j-1} & \\
& -\alpha r_x p_w z_{i,j-1} - \alpha p_x p_w z_{i-1,j-1} = 1, & i,j \geq 2, \\
& y_{i,j} \geq 0, & i,j \geq 0, \\
& z_{i,j} \geq 0, & i,j \geq 1.
\end{array}
$$

5.3 Complementary Slackness Conditions

Based on these primal and dual problems, we can write CS conditions. We skip to write all conditions but the important one:

$$
\begin{aligned}
(v_{i,j} \quad & - \quad (\alpha(p_x p_w v_{i+1,j+1} + r_x p_w v_{i,j+1} + q_x p_w v_{i-1,j+1} \\
& + \quad p_x r_w v_{i+1,j} + r_x r_w v_{i,j} + q_x r_w v_{i-1,j} \\
& + \quad p_x q_w v_{i+1,j-1} + r_x q_w v_{i,j-1} + q_x q_w v_{i-1,j-1}))z_j = 0, i,j \geq 0
\end{aligned}
$$

Crucial part to solve this problem is to solve the following second order homogeneous difference equation

$$\begin{aligned} v_{i,j} &- (\alpha(p_x p_w v_{i+1,j+1} + r_x p_w v_{i,j+1} + q_x p_w v_{i-1,j+1} \\ &+ p_x r_w v_{i+1,j} + r_x r_w v_{i,j} + q_x r_w v_{i-1,j} \\ &+ p_x q_w v_{i+1,j-1} + r_x q_w v_{i,j-1} + q_x q_w v_{i-1,j-1}) = 0, i, j \geq 0 \end{aligned}$$

with the boundary conditions:

$$v_{0,0} = 0,$$

$$v_{i^*,j^*} = f_{i^*,j^*}.$$

We could not solve this equation to come up with a similar closed form formula for v_{i^*,j^*}, i^*, and j^*. We leave this part as a further research area. Instead, we will use the primal problem and explore exercise regions for maximum, minimum, average, and spread type multiple options. It will reveal the optimal exercising strategy for the option holder by telling at which (i, j) stock price pairs (s)he should exercise the option.

5.4 Exercise Regions

An increasing number of contractual rewards and obligations, held by economic entities, involve options that are written on multiple underlying assets. Some of the payoffs that have become fairly common include options on the maximum of several asset prices (max-options), options on the minimum of several prices (min-options), options on an average of prices (portfolio options) and options on the difference between two prices (spread options)[17]. Contingent claims with these characteristics can be found on financial exchanges, in over-the-counter transactions, in cash flows generated by investment decisions of firms and in executive compensation plans.

For options written on single assets, we have shown that there exists a critical point after which it is optimal to exercise the option. Intuition suggests that the same must be true for multiple type options. However, the determination of optimal strategy for multiple type options is not as straightforward as in the single asset case. The presence of multiple prices, that jointly determine the exercise region, is an obvious complication. Some of the basic insights from the single asset type options do not carry over when several prices affect the payoff.

Let us give an example for such a situation. The fundamental intuition from single asset case is that for large stock price values, it is optimal to exercise the option. However, if we study maximum call options, they have a property called *diagonal property*. On the diagonal of the stock price pairs matrix, stock prices are equal to each other. For the sake of simplicity, let us assume that both stocks follow the same simple random walk with both up and down probabilities are equal to 0.5. Then, the probability of an increase in any one of the underlying asset prices over the next increment of time is roughly equal to 0.5. But the probability of an increase in the maximum of two prices is about 0.75. The max-option holder has therefore much better chance of improving his/her payoff by waiting, than the holder of an option on a single underlying asset. Even if the stock prices are high, it might not be optimal to exercise the option on the diagonal for max-call options.

The multivariate nature of these claims ultimately results in complex decision making. Precise description of the exercise regions become useful for accurate decision making, as well as for pricing and risk management purposes.

We solve the finite approximation of our infinite dimensional model for some chosen parameters and explore the exercise and continuation region for four different multiple type options. Note that 1's in the figures represent the stock price pairs at which the options should be exercised, meaning that collection of 1's in the figures stands for exercise region. Empty regions stand for continuation region in which option holder should keep the option.

5.4.1 Maximum Call Options

The payoff function of maximum call options is represented as:

$$f(x_i, w_j) = max\{max\{x_i, w_j\} - S, 0\}.$$

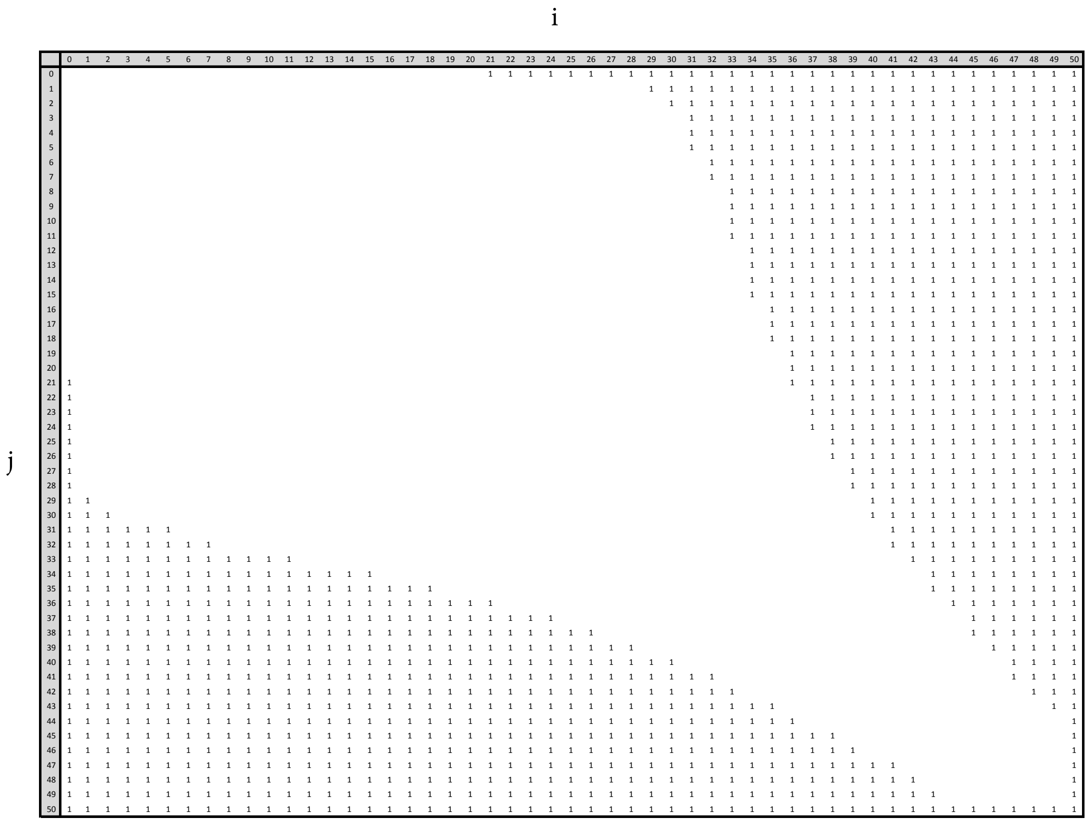

Figure 5.1: Exercise Region for Maximum Call Options ($p = 0.35$, $q = 0.35$, $\alpha = 0.998$, $S = 2$, $\Delta x = 0.1$).

5.4.2 Minimum Call Options

The payoff function of minimum call options is as follows:

$$f(x_i, w_j) = max\{min\{x_i, w_j\} - S, 0\}.$$

i

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
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Figure 5.2: Exercise Region for Minimum Call Options ($p = 0.35$, $q = 0.35$, $\alpha = 0.998$, $S = 2$, $\Delta x = 0.1$).

5.4.3 Average Call Options

The payoff function of (arithmetic) average call options is:

$$f(x_i, w_j) = max\{average\{x_i, w_j\} - S, 0\}.$$

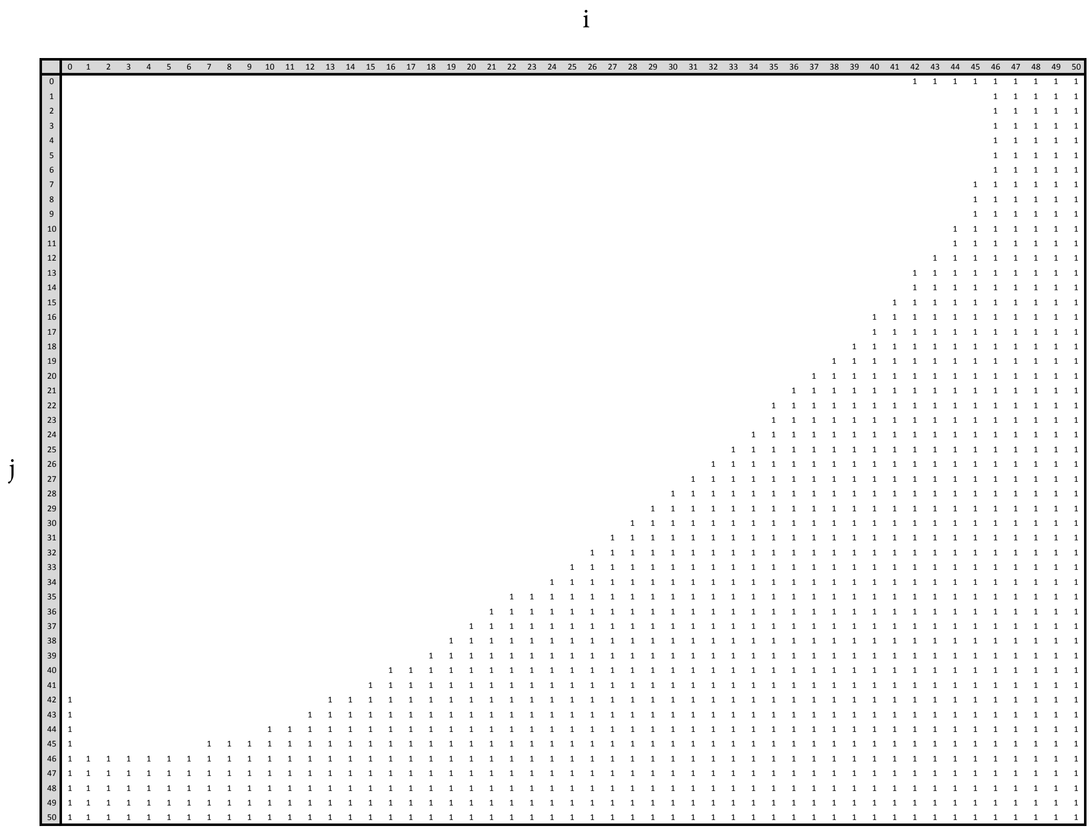

Figure 5.3: Exercise Region for Average Call Options ($p = 0.35$, $q = 0.35$, $\alpha = 0.998$, $S = 2$, $\Delta x = 0.1$).

5.4.4 Spread Options

Spread options have the following payoff structure:

$$f(x_i, w_j) = max\{x_i - w_j - S, 0\}.$$

i

j

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
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Figure 5.4: Exercise Region for Spread Call Options ($p = 0.35$, $q = 0.35$, $\alpha = 0.998$, $S = 2$, $\Delta x = 0.1$).

Chapter 6

Conclusion

In this thesis, we have studied the optimal stopping problem of perpetual American call options. American options can be exercised any time in the option contract period and perpetual American options have no maturity date. The optimal stopping problem of perpetual American options is to find set of stock price states that the option holder should exercise the option to get maximum gain. We assume that stock price follows discrete time discrete state Markov process, especially triple random walk where stock can go up, down, or remain same with some probabilities. The problem has been modeled with an infinite dimensional linear program and it has been solved to find a closed form formula to find exercise region by the linear programming duality and solution of difference equations.

This thesis mainly consists of two parts. In the first part, we study collar type perpetual American call options and provide a closed form formula to calculate critical point j^* after which it is optimal to exercise the option. In addition, sensitivity analysis on the problem has been provided to see the impact of parameter changes on the exercise region.

In the second part, we study a more complex problem where the option is written on multiple stocks, called multiple type options. The problem has been modeled again by infinite dimensional linear program, yet for this type of problem, we could not come up with a closed form formula to define the exercise region. Instead, we study 4 different types of multiple options; maximum, minimum, average, and spread options and explore the exercise regions by using the finite approximation of the linear program.

Optimal stopping problem on American options is a well-known problem in the literature. However, approaching this problem in linear programming perspective is fairly new approach. In this subject, ground work has been done in [15] and we have further shown that the study can be extended to different random walk models and option types. The study also is extended to options on multiple assets.

We will conclude the thesis by providing further research potentials arising from the work that has been done in this thesis. The methodology conducted to solve collar type options can be applied to different single asset options. It only requires a little modification on the boundary conditions during the solution process of difference equations, yet a formal study is needed. Another important extension would be to solve the difference equation that remained unsolved to come up closed form solution for multiple type options. It requires deeper differential equation knowledge compared to single asset type options case.

Bibliography

[1] J. C. Hull, *Options, Futures, and Other Derivatives.* Pearson Prentice Hall, 2005.

[2] D. Hobson, “A survey of mathematical finance,” *Proceedings: Mathematical, Physical and Engineering Sciences*, vol. 460, no. 2052, pp. 3369–3401, 2004.

[3] R. Myneni, “The pricing of the American option,” *The Annals of Applied Probability*, vol. 2, no. 1, pp. 1–23, 1992.

[4] L. Bachelier, *Theory of Speculation (translation of 1900 French edition).* The M.I.T. Press, 1964. Republished in P.H. Cootner, editor, *The Random Character of Stock Market Prices.*

[5] P. Samuelson, “Rational theory of warrant pricing,” *Industrial Management Review*, vol. 6, pp. 13–31, 1965.

[6] F. Black and M.Scholes, “The pricing of options and corporate liabilities,” *Journal of Political Economy*, vol. 81, no. 3, pp. 637–654, 1972.

[7] R. Merton, “Theory of rational option pricing,” *The Bell Journal of Economics and Management Science*, vol. 4, no. 1, pp. 141–183, 1973.

[8] P. A. Samuelson, “Rational theory of warrant pricing,” *Industrial Management Review*, vol. 6, pp. 13–31, 1965. Appendix by H. P. McKean, pp. 32–39.

[9] P. van Moerbeke, “On optimal stopping and free boundary problems,” *Archive for Rational Mechanics and Analysis*, vol. 60, pp. 101–148, 1976.

[10] I. Karatzas, "On the pricing of American options," *Applied Mathematics and Optimization*, vol. 17, pp. 37–60, 1988.

[11] D. Wong, *Generalized Optimal Stopping Problems and Financial Markets.* Addison Wesley Longman, 1996.

[12] E. B. Dynkin and A. A. Yushkevich, *Markov Processes: Theorems and Problems.* Plenum Press, 1969. Translated from Russian by James S. Wood.

[13] E. Çınlar, *Introduction to Stochastic Processes.* Prentice-Hall, 1975.

[14] M. Taksar, "Infinite dimensional linear programming approach to singular stochastic control," *SIAM Journal on Control and Optimization*, vol. 35, no. 2, pp. 604–625, 1997.

[15] R. Vanderbei and M. Ç. Pınar, "Pricing American perpetual warrants by linear programming," *SIAM Review*, vol. 51, no. 4, pp. 767–782, 2009.

[16] A. Pascucci and W. Runggaldier, *Financial Mathematics: Theory and Problems for Multi-period Models.* Springer, 2012.

[17] J. Detemple, *American-Style Derivatives: Valuation and Computation.* Chapman and Hall/CRC, 2006.

Appendix A

GAMS code for the model of collar type options

```
Set J/0*20/;

Parameter mesh(j);
mesh(j) = 0;
mesh(j) = 0.1*(ord(j)-1);

Parameters strike1, strike2;

strike1 = 0.5;
strike2 = 1.5;

Parameters f(j), alpha, p, q;

alpha = 0.998;
p = 0.35 ;
q = 0.35 ;

f(j) = min(max(mesh(j),strike1), strike2);
```

```
Variables z ;

positive variables v(j);

Equations obj, eq1(j), eq2(j);

obj.. z =e= sum(j,v(j));
eq1(j).. v(j) =g= f(j);
eq2(j)$(ord(j) gt 1).. v(j) =g= alpha*(p*v(j+1) + q*v(j-1)
+ (1-p-q)*v(j));

Model collar /all/;

Solve collar minimizing z using lp;

execute_unload "collar.gdx" v.L

execute 'gdxxrw.exe collar.gdx var = v.L'
```

Appendix B

GAMS code for the model of maximum call options

```
Set i /0*50/ ;
Set j /0*50/ ;

Parameter x(i) ;
Parameter w(j) ;

Parameter deltax ;
Parameter deltaw ;

deltax = 0.1 ;
deltaw = 0.1 ;

x(i) = deltax * (ord(i)-1) ;
w(j) = deltaw * (ord(j)-1) ;

Display x ;
Display w ;
```

```
Parameter strike ;
strike = 2 ;

Parameter alpha ;
alpha = 0.998 ;

Parameter f(i,j) ;
f(i,j) = max(0, max(x(i), w(j)) - strike) ;
Display f;

Variable z ;
Positive Variable v(i,j) ;

Parameters px, rx, qx ;
Parameters pw, rw, qw ;

px = 0.35 ;
qx = 0.35 ;
rx = 1 - (px + qx) ;

pw = 0.35 ;
qw = 0.35 ;
rw = 1 - (pw + qw) ;

Equations obj, eq1(i,j), eq2(i,j) ;

obj.. z =e= sum((i,j), v(i,j)) ;
eq1(i,j).. v(i,j) =g= f(i,j) ;
eq2(i,j)$(ord(i) gt 1 and ord(j) gt 1).. v(i,j) =g=
alpha * (px*pw*v(i+1,j+1)+ rx*pw*v(i,j+1) + qx*pw*v(i-1,j+1)
+ px*rw*v(i+1,j) + rx*rw*v(i,j)   + qx*rw*v(i-1,j)
+ px*qw*v(i+1,j-1) + rx*qw*v(i,j-1) + qx*qw*v(i-1,j-1));
```

```
Model maxcall /all/;

Solve maxcall minimizing z using lp;

execute_unload "maxcall.gdx" v.L, f ;

execute 'gdxxrw.exe maxcall.gdx var = v.L rng valuefunc!A1:AZ52' ;
```