CHARACTERIZATION OF ENVY FREE SOLUTIONS FOR QUEUING PROBLEMS

A Master's Thesis

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by

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In Partial Fulfilment of the Requirements for the Degree of
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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

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September 2006

In this study we are working on queuing problems. In our model a solution to a queuing problem is an ordering of agents and a transfer vector where the sum of the transfers of agents is equal to zero. Hence a queuing problem is a double, where we have a finite set of agents and a profile of payoff functions of agents which represent their preferences on their orderings and transfers. We are assuming that the payoff functions of agents are quasi-linear on transfers. Our main aim is to find envy free solutions for queuing problems. Since payoff functions of agents are quasi-linear envy freeness implies Pareto efficiency. For problems where there are less than five agents, we show that the set of envy free solutions is not empty and we are able to characterize the envy free solutions. We conjecture that our results may be extended to general case similar to our extension from three person case to four person case. When we assume that a queuing problem satisfies order preservation property we are able to characterize envy free solutions with a solution concept that we introduce in this study.

Keywords: Queuing Problems, No-envy.

ÖZET

SIRALAMA PROBLEMLERİNDE KISKANÇLIGI ÖNLEYEN ÇÖZÜMLERİN KARAKTERİZASYONU

Esmerok, İbrahim Barıs

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Bu çalısmada sıralama problemlerinde kişiler arasında kıskançlığı önleyen çözümler üzerinde çalışılmıştır. Bizim çalışmamızda kişi kümeleri sabit tutulmuş ve kişilerin sıralamalar ve transferler üzerinde ki tercihlerini gösteren değer fonksiyonları transferlerde quasi-linear kabul edilmiştir. Değer fonksiyonları transferlerde quasi-linear varsayıldığı için bir çözümün kıskançlığı önleyen bir çözüm olması onun Pareto en iyi olmasını da yanında getirmektedir. Bu çalışmada dört ya da dörtten daha az kişinin ele alındığı problemlerde kıskançlığı önleyen çözümler kümesinin boş olmadığı ve bu çözümlerin karakterize edilebileceği gösterilmektedir. Ele alınan problemlerin sıralamayı koruma özelliğini sağlaması durumunda ise kıskançlığı önleyen çözümlerin karaterizaysonu bu çalışmada önerilen çözüm kavramı sayesinde sağlanabilmektedir.

Anahtar Kelimeler: Sıralama Problemleri, kıskançlığı önleyen çözümler

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TABLE OF CONTENTS

Chapter 1

INTRODUCTION

Getting into queues and waiting for other people to complete their jobs to be able to execute one's own job is a familiar phenomenon to every one. Think of a facility which is capable of processing a certain job. People apply to the facility with jobs which last for a a certain amount of time. The total amount of time needed to execute jobs of all applicants could be greater than the capacity of the facility. So a facility may fail to process all of the jobs, or some of the jobs would be partially processed by the facility. Everyone would like to have their jobs processed as much as possible. For example in libraries there are some materials which students would like to use extensively especially when examinations are coming, but libraries usually do have a copy or two. In such cases libraries usually make this material available only in library use and put time quotas to students. Most of the students in such times are able to use the material less than they have planned. Thus whenever a group of people aim to use such facility the available usage time for each individual should be indicated. Most of the time facilities process jobs one by one so one may have to wait for other person's job to be processed. Everyone would like to wait as little as possible. Most of the computer laboratories has only one super computer. Students in laboratory sessions would like to use the super computer but the length of the session would be less than the total amount of time that students request to use the computer. In this case an arrangement of computer usage times is necessary. Since there is only one computer, students should use the computer one by one. Hence, besides an arrangement of computer usage times a computer using schedule, an ordering, is needed too. Whenever such a problem is faced, the planner of the facility should come up with how much of the requested jobs of each agent will be processed and in which order they will be processed.

The scenarios in the previous paragraph are examples of queuing problems. A problem is considered as a queuing problem whenever there is a facility which is capable of processing jobs one by one and there is a group of agents who has requested jobs from the facility. In the literature queuing problems are studied under different models. The difference between these models are usually on whether the facility has a limited capacity or not, and whether individuals have identical job lengths or not. In the model that Maniquet (2003), Chun (2004*a*), and Kayi & Ramaekers (2006) studies the capacity of the facility is unlimited, and job lengths of agents are identical. In Moulin (2004), Moulin (2005), and Özsoy (2005)'s model again the capacity of the facility is unlimited, but this time agents may have different job lengths. At first glance the first class of problems may be perceived as the subset of the second one, but the assumptions on preferences of agents in these two models are not in line with this perception. In the second model agents account for the time they wait similarly, it costs a dollar for each unit of time that they wait for. In the first model waiting cost of a unit of time could be different for the agents. Hence the first model considers more general preferences than the second one. Another important aspect of both of the models is that all jobs should be fully processed.

As an example to the first model we can think of a group of tourists who joins to a travel agency's trip to a certain touristic site. When this group arrives to their hotel they should be checked in to the hotel. Checking in procedure is more or less a standard procedure, so all tourists' requested job lengths are identical. Even though their requested job lengths are equal some individuals may think that after a long trip to the destination waiting for others to check in is more costly than others think, so waiting cost of a unit of time is different for everyone. An uncompleted checking in procedure is meaningless hence everyone's job should be fully processed.

A situation where a group of wounded men are brought to a hospital from a combat zone can be thought of an example for the second model. The time needed for minimal amount of medical treatment for each soldier would differ so requested job lengths of agents are different than each other. Not even getting the minimal amount of medical treatment does not mean much for a wounded soldier so partially processing a job is meaningless. Since all agents are in an urgent situation it can be considered that they should all get the treatment as quick as possible so they account the time they wait similarly.

Moulin (2004), Moulin (2005), and Özsoy (2005) characterize solution concepts which are immune to certain maneuvers by a group or groups of agents. The motivation behind these maneuvers are agents' aim to get higher benefits. *Merging*, *splitting*, and *partial transferring of jobs* are among these maneuvers. Before preceding to the results of their articles an understanding of these maneuvers is beneficial.

Merging is the case where a group of people choose an individual from their group; that individual declares a job length which is equal to the sum of the job lengths of the individuals in that group. Obviously if a scheduling rule favors longer jobs in ordering the agents, merging will be beneficial.

Splitting is the converse of merging. If an agent with a positive job length finds a group of agents who does not have any jobs to do or in other words who have job lengths equal to zero, the agent with the positive job length divides his job length into smaller time slots and distributes them to agents who have job lengths equal to zero. They all declare that they have such jobs to do. In this case if a scheduling rule favors shorter jobs, splitting would be beneficial under that rule.

Partial transferring of jobs is sort of in between merging and splitting. In this case a group of agents reallocate their job lengths, and try to benefit from it. It is clear that some agents can take more advantegous positions in the queue by using such maneuvers. Thus characterizing a scheduling rule which is invulnerable to such maneuvers is desirable.

Moulin (2004), Moulin (2005), focuses on these maneuvers. Before discussing Moulin's findings it would be wise to make a distinction between his works on this subject. Moulin makes a main distinction between types of scheduling rules. A scheduling rule is deterministic if a planner assigns a schedule to each problem. Under deterministic rules agents know their exact places in the queue. A scheduling rule is non-deterministic or randomized if a planner assigns a probability distribution over all possible orderings. Under randomized rules agents do not know their exact places in the queue but they know with which probability they will be in any place of the queue.

Moulin (2005) deals with deterministic rules. He assumes that users have identical linear waiting costs. In this context, if an ordering minimizes the sum of waiting times of agents it is called an efficient schedule. To minimize the sum of waiting times an ordering should schedule shorter jobs earlier than longer jobs. Moulin points out that any efficient deterministic scheduling rule will fail to satisfy merge-proofness or split-proofness. To overcome this drawback he introduces scheduling fees or in other words monetary transfers to the model.

Moulin defines monetary transfers as taking money from some individuals and giving money to others. He designs this transfers in a way that sum of tranfers is equal to zero, hence situations where money collected is greater than money allocated or vice versa are ruled out. After adding possibility of transfers to the model agents start to consider the time they will wait until their job is executed and their monetary transfer. To be able to form a link between waiting times of individuals and their transfers Moulin makes further assumptions. He assumes that each individual is incuring a waiting cost of a dollar per unit of time until his job is completed. Thus the cost of an individual is his waiting time minus his transfer.

Moulin introduces the concepts of scheduling mechanisms and scheduling methods. A mechanism assigns an ordering of the agents and a tranfer vector to each problem. A method assigns a profile of net waiting costs to each problem. The sum of the waiting costs is equal to the sum of the requested job lengths of agents plus sum of the net waiting times of each agents. Net waiting time of an agent in a certain ordering is the sum of the job lengths of agents who preceed him in that ordering. Since a mechanism assigns an ordering of the agents and a tranfer vector, by adding the transfer of an agent to his waiting time we can find his waiting cost and by subtracting his requested job length from his waiting cost we reach to his net waiting cost. So to any given mechanism a method can be associated.

Equal treatment of equals, and *continuity* are among nice properties which a scheduling rule may satisfy. A mechanism satisfies *equal treatment of equals* (ETE) if the requested job lengths of any two agents are equal then their induced waiting costs are equal. A mechanism is *continuous* if mapping from requested job lengths to waiting costs is continuous.

Moulin shows that when there are more than four agents any efficient mechanism satisfying equal treatment of equals cannot be both merge-proof and split-proof. Replacing equal treatment of equals with continuoutiy leads to the same outcome. In other words if one wants a mechanism to be both merge-proof and split-proof then this mechanism will fail to satisfy both equal treament of equals and continuity.

Another contribution of this paper is about relationships between merge-proofness, split-proofness and some equity conditions. These equity conditions are:

Stand alone bound**;** A method satisfies stand alone bound if for any agent, the total waiting cost plus his transfer is greater than or equal to his requested job length.

Ranking**:** A method satisfies ranking if agent i's requested job length is less than or equal to agent j's requested job length then agent i's cost is less than or equal to agent j's cost.

Monotonicity**:** A method satisfies monotonicity if requested job length of an agent increases, his cost increases or stays the same.

No charge for null jobs**:** A method satisfies no charge for null jobs if an agent's requested job length approaches to zero his cost approaches to zero.

Finite liability**:** A method satisfies finite liability if all agents' costs are strictly less than infinity.

Moulin states that whenever there are more than four potential users; any efficient, continuous, split-proof mechanism which satisfies equal treatment of equals fails to satisfy stand alone bound, ranking, and monotonicity. Furthermore, if the set of potential users is infinite it fails to satisfy no charge for null jobs and finite liability.

Moulin defines a continuous function $\Theta : \mathbb{R}^2_{++} \to \mathbb{R}$ for all $a, b \in \mathbb{R}_+$ $\Theta(a,b) + \Theta(b,a) = \min\{a,b\}$. He designs two methods named S^+ and S^-. These two methods are Θ-separable scheduling methods. Respectively they are associated with Θ^+ and Θ^- which we define as : for all $a, b \in \mathbb{R}_+$

$$\Theta^+(a,b) = \frac{1}{2}\min\{a,b\}$$

$$\Theta^-(a,b) = \frac{1}{2}\max\{a,b\}.$$

It turns out that S^+ is merge-proof, and S^- is split-proof. Hence S^- violates the equity tests which are stated above, but S^+ satisfies all of them.

Moulin works on partial transfer of jobs mainly by taking into account the cases where transfers are only between two agents. He first defines ϵ-shrink and ϵ-spread of a problem. In forming the *ϵ-shrink* of a problem, take two agents and keep the requested job lengths of the other agents constant, partially transfer some job from the one of the two agents who has a higher job length to the other one in a way that the one who has a higher job length in the beginning still has a higher job length. The *ϵ-spread* of a problem is similar to the *ϵ-shrink* of a problem. In *ϵ-spread* of a problem the transfer is from the agent with a smaller job length to the one who has a higher job length. A mechanism is *pairwise shrink-proof* whenever for all problems, and for all ϵ-shrinks of that problem, total waiting costs of agents in the original problem is less than or equal to the total waiting costs of individuals in the transformed problem minus ϵ. A mechanism is *pairwise spread-proof* whenever for all problems, and for all ϵ-spreads of that problem, total waiting costs of agents in the original problem is less than or equal to the total waiting costs of agents in the transformed problem plus ϵ plus sum of the requested job lengths of agnets in between agents who has created ϵ-spread. A mechanism is called *pairwise transfer-proof* if it is both pairwise shrink-proof and pairwise transfer-proof. Moulin characterizes the mechanism S^+ by efficiency, continuity, pairwise transfer-proofness, and by the stand alone bound or no charge for null jobs. As a final contribution he states that if there are more than three users any efficient and continuos mechanism is vulnerable to job transfers involving two or three agents.

Moulin (2004) studies randomized rules. Just like Moulin (2005) there are infinite number of potential users, and a group of people from that group uses the facility. Each agent has a non-negative requested job length. The crucial difference between Moulin (2004) and Moulin (2005) is that in Moulin (2005) there are no transfers in the model instead there is a probability distribution over the set of possible orderings. A scheduling rule assigns a probability distribution over the set of orderings to every problem. Naturally agents in this model consider the expected completion time of their jobs. For each agent having a smaller expected waiting time is better than a longer one. A scheduling method assigns a profile of expected waiting times to each problem.

Moulin defines *quasi-proportional rules.* He defines ω which is a function on R_+ such that $\omega(0) = 0$ and for all $a > 0 \quad \omega(a) > 0$. Let x be the vector of requested job lengths of agents, for a requested job length vector such that all of the agents have job lengths greater than zero and for all $z \geq 0$ a ω-quasi-proportional rule draws as many as the cardinalty of the users independent random variables Z_i with respective cumulative distribution functions $F_i(z) = \min\{z^{\omega(x_i)}, 1\}$ for all $z \geq 0$ and orders jobs accordingly to the realization of these variables by breaking ties arbitrarly with uniform probability. When $x \geq 0$ and there are some components of x equal to 0 these agents with zero requested job lengths are ordered before the ones with positive requested job lengths.

If for all $a > 0$ we have $\omega(a) = a$ then this ω-quasi-proportional rule is the proportional rule. Proportional rule assigns a probability distribution to set of orderings. The probability of agent i being last ordered is calculated by dividing the requested job length of agent i to the sum of requested job lengths of all agents. The probability that agent j ordered just before last ordered agent i is equal to the requested job length of agent j over sum of the requested job lengths of all agents besides agent i. Hence for a given ordering the probability that agent l being in his place in that given ordering is equal to the requested job length of agent l over the sum of requested job lengths of agents preceding agent l plus the requested job length of agent l. So the probability of occurance of an ordering is calculated by multiplying probabilities of each agent being in his position in that ordering. Other well known rules can also be expressed as special cases of ω-quasi-proportional rules. If for all $a > 0$ we have $\omega(a) = 1$ then this ω-quasi-proportional rule is the uniform rule where probability of each ordering is equal. If for all $a > 0$ we have $\omega(a) = a^{\alpha}$, where $\alpha \in \mathbb{R}$. When the limit of α goes to $+\infty$ then this ω-quasi-proportional rule is the shortest job first rule where only possible orderings are the ones in which jobs are ordered from shortest job to longest one with equal probability. When the limit of α goes to $-\infty$ then this ω-quasi-proportional rule is the longest job first rule where only possible orderings are the ones in which jobs are ordered from longest job to shortest one with equal probability.

Moulin defines recursivity, a scheduling rule is *recursive* if for all problems there exists a probability distribution over set of possible orderings such that probability of an order is equal to probability that last ordered agent is ordered last times the porbabilty of occurance of the rest of the queue. The second one is seperability, a scheduling rule is *separable* if for all problems and all subsets of the set of users of the facility the random ordering of the jobs in the subset of users is independent of the jobs outside the subset. The third one is the well known anonymity axiom. A scheduling rule is *anonoymous* if agents' names are not important, i.e., the only important characteristic of a participant is his requested job length, the last one is the *positivity property* which may be interpreted as: for any requested job length profile any agent has a chance to precede any other agent. Moulin characterizes quasi-proportional rules by a combination of seperability, recursivity, anonymity, and the positivity property

In Moulin (2005) merge-proofness and split-proofness were the main objects of attention. Just like Moulin (2005), Moulin (2004) handles these maneuvers. Moulin defines demand monotonicity and explores merge-proofness in terms of demand monotonicity and previously defined properties. A scheduling method is *demand monotonic* if for all problems and for all agents expected wait of an agent minus his requested job length is non decresing in his requested job length. In other words whenever an agent's requested job length increses his net expected waiting time does not decrease. A scheduling rule is called demand monotonic whenever the associated method is demand monotonic. Moulin proposes that:

1. An ω-separable scheduling method is demand monotonic if and only if the probability that agent i follows agent j where agent j has a positive requested job length is non decreasing in agent i's requested job length.

2. An ω-separable scheduling rule is demand monotonic if and only if ω is nondecreasing.

As his main result Moulin states that an anonymous, demand monotonic, and separable scheduling method is merge-proof, so a demand monotonic quasi-proportional rule is merge-proof.

A method is *merge-invariant* if for all problems expected waiting times of agents before merging is equal to the expected waiting times of agents after merging. Uniform method is the only merge-invariant scheduling method.

Moulin explores split-proofness, and he makes characterizations of separable split-proof scheduling rules, and recursive scheduling rules. He states that a separables scheduling rule is split-proof if and only if the corresponding method is split-proof and a technical property is satisfied. Moulin adds that a ω-quasi-proportional rule is split-proof if and only if ω is subadditive. A recursive scheduling rule is split-proof when one of the agents splits his job into pieces the probability that another job is scheduled last does not become greater.

Moulin defines a new property, ranking. A scheduling method meets *ranking* if an agent i's requested job length is greater than or equal to agent j's requested job length then agent i's expected net waiting time is greater than or equal to agent j's expected net waiting time. Moulin states that an anonymous ω-separable method meets ranking if and only if the probability that agent i follows agent j where agent j has a positive requested job length is non decreasing in agent i's job length and it is less than or equal to agent i's requested job length over agent i's requested job length plus other agent's requested job length.

As a final contribution Moulin introduces split-invariance property, a scheduling rule is *split-invariant* if for all problems and all agents the agent who splits his job does not gain or lose with respect to not splitting his job. Moulin states that

1. If a separable scheduling rule is split-invariant then its method is proportional.

2. The proportional rule is the only recursive and split-invariant scheduling rule.

Özsoy (2005) considers another maneuver named coordinated splitting. By *coordinated splitting* it is meant that at least two different agents split their jobs and make a contract between themselves. This contract may be imagined as agents switch their places in the queue with a certain probability in favor of one of the agents. Whenever only two agents take action in coordinated splitting it is called pairwise splitting.

The main result of this article is that whenever there are infinite number of potential users. A separable and anonymous scheduling rule is demand monotonic and coordinated split-proof if and only if its corresponding method is uniform. zsoy introduces a new property named population monotonicity. By *population monotonicity* it is ment that for all problems and for all agents if a new agent joins to the group the expected costs of all agents in the original problem are less than or equal to their costs in the new problem. She states another theorem: an anonymous, separable, and population monotonic scheduling rule is demand monotonic and coordinated split-proof if and only if its associated method is uniform.

No-envy is a well known criterion of fairness. By no-envy it is meant that non of the agents prefer the the ordering and the transfer of another agent to his own ordering and transfer. The concept of no-envy is applicable to queuing problems. Youngsub Chun in Chun (2004*b*) works on this subject. Before discussing this article mentioning Maniquet (2003) will be enlightening.

In Maniquet (2003) just like the models in previous articles there is a group of agents who wants to use a facility which has an unlimited capacity and transfers are designed in the same way. Agents should use the facility one by one. Hence the planner of the facility should order the agents and assign monetary transfers between them. Unlike the previous models job lengths of agents are same, and each person is identified with an impatience level. The impatience level of an agent is the cost he incurs while waiting for another agent. The waiting cost of an agent is the number of people he waits times his impatience level. A solution to a queuing problem in this model is an ordering of agents and a profile of transfers. Each agent's payoff is linear and accounted by subtracting his waiting cost from his transfer.

An allocation is an ordering of agents and a profile of transfers. An allocation is *feasible* if no two people get the same place in the queue and sum of transfers is less than or equal to zero. It is *efficient* if the sum of waiting costs of individuals in the induced ordering is less than or equal to waiting costs of individuals in any other possible ordering and sum of transfers is equal to zero. Hence for an allocation to be efficient the induced ordering should order agents starting from the one who has the greatest impatience level to the one who has the lowest impatience level by breaking ties arbitrarily.

Maniquet forms a link between well-known concept from cooperative games *Shapley value* and queuing problems. The spirit of Shapley value is to account an agent's marginal contribution to a coalition. Take a group of people if they are producing something. Their production is the value of their group, and this value can be distributed between the individuals forming the group. Shapley value for a member of the group is the weighted sum of that agent's marginal contribution to each coalition. A coalition is a subgroup of agents in that group.

Maniquet defines valuations for coalitions to be able to use Shapley value concept. There would be different valuations for coalitions. He introduces two different valuation methods. First he takes a coalition and assumes that this coalition is served earlier than the rest of the agents. He takes an efficient queue in that coalition and sums up the waiting costs of the agents in that coalition according that efficient queue. He calls that sum the value of that coalition. In his second approach, he orders the entire group efficiently then sums up the waiting costs of the agents in the colaition according to that efficient ordering and he adds the incremental cost that coalition imposes on non-members. He calls this sum the value of the coalition. The cost incremented by a coalition to a nonmember of a coalition is sum of the requested job lengths of coalition members that preceed that nonmember. The sum of that costs is the incremental cost that coalition imposes on non-members. By using this valuations of coalitions Maniquet finds Shapley values of agents. Maniquet shows that this two approaches of valuation is dual hence Shapley values will be equal for each agent in both approaches. He states that if an allocation gives agents the payoffs corresponding to Shapley value then induced ordering will be efficient and induced transfer for each agent will be equal to half of his own waiting cost minus sum of half of the impatience levels of the agents that follow him.

As a final contribution Maniquet defines nine axioms and characterizes Shapley value in queuing problems in terms of those axioms. Effciency and feasibility are in those nine axioms. The others are:

***Pareto indifference*:** A solution concept satisfies *Pareto indifference* if it selects all the allocations which give the same payoff to all agents with an allocation which that solution concept has already chosen.

Anonymity: A solution concept satisfies *anonymity* if changing the names of agents does not effect the selection of that concept.

Equal treatment of equals: A solution concept satisfies *equal treatment of equals* whenever the payoffs of the agents with same impatience levels are equal.

Identical preferences lower bound: A solution concept satisfies *identical preferences lower bound* if for any queuing problem agent i's payoff is greater than or equal to the payoff he gets under the problem where all other agents' impatience levels are equal to agent i's impatience level.

Negative cost monotonicity: A solution concept satisfies *negative cost monotonicity* if agent i becomes more impatient while other agents' impatience levels stays the same then all the agents besides agent i gets better-off.

Independence of preceding costs: A solution concept satisfies *independence of preceding costs* if agent i's impatience level increases while the other agents' impatience levels stays the same, the payoffs of the agents who are queued after agent i in the original problem are not effected.

Last agent equal responsibility: A solution concept satisfies *last agent equal responsibility* if the agent who is queued last in the original problem leaves the set of users of the facility, the remaining agents' relative place in the queue stays the same and they share his transfer equally in the solution of the new problem.

Maniquet's charaterizetion result is the equivalance of the statements for a solution concept:

i. For all problems, solution concept Θ selects all the allocations assigning to the agents payoffs corresponding to the Shapley value of the queuing problem where coaltions are valued in Maniquet's way.

ii. Θ satisfies efficiency, anonymity, equal treatment of equals, and independence of preceding costs.

iii. Θ satisfies feasibility, Pareto indifference, identical preferences lower bound, negative cost monotonicity, and last agent equal responsibility.

iv. Among the solution concepts satisfying feasibility, Pareto indifference, identical preferences lower bound, and last agent equal responsibility, Θ minimizes the sum of absolute values of the transfers.

In Chun (2004*a*) the model in Maniquet (2003) is used but different valuation of coalitions is introduced. Chun takes a coalition and assumes that the coalition is served later than rest of the group. He considers that the coalition in itself is queued efficiently then he sums up the waiting costs of the agents who are in coalition. Like in Maniquet's approach this sum is used as the value of a coalition to find Shapley values of the agents. Similar to previous paper if an allocation gives agents the payoffs corresponding to Shapley value then induced ordering is efficient and this time induced transfer for each agent will be equal to sum of the half of impatience levels of agents that precede him minus number of people waiting him times half of his own impatience level. The only diffence between these two papers is whether the coalition is served first or last. Again like Maniquet, Chun characterizes his rule with nine axioms which are similar to Maniquet's. Like his approach his axioms are in a way reverse of Maniquet's. The first six axioms of Chun are same as Maniquet's first six axioms but he uses positive cost monotonicity instead of negative cost monotonicity, independence of following costs instead of indepence of preceding costs, first agent equal responsibility instead of last agent equal responsibility.

***Positive cost monotonicity*:** A solution concept satisfies *positive cost monotonicity* if agent i becomes more impatient while other agents' impatience levels stays the same then all the agents besides agent i gets worse-off.

***Independence of following costs*:** A solution concept satisfies *independence of following costs* if agent i's impatience level increases while the other agents' impatience levels stays the same, the payoffs of the agents who are queued before agent i in the original problem are not effected.

***First agent equal responsibility*:** A solution concept satisfies *first agent equal responsibility* if the agent who is queued first in the original problem leaves the set of users of the facility, the remaining agents' relative place in the queue stays the same and they share his transfer equally in the solution of the new problem.

Maniquet's charaterizetion result is the equivalance of the statements for a solution concept:

i. For all problems, solution concept Θ selects all the allocations assigning to the agents payoffs corresponding to the Shapley value of the queuing problem where coaltions are valued in Chun's way.

ii. Θ satisfies efficiency, Pareto indifference, equal treatment of equals, and independence of following costs.

iii. Θ satisfies feasibility, Pareto indifference, identical preferences lower bound, positive cost monotonicity, and first agent equal responsibility.

iv. Among the solution concepts satisfying feasibility, Pareto indifference, identical preferences lower bound, and first agent equal responsibility, Θ maximizes the sum of absolute values of the transfers.

Chun (2004b) considers the same model as in Maniquet (2003), and Chun (2004a). Chun characterizes solution concepts satisfying Pareto efficiency, and envy freeness. Due to simplicity of payoff functions in this setup, they are characterized by ordering agents starting from the one with the highest impatience level to the one with lowest impatience level; and arranging transfers in a way that difference between transfers of two agents where one follows the other is less than or equal to the impatience level of the one who is ahead and greater than or equal to the one who is behind.

The other contributions of Chun in this article are negative results. It is stated in Chun that when there are more than two agents there is no solution concept which is Pareto efficient, envy-free, negative or positive cost monotonic. Again when there are more than two agents there is no solution concept satisfying Pareto efficiency, no-envy, and independence of following or preceding costs.

After reaching such negative results Chun modificates the concept of no-envy as backward, and forward no envy. A solution concept satisfies *backward no-envy* if none of the agents envy another agent who has a lower impatience level. A solution concept satidfies forward no-envy if none of the agents envy another agent who has higher impatience level. Chun states that Maniquet rule satisfies backward no-envy and his rule satisfies forward no-envy.

Kayi & Ramaekers (2006) concetrates on strategy-proffness in queuing problem. A solution concept satidfies *strategy-proofness* if an agent revels a different impatience level instead of his real impatience level he does not get better-off.

Kayi & Ramaekers (2006) introduces non-bossiness. A solution concept satisfies *non-bossiness* if there is no agent who can effect the allocation of other agents by announcing a different impatiance level for himself without affecting his own allocation. They state that there is no solution concept satisfying Pareto efficiency, strategy proofness, and non-bossiness.

Besides this negative result Kayi & Ramaekers (2006) revisits *Mitra and Sen's rule* which is proposing efficient orderings and letting transfer of agent i equal to sum of impatience levels of agents who are ordered before agent i times number of people they wait for divided by number of agents minus two minus sum of impatience levels of agents who are ordered after agent i times number of agents they wait starting from agent i divided by number of agents minus two. Mitra and Sen's rule is strategy-proof. Kayi and Ramaekers concetrates on properties of Mitra and Sen's rule in the rest of the paper.

I am considering a different model than the models that has been discussed above. Under my model the capacity of the facility is unlimited. This is in line with all of the models that we have discussed so far. Like Maniquet (2003)'s, Chun (2004*a*)'s, and Kayi & Ramaekers (2006)'s articles the job lengths of agents are identical. The difference in my model is about the preferences of agents. I am only assuming that all of the agents prefer being in an earlier position in an ordering to being in later position. Hence, waiting cost of a unit of time may be different for all agents, and for an agent the cost of waiting the same amount of time in a different place in the queue may be different.(for example an agent may think that, waiting the first ordered agent while being second ordered is more costly than waiting the fourth ordered agent while being fifth ordered) I am adding transfers to the model just like the previous authors has done. My main aim in my study is to characterize the envy free solutions. Besides I am imposing a another assumption on preferences of agents ,namely order preservation property, which still keeps the preferences of the agents more general than previous papers' models. Order preservation property says that if agent i values the difference between two consecutive positions more than agent j does, then agent i will keep valuing the difference between consecutive positions which come later in the queue greater than or equal to agent j values. In the set of problems where order preservation property is satisfied I again search for the characterization of envy free solutions.

Chapter 2

MODEL AND RESULTS

2.1 THE MODEL AND BASIC DEFINITIONS

Let $N = \{1, \dots, n\}$ be a finite set of agents where $n = |N|$. The agents in the set N request from a facility to process their jobs. We will assume that the job lengths of all agents are equal, the facility has unlimited capacity, and it can process jobs one at a time. Let $\mathbb{N}_n$ denote the positive integers up to n. An ordering of the agents in N is one-to-one function $\sigma : N \to \mathbb{N}_n$. Let $\Pi(N)$ denote the of set orderings of N. For any $i \in N$ and for any $\sigma \in \Pi(N)$, σ_i is the position of agent i in the ordering σ. For any $a \in \mathbb{N}_n$, $\sigma^{-1}(a)$ denotes the agent who is placed in the a^{th} position in the ordering σ. Let $t \in \mathbb{R}^n$ denote the vector of transfers. If for some $i \in N$ we have $t_i > 0$, then it means that agent i is receiving t_i units of money; if $t_i < 0$, then it means that agent i is paying t_i units of money. The sum of the transfers of the agents, $\sum_{i \in N} t_i$, is the budget of the facility. If $\sum_{i \in N} t_i = 0$, then we say that the budget is balanced, hence in total money collected is equal to money distributed. Let

BB denote the set of budget balanced transfer vectors, i.e.,

$$BB = \{t \in \mathbb{R}^n | \sum_{i \in N} t_i = 0\}.$$

Now we are faced with the problem of ordering agents and arranging transfers between them. Hence a pair $(\sigma, t) \in \Pi(N) \times BB$ is a solution to the problem faced by the group N. We are assuming that agents only care about their positions in the queue and their transfers. For any $i \in N$, let u_i denote the payoff function of agent i on $\mathbb{N}_n \times \mathbb{R}$. We will assume that $u_i : \mathbb{N}_n \times \mathbb{R} \to \mathbb{R}$ is such that $u_i(\sigma_i, t_i) = v_i(\sigma_i) + t_i$. Where for any $i \in N$, $v_i : \mathbb{N}_n \to \mathbb{R}$ is such that, for any $a, b \in \mathbb{N}_n$ if $a \leq b$, then $v_i(a) \geq v_i(b)$. Thus for any $i \in N$, u_i is quasi-linear in transfers and all of the agents weakly prefer an earlier position to a later position. For any $i \in N$ define $d_i : \mathbb{N}_n \times \mathbb{N}_n \to \mathbb{R}$ as: for any $a, b \in \mathbb{N}_n$, $d_i(a, b) = v_i(a) - v_i(b)$. Thus $d_i(a, b)$ gives the difference between the valuation of agent i from being in position a and being in position b.

Hence (N, u) is a queuing problem where N is a finite set of agents and $u = (u_i)_{i \in N}$ is a profile of payoff functions. A pair $(\sigma, t) \in \Pi(N) \times BB$ is a solution to a queuing problem. Let Q be the set of all queuing problems. A solution concept F is defined as $F : Q \to 2^{\Pi(N) \times \mathbb{R}^n}$.

At a given solution if an agent i prefers the ordering and transfer of agent j to his own ordering and transfer then it means that agent i envies agent j. If for a given solution none of the agents envy any other agent then we say that this solution is an envy free solution.

Definition 1 (Envy freeness, NE) *For any $(N, u) \in Q$, a solution $(\sigma, t) \in \Pi(N) \times BB$ is envy free if and only if for any $i, j \in N$, $u_i(\sigma_i, t_i) \geq u_i(\sigma_j, t_j)$. We use $NE(N, u)$ to denote the set of envy free solutions of the problem (N, u).*

A solution (σ, t) Pareto dominates another solution (σ', t'), if all of the agents weakly prefer their orderings and transfers under (σ, t) to their orderings and transfers under (σ', t'), and there is at least one agent who strictly prefers his ordering and transfer under (σ, t) to his ordering and transfer under (σ', t'). If a solution is not Pareto dominated by any other solution then it is a Pareto efficient solution.

Definition 2 (Pareto efficiency, PE) *For any $(N, u) \in Q$, a solution $(\sigma, t) \in \Pi(N) \times BB$ is Pareto efficient if and only if there does not exist $(\sigma', t') \in \Pi(N) \times BB$ such that for any $i \in N$, $u_i(\sigma'_i, t'_i) \geq u_i(\sigma_i, t_i)$ and there exists $j \in N$ with $u_i(\sigma'_j, t'_j) > u_i(\sigma_j, t_j)$. We use $PE(N, u)$ to denote the Pareto efficient solutions of the problem (N, u).*

For any given problem, an ordering is a maximal ordering if it maximizes the sum of valuations of agents over the set of orderings. The set of maximal orderings for the problem (N, u) will be denoted with $\Sigma_{max}(N, u)$, i.e.,

$$\Sigma_{max}(N, u) = \{\sigma \in \Pi(N) | \forall \sigma' \in \Pi(N) : \sum_{i \in N} v_i(\sigma_i) \geq \sum_{i \in N} v_i(\sigma'_i)\}.$$

Maximal orderings will play crucial roles in our results.

2.2 RESULTS

In the next lemma we will show that for any given problem, if a solution is an envy free solution then the ordering of agents should be a maximal ordering.

Lemma 1 *For all $(N, u) \in Q$, if $\sigma \in \Pi(N) \backslash \Sigma_{max}(N, u)$, then for all $t \in BB$, $(\sigma, t) \notin NE(N, u)$.*

Proof Let $\sigma \in \Pi(N)$ be such that, there exists $\sigma' \in \Pi(N)$ where $\sum_{i \in N} v_i(\sigma_i) < \sum_{i \in N} v_i(\sigma'_i)$. Assume that there exists $t \in BB$ such that $(\sigma, t) \in NE(N, u)$. So

$$\forall i, j \in N \quad u_i(\sigma_i, t_i) \geq u_i(\sigma_j, t_j) \ .$$

Therefore

$$\forall i, j \in N \quad v_i(\sigma_i) + t_i \geq v_i(\sigma_j) + t_j \ \ .$$

Note that for all $i \in N$ there exists $j \in N$ such that $\sigma'_i = \sigma_j$, which is given by $\sigma^{-1}(\sigma'_i)$. Hence

$$\forall i \in N \quad v_i(\sigma_i) + t_i \geq v_i(\sigma'_i) + t_{\sigma^{-1}(\sigma'_i)} \ \ .$$

Which implies

$$\sum_{i \in N} v_i(\sigma_i) + \sum_{i \in N} t_i \geq \sum_{i \in N} v_i(\sigma'_i) + \sum_{i \in N} t_{\sigma^{-1}(\sigma'_i)} \ \ .$$

Note that

$$\sum_{i \in N} t_{\sigma^{-1}(\sigma'_i)} = \sum_{i \in N} t_i \ \ .$$

So

$$\sum_{i \in N} v_i(\sigma_i) + \sum_{i \in N} t_i \geq \sum_{i \in N} v_i(\sigma'_i) + \sum_{i \in N} t_i \ \ .$$

Hence

$$\sum_{i \in N} v_i(\sigma_i) \geq \sum_{i \in N} v_i(\sigma'_i) \ \ .$$

Which contradicts with $\sum_{i \in N} v_i(\sigma_i) < \sum_{i \in N} v_i(\sigma'_i)$. Hence $(\sigma, t) \notin NE(N, u)$. $\square$

By previous lemma we have seen that any envy free solution uses a maximal ordering to order the agents. By using this fact in the next lemma we will show that any envy free solution is also a Pareto efficient solution.

Lemma 2 *For all* $(N, u) \in Q$*, if* $(\sigma, t) \in NE(N, u)$ *then* $(\sigma, t) \in PE(N, u)$

Proof Let $(\sigma, t) \in NE(N, u)$ by Lemma 1

$$\forall \sigma' \in \Pi(N) \quad \sum_{i \in N} v_i(\sigma_i) \geq \sum_{i \in N} v_i(\sigma'_i).$$

Assume that there exists $(\overline{\sigma}, \overline{t}) \in \Pi(N) \times BB$ which Pareto dominates (σ, t). So

$$\forall i \in N \quad v_i(\overline{\sigma}_i) + \overline{t}_i \geq v_i(\sigma_i) + t_i \ \ ,$$

and

$$\exists j \in N: \quad v_j(\overline{\sigma}_j) + \overline{t}_j \geq v_j(\sigma_j) + t_j \ \ .$$

Hence

$$\sum_{i \in N} v_i(\overline{\sigma}_i) + \sum_{i \in N} \overline{t}_i > \sum_{i \in N} v_i(\sigma_i) + \sum_{i \in N} t_i.$$

Note that

$$\sum_{i \in N} \overline{t}_i = \sum_{i \in N} t_i = 0 \ \ .$$

So

$$\sum_{i \in N} v_i(\overline{\sigma}_i) > \sum_{i \in N} v_i(\sigma_i) \ \ .$$

Which contradicts with $\sum_{i \in N} v_i(\overline{\sigma}_i) \leq \sum_{i \in N} v_i(\sigma_i)$, concludes the proof. □

The next lemma states that if a solution is an envy free solution, then there is a link between the difference between the transfers of agents and their valuations on orderings. This result is due to payoff functions of agents being quasi-linear in transfers.

Lemma 3 *If* $(\sigma, t) \in NE(N, u)$ *then for all* $i, j \in N$ *such that* $\sigma_i < \sigma_j$ $d_i(\sigma_i, \sigma_j) \geq t_j - t_i \geq d_j(\sigma_i, \sigma_j)$

Proof Let $(\sigma, t) \in NE(N, u)$, then for all $i, j \in N$ such that $\sigma_i < \sigma_j$ we have

$$v_i(\sigma_i) + t_i - v_i(\sigma_j) - t_j \geq 0 \ \ ,$$

$$v_j(\sigma_j) + t_j - v_j(\sigma_i) - t_i \geq 0 \ \ .$$

These two inequalities imply

$$v_i(\sigma_i) - v_i(\sigma_j) \geq t_j - t_i \geq v_j(\sigma_i) - v_j(\sigma_j) \ \ .$$

Hence

$$d_i(\sigma_i, \sigma_j) \geq t_j - t_i \geq d_j(\sigma_i, \sigma_j) \ .$$

□

As the previous lemma underlines the difference between the valuations of agents on different positions in the queue is very curicial in constructing the transfers. The next property is a restriction on the payoff functions of the individuals. It says that, if agent i values the difference between two consecutive positions more than the agent j does, then agent i will keep valuing the difference between consecutive positions which come later in the queue greater than or equal to agent j values these consecutive positions.

Definition 3 (Order Preservation Property, OPP**)** *A problem* $(N, u) \in Q$ *is said to satisfy order preservation property, if and only if for all* $k \in \mathbb{N}_n$ *and for all* $i, j \in N$ *if* $v_i(k) - v_i(k+1) \geq v_j(k) - v_j(k+1)$ *then for all* $l \in \mathbb{N}_n$ *such that* $l \geq k$ *we have* $v_i(l) - v_i(l+1) \geq v_j(l) - v_j(l+1)$.

Next we define a solution concept F. With F we order agents recursively, by placing an agent in the first position if this agent's difference between his valuation on being in the first position and being in the second position is greater than or equal to other agents' differences between their valuations on being in the first position and being in the second position. We keep placing agents to the next positions by consulting the same reasoning. After arranging an ordering we construct transfers according to chosen ordering, while constructing transfers we are keeping in mind the wisdom from Lemma 3.

Define F: $Q \to 2^{\Pi(N)\times\mathbb{R}^n}$ such that first order agents recursively by the following algorithm:

$$
\begin{aligned}
\sigma^{-1}(1) \in S_1 &= \{i \in N | \forall j \in N : d_i(1,2) \geq d_j(1,2)\} \ , \\
\sigma^{-1}(2) \in S_2 &= \{i \in N \setminus \{\sigma^{-1}(1)\} | \forall j \in N \setminus \{\sigma^{-1}(1)\} : \\
&\qquad d_i(2,3) \geq d_j(2,3)\} \ , \\
\vdots \qquad & \vdots \\
\sigma^{-1}(k) \in S_k &= \{i \in N \setminus \{\sigma^{-1}(1), \ldots, \sigma^{-1}(k-1)\} | \\
&\qquad \forall j \in N \setminus \{\sigma^{-1}(1), \ldots, \sigma^{-1}(k-1)\} : \\
&\qquad d_i(k,k+1) \geq d_j(k,k+1)\} \ , \\
\vdots \qquad & \vdots \\
\sigma^{-1}(n-1) \in S_{n-1} &= \{i \in N \setminus \{\sigma^{-1}(1), \ldots, \sigma^{-1}(n-2)\} | \\
&\qquad \forall j \in N \setminus \{\sigma^{-1}(1), \ldots, \sigma^{-1}(n-2)\} : \\
&\qquad d_i(n-1,n) \geq d_j(n-1,n)\} \ , \\
\sigma^{-1}(n) \in S_n &= \{i \in N | N \setminus \{\sigma^{-1}(1), \ldots, \sigma^{-1}(n-1)\}\} \ .
\end{aligned}
$$

Note that there might be more than one ordering.

Without loss of generality assume that for any $i \in N$, $\sigma_i = i$.

Transfers are designed as: for $k \in \{1, \ldots, n-1\}$

if $|S_k| > 1$, then

$$t_{k+1} - t_k = d_k(k,k+1);$$

if $|S_k| = 1$, then

$$t_{k+1} - t_k \in [\max_{l \in S_{k+1}} d_l(k,k+1), d_k(k,k+1))];$$

and $\sum_{i \in N} t_i = 0$.

The next theorem states that, if a queuing problem satisfies the order preservation property, then we are able to find all of the envy free solutions of this problem by using the solution concept F, furthermore all of the solutions of F are envy free solutions of this problem.

Theorem 1 *For all $(N, u) \in Q$, if (N, u) satisfies order preservation property, then $(\sigma, t) \in NE(N, u)$ if and only if $(\sigma, t) \in F(N, u)$.*

Proof ($\Leftarrow$) First we will prove that if $(\sigma, t) \in F(N, u)$, then $(\sigma, t) \in NE(N, u)$. Let $(\sigma, t) \in F(N, u)$. Let $i, j \in N$ and assume that $\sigma_i < \sigma_j$. Without loss of generality let $\sigma_i = i, \sigma_j = j$. So

$$t_j - t_i = \sum_{l=0}^{j-i-1} (t_{i+l+1} - t_{i+l}) \ .$$

We know from the construction of transfers via F that for all $l \in \{0, \ldots, j-i-1\}$,

$$t_{i+j+1} - t_{i+l} \leq d_{i+l}(i+l, i+l+1) \ .$$

Which implies

$$t_j - t_i \leq \sum_{l=0}^{j-i-1} d_{i+l}(i+l, i+l+1).$$

By using the definition of $d_i(\cdot, \cdot)$ we reach to

$$v_i(i) - v_i(j) = \sum_{l=0}^{j-i-1} (v_i(i+l) - v_i(i+l+1)) = \sum_{l=0}^{j-i-1} d_i(i+l, i+l+1).$$

From the construction of F we know that $i \in S_i$ so for all $l \in \{0, \ldots, j-i-1\}$,

$$d_i(i, i+1) \geq d_{i+l}(i, i+1) \ .$$

Since order preservation property is satisfied for all $l \in \{0, \ldots, j-i-1\}$,

$$d_i(i+l, i+l+1) \geq d_{i+l}(i+l, i+l+1) \ .$$

So

$$v_i(i) - v_i(j) = \sum_{l=0}^{j-i-1} d_i(i+l, i+l+1) \geq \sum_{l=0}^{j-i-1} d_{i+l}(i+l, i+l+1) \geq t_j - t_i \ .$$

Therefore

$$u_i(i, t_i) - u_i(j, t_j) = v_i(i) - v_i(j) - (t_j - t_i) \geq 0 \ .$$

Hence agent i does not envy agent j, i.e., None of the agents envy any other agent who is ordered after himself.

Now we will show that agent j does not envy agent i. Recall that

$$t_j - t_i = \sum_{l=0}^{j-i-1} (t_{i+l+1} - t_{i+l}).$$

For $l \in \{0, \ldots, j-i-1\}$,

if $|S_{i+l}| > 1$, then

$$t_{i+l+1} - t_{i+l} = d_{i+l}(i+l, i+l+1) \ .$$

If $j \in S_{i+l}$, then

$$d_{i+l}(i+l, i+l+1) = d_j(i+l, i+l+1) \ .$$

If $j \notin S_{i+l}$, then

$$d_{i+l}(i+l, i+l+1) > d_j(i+l, i+l+1) \ .$$

If $|S_{i+l}| = 1$, then $i+l = S_{i+l}$ and $j > i+l$ hence $i+l \neq j$. Then

$$t_{i+l+1} - t_{i+l} \in [\max_{k \in S_{i+l+1}} d_k(i+l, i+l+1), d_{i+l}(i+l, i+l+1)] \ .$$

If $j \in S_{i+l+1}$, then

$$\max_{k \in S_{i+l+1}} d_k(i+l, i+l+1) \geq d_j(i+l, i+l+1) \ .$$

If $j \notin S_{i+l+1}$, then assume that

$$d_j(i+l, i+l+1) > \max_{k \in S_{i+l+1}} d_k(i+l, i+l+1) \ .$$

Since order preservation property is satisfied, for all $k \in S_{i+l+1}$,

$$d_j(i+l+1, i+l+2) \geq d_k(i+l+1, i+l+2) \ .$$

Then by definition $j \in S_{i+l+1}$ which contradicts with $j \notin S_{i+l+1}$. So

$$\max_{k \in S_{i+l+1}} d_k(i+l, i+l+1) \geq d_j(i+l, i+l+1) \ .$$

Therefore

$$t_j - t_i = \sum_{l=0}^{j-i-1} (t_{i+l+1} - t_{i+l}) \geq \sum_{l=0}^{j-i-1} d_j(i+l, i+l+1).$$

By definition

$$v_j(i) - v_j(j) = \sum_{l=0}^{j-i-1} d_j(i+l, i+l+1) \ .$$

Which implies

$$v_j(i) - v_j(j) = \sum_{l=0}^{j-i-1} d_j(i+l, i+l+1) \leq t_j - t_i \ .$$

Therefore

$$u_j(j, t_j) - u_j(i, t_i) = -(v_j(i) - v_j(j)) + t_j - t_i \geq 0 \ .$$

Hence agent j does not envy agent i, i.e., none of the agents envy any other agent who is ordered before himself. So $(\sigma, t) \in NE(N, u)$.

($\Rightarrow$) Now we will prove that if $(\sigma, t) \in NE(N, u)$, then $(\sigma, t) \in F(N, u)$. Let $(\sigma, t) \in NE(N, u)$. Assume that $(\sigma, t) \notin F(N, u)$.

Part 1) Assume that we cannot construct σ with the ordering algorithm which is defined in page 26. So there exists $i, j \in N$ such that for some $k \in \{1, \ldots, n-1\} \quad i \in S_k,\ j \notin S_k$ but $\sigma_j = k$ hence $\sigma_i = l > k$. Since $i \in S_k,\ j \notin S_k$ we have $d_i(k, k+1) > d_j(k, k+1)$. Order preservation property is satisfied, so for any $n \in \{1, \ldots, l-k-1\}$

$$d_i(k+n, k+n+1) \geq d_j(k+n, k+n+1) \ .$$

Which implies

$$d_i(k, l) = \sum_{n=0}^{l-k-1} d_i(k+n, k+n+1) > \sum_{n=0}^{l-k-1} d_j(k+n, k+n+1) = d_j(k, l) \ .$$

Hence $d_i(k, l) > d_j(k, l)$. Since $(\sigma, t) \in NE(N, u)$ we have

$$v_i(l) + t_i - v_i(k) - t_j \geq 0 \ ,$$

and

$$v_j(k) + t_j - v_j(l) - t_i \geq 0 \ .$$

Therefore

$$v_j(k) - v_j(l) \geq t_i - t_j \geq v_i(k) - v_i(l) \ .$$

Hence

$$d_j(k, l) \geq d_i(k, l) \ .$$

Which contradicts with $d_i(k, l) > d_j(k, l)$. So we can construct σ with the ordering algorithm which is defined in page 26.

Part 2) Even though the ordering of agents is constructed by the algorithm which is defined in page 26, the transfers are not defined as it is defined in page 26.

A) For some $k \in \{1, \ldots, n-1\}$ if $|S_k| > 1$.

Assume that $t_{k+1} - t_k \neq d_k(k, k+1)$, let $i, j \in S_k$ and with out loss of generality let $\sigma_j = k$ so $\sigma_i = l > k$. For sake of simplicity rename agent j as agent k and agent i as agent l. Since $k, l \in S_k$

$$d_k(k, k+1) = d_l(k, k+1) \ .$$

By order preservation property for any $n \in \{1, \ldots, |N| - k - 1\}$,

$$d_k(k+n, k+n+1) = d_l(k+n, k+n+1) \ .$$

So

$$d_k(k, l) = \sum_{n=0}^{l-k-1} d_k(k+n, k+n+1) = \sum_{n=0}^{l-k-1} d_l(k+n, k+n+1) = d_l(k, l) \ .$$

Since $(\sigma, t) \in NE(N, u)$, by Lemma 3 we have

$$d_k(k, l) \geq t_l - t_k \geq d_l(k, l) \ .$$

Therefore

$$t_l - t_k = \sum_{n=0}^{l-k-1} d_k(k+n, k+n+1) \ .$$

Hence

$$t_l - t_{k-1} - t_k = d_k(k, k+1) + \sum_{n=1}^{l-k-1} d_k(k+n, k+n+1) \ .$$

But by assumption

$$t_{k+1} - t_k \neq d_k(k, k+1) \ .$$

And by Lemma 3

$$t_{k+1} - t_k < d_k(k, k+1) \ .$$

Then

$$t_l - t_{k+1} > \sum_{n=1}^{l-k-1} d_k(k+n, k+n+1).$$

i) If $k+1 = l$ then $0 > 0$ is a contradiction.

ii) If $k+1 < l$ by order preservation property

$$\sum_{n=1}^{l-k-1} d_{k+n}(k+n, k+n+1) \leq \sum_{n=1}^{l-k-1} d_k(k+n, k+n+1) \ .$$

Therefore

$$t_l - t_{k+1} > \sum_{n=1}^{l-k-1} d_{k+n}(k+n, k+n+1) \ .$$

Then there exists $m \in \{k+1, \ldots, l-1\}$ such that

$$t_{m+1} - t_m > d_m(m, m+1) \ .$$

But by Lemma 3

$$t_{m+1} - t_m \leq d_m(m, m+1) \ .$$

Since $(\sigma, t) \in NE(N, u)$ we have

$$t_{m+1} - t_m > d_m(m, m+1) \ .$$

Which contradicts with

$$t_{m+1} - t_m \leq d_m(m, m+1).$$

Hence $t_{k+1} - t_k = d_k(k, k+1)$.

B) If $|S_k| = 1$. Assume that

$$t_{k+1} - t_k \notin [\max_{j \in S_{k+1}} d_j(k, k+1), d_k(k, k+1)] \ .$$

By Lemma 3

$$t_{k+1} - t_k \leq d_k(k, k+1) \ .$$

Hence it should be the case that

$$t_{k+1} - t_k < \max_{j \in S_{k+1}} d_j(k, k+1) \ .$$

Take

$$l \in \{i \in S_{k+1} | \forall j \in S_{k+1} \quad d_i(k, k+1) \geq d_j(k, k+1)\} \ .$$

If $l = k + 1$, then

$$t_{k+1} - t_k < d_{k+1}(k, k+1) \ .$$

Since $(\sigma, t) \in NE(N, u)$ and again by Lemma 3

$$t_{k+1} - t_k \geq d_{k+1}(k, k+1) \ .$$

Which is a contradiction.

If $l > k + 1$, then $k + 1, l \in S_{k+1}$ so

$$d_l(k+1, k+2) = d_{k+1}(k+1, k+2) \ .$$

Since order preservation property is satisfied for all $n \in \{0, \ldots, l - k - 2\}$

$$d_l(k+1+n, k+n+2) = d_{k+1}(k+1+n, k+n+2) \ .$$

Hence

$$d_l(k+1,l) = \sum_{n=0}^{l-k-2} d_l(k+n+1,k+n+2) = \sum_{n=0}^{l-k-2} d_{k+1}(k+n+1,k+n+2) = d_{k+1}(k+1,l) \ .$$

By no envy

$$t_l - t_{k+1} = d_l(k+1,l) \ .$$

Therefore

$$t_l - t_k < d_l(k,l) \ .$$

Since $(\sigma,t) \in NE(N,u)$ again by Lemma 3

$$t_l - t_k \geq d_l(k,l) \ .$$

which contradicts with

$$t_l - t_k < d_l(k,l) \ .$$

Hence $t_{k+1} - t_k \in [\max_{j \in S_{k+1}} d_j(k,k+1), d_k(k,k+1)]$. So $(\sigma,t) \in F(N,u)$. $\square$

In the rest of the paper we are not assuming order preservation property any more. The next lemma is a trivial statement but it will be very useful in finding further results

Lemma 4 *For all $(N,u) \in Q \quad \sigma \in \Sigma_{max}(N,u)$ if and only if for all $\sigma' \in \Pi(N \quad \sum_{i\in N} d_i(\sigma_i,\sigma_i') \geq 0$.*

Proof Let $\sigma \in \Sigma_{max}(N,u)$ and $\sigma' \in \Pi(N)$ then

$$\sum_{i\in N} v_i(\sigma_i) \geq \sum_{i\in N} v_i(\sigma_i') \ .$$

Hence

$$\sum_{i\in N} (v_i(\sigma_i) - v_i(\sigma_i')) \geq 0 \ .$$

By definition

$$\forall i \in N \quad d_i(\sigma_i,\sigma_i') = v_i(\sigma_i) - v_i(\sigma_i') \ .$$

Therefore

$$\sum_{i \in N} d_i(\sigma_i, \sigma'_i) = \sum_{i \in N} (v_i(\sigma_i) - v_i(\sigma'_i)) \geq 0 \ .$$

Let $\sigma \in \Pi(N)$ be such that for all $\sigma' \in \Pi(N, u)$ we have $\sum_{i \in N} d_i(\sigma_i, \sigma'_i) \geq 0$. Assume that there exists $\sigma' \in \Pi(N, u)$ such that $\sum_{i \in N} v_i(\sigma'_i) > \sum_{i \in N} v_i(\sigma_i)$. So

$$\sum_{i \in N} (v_i(\sigma'_i) - v_i(\sigma_i)) > 0.$$

Thus

$$\sum_{i \in N} d_i(\sigma'_i, \sigma_i) > 0.$$

Which implies

$$\sum_{i \in N} d_i(\sigma_i, \sigma'_i) < 0.$$

But it contradicts with

$$\sum_{i \in N} d_i(\sigma_i, \sigma'_i) \geq 0.$$

This concludes the proof.

□

The next lemma states that for any queuing problem if we take a maximal ordering and change the positions of agents, then the total losses of agents who move to a later position in the queue is greater than the total gains of the agents who move to an earlier position in the queu.

Lemma 5 *For all $(N, u) \in Q$ if $\sigma \in \Sigma_{max}(N, u)$ then for any sequence of agents $(i_r)_{i=1}^k$ such that $i_1 = i_k$ and for any $l, j \in \{1, \ldots, k-1\} \quad i_l \neq i_j$ we have*

$$\sum_{\substack{r=1: \\ \sigma_{i_r} < \sigma_{i_{r+1}}}}^{k-1} d_{i_r}(\sigma_{i_r}, \sigma_{i_{r+1}}) \geq \sum_{\substack{r=1: \\ \sigma_{i_r} \geq \sigma_{i_{r+1}}}}^{k-1} d_{i_r}(\sigma_{i_{r+1}}, \sigma_{i_r})$$

Proof Let $\sigma \in \Sigma_{max}(N, u)$ and let $(i_r)_{i=1}^k$ be a sequence of agents such that $i_1 = i_k$ and for all $l, j \in \{1, \ldots, k-1\}, \quad i_l \neq i_j$. Construct the ordering σ' as: $l \in \{1, \ldots, k-1\}, \quad \sigma'_l = \sigma_{1+l}$ and for all $i \in N \setminus \{(i_r) : r \in \{1, \ldots, k\}\}$,

$\sigma'_i = \sigma_i$. Now σ' is another ordering. Since $\sigma \in \Sigma_{max}(N,u)$ by Lemma 4

$$\sum_{i \in N} d_i(\sigma_i, \sigma'_i) \geq 0 \ .$$

Which implies

$$\sum_{\substack{i=1: \\ \sigma_i < \sigma'_i}}^{n} d_i(\sigma_i, \sigma'_i) \geq \sum_{\substack{i=1: \\ \sigma_i \geq \sigma'_i}}^{n} d_i(\sigma'_i, \sigma_i) \ .$$

Due to the construction of σ'

$$\sum_{\substack{i=1: \\ \sigma_i < \sigma'_i}}^{n} d_i(\sigma_i, \sigma'_i) = \sum_{\substack{r=1: \\ \sigma_{i_r} < \sigma_{i_{r+1}}}}^{k-1} d_{i_r}(\sigma_{i_r}, \sigma_{i_{r+1}}) \ ,$$

and

$$\sum_{\substack{i=1: \\ \sigma_i \geq \sigma'_i}}^{n} d_i(\sigma'_i, \sigma_i) = \sum_{\substack{r=1: \\ \sigma_{i_r} \geq \sigma_{i_{r+1}}}}^{k-1} d_{i_r}(\sigma_{i_{r+1}}, \sigma_{i_r}) \ .$$

Hence

$$\sum_{\substack{r=1: \\ \sigma_{i_r} < \sigma_{i_{r+1}}}}^{k-1} d_{i_r}(\sigma_{i_r}, \sigma_{i_{r+1}}) \geq \sum_{\substack{r=1: \\ \sigma_{i_r} \geq \sigma_{i_{r+1}}}}^{k-1} d_{i_r}(\sigma_{i_{r+1}}, \sigma_{i_r}) \ .$$

This concludes the proof. □

Lemma 6 *For any $(N,u) \in Q$ if $|N| = 3$, then $NE(N,u) \neq \emptyset$ and $(\sigma,t) \in NE(N,u)$ if and only if $\sigma \in \Sigma_{max}(N,u)$ and $t_2 - t_1 \in [d_2(1,2), d_1(1,2)]$, $t_3 - t_2 \in [d_3(2,3), d_2(2,3)]$, $t_3 - t_1 \in [\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$, and $t_1 + t_2 + t_3 = 0$.*

Proof Let $\sigma \in \Sigma_{max}(N,u)$ and without loss of generality let $\sigma = (1,2,3)$. From Lemma 5 we know that

$$i) \quad d_2(1,2) \leq d_1(1,2)$$

$$ii) \quad d_3(2,3) \leq d_2(2,3)$$

$$iii) \quad d_3(1,3) \leq d_1(1,3)$$

If we add i to ii we obtain

$$iv) \quad d_2(1,2) + d_3(2,3) \leq d_1(1,2) + d_2(2,3)$$

Again from Lemma 5

$$d_3(1,3) \leq d_1(1,2) + d_2(2,3) \quad \text{and} \quad d_2(1,2) + d_3(2,3) \leq d_1(1,3)$$

Hence

$$[d_3(1,3), d_1(1,3)] \bigcap [d_2(1,2) + d_3(2,3), d_1(1,2) + d_2(2,3)] \neq \emptyset$$

so for any

$$z \in [\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

there exists

$$x \in [d_2(1,2), d_1(1,2)]$$

and

$$y \in [d_3(2,3), d_2(2,3)]$$

such that $z = x + y$. Let $(t_1, t_2, t_3) \in \mathbb{R}^3$ be such that

$$t_2 - t_1 = x\,, \qquad t_3 - t_2 = y \quad \text{and } t_1 + t_2 + t_3 = 0 \ .$$

Then, for agent 1

$$u_1(1,t_1) - u_1(2,t_2) = d_1(1,2) - x \geq 0$$
$$u_1(1,t_1) - u_1(3,t_3) = d_1(1,3) - (x+y) \geq 0 \ .$$

Thus agent 1 does not envy other agents.

For agent 2

$$u_2(2,t_2) - u_2(1,t_1) = -d_2(1,2) + x \geq 0$$
$$u_2(2,t_2) - u_2(3,t_3) = d_2(2,3) - y \geq 0 \ .$$

So agent 2 does not envy other agents.

For agent 3

$$u_3(3,t_3) - u_3(1,t_1) = -d_3(1,3) + x + y \geq 0$$
$$u_3(3,t_3) - u_3(2,t_2) = -d_3(2,3) + y \geq 0 \ .$$

so agent 3 does not envy other agents.

Hence $(\sigma, t = (t_1, t_1 + x, t_1 + x + y)) \in NE(N,u)$

Let $(\sigma, t) \in NE(N,u)$. By Lemma 1 $\sigma \in \Sigma_{max}(N,u)$. For sake of simplicity let $\sigma = (1,2,3)$, by Lemma 3

$$t_2 - t_1 \in [d_2(1,2), d_1(1,2)]$$

$$t_3 - t_2 \in [d_3(2,3), d_2(2,3)]$$

$$t_3 - t_1 \in [d_3(1,3), d_1(1,3)] \ .$$

So there are only two cases to check. Assume that

$$t_2 - t_1 \in [d_2(1,2), d_1(1,2)],$$

$$t_3 - t_2 \in [d_3(2,3), d_2(2,3)],$$

$$t_3 - t_1 \notin [\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

Case 1) If $d_3(1,3) \leq t_3 - t_1 < d_2(1,2) + d_3(2,3)$

$$t_3 - t_1 = t_3 - t_2 + t_2 - t_1$$

and

$$t_3 - t_2 \geq d_3(2,3)$$

$$t_2 - t_1 \geq d_2(1,2)$$

so

$$t_3 - t_1 \geq d_2(1,2) + d_3(2,3)$$

is in contradiction with

$$t_3 - t_1 < d_2(1,2) + d_3(2,3)$$

Case 2) If $d_1(1,3) \geq t_3 - t_1 > d_1(1,2) + d_2(2,3)$

$$t_3 - t_1 = t_3 - t_2 + t_2 - t_1$$

and

$$t_3 - t_2 \leq d_2(2,3)$$

$$t_2 - t_1 \leq d_1(1,2)$$

so

$$t_3 - t_1 \leq d_1(1,2) + d_2(2,3)$$

is in contradiction with

$$t_3 - t_1 > d_1(1,2) + d_2(2,3)$$

$\square$

In next lemma we will carry our results to the case when $n = 4$.

Lemma 7 *For any $(N,u) \in Q$ if $|N| = 4$, then $NE(N,u) \neq \emptyset$ and $(\sigma, t) \in NE(N,u)$ if and only if $\sigma \in \Sigma max(N,u)$ and $t_2 - t_1 \in [d_2(1,2), d_1(1,2)]$, $t_3 - t_2 \in [d_3(2,3), d_2(2,3)]$, $t_4 - t_3 \in [d_4(3,4), d_3(3,4)]$, $t_3 - t_1 \in [\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$, $t_4 - t_2 \in [\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}]$,*

$$t_4 - t_1 \in [\max\{\max\{d_3(2,3) + d_4(3,4), d_4(2,4), d_3(1,3) - d_1(1,2) + d_4(3,4)\} +$$

$$\max\{d_2(1,2), d_3(1,3) - d_2(2,3), d_3(1,3) + d_4(3,4) - d_2(2,4)\}, d_4(1,4)\},$$

$$\min\{\min\{d_2(2,3) + d_3(3,4), d_2(2,4), d_1(1,3) - d_2(1,2) + d_3(3,4)\} +$$

$$\min\{d_1(1,2), d_1(1,3) - d_3(2,3), d_1(1,3) + d_3(3,4) - d_4(2,4)\}, d_1(1,4)\}]$$

such that $t_1 + t_2 + t_3 = 0$.

From Lemma 6 we know that if there are only three agents we can find envy free solutions. In the proof of Lemma 7 we will use this information. We start by considering first three and last three agents in four person problem seperately and by Lemma 6 we know that there is a set of envy free solutions to both first three agents and last three agents problems. Note that there are some agents which are both in first three and last three agents, agent 2 and agent 3. In the next step we check if the intersection of solutions for the transfer difference between agents 2 and 3 coming from the solutions to first three agents' and last three agents' problems is nonempty or not. It turns out that it is nonempty, hence we are able to connect two pieces of information. The last thing we do is to check if there are solutions which make agent 1 and agent 4 envy free between themselves.

Proof Let $\sigma \in \Sigma_{max}(N, u)$ and without loss of generality let $\sigma = (1, 2, 3, 4)$. From Lemma 6 we know that there are x, y and y, z pairs satisfying

i)

$$x \in [d_2(1,2), d_1(1,2)]$$

$$y \in [d_3(2,3), d_2(2,3)]$$

and

$$x + y \in [\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

ii)

$$y \in [d_3(2,3), d_2(2,3)]$$

$$z \in [d_4(3,4), d_3(3,4)]$$

and

$$y + z \in [\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}]$$

Let

$$Y = \{y \in [d_3(2,3), d_2(2,3)] \mid \exists x \in [d_2(1,2), d_1(1,2)] :$$

$$x + y \in [\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

If $\max(Y) = d_2(2,3)$ then by Lemma 5

$$d_3(2,3) + d_4(3,4) \leq d_2(2,3) + d_3(3,4)$$

$$d_4(2,4) \leq d_2(2,3) + d_3(3,4)$$

If $\max(Y) \neq d_2(2,3)$ then $d_2(2,3) \notin Y$, we know from Lemma 5 that

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} \leq d_2(2,3) + d_1(1,2)$$

and

$$d_2(2,3) + d_2(1,2) \leq d_2(2,3) + d_1(1,2)$$

then it should be the case that

$$\min\{d_1(1,2) + d_2(2,3), d_1(1,3)\} < d_2(2,3) + d_2(1,2)$$

else $d_2(2,3) \in Y$. So it is the case that

$$d_1(1,3) < d_2(2,3) + d_2(1,2) \leq d_1(1,2) + d_2(2,3).$$

Since

$$d_1(1,3) - d_2(1,2) + d_2(1,2) = d_1(1,3),$$

then

$$d_1(1,3) - d_2(1,2) \in Y,$$

so

$$\max(Y) \geq d_1(1,3) - d_2(1,2).$$

Assume that

$$\max(Y) > d_1(1,3) - d_2(1,2)$$

then

$$\max(Y) + d_2(1,2) > d_1(1,3)$$

hence $\max(Y) \notin Y$ which is a contradiction. Hence

$$\max(Y) \leq d_1(1,3) - d_2(1,2)$$

so

$$\max(Y) = d_1(1,3) - d_2(1,2)$$

then by Lemma 5

$$d_3(2,3) + d_4(3,4) \leq d_1(1,3) - d_2(1,2) + d_3(3,4)$$

$$d_4(2,4) \leq d_1(1,3) - d_2(1,2) + d_3(3,4)$$

so

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} \leq \max\{d_2(2,3), d_1(1,3) - d_2(1,2)\} + d_3(3,4)$$

hence

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} \leq \max(Y) + d_3(3,4) \ .$$

If $\min(Y) = d_3(2,3)$ then by Lemma 5

$$d_2(2,3) + d_3(3,4) \geq d_3(2,3) + d_4(3,4)$$

$$d_2(2,4) \geq d_3(2,3) + d_4(3,4)$$

If $\min(Y) \neq d_3(2,3)$ then $d_3(2,3) \notin Y$, we know from Lemma 5 that

$$\min\{d_1(1,2) + d_2(2,3), d_1(1,3)\} \geq d_3(2,3) + d_2(1,2)$$

and

$$d_3(2,3) + d_1(1,2) \geq d_3(2,3) + d_2(1,2)$$

then it should be the case that

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} > d_3(2,3) + d_1(1,2)$$

else $d_3(2,3) \in Y$. So it is the case that

$$d_3(1,3) > d_3(2,3) + d_1(1,2) \geq d_2(1,2) + d_3(2,3).$$

Since

$$d_3(1,3) - d_1(1,2) + d_1(1,2) = d_3(1,3),$$

then

$$d_3(1,3) - d_1(1,2) \in Y,$$

so

$$\min(Y) \leq d_3(1,3) - d_1(1,2).$$

Assume that

$$\min(Y) < d_3(1,3) - d_1(1,2)$$

then

$$\min(Y) + d_1(1,2) < d_3(1,3)$$

hence $\min(Y) \notin Y$ which is a contradiction. Hence

$$\min(Y) \geq d_3(1,3) - d_1(1,2)$$

so

$$\min(Y) = d_3(1,3) - d_1(1,2)$$

then by Lemma 5

$$d_2(2,3) + d_3(3,4) \geq d_3(1,3) - d_1(1,2) + d_4(3,4)$$

$$d_2(2,4) \geq d_3(1,3) - d_1(1,2) + d_4(3,4)$$

so

$$\min\{d_2(2,3) + d_3(3,4), d_2(2,4)\} \geq \min\{d_3(2,3), d_3(1,3) - d_1(1,2)\} + d_4(3,4)$$

hence

$$\min\{d_2(2,3) + d_3(3,4), d_2(2,4)\} \geq \min(Y) + d_4(3,4).$$

Hence

$$[\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}] \bigcap$$

$$[\min\{d_3(2,3), d_3(1,3) - d_1(1,2)\} + d_4(3,4),$$

$$\max\{d_2(2,3), d_1(1,3) - d_2(1,2)\} + d_3(3,4)] \neq \emptyset$$

so

$$[\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}] \bigcap$$

$$[\min(Y) + d_4(3,4), \max(Y) + d_3(3,4)] \neq \emptyset$$

so there exists a triple $x, y, z \in R_+$ such that

$$x \in [d_2(1,2), d_1(1,2)]$$

$$y \in [d_3(2,3), d_2(2,3)]$$

$$z \in [d_4(3,4), d_3(3,4)]$$

$$x + y \in [\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

$$y + z \in [\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}]$$

holds. For any such triple

$$y + z \in [\max\{d_3(2,3) + d_4(3,4), d_4(2,4), d_3(1,3) - d_1(1,2) + d_4(3,4)\},$$

$$\min\{d_2(2,3) + d_3(3,4), d_2(2,4), d_1(1,3) - d_2(1,2) + d_3(3,4)\}].$$

Let X be the set of x which satisfy above requirement with a (y,z) couple. $X = \{x \in [d_2(1,2), d_1(1,2)] \mid y \in [d_3(2,3), d_2(2,3)], z \in [d_4(3,4), d_3(3,4)] : x+y \in [\max\{d_2(1,2)+d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2)+d_2(2,3), d_1(1,3)\}], y+z \in [\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}]\}$

If $\min(X) = d_2(1,2)$ then by Lemma 5

$$d_2(1,2) + d_3(2,3) + d_4(3,4) \leq d_1(1,4)$$

$$d_2(1,2) + d_4(2,4) \leq d_1(1,4)$$

$$d_2(1,2) + d_3(1,3) - d_1(1,2) + d_4(3,4) \leq d_1(1,4)$$

If $\min(X) \neq d_2(1,2)$ note that $d_2(1,2) + d_3(2,3) \leq \min\{d_1(1,3), d_1(1,2) + d_2(2,3)\}$ so

1) for all $y \in [d_3(2,3), d_2(2,3)]$

$$y + d_2(1,2) < \max\{d_3(1,3), d_2(1,2) + d_3(2,3)\}$$

hence

$$d_2(2,3) + d_2(1,2) < \max\{d_3(1,3), d_2(1,2) + d_3(2,3)\}$$

so

$$d_2(1,2) + d_3(2,3) \leq d_2(2,3) + d_2(1,2) < d_3(1,3)$$

assume that

$$\min(X) < d_3(1,3) - d_2(2,3)$$

then

$$\min(X) + d_2(2,3) < d_3(1,3)$$

so

$$\min(X) \notin X$$

which is a contradiction. Hence

$$\min(X) \geq d_3(1,3) - d_2(2,3)$$

a) If $\min(X) = d_3(1,3) - d_2(2,3)$ then by Lemma 5

$$d_3(1,3) - d_2(2,3) + d_3(2,3) + d_4(3,4) \leq d_1(1,4)$$

$$d_3(1,3) - d_2(2,3) + d_4(2,4) \leq d_1(1,4)$$

$$d_3(1,3) - d_2(2,3) + d_3(1,3) - d_1(1,2) + d_4(3,4) \leq d_1(1,4)$$

b) If $\min(X) \neq d_3(1,3) - d_2(2,3)$ note that letting $y = d_2(2,3)$

$$d_3(1,3) - d_2(2,3) + d_2(2,3) = d_3(1,3) \in$$

$$[\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

since

$$d_2(1,2) + d_3(2,3) < d_3(1,3)$$

so for any $y \in [d_3(2,3), d_2(2,3)]$ such that

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} - d_3(1,3) + d_2(2,3) \leq y$$

$$\leq \min\{d_1(1,2)+d_2(2,3), d_1(1,3)\} - d_3(1,3) + d_2(2,3)$$

and any $z \in [d_4(3,4), d_3(3,4)]$

$$y + z \notin [\max\{d_3(2,3)+d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3)+d_3(3,4), d_2(2,4)\}]$$

by Lemma 5

$$i) d_1(1,3) - d_3(1,3) + d_2(2,3) + d_3(3,4) \geq d_3(2,3) + d_4(3,4)$$

$$ii) d_1(1,3) - d_3(1,3) + d_2(2,3) + d_3(3,4) \geq d_4(2,4)$$

$$iii) d_1(1,2) + d_2(2,3) - d_3(1,3) + d_2(2,3) + d_3(3,4) \geq d_3(2,3) + d_4(3,4)$$

$$iv) d_1(1,2) + d_2(2,3) - d_3(1,3) + d_2(2,3) + d_3(3,4) \geq d_4(2,4)$$

so

$$\min\{d_1(1,2)+d_2(2,3), d_1(1,3)\} - d_3(1,3) + d_2(2,3) + d_3(3,4) \geq$$

$$\max\{d_3(2,3)+d_4(3,4), d_4(2,4)\}$$

hence it should be the case that

$$\min\{d_2(2,3)+d_3(3,4), d_2(2,4)\} <$$

$$\max\{d_2(1,2)+d_3(2,3), d_3(1,3)\} - d_3(1,3) + d_2(2,3) + d_4(3,4)$$

by Lemma 5

$$i) d_2(2,3) + d_3(3,4) \geq d_3(1,3) - d_3(1,3) + d_2(2,3) + d_4(3,4)$$

since in our case $d_3(1,3) > d_2(1,2) + d_3(2,3)$ and by Lemma 5

$$ii) d_2(2,3) + d_3(3,4) \geq d_2(1,2) + d_3(2,3) - d_3(1,3) + d_2(2,3) + d_4(3,4)$$

since in our case $d_3(1,3) \geq d_2(1,2) + d_2(2,3)$ and by Lemma 5

$$iii) d_2(2,4) \geq d_2(1,2) + d_3(2,3) - d_3(1,3) + d_2(2,3) + d_4(3,4)$$

so it should be

$$iv) d_2(2,4) < d_3(1,3) - d_3(1,3) + d_2(2,3) + d_4(3,4).$$

Assume that

$$\min(X) < d_3(1,3) + d_4(3,4) - d_2(2,4)$$

then

$$d_2(2,4) < d_3(1,3) - \min(X) + d_4(3,4)$$

so

$$\min(X) \notin X$$

which is a contradiction. Hence

$$\min(X) \geq d_3(1,3) + d_4(3,4) - d_2(2,4)$$

A) In our case

$$d_2(1,2) < d_3(1,3) - d_2(2,3)$$

and

$$d_3(1,3) - d_2(2,3) < d_3(1,3) + d_4(3,4) - d_2(2,4)$$

hence

$$d_2(1,2) < d_3(1,3) + d_4(3,4) - d_2(2,4)$$

and by Lemma 5

$$d_3(1,3) + d_4(3,4) - d_2(2,4) \leq d_1(1,2)$$

hence

$$d_3(1,3) + d_4(3,4) - d_2(2,4) \in [d_2(1,2), d_1(1,2)]$$

B)

a) In our case

$$d_3(1,3) - d_2(2,3) < d_3(1,3) + d_4(3,4) - d_2(2,4)$$

so

$$d_3(1,3) < d_3(1,3) + d_4(3,4) - d_2(2,4) + d_2(2,3)$$

again in our case

$$d_2(1,2) + d_3(2,3) < d_3(1,3)$$

so

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} < d_3(1,3) + d_4(3,4) - d_2(2,4) + d_2(2,3)$$

b) By Lemma 5

$$d_3(1,3) + d_4(3,4) - d_2(2,4) + d_3(2,3) \leq \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}$$

so

$$[\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}] \bigcap$$

$$[d_3(1,3)+d_4(3,4)-d_2(2,4)+d_3(2,3), d_3(1,3)+d_4(3,4)-d_2(2,4)+d_2(2,3)] \neq \emptyset$$

hence there exists $y \in [d_3(2,3), d_2(2,3)]$ such that

$$d_3(1,3) + d_4(3,4) - d_2(2,4) + y \in$$

$$[\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

C) a) by Lemma 5

$$i) d_1(1,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_3(3,4) \geq d_3(2,3) + d_4(3,4)$$

$$ii) d_1(1,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_3(3,4) \geq d_4(2,4)$$

$$iii) d_1(1,2) + d_2(2,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_3(3,4) \geq d_3(2,3) + d_4(3,4)$$

$$iv) d_1(1,2) + d_2(2,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_3(3,4) \geq d_4(2,4)$$

so

$$\min\{d_1(1,2) + d_2(2,3), d_1(1,3)\} - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_3(3,4) \geq$$

$$\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}$$

b) since $-d_3(1,3) - d_4(3,4) + d_2(2,4) < -d_3(1,3) + d_2(2,3)$

$$i) d_2(2,3) + d_3(3,4) \geq d_3(1,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_4(3,4)$$

$$ii) d_2(2,3) + d_3(3,4) \geq d_2(1,2) + d_3(2,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_4(3,4)$$

$$iii) d_2(2,4) \geq d_2(1,2) + d_3(2,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_4(3,4)$$

$$iv) d_2(2,4) = d_3(1,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_4(3,4)$$

so

$$\min\{d_2(2,3) + d_3(3,4), d_2(2,4)\} \geq$$

$$\max\{d_2(1,2)+d_3(2,3),d_3(1,3)\}-d_3(1,3)-d_4(3,4)+d_2(2,4)+d_4(3,4)$$

hence$d_3(1,3)+d_4(3,4)-d_2(2,4)\in X$ so $\min(X)=d_3(1,3)+d_4(3,4)-d_2(2,4)$. By Lemma 5

$$d_3(1,3)+d_4(3,4)-d_2(2,4)+d_3(2,3)+d_4(3,4)\leq d_1(1,4)$$

$$d_3(1,3)+d_4(3,4)-d_2(2,4)+d_4(2,4)\leq d_1(1,4)$$

$$d_3(1,3)+d_4(3,4)-d_2(2,4)+d_3(1,3)-d_1(1,2)+d_4(3,4)\leq d_1(1,4)$$

2) There exist $y\in[d_3(2,3),d_2(2,3)]$ such that

$$\max\{d_2(1,2)+d_3(2,3),d_3(1,3)\}\leq$$

$$y+d_2(1,2)\leq\min\{d_1(1,2)+d_2(2,3),d_1(1,3)\}$$

hence

$$\max\{d_2(1,2)+d_3(2,3),d_3(1,3)\}\leq d_2(2,3)+d_2(1,2)$$

so in this case

$$d_3(1,3)\leq d_2(2,3)+d_2(1,2)$$

for any $y\in[d_3(2,3),d_2(2,3)]$ such that

$$\max\{d_2(1,2)+d_3(2,3),d_3(1,3)\}-d_2(1,2)\leq$$

$$y\leq\min\{d_1(1,2)+d_2(2,3),d_1(1,3)\}-d_2(1,2)$$

and any $z\in[d_4(3,4),d_3(3,4)]$

$$y+z\notin[\max\{d_3(2,3)+d_4(3,4),d_4(2,4)\},\min\{d_2(2,3)+d_3(3,4),d_2(2,4)\}]$$

by Lemma 5

$$i) d_1(1,3) - d_2(1,2) + d_3(3,4) \geq d_3(2,3) + d_4(3,4)$$

$$ii) d_1(1,3) - d_2(1,2) + d_3(3,4) \geq d_4(2,4)$$

$$iii) d_1(1,2) + d_2(2,3) - d_2(1,2) + d_3(3,4) \geq d_3(2,3) + d_4(3,4)$$

$$iv) d_1(1,2) + d_2(2,3) - d_2(1,2) + d_3(3,4) \geq d_4(2,4)$$

so

$$\min\{d_1(1,2) + d_2(2,3), d_1(1,3)\} - d_2(1,2) + d_3(3,4) \geq$$

$$\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}$$

hence it should be the case that

$$\min\{d_2(2,3) + d_3(3,4), d_2(2,4)\} <$$

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} - d_2(1,2) + d_4(3,4)$$

since in this case $d_3(1,3) \leq d_2(2,3) + d_2(1,2)$ and by Lemma 5

$$i) d_2(2,3) + d_3(3,4) \geq d_3(1,3) - d_2(1,2) + d_4(3,4)$$

by Lemma 5

$$ii) d_2(2,3) + d_3(3,4) \geq d_2(1,2) + d_3(2,3) - d_2(1,2) + d_4(3,4)$$

$$iii) d_2(2,4) \geq d_2(1,2) + d_3(2,3) - d_2(1,2) + d_4(3,4)$$

so it should be

$$iv) d_2(2,4) < d_3(1,3) - d_2(1,2) + d_4(3,4).$$

Assume that

$$\min(X) < d_3(1,3) + d_4(3,4) - d_2(2,4)$$

then

$$d_2(2,4) < d_3(1,3) - \min(X) + d_4(3,4)$$

so

$$\min(X) \notin X$$

which is a contradiction. Hence

$$\min(X) \geq d_3(1,3) + d_4(3,4) - d_2(2,4)$$

A) In our case

$$d_2(1,2) < d_3(1,3) + d_4(3,4) - d_2(2,4)$$

and by Lemma 5

$$d_3(1,3) + d_4(3,4) - d_2(2,4) \leq d_1(1,2)$$

hence

$$d_3(1,3) + d_4(3,4) - d_2(2,4) \in [d_2(1,2), d_1(1,2)]$$

B)

a) In this case

$$d_3(1,3) - d_2(2,3) \leq d_2(1,2) < d_3(1,3) + d_4(3,4) - d_2(2,4)$$

so

$$d_3(1,3) < d_3(1,3) + d_4(3,4) - d_2(2,4) + d_2(2,3)$$

again in our case

$$d_2(1,2) + d_3(2,3) \leq d_2(2,3) + d_2(1,2)$$

so

$$d_2(1,2) + d_3(2,3) - d_2(2,3) \leq d_2(1,2)$$

and

$$d_2(1,2) < d_3(1,3) + d_4(3,4) - d_2(2,4)$$

hence

$$d_2(1,2) + d_3(2,3) < d_3(1,3) + d_4(3,4) - d_2(2,4) + d_2(2,3)$$

so

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} < d_3(1,3) + d_4(3,4) - d_2(2,4) + d_2(2,3)$$

b) By Lemma 5

$$d_3(1,3) + d_4(3,4) - d_2(2,4) + d_3(2,3) \leq \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}$$

so

$$[\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}] \bigcap$$

$$[d_3(1,3)+d_4(3,4)-d_2(2,4)+d_3(2,3), d_3(1,3)+d_4(3,4)-d_2(2,4)+d_2(2,3)] \neq \emptyset$$

hence there exists $y \in [d_3(2,3), d_2(2,3)]$ such that

$$d_3(1,3) + d_4(3,4) - d_2(2,4) + y \in$$

$$[\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

C) a) by Lemma 5

$$i)d_1(1,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_3(3,4) \geq d_3(2,3) + d_4(3,4)$$

$$ii)d_1(1,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_3(3,4) \geq d_4(2,4)$$

$$iii)d_1(1,2)+d_2(2,3)-d_3(1,3)-d_4(3,4)+d_2(2,4)+d_3(3,4) \geq d_3(2,3)+d_4(3,4)$$

$$iv)d_1(1,2) + d_2(2,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_3(3,4) \geq d_4(2,4)$$

so

$$\min\{d_1(1,2) + d_2(2,3), d_1(1,3)\} - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_3(3,4) \geq$$

$$\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}$$

b) since $-d_3(1,3) - d_4(3,4) + d_2(2,4) < -d_2(1,2)$

$$i)d_2(2,3) + d_3(3,4) \geq d_3(1,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_4(3,4)$$

$$ii)d_2(2,3)+d_3(3,4) \geq d_2(1,2)+d_3(2,3)-d_3(1,3)-d_4(3,4)+d_2(2,4)+d_4(3,4)$$

$$iii)d_2(2,4) \geq d_2(1,2) + d_3(2,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_4(3,4)$$

$$iv)d_2(2,4) = d_3(1,3) - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_4(3,4)$$

so

$$\min\{d_2(2,3) + d_3(3,4), d_2(2,4)\} \geq$$

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} - d_3(1,3) - d_4(3,4) + d_2(2,4) + d_4(3,4)$$

hence$d_3(1,3)+d_4(3,4)-d_2(2,4) \in X$ so $\min(X) = d_3(1,3)+d_4(3,4)-d_2(2,4)$. By Lemma 5

$$d_3(1,3) + d_4(3,4) - d_2(2,4) + d_3(2,3) + d_4(3,4) \leq d_1(1,4)$$

$$d_3(1,3) + d_4(3,4) - d_2(2,4) + d_4(2,4) \leq d_1(1,4)$$

$$d_3(1,3) + d_4(3,4) - d_2(2,4) + d_3(1,3) - d_1(1,2) + d_4(3,4) \leq d_1(1,4)$$

so we have reached to the conclusion that

$$\max\{d_3(2,3) + d_4(3,4), d_4(2,4), d_3(1,3) - d_1(1,2) + d_4(3,4)\}+$$

$$\max\{d_2(1,2), d_3(1,3) - d_2(2,3), d_3(1,3) + d_4(3,4) - d_2(2,4)\} \leq d_1(1,4).$$

If $\max(X) = d_1(1,2)$ then by Lemma 5

$$d_1(1,2) + d_2(2,3) + d_3(3,4) \geq d_4(1,4)$$

$$d_1(1,2) + d_2(2,4) \geq d_4(1,4)$$

$$d_1(1,2) + d_1(1,3) - d_2(1,2) + d_3(3,4) \geq d_4(1,4)$$

If $\max(X) \neq d_1(1,2)$ note that $d_1(1,2)+d_2(2,3) \geq \max\{d_3(1,3), d_2(1,2)+ d_3(2,3)\}$ so

1) for all $y \in [d_3(2,3), d_2(2,3)]$

$$y + d_1(1,2) > \min\{d_1(1,3), d_1(1,2) + d_2(2,3)\}$$

hence

$$d_3(2,3) + d_1(1,2) > \min\{d_1(1,3), d_1(1,2) + d_2(2,3)\}$$

so

$$d_1(1,2) + d_2(2,3) \geq d_3(2,3) + d_1(1,2) > d_3(1,3)$$

assume that

$$\max(X) > d_1(1,3) - d_3(2,3)$$

then

$$\max(X) + d_3(2,3) < d_1(1,3)$$

so

$$\max(X) \notin X$$

which is a contradiction. Hence

$$\max(X) \leq d_1(1,3) - d_3(2,3)$$

a) If $\max(X) = d_1(1,3) - d_3(2,3)$ then by Lemma 5

$$d_1(1,3) - d_3(2,3) + d_2(2,3) + d_3(3,4) \geq d_4(1,4)$$

$$d_1(1,3) - d_3(2,3) + d_2(2,4) \geq d_4(1,4)$$

$$d_1(1,3) - d_3(2,3) + d_1(1,3) - d_2(1,2) + d_3(3,4) \geq d_4(1,4)$$

b) If $\max(X) \neq d_1(1,3) - d_3(2,3)$ note that letting $y = d_3(2,3)$

$$d_1(1,3) - d_3(2,3) + d_3(2,3) = d_1(1,3) \in$$

$$[\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

since

$$d_1(1,2) + d_2(2,3) > d_3(1,3)$$

so for any $y \in [d_3(2,3), d_2(2,3)]$ such that

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} - d_1(1,3) + d_3(2,3) \leq y$$

$$\leq \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\} - d_1(1,3) + d_3(2,3)$$

and any $z \in [d_4(3,4), d_3(3,4)]$

$$y + z \notin [\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}]$$

by Lemma 5

$$i) d_3(1,3) - d_1(1,3) + d_3(2,3) + d_4(3,4) \leq d_2(2,3) + d_3(3,4)$$

$$ii) d_3(1,3) - d_1(1,3) + d_3(2,3) + d_4(3,4) \leq d_2(2,4)$$

$$iii) d_2(1,2) + d_3(2,3) - d_1(1,3) + d_3(2,3) + d_4(3,4) \leq d_2(2,3) + d_3(3,4)$$

$$iv) d_2(1,2) + d_3(2,3) - d_1(1,3) + d_3(2,3) + d_4(3,4) \leq d_2(2,4)$$

so

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} - d_1(1,3) + d_3(2,3) + d_4(3,4) \leq$$

$$\min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}$$

hence it should be the case that

$$\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\} >$$

$$\min\{d_1(1,2) + d_2(2,3), d_1(1,3)\} - d_1(1,3) + d_3(2,3) + d_3(3,4)$$

by Lemma 5

$$i) d_3(2,3) + d_4(3,4) \leq d_1(1,3) - d_1(1,3) + d_3(2,3) + d_3(3,4)$$

since in our case $d_1(1,3) < d_1(1,2) + d_2(2,3)$ and by Lemma 5

$$ii) d_3(2,3) + d_4(3,4) \leq d_1(1,2) + d_2(2,3) - d_1(1,3) + d_3(2,3) + d_3(3,4)$$

since in our case $d_1(1,3) < d_1(1,2) + d_3(2,3)$ and by Lemma 5

$$iii) d_4(2,4) \leq d_1(1,2) + d_2(2,3) - d_1(1,3) + d_3(2,3) + d_3(3,4)$$

so it should be

$$iv) d_4(2,4) > d_1(1,3) - d_1(1,3) + d_3(2,3) + d_3(3,4).$$

Assume that

$$\max(X) > d_1(1,3) + d_3(3,4) - d_4(2,4)$$

then

$$d_4(2,4) > d_1(1,3) - \max(X) + d_3(3,4)$$

so

$$\max(X) \notin X$$

which is a contradiction. Hence

$$\max(X) \leq d_1(1,3) + d_3(3,4) - d_4(2,4)$$

A) In our case

$$d_1(1,2) > d_1(1,3) - d_3(2,3)$$

and

$$d_1(1,3) - d_3(2,3) > d_1(1,3) + d_2(3,4) - d_4(2,4)$$

hence

$$d_1(1,2) > d_1(1,3) + d_3(3,4) - d_4(2,4)$$

and by Lemma 5

$$d_1(1,3) + d_3(3,4) - d_4(2,4) \geq d_2(1,2)$$

hence

$$d_1(1,3) + d_3(3,4) - d_4(2,4) \in [d_2(1,2), d_1(1,2)]$$

B)

a) In our case

$$d_1(1,3) - d_3(2,3) > d_1(1,3) + d_3(3,4) - d_4(2,4)$$

so

$$d_1(1,3) > d_1(1,3) + d_3(3,4) - d_4(2,4) + d_3(2,3)$$

again in our case

$$d_1(1,2) + d_2(2,3) > d_1(1,3)$$

so

$$\min\{d_1(1,2) + d_2(2,3), d_1(1,3)\} > d_1(1,3) + d_3(3,4) - d_4(2,4) + d_3(2,3)$$

b) By Lemma 5

$$d_1(1,3) + d_3(3,4) - d_4(2,4) + d_2(2,3) \geq \max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}$$

so

$$[\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}] \bigcap$$

$$[d_1(1,3)+d_3(3,4)-d_4(2,4)+d_3(2,3), d_1(1,3)+d_3(3,4)-d_4(2,4)+d_2(2,3)] \neq \emptyset$$

hence there exists $y \in [d_3(2,3), d_2(2,3)]$ such that

$$d_1(1,3) + d_3(3,4) - d_4(2,4) + y \in$$

$$[\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

C) a) by Lemma 5

$$i) d_3(1,3) - d_1(1,3) - d_3(3,4) + d_4(2,4) + d_4(3,4) \leq d_2(2,3) + d_3(3,4)$$

$$ii) d_3(1,3) - d_1(1,3) - d_3(3,4) + d_4(2,4) + d_4(3,4) \leq d_2(2,4)$$

$$iii) d_2(1,2) + d_3(2,3) - d_1(1,3) - d_3(3,4) + d_4(2,4) + d_4(3,4) \leq d_2(2,3) + d_3(3,4)$$

$$iv) d_2(1,2) + d_3(2,3) - d_1(1,3) - d_3(3,4) + d_4(2,4) + d_4(3,4) \leq d_2(2,4)$$

so

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} - d_1(1,3) - d_3(3,4) + d_4(2,4) + d_4(3,4) \leq$$

$$\min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}$$

b) since $-d_1(1,3) - d_3(3,4) + d_4(2,4) > -d_1(1,3) + d_3(2,3)$

$$i) d_3(2,3) + d_4(3,4) \leq d_1(1,3) - d_1(1,3) - d_3(3,4) + d_4(2,4) + d_3(3,4)$$

$$ii) d_3(2,3) + d_4(3,4) \leq d_1(1,2) + d_1(2,3) - d_1(1,3) - d_3(3,4) + d_4(2,4) + d_3(3,4)$$

$$iii) d_4(2,4) \leq d_1(1,2) + d_2(2,3) - d_1(1,3) - d_3(3,4) + d_4(2,4) + d_3(3,4)$$

$$iv) d_4(2,4) = d_1(1,3) - d_1(1,3) - d_3(3,4) + d_4(2,4) + d_3(3,4)$$

so

$$\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\} \leq$$

$$\min\{d_1(1,2) + d_2(2,3), d_1(1,3)\} - d_1(1,3) - d_3(3,4) + d_4(2,4) + d_3(3,4)$$

hence$d_1(1,3) + d_3(3,4) - d_4(2,4) \in X$ so $\max(X) = d_1(1,3) + d_3(3,4) - d_4(2,4)$. By Lemma 5

$$d_1(1,3) + d_3(3,4) - d_4(2,4) + d_2(2,3) + d_3(3,4) \geq d_4(1,4)$$

$$d_1(1,3) + d_3(3,4) - d_4(2,4) + d_2(2,4) \geq d_4(1,4)$$

$$d_1(1,3) + d_3(3,4) - d_4(2,4) + d_1(1,3) - d_2(1,2) + d_3(3,4) \geq d_4(1,4)$$

2) There exist $y \in [d_3(2,3), d_2(2,3)]$ such that

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} \leq$$

$$y + d_1(1,2) \leq \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}$$

hence

$$\min\{d_1(1,2) + d_2(2,3), d_1(1,3)\} \geq d_3(2,3) + d_1(1,2)$$

so in this case

$$d_1(1,3) \geq d_3(2,3) + d_1(1,2)$$

for any $y \in [d_3(2,3), d_2(2,3)]$ such that

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} - d_1(1,2) \leq$$

$$y \leq \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\} - d_1(1,2)$$

and any $z \in [d_4(3,4), d_3(3,4)]$

$$y + z \notin [\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}]$$

by Lemma 5

$$i) d_3(1,3) - d_1(1,2) + d_4(3,4) \leq d_2(2,3) + d_3(3,4)$$

$$ii) d_3(1,3) - d_1(1,2) + d_4(3,4) \leq d_2(2,4)$$

$$iii) d_2(1,2) + d_3(2,3) - d_1(1,2) + d_4(3,4) \leq d_2(2,3) + d_3(3,4)$$

$$iv) d_2(1,2) + d_3(2,3) - d_1(1,2) + d_4(3,4) \leq d_2(2,4)$$

so

$$\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\} - d_1(1,2) + d_4(3,4) \leq$$

$$\min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}$$

hence it should be the case that

$$\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\} >$$

$$\min\{d_1(1,2) + d_2(2,3), d_1(1,3)\} - d_1(1,2) + d_3(3,4)$$

since in this case $d_1(1,3) \geq d_3(2,3) + d_1(1,2)$ and by Lemma 5

$$i) d_3(2,3) + d_4(3,4) \leq d_1(1,3) - d_1(1,2) + d_3(3,4)$$

by Lemma 5

$$ii) d_3(2,3) + d_4(3,4) \leq d_1(1,2) + d_2(2,3) - d_1(1,2) + d_3(3,4)$$

$$iii) d_4(2,4) \leq d_1(1,2) + d_2(2,3) - d_1(1,2) + d_3(3,4)$$

so it should be

$$iv) d_4(2,4) > d_1(1,3) - d_1(1,2) + d_3(3,4).$$

Assume that

$$\max(X) > d_1(1,3) + d_3(3,4) - d_4(2,4)$$

then

$$d_4(2,4) > d_1(1,3) - \max(X) + d_3(3,4)$$

so

$$\max(X) \notin X$$

which is a contradiction. Hence

$$\max(X) \leq d_1(1,3) + d_3(3,4) - d_4(2,4)$$

A) In our case

$$d_1(1,2) \geq d_1(1,3) + d_3(3,4) - d_4(2,4)$$

and by Lemma 5

$$d_1(1,3) + d_3(3,4) - d_4(2,4) \geq d_2(1,2)$$

hence

$$d_1(1,3) + d_3(3,4) - d_4(2,4) \in [d_2(1,2), d_1(1,2)]$$

B)

a) In this case

$$d_1(1,3) - d_3(2,3) \geq d_1(1,2) > d_1(1,3) + d_3(3,4) - d_4(2,4)$$

so

$$d_1(1,3) > d_1(1,3) + d_3(3,4) - d_4(2,4) + d_3(2,3)$$

again in our case

$$d_1(1,2) \geq d_1(1,3) + d_3(3,4) - d_4(2,4)$$

and by Lemma 5

$$d_2(2,3) \geq d_3(2,3)$$

so

$$d_1(1,2) + d_2(2,3) \geq d_1(1,3) + d_3(3,4) - d_4(2,4) + d_3(2,3)$$

so

$$\min\{d_1(1,2)+d_2(2,3), d_1(1,3)\} \geq d_1(1,3)+d_3(3,4)-d_4(2,4)+d_3(2,3)$$

b)b) By Lemma 5

$$d_1(1,3)+d_3(3,4)-d_4(2,4)+d_2(2,3) \geq \max\{d_2(1,2)+d_3(2,3), d_3(1,3)\}$$

so

$$[\max\{d_2(1,2)+d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2)+d_2(2,3), d_1(1,3)\}] \bigcap$$

$$[d_1(1,3)+d_3(3,4)-d_4(2,4)+d_3(2,3), d_1(1,3)+d_3(3,4)-d_4(2,4)+d_2(2,3)] \neq \emptyset$$

hence there exists $y \in [d_3(2,3), d_2(2,3)]$ such that

$$d_1(1,3)+d_3(3,4)-d_4(2,4)+y \in$$

$$[\max\{d_2(1,2)+d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2)+d_2(2,3), d_1(1,3)\}]$$

C) a) by Lemma 5

$$i) d_3(1,3)-d_1(1,3)-d_3(3,4)+d_4(2,4)+d_4(3,4) \leq d_2(2,3)+d_3(3,4)$$

$$ii) d_3(1,3)-d_1(1,3)-d_3(3,4)+d_4(2,4)+d_4(3,4) \leq d_2(2,4)$$

$$iii) d_2(1,2)+d_3(2,3)-d_1(1,3)-d_3(3,4)+d_4(2,4)+d_4(3,4) \leq d_2(2,3)+d_3(3,4)$$

$$iv) d_2(1,2)+d_3(2,3)-d_1(1,3)-d_3(3,4)+d_4(2,4)+d_4(3,4) \leq d_2(2,4)$$

so

$$\max\{d_2(1,2)+d_3(2,3), d_3(1,3)\}-d_1(1,3)-d_3(3,4)+d_4(2,4)+d_4(3,4) \leq$$

$$\min\{d_2(2,3)+d_3(3,4), d_2(2,4)\}$$

b) since $-d_1(1,3)-d_3(3,4)+d_4(2,4) > -d_1(1,3)+d_3(2,3)$

$$i) d_3(2,3)+d_4(3,4) \leq d_1(1,3)-d_1(1,3)-d_3(3,4)+d_4(2,4)+d_3(3,4)$$

$$ii) d_3(2,3)+d_4(3,4) \leq d_1(1,2)+d_1(2,3)-d_1(1,3)-d_3(3,4)+d_4(2,4)+d_3(3,4)$$

$$iii) d_4(2,4) \leq d_1(1,2)+d_2(2,3)-d_1(1,3)-d_3(3,4)+d_4(2,4)+d_3(3,4)$$

$$iv) d_4(2,4) = d_1(1,3)-d_1(1,3)-d_3(3,4)+d_4(2,4)+d_3(3,4)$$

so

$$\max\{d_3(2,3)+d_4(3,4), d_4(2,4)\} \leq$$

$$\min\{d_1(1,2)+d_2(2,3), d_1(1,3)\}-d_1(1,3)-d_3(3,4)+d_4(2,4)+d_3(3,4)$$

hence$d_1(1,3)+d_3(3,4)-d_4(2,4) \in X$ so $\max(X) = d_1(1,3)+d_3(3,4)-d_4(2,4)$. By Lemma 5

$$d_1(1,3)+d_3(3,4)-d_4(2,4)+d_2(2,3)+d_3(3,4) \geq d_4(1,4)$$

$$d_1(1,3)+d_3(3,4)-d_4(2,4)+d_2(2,4) \geq d_4(1,4)$$

$$d_1(1,3)+d_3(3,4)-d_4(2,4)+d_1(1,3)-d_2(1,2)+d_3(3,4) \geq d_4(1,4)$$

so we have reached to the conclusion that

$$\min\{d_2(2,3)+d_3(3,4), d_2(2,4), d_1(1,3)-d_2(1,2)+d_3(3,4)\}+$$

$$\min\{d_1(1,2), d_1(1,3)-d_3(2,3), d_1(1,3)+d_3(3,4)-d_4(2,4)\} \geq d_4(1,4).$$

Hence there exists $x, y, z \in \mathbb{R}_+$ such that

$$x \in [d_2(1,2), d_1(1,2)]$$

$$y \in [d_3(2,3), d_2(2,3)]$$

$$z \in [d_4(3,4), d_3(3,4)]$$

$$x + y \in [\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

$$y + z \in [\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}]$$

$$x+y+z \in [\max\{\max\{d_3(2,3)+d_4(3,4), d_4(2,4), d_3(1,3)-d_1(1,2)+d_4(3,4)\}+$$

$$\max\{d_2(1,2), d_3(1,3) - d_2(2,3), d_3(1,3) + d_4(3,4) - d_2(2,4)\}, d_4(1,4)\},$$

$$\min\{\min\{d_2(2,3) + d_3(3,4), d_2(2,4), d_1(1,3) - d_2(1,2) + d_3(3,4)\}+$$

$$\min\{d_1(1,2), d_1(1,3) - d_3(2,3), d_1(1,3) + d_3(3,4) - d_4(2,4)\}, d_1(1,4)\}]$$

Let $t_2 - t_1 = x$, $t_3 - t_2 = y$, $t_4 - t_3 = z$, such that $\sum_{i=1}^{4} t_i = 0$ hence for agent 1

$$u_1(1, t_1) - u_1(2, t_2) = d_1(1,2) - x \geq 0$$

$$u_1(1, t_1) - u_1(3, t_3) = d_1(1,3) - (x + y) \geq 0$$

$$u_1(1, t_1) - u_1(4, t_4) = d_1(1,4) - (x + y + z) \geq 0$$

so agent 1 does not envy other agents.

For agent 2

$$u_2(2, t_2) - u_2(1, t_1) = -d_2(1,2)x \geq 0$$

$$u_2(2, t_2) - u_2(3, t_3) = d_2(2,3) - y \geq 0$$

$$u_2(2, t_2) - u_2(4, t_4) = d_2(2,4) - (y + z) \geq 0$$

so agent 2 does not envy other agents.

For agent 3

$$u_3(3,t_3)-u_3(1,t_1)=-d_3(1,3)x+y\geq 0$$

$$u_3(3,t_3)-u_3(2,t_2)=-d_3(2,3)y\geq 0$$

$$u_3(3,t_3)-u_3(4,t_4)=d_3(3,4)-z\geq 0$$

so agent 3 does not envy other agents.

For agent 4

$$u_4(4,t_4)-u_4(1,t_1)=-d_4(1,4)x+y+z\geq 0$$

$$u_4(4,t_4)-u_4(2,t_2)=-d_4(2,4)y+z\geq 0$$

$$u_4(4,t_4)-u_4(3,t_3)=-d_4(3,4)z\geq 0$$

so agent 4 does not envy other agents.

Let $(\sigma,t)\in NE(N,u)$ so by Lemma 1 $\sigma\in\Sigma_{max}(N,u)$. For sake of simplicity let $\sigma=(1,2,3,4)$, by Lemma 3

$$\forall i\in\{1,2\}\quad t_{i+1}-t_i\in[d_{i+1}(i,i+1),d_i(i,i+1)].$$

So

$$t_2-t_1\in[d_2(1,2),d_1(1,2)],$$

$$t_3-t_2\in[d_3(2,3),d_2(2,3)],$$

$$t_4-t_3\in[d_4(3,4),d_3(3,4)],$$

$$t_3-t_1\in[d_3(1,3),d_1(1,3)],$$

$$t_4-t_2\in[d_4(2,4),d_2(2,4)],$$

$$t_4-t_1\in[d_4(1,4),d_1(1,4)].$$

Assume that

$$t_2 - t_1 \in [d_2(1,2), d_1(1,2)],$$

$$t_3 - t_2 \in [d_3(2,3), d_2(2,3)],$$

$$t_3 - t_1 \notin [\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

Case 1) If $d_3(1,3) \leq t_3 - t_1 < d_2(1,2) + d_3(2,3)$

$$t_3 - t_1 = t_3 - t_2 + t_2 - t_1$$

and

$$t_3 - t_2 \geq d_3(2,3)$$

$$t_2 - t_1 \geq d_2(1,2)$$

so

$$t_3 - t_1 \geq d_2(1,2) + d_3(2,3)$$

is in contradiction with

$$t_3 - t_1 < d_2(1,2) + d_3(2,3).$$

Case 2) If $d_1(1,3) \geq t_3 - t_1 > d_1(1,2) + d_2(2,3)$

$$t_3 - t_1 = t_3 - t_2 + t_2 - t_1$$

and

$$t_3 - t_2 \leq d_2(2,3)$$

$$t_2 - t_1 \leq d_1(1,2)$$

so

$$t_3 - t_1 \leq d_1(1,2) + d_2(2,3)$$

is in contradiction with

$$t_3 - t_1 > d_1(1,2) + d_2(2,3).$$

Assume that

$$t_3 - t_2 \in [d_3(2,3), d_2(2,3)],$$

$$t_4 - t_3 \in [d_4(3,4), d_3(3,4)],$$

$$t_4 - t_2 \notin [\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}]$$

Case 1) If $d_4(2,4) \leq t_4 - t_2 < d_3(2,3) + d_4(3,4)$

$$t_4 - t_2 = t_4 - t_3 + t_3 - t_2$$

and

$$t_4 - t_3 \geq d_4(3,4)$$

$$t_3 - t_2 \geq d_3(2,3)$$

so

$$t_4 - t_2 \geq d_3(2,3) + d_4(3,4)$$

is in contradiction with

$$t_4 - t_2 < d_3(2,3) + d_4(3,4).$$

Case 2) If $d_2(2,4) \geq t_4 - t_2 > d_2(2,3) + d_3(3,4)$

$$t_4 - t_2 = t_4 - t_3 + t_3 - t_2$$

and

$$t_4 - t_3 \le d_3(3,4)$$

$$t_3 - t_2 \le d_2(2,3)$$

so

$$t_4 - t_2 \le d_2(2,3) + d_3(3,4)$$

is in contradiction with

$$t_4 - t_2 > d_2(2,3) + d_3(3,4).$$

Assume that

$$t_2 - t_1 \in [d_2(1,2), d_1(1,2)]$$

$$t_3 - t_2 \in [d_3(2,3), d_2(2,3)]$$

$$t_4 - t_3 \in [d_4(3,4), d_3(3,4)]$$

$$t_3 - t_1 \in [\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

$$t_4 - t_2 \in [\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}]$$

$$t_4 - t_1 \notin [\max\{\max\{d_3(2,3) + d_4(3,4), d_4(2,4), d_3(1,3) - d_1(1,2) + d_4(3,4)\} +$$

$$\max\{d_2(1,2), d_3(1,3) - d_2(2,3), d_3(1,3) + d_4(3,4) - d_2(2,4)\}, d_4(1,4)\},$$

$$\min\{\min\{d_2(2,3) + d_3(3,4), d_2(2,4), d_1(1,3) - d_2(1,2) + d_3(3,4)\} +$$

$$\min\{d_1(1,2), d_1(1,3) - d_3(2,3), d_1(1,3) + d_3(3,4) - d_4(2,4)\}, d_1(1,4)\}]$$

Case 1)

$$t_2 - t_1 \in [d_2(1,2), d_1(1,2)]$$

$$t_3 - t_2 \in [d_3(2,3), d_2(2,3)]$$

$$t_4 - t_3 \in [d_4(3,4), d_3(3,4)]$$

$$t_3 - t_1 \in [\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

$$t_4 - t_2 \in [\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}]$$

but

$$d_4(1,4) \le t_4 - t_1 < \max\{\max\{d_3(2,3) + d_4(3,4), d_4(2,4), d_3(1,3) - d_1(1,2) + d_4(3,4)\} +$$

$$\max\{d_2(1,2), d_3(1,3) - d_2(2,3), d_3(1,3) + d_4(3,4) - d_2(2,4)\}$$

1) If

$$t_4 - t_1 < d_3(2,3) + d_4(3,4) + d_2(1,2)$$

we know that

$$t_4 - t_1 = t_4 - t_3 + t_3 - t_2 + t_2 - t_1$$

and

$$t_2 - t_1 \ge d_2(1,2),$$

$$t_3 - t_2 \ge d_3(2,3),$$

$$t_4 - t_3 \ge d_4(3,4).$$

Hence

$$t_4 - t_1 \ge d_3(2,3) + d_4(3,4) + d_2(1,2)$$

contradicts with

$$t_4 - t_1 < d_3(2,3) + d_4(3,4) + d_2(1,2).$$

2) If

$$t_4 - t_1 < d_3(2,3) + d_4(3,4) + d_3(1,3) - d_2(2,3)$$

then

$$t_4 - t_3 + t_3 - t_2 + t_2 - t_1 < d_3(2,3) + d_4(3,4) + d_3(1,3) - d_2(2,3)$$

by Lemma 3

$$t_4 - t_3 \geq d_4(3,4),$$

$$t_3 - t_2 \geq d_3(2,3).$$

So

$$t_2 - t_1 < d_3(1,3) - d_2(2,3)$$

so

$$t_2 - t_1 + d_2(2,3) < d_3(1,3)$$

by Lemma 3

$$t_3 - t_2 \leq d_2(2,3)$$

so

$$t_3 - t_1 \leq t_2 - t_1 + d_2(2,3) < d_3(1,3)$$

but

$$t_3 - t_1 < d_3(1,3)$$

contradicts with Lemma 3.

3) If

$$t_4 - t_1 < d_3(2,3) + d_4(3,4) + d_3(1,3) + d_4(3,4) - d_2(2,4)$$

then

$$t_4 - t_3 + t_3 - t_2 + t_2 - t_1 < d_3(2,3) + d_4(3,4) + d_3(1,3) + d_4(3,4) - d_2(2,4)$$

by Lemma 3

$$t_4 - t_3 \geq d_4(3,4),$$

$$t_3 - t_2 \geq d_3(2,3).$$

So

$$t_2 - t_1 < d_3(1,3) + d_4(3,4) - d_2(2,4)$$

so

$$t_2 - t_1 + d_2(2,4) < d_3(1,3) + d_4(3,4)$$

by Lemma 3

$$t_4 - t_2 \leq d_2(2,4)$$

so

$$t_4 - t_1 \leq t_2 - t_1 + d_2(2,4) < d_3(1,3) + d_4(3,4)$$

hence either

$$t_4 - t_3 < d_4(3,4)$$

or

$$t_3 - t_1 < d_3(1,3)$$

but by Lemma 3 it is both

$$t_4 - t_3 \geq d_4(3,4)$$

$$t_3 - t_1 \geq d_3(1,3)$$

which is a contradiction.

4) If

$$t_4 - t_1 < d_4(2,4) + d_2(1,2)$$

we know that

$$t_4 - t_1 = t_4 - t_3 + t_3 - t_2 + t_2 - t_1$$

and by Lemma 3

$$t_2 - t_1 \geq d_2(1,2),$$

$$t_4 - t_2 \geq d_4(2,4).$$

Hence

$$t_4 - t_1 \geq d_4(2,4) + d_2(1,2)$$

contradicts with

$$t_4 - t_1 < d_4(2,4) + d_2(1,2).$$

5) If

$$t_4 - t_1 < d_4(2,4) + d_3(1,3) - d_2(2,3)$$

then

$$t_4 - t_3 + t_3 - t_2 + t_2 - t_1 < d_4(2,4) + d_3(1,3) - d_2(2,3)$$

by Lemma 3

$$t_4 - t_2 \geq d_4(2,4).$$

So

$$t_2 - t_1 < d_3(1,3) - d_2(2,3)$$

so

$$t_2 - t_1 + d_2(2,3) < d_3(1,3)$$

by Lemma 3

$$t_3 - t_2 \leq d_2(2,3)$$

so

$$t_3 - t_1 \leq t_2 - t_1 + d_2(2,3) < d_3(1,3)$$

but

$$t_3 - t_1 < d_3(1,3)$$

contradicts with Lemma 3.

6) If

$$t_4 - t_1 < d_4(2,4) + d_3(1,3) + d_4(3,4) - d_2(2,4)$$

then

$$t_4 - t_3 + t_3 - t_2 + t_2 - t_1 < d_4(2,4) + d_3(1,3) + d_4(3,4) - d_2(2,4)$$

by Lemma 3

$$t_4 - t_2 \geq d_4(2,4).$$

So

$$t_2 - t_1 < d_3(1,3) + d_4(3,4) - d_2(2,4)$$

so

$$t_2 - t_1 + d_2(2,4) < d_3(1,3) + d_4(3,4)$$

by Lemma 3

$$t_4 - t_2 \leq d_2(2,4)$$

so

$$t_4 - t_1 \leq t_2 - t_1 + d_2(2,4) < d_3(1,3) + d_4(3,4)$$

hence either

$$t_4 - t_3 < d_4(3,4)$$

or

$$t_3 - t_1 < d_3(1,3)$$

but by Lemma 3 it is both

$$t_4 - t_3 \geq d_4(3,4)$$

$$t_3 - t_1 \geq d_3(1,3)$$

which is a contradiction.

7) If

$$t_4 - t_1 < d_3(1,3) - d_1(1,2) + d_4(3,4) + d_2(1,2)$$

then

$$t_4 - t_3 + t_3 - t_2 + t_2 - t_1 < d_3(1,3) - d_1(1,2) + d_4(3,4) + d_2(1,2)$$

by Lemma 3

$$t_2 - t_1 \geq d_2(1,2).$$

So

$$t_4 - t_2 < d_3(1,3) - d_1(1,2) + d_4(3,4)$$

so

$$t_4 - t_2 + d_1(1,2) < d_3(1,3) + d_4(3,4)$$

by Lemma 3

$$t_2 - t_1 \leq d_1(1,2)$$

hence

$$t_4 - t_1 \leq t_4 - t_2 + d_1(1,2) < d_3(1,3) + d_4(3,4)$$

hence either

$$t_4 - t_3 < d_4(3,4)$$

or

$$t_3 - t_1 < d_3(1,3)$$

but by Lemma 3 it is both

$$t_4 - t_3 \geq d_4(3,4)$$

$$t_3 - t_1 \geq d_3(1,3)$$

which is a contradiction.

8) If

$$t_4 - t_1 < d_3(1,3) - d_1(1,2) + d_4(3,4) + d_3(1,3) - d_2(2,3)$$

then

$$t_4 - t_3 + t_3 - t_2 + t_2 - t_1 < d_3(1,3) - d_1(1,2) + d_4(3,4) + d_3(1,3) - d_2(2,3)$$

by Lemma 3

$$t_3 - t_1 \geq d_3(1,3).$$

So

$$t_4 - t_3 < d_3(1,3) - d_1(1,2) + d_4(3,4) - d_2(2,3)$$

so

$$t_4 - t_3 + d_1(1,2) + d_2(2,3) < d_3(1,3) + d_4(3,4)$$

by Lemma 3

$$t_2 - t_1 \leq d_1(1,2),$$

and

$$t_3 - t_2 \leq d_2(2,3).$$

Hence

$$t_4 - t_1 \leq t_4 - t_3 + d_1(1,2) + d_2(2,3) < d_3(1,3) + d_4(3,4)$$

hence either

$$t_4 - t_3 < d_4(3,4)$$

or

$$t_3 - t_1 < d_3(1,3)$$

but by Lemma 3 it is both

$$t_4 - t_3 \geq d_4(3,4)$$

$$t_3 - t_1 \geq d_3(1,3)$$

which is a contradiction.

9) If

$$t_4 - t_1 < d_3(1,3) - d_1(1,2) + d_4(3,4) + d_3(1,3) + d_4(3,4) - d_2(2,4)$$

then

$$t_4 - t_3 + t_3 - t_2 + t_2 - t_1 < d_3(1,3) - d_1(1,2) + d_4(3,4) + d_3(1,3) + d_4(3,4) - d_2(2,4)$$

by Lemma 3

$$t_3 - t_1 \geq d_3(1,3),$$

$$t_4 - t_3 \geq d_4(3,4).$$

So

$$0 < d_3(1,3) - d_1(1,2) + d_4(3,4) - d_2(2,4)$$

hence

$$d_1(1,2) + d_2(2,4) < d_3(1,3) + d_4(3,4)$$

but by Lemma 5

$$d_1(1,2) + d_2(2,4) \geq d_3(1,3) + d_4(3,4)$$

which is a contradiction.

Case 2)

$$t_2 - t_1 \in [d_2(1,2), d_1(1,2)]$$

$$t_3 - t_2 \in [d_3(2,3), d_2(2,3)]$$

$$t_4 - t_3 \in [d_4(3,4), d_3(3,4)]$$

$$t_3 - t_1 \in [\max\{d_2(1,2) + d_3(2,3), d_3(1,3)\}, \min\{d_1(1,2) + d_2(2,3), d_1(1,3)\}]$$

$$t_4 - t_2 \in [\max\{d_3(2,3) + d_4(3,4), d_4(2,4)\}, \min\{d_2(2,3) + d_3(3,4), d_2(2,4)\}]$$

but

$$d_1(1,4) \geq t_4 - t_1 > \min\{\min\{d_2(2,3) + d_3(3,4), d_2(2,4), d_1(1,3) - d_2(1,2) + d_3(3,4)\} +$$

$$\min\{d_1(1,2), d_1(1,3) - d_3(2,3), d_1(1,3) + d_3(3,4) - d_4(2,4)\}$$

The proof of this part is very similar to Case 1).

□

I conjecture that this result may be carried to the general case inductively.

Chapter 3

CONCLUSION

In our model the capacity of the facility is unlimited just like Maniquet (2003)'s, Chun (2004*a*)'s, Kayi & Ramaekers (2006)'s, Moulin (2004)'s, Moulin (2005)'s, and Özsoy (2005)'s models. The job lengths of individuals are identical like Maniquet (2003)'s, Chun (2004*a*)'s, and Kayi & Ramaekers (2006)'s models. The difference between our model and other models is that we are considering more general preferences for agents than other models do. In Moulin (2004)'s, Moulin (2005)'s, and Özsoy (2005)'s models all of the agents value cost of waiting for a unit of time similarly but in our model these valuations may be different. In Maniquet (2003)'s, Chun (2004*a*)'s, and Kayi & Ramaekers (2006)'s models each agent has an impatience level hence the difference between any two consecutive positions is equal for an agent. In our model an agent may value the difference between two different consecutive positions differently.

Our first result is that if a solution is an envy free solution, then the ordering of agents should be a maximal ordering. Since numeric representation of preferences of agents are quasi-linear in transfers any envy free solution is also a Pareto efficient solution. If order preservation property is satisfied, then we are able to characterize envy free solutions by using the solution concept F. Even if order preservation property is not satisfied I conjecture that set of envy free solutions are not empty. Up to four agents we have characterized the envy free solutions. Generalizing this result is an open problem. I think that in generalizing this result the properties of maximal orderings which are explored in this study will play a key role.

A natural extension of this study is limiting the capacity of the facility, and studying with non-identical job lengths. Taking more general preferences of agents might be another option. Since envy freeness is a fairness notion, one might study other fairness notions such as identical preferences lower bound under this or under a more general model. Best of my knowledge there is not much solution concepts defined for queuing problems. We have introduced the solution concept F in this study, one might come up with different solution concepts for queuing problems.

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