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THE LATTICE OF PERIODS OF A GROUP ACTION AND ITS TOPOLOGY

A THESIS

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FOR THE DEGREE OF

MASTER OF SCIENCE

By

Hüseyin Acan

July, 2006

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

THE LATTICE OF PERIODS OF A GROUP ACTION AND ITS TOPOLOGY

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July, 2006

In this thesis, we study the topology of the poset obtained by removing the greatest and least elements of lattice of periods of a group action. For a G-set X where G is a finite group, the lattice of periods is defined as the image of the map from the subgroup lattice of G to the partition lattice of X which sends a subgroup H of G to the partition of X whose blocks are the H-orbits of X. We study the homotopy type of the associated simplicial complex. When the group G belongs to one of the families dihedral group of order 2^n, dihedral group of order $2p^n$ where p is an odd prime, semi-dihedral group, or quaternion group and the set X is transitive, we find the homotopy type of the corresponding poset. If G is the dihedral group of order 2^n or one of semidihedral and quaternion groups, we find that the homotopy type of the complex is either contractible or has the homotopy type of three points. In the case of dihedral group of order $2p^n$, the associated complex is either contractible or it has the homotopy type of p points or it has the homotopy type of $p+1$ points.

Keywords: lattice of periods, poset topology.

ÖZET

YÖRÜNGE LATİSLERİ VE ONLARIN TOPOLOJİLERİ

Hüseyin Acan
Matematik, Yüksek Lisans
Tez Yöneticisi: Yard. Doç. Dr. Ergün Yalçın
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Bu tezde yörünge latisinden en büyük ve en küçük elemanların çıkarılmasıyla elde edilen kısmi sıralı kümelerin topolojisini çalıştık. G sonlu bir grup ve X sonlu bir G-kümesi olsun. G'nin altgruplarının latisini $L(G)$ ile ve X'in bölüntü (parçalama) latisini $\Pi(X)$ ile gösterelim. Verilen bir altgrubu onun herbir X yörüngesini bir blok olarak kabul eden bölüntüye götüren fonksiyonun görüntü kümesine yörünge latisi deniyor. Biz bu latisten elde edilen kısmi sıralı kümeye karşılık gelen simpleksler kompleksinin homotopi çeşidini inceledik. Eğer G, eleman sayısı 2^n veya $2p^n$ (p asal) olan bir dihedral grup, bir yarı-dihedral grup veya bir quaternion grup ise, oluşacak kısmi sıralı kümenin homotopi çeşidini tam olarak hesaplıyoruz. G grubu eleman sayısı 2^n olan bir dihedral grup, bir yarı-dihedral grup veya bir quaternion grup ise, oluşan simplekler kompleksi, G-kümesi X'in eleman sayısına bağlı olarak ya bir noktaya büzülebilir bir kompleks oluyor ya da 3 tane noktanın homotopi çeşidine sahip oluyor. Eğer G, eleman sayısı $2p^n$ olan dihedral grup ise üç farklı durum söz konusu: Kompleks ya bir noktaya büzülebilir oluyor, ya p tane noktanın homotopi çeşidine sahip oluyor ya da $p+1$ tane noktanın homotopi çeşidine sahip oluyor.

Anahtar sözcükler: yörünge latisi, kısmi sıralı küme topolojisi.

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Contents

Chapter 1

Introduction

In [6] and [7], G. C. Rota introduced the *lattice of periods* of a group action. It is constructed from a finite group G and a G-set X. An element of the lattice of periods is a partition of X whose blocks are the orbits of some subgroup H of G. Formally, we have a map η from the subgroup lattice $L(G)$ of G to the partition lattice $\Pi(X)$ of X. This map sends a subgroup H of G to the partition of X whose blocks are the H orbits of G. The image of η is a lattice with the ordering inherited from the partition lattice of X and it is called the lattice of periods of a group action. It is denoted by $\Gamma(G, X)$.

It is clear that the image of η is a subposet of the partition lattice $\Pi(X)$. However, it does not have to be a sublattice of $\Pi(X)$. Although the join (taken in $\Pi(X)$) of any two elements of $Im(\eta)$ is again in $Im(\eta)$, the meet of two elements of $Im(\eta)$ may not lie in it.

In [2], W. Doran gives some characterizations of the isomorphism classes of the lattice of periods for a group G. The main theorem of [2] states that for a finite group G and a G-set X, the corresponding lattice of periods depends on the support of the complex representation $\mathbb{C}X$. The support of a representation is the set of complex characters which appear in the representation. In Section 2.2 we give an alternative proof for this theorem.

In this thesis we cosider the topology of the lattice of periods. Recall that, given a poset P, associated to it there is a simplicial complex $\Delta(P)$ where the faces (simplices) of $\Delta(P)$ corresponds to the chains in P. By this way, every poset can be seen as a topological object. We study the topology of the poset obtained by removing the least and the greatest elements of the lattice. We denote this poset by $\Gamma_0(G, X)$. Indeed, any poset with an element which is comparable to any other element is contractible to that element. So, a (finite) lattice is contractible since it has a greatest element and a least element. Hence, it is more interesting to consider the lattice without the least element and the greatest element.

We consider the lattice of periods generated by transitive G-sets where G belongs to one of the following families: Dihedral groups of order $2p^n$ where p is an odd prime, dihedral groups of order 2^n, semidihedral groups, and quaternion groups. In all these cases, we find that the poset we are interested in is either contractible or has the homotopy type of disjoint union of points. When G is a member of the last three families, we show that the poset (if not empty) is either contractible or has the homotopy type of 3 points. When G is a dihedral group of order $2p^n$, the poset is either contractible or has the homotopy type of p points or has the homotopy type of $p+1$ points.

We also find some more general results. The poset $\Gamma_0(G, X)$ is homotopy equivalent to the poset obtained by removing the least and the greatest elements of the quotient lattice $L(G)/\ker\eta$. This is equivalent to saying that the poset $\Gamma_0(G, X)$ is homotopy equivalent to the poset obtained from $L(G)$ by removing the block of G in $\ker\eta$ and the block of the trivial subgroup in $\ker\eta$. The main ingredient for the proof is the theorem known as Quillen Fiber Lemma which states that two posets P and Q are homotopy equivalent if there is a poset map $f : P \to Q$ such that the preimage of the elements which is smaller than or equal to q is contractible for each $q \in Q$, i.e., if $f^{-1}(Q_{\leq q})$ is contractible for any $q \in Q$.

Every transitive G-set is G-isomorphic to G/H for some subgroup H of G where the action on G/H is given by left multiplication. Our attention will be on transitive sets in the next chapters. Assume that G is a finite group and H is a subgroup of it. Assume further that N is a normal subgroup of G which is contained in H. In this case we find that $\Gamma(G, G/H)$ and $\Gamma(G/N, G/N\big/H/N)$ are isomorphic lattices. This result enables us to deal with smaller groups.

The rest of the thesis is organized as follows:

Chapter 2 has three parts. In the first part we give background material for posets and lattices. In the second part, we define the lattice of periods and give general properties of it. In the last part of the chapter, how to construct all possible lattices of periods for a given group G is described. Most of the material in the last two parts of Chapter 2 is due to Doran [2].

In Chapter 3, we start with the topological notions. Then we give some well known results about the poset topology such as the Quillen Fiber Lemma and give a homotopy equivalence for the poset $\Gamma_0(G, X)$.

In Chapter 4, we calculate the homotopy type of $\Gamma_0(G, X)$ for various transitive G-sets for 2-groups belonging to the families of dihedral, semi-dihedral, and quaternion groups. The results for semi-dihedral and quaternion groups mainly follow from the dihedral case. We also calculate the homotopy type of the lattice for dihedral group of order $2p^n$ where p is an odd prime.

Chapter 2

General Properties of Lattice of Periods

In the first part of this chapter we will give background material on lattices.

In the second part we will define the lattice of periods for a G-set X where G is a finite group. The lattice of periods of a group action was first introduced by G.C. Rota in [6] and [7].

Some general properties of lattice of periods will follow. The most important result of this section is Theorem 2.2.22. It says that, the set of irreducible characters which appear in the character of permutation module $\mathbb{C}X$ uniquely determines the lattice of periods.

In the last part of the chapter, we will give an algorithm for constructing all possible lattice of periods for a given group G. Most of the results in this chapter are due to Doran [2].

2.1 Background Material on Lattices

In this section, necessary definitions and background material on posets and lattices will be given. The material in this section is standard and can be found in any lattice theory book but we follow mostly [3].

A *partially ordered set* or a *poset* $(P, \leq)$ is a nonempty set P together with a binary relation $\leq$ satisfying the first three properties of the following:

1. Reflexivity: $a \leq a$ for any $a \in P$
2. Antisymmetry: $a \leq b$ and $b \leq a$ together imply that $a = b$ for $a, b \in P$.
3. Transitivity: $a \leq b$ and $b \leq c$ together imply that $a \leq c$ for $a, b, c \in P$.
4. Linearity: $a \leq b$ or $b \leq a$ for $a, b \in P$

The binary relation mentioned in the definition of the poset is called the ordering (of P). If two distinct elements a, b in a poset P is related by $a \leq b$ then we say that a is smaller than b, or b is greater than a. If a poset P satisfies the linearity property then it is called a *totally ordered set* or a *toset* (also called *fully ordered set, linearly ordered set*). The most natural examples of totally ordered sets are $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ with the usual $\leq$ relation. For a set A, the set of all subsets of A is called the power set of A and denoted by $\mathcal{P}(A)$. Any subset of the power set $\mathcal{P}(A)$ is a poset with containment ordering : $X \leq Y$ if and only if $X \subseteq Y$ for $X, Y \in \mathcal{P}(A)$. Usually, the ordering $\leq$ is omitted in the notation and just P is used instead of $(P, \leq)$.

Let P be a poset and Q be a nonempty subset of P. Then there is a natural ordering $\leq_Q$ on Q induced by the ordering $\leq$ in P as follows: for $a, b \in Q$, $a \leq_Q b$ if and only if $a \leq b$. We call $(Q, \leq_Q)$ or simply Q a *subposet* of P.

If a and b are elements of a poset P, they are called comparable if $a \leq b$ or $b \leq a$. They are called incomparable otherwise. If a subposet C of a poset P is consisting of pairwise comparable elements then it is called a *chain*. In other words, a chain C is a totally ordered subposet of a poset P. The length $l(C)$ of a finite chain C is one less than the elements of it, i.e., $l(C) = |C| - 1$. A subposet A of a poset P is called antichain if it consists of elements pairwise incomparable.

Given a poset P and two elements $a, b \in P$, $[a, b]$ denotes the set of elements of P between a and b, i.e., $[a, b] = \{c \in P : a \leq c \leq b\}$. In particular, if b is not greater than or equal to a, then $[a, b] = \emptyset$.

Let S be a subset of P and $a \in P$. If $s \leq a$ for each $s \in S$ then a is called an upper bound for S. It is called the *least upper bound* of S or *supremum* of S if for any upper bound b of S we have $a \leq b$. It is denoted by $supS$. Similarly, any element c of P is called a lower bound for S if $c \leq s$ for any $s \in S$. An element d of P is called the *greatest lower bound* of S or *infimum* of S if it is greater than any other lower bound of S, i.e., $c \leq d$ for any lower bound c of S. The infimum of S is denoted by $infS$.

Proposition 2.1.1. *Assume that P is a poset and S is a subset of it. If $supS$ exists in P, then it is unique. Similarly, if $infS$ exists in P, then it is unique.*

Proof. Assume that a and b are two least upper bounds for S. By definition, $a \leq b$ but also $b \leq a$. This is possible only if $a = b$. The uniqueness of greatest lower bound is shown similarly. □

Definition 2.1.2. Let P be a poset. An element a in P is called a *minimal element* if there is no $a \neq x \in P$ such that $x \leq a$. An element b in P is called a *maximal element* if there is no $b \neq y \in P$ such that $b \leq y$.

Let P be a poset. The dual poset of P is denoted by P^d and constructed as follows: The elements of P^d is the same as the elements of P and $a \leq b$ in P^d if and only if $b \leq a$ in P. The dual of P^d is the same poset as P. So, if P is the dual poset of Q then also Q is the dual poset of P. The minimal elements in P become the maximal elements in P^d and vice versa. Similarly, the greatest and least elements interchange in two posets. The supremums interchange with infimums, upper bounds interchange with lower bounds.

Assume that P and Q are two posets. A map $f : P \to Q$ is called *order preserving* if $a \leq b$ in P implies that $f(a) \leq f(b)$ in Q. Such a map is also called a *poset map*. Two posets P and Q are said to be *isomorphic* if there is an order preserving bijective map $f : P \to Q$ such that the inverse map f^{-1} is also order preserving. If P and Q are isomorphic posets we write $P \cong Q$.

Definition 2.1.3. A lattice L is a poset such that $inf\{a, b\}$ and $sup\{a, b\}$ exist for any pair of elements a and b. It is equivalent to saying that for any finite nonempty subset S of L, the greatest lower bound $infS$ and the least upper bound $supS$ exist in L.

Lemma 2.1.4. *If L is a finite lattice, then there is an element which is smaller than all the other elements; it is called the least element of L and denoted by $\widehat{0}$. There is also an element which is greater than all the other elements; it is called the greatest element and denoted by $\widehat{1}$.*

Proof. It is easy to see that $infL$ is smaller than all the other elements. Similarly, $supL$ is greater than all the other elements. □

Let L be a lattice and $a, b \in L$ are two elements. Then, $a \wedge b$ denotes the infimum of a and b, and $a \vee b$ denotes the supremum of a and b, i.e., $a \wedge b = inf\{a, b\}$ and $a \vee b = sup\{a, b\}$. The notation $\wedge$ is called the *meet* and $\vee$ is called the *join*. We call $a \wedge b$ the meet of a and b. Similarly, we call $a \vee b$ the join of a and b. These notions can be generalized to arbitrary subsets of L. For any subset S of L, we will use $\bigwedge S$ instead of $infS$ and $\bigvee S$ instead of $supS$. If S is empty then we take $\bigwedge S = \widehat{1}$ and $\bigvee S = \widehat{0}$.

Definition 2.1.5. Suppose that L is a finite lattice. The minimal elements in $L - \{\widehat{0}, \widehat{1}\}$ are called atoms and maximal elements are called coatoms.

Definition 2.1.6. Let L be a lattice and K be a subposet of L. If $a \wedge b \in K$ and $a \vee b \in K$ for every $a, b \in K$ then K is called a *sublattice* of L.

Remark 2.1.7. It is possible that a subposet K of a lattice L is a lattice (with the same ordering) but not a sublattice of L. For example, let $A = \{1, 2, 3\}$, $X = \{1, 2\}$, and $Y = \{2, 3\}$. The power set $\mathcal{P}(A)$ is a lattice with the containment ordering, i.e., $B \leq C$ iff $B \subseteq C$ for $B, C \in \mathcal{P}(A)$. The subposet $\{X, Y, \emptyset, A\}$ of $\mathcal{P}(A)$ is a lattice but it is not a sublattice of $\mathcal{P}(A)$ since $X \wedge Y = \{2\}$ is not an element of this subposet.

Definition 2.1.8. Let L be a finite lattice and L^* denote the sublattice of L consisting meets of arbitrary set of coatoms L, i.e.,

$$L^* = \{\bigwedge I : I \text{ is a subset of coatoms in } L\}.$$

The order of L^* is inherited from L. The meet of empty set is $\widehat{1}$ by definition, so $\widehat{1}$ is an element of L^*. We will call L^* the *meet sublattice* of L. The *join sublattice* of L is defined similarly; replace the coatoms with atoms and meets with joins in the definition of the meet sublattice. The element $\widehat{0}$ is in the join sublattice of L since the join of empty set gives $\widehat{0}$.

If the least upper bound exists for any set of elements in a poset P then it is called a *join semilattice*. Similarly, if the greatest lower bound exists for any set of elements in P then it is called a *meet semilattice*. A poset L is a lattice if and only if it is both a join semilattice and a meet semilattice.

Lemma 2.1.9. *A join semilattice P with a least element is a lattice. Similarly, a meet semilattice Q with a greatest element is a lattice.*

Proof. Since P has a well defined join we need only to show it has a well defined meet. Let S be a subset of P and $Glb(S)$ denotes the set of lower bounds of S. This set is not empty since it contains the least element. Then the join of $Glb(S)$ is the greatest lower bound (meet) of S. The second claim has a similar proof. □

2.2 The Lattice of Periods of a Group Action

A partition π of a set X is a collection of disjoint nonempty subsets of S such that their union is X, i.e., $X = \bigcup_{i\in I} X_i$ and $X_i \bigcap X_j = \emptyset$ for any $i, j \in I$. The X_i's are called the blocks of the partition. In this work, X will always denote a finite set and hence the index set is always finite. We will denote by $X_1|X_2|...|X_n$ a partition whose blocks are $X_1, X_2, \ldots, X_n$. One can define an ordering $\geq$ on the set of partitions of X such that $A_1|...|A_s \geq B_1|...|B_r$ if each B_i is a subset of some A_j. This ordering is called the refinement ordering. All the partitions of X

form a lattice with this ordering which is called the *lattice of partitions* of X and denoted by $\Pi(X)$. The partition lattice of the set $\{1, 2, \ldots, n\}$ is denoted by Π_n.

For $\pi \in \Pi(X)$, the notation $a \sim_\pi b$ is used to denote that a and b are in the same block of π.

For a group G, we denote the subgroup lattice of G by $L(G)$. The elements of $L(G)$ are the subgroups of G and they are ordered by containment. All the groups that we consider are finite groups.

Definition 2.2.1 ([4])**.** Let G be a group and X be a nonempty set. Assume that for each $g \in G$ and $x \in X$ there is defined a unique element $g \cdot x \in X$ such that,

(i) $1 \cdot x = x$ for every $x \in X$ and,

(ii) $x \cdot gh = (x \cdot g) \cdot h$ for every $x \in X$ and $g, h \in G$.

Then we say that G acts on X or $\cdot$ is an action of G on X. A set X together with a G-action is called a G-set.

Definition 2.2.2. Let the finite group G act on the finite set X. Let $\eta : L(G) \to \Pi(X)$ be such that $H \mapsto A_1|...|A_s$ where $a \sim_{\eta(H)} b$ if and only if $a = g \cdot b$ for some $g \in H$. The image of η forms a subposet with the order inherited from the partition lattice of X. Actually, it forms a lattice which, in general, is not a sublattice of $\Pi(X)$. This lattice is called the *lattice of periods* of the G-action on X. We will denote it by $\Gamma(G, X)$. If the group G and the set X is clear in the context we will use the term '*lattice of periods*' for short.

Remark 2.2.3. Unless otherwise stated the map η will always denote the map defined above throughout this thesis.

Example 2.2.4. Let S_3 acts on the set $\{1, 2, 3\}$ in the usual way. Then,

$$\begin{aligned}
&\eta(\langle id \rangle) \mapsto 1|2|3 \\
&\eta(\langle (12) \rangle) \mapsto 12|3 \\
&\eta(\langle (13) \rangle) \mapsto 13|2 \\
&\eta(\langle (23) \rangle) \mapsto 1|23 \\
&\eta(\langle (123) \rangle) \mapsto 123 \\
&\eta(S_3) \mapsto 123.
\end{aligned}$$

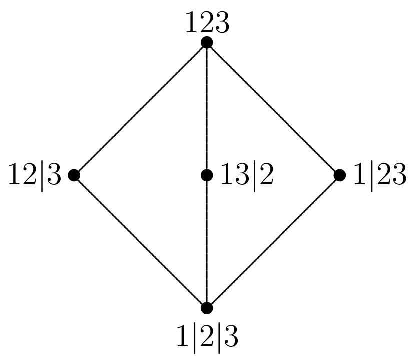

Figure 2.1: $\Gamma(S_3, \{1,2,3\})$

In general, when S_n acts on the set $[n] = \{1, 2, ..., n\}$, the resulting lattice of periods is Π_n. Indeed, for any partition $\pi = A_1|A_2|...|A_s$ one can take the subgroup of Π_n generated by the cycles $C_1, C_2, ..., C_s$ where C_i has the elements of A_i for $i = 1, 2, \ldots, s$, then the image of this subgroup is π.

Now we will present some properties of the map η and the poset $\Gamma(G, X)$.

Proposition 2.2.5. *The map η is order preserving.*

Proof. Let $H_1 \le H_2$. Assume that a and b are in the same block in $\eta(H_1)$. Then, $a = hb$ for some $h \in H_1$. Since $h \in H_1$ implies $h \in H_2$, the elements a and b must be in the same block in $\eta(H_2)$. Thus, $\eta(H_1) \le \eta(H_2)$. $\square$

Proposition 2.2.6. *The map η preserves joins. That is*

$$\eta(H \vee K) = \eta(H) \vee \eta(K),$$

where the first join takes place in $L(G)$ and the second in $\Pi(X)$.

Proof. By the previous proposition $\eta(H) \le \eta(H \vee K)$. Similarly, $\eta(K) \le \eta(H \vee K)$. Hence, $\eta(H) \vee \eta(K) \le \eta(H \vee K)$. Now let a and b are in the same block in $\eta(H \vee K)$. We need to show that they are in the same block in $\eta(H) \vee \eta(K)$. First, note that $H \vee K$ is a subgroup consisting of elements of the form $h_1k_1h_2k_2 \ldots h_nk_n$ where $h_i \in H$ and $k_i \in K$ for $i = 1, 2, \ldots, n$. If a and b are in the same block in $\eta(H \vee K)$ then $a = h_1k_1 \ldots h_nk_n \cdot b$.

$$a \sim_{\eta(H)} k_1h_2k_2 \cdots h_nk_nb$$

$$k_1 h_2 \cdots h_n k_n b \sim_{\eta(K)} h_2 k_2 \cdots k_n b$$
$$\vdots$$
$$k_n b \sim_{\eta(K)} b.$$

Hence a and b are in the same block in $\eta(H) \vee \eta(K)$. □

Remark 2.2.7. The map η does not necessarily preserve meets. So, $\Gamma(G, X)$ is not necessarily a sublattice of $\Pi(X)$. For instance, in Example 2.2.4,

$$\eta(\langle(12)\rangle \wedge \langle(123)\rangle) = \eta(\langle id\rangle) = 1|2|3$$

whereas

$$\eta(\langle(12)\rangle) \wedge \eta(\langle(123)\rangle) = 12|3 \wedge 123 = 12|3.$$

Corollary 2.2.8. $\eta(H) = \bigvee_{g \in H} \eta(\langle g\rangle)$

Proof. This is clear since $H = \bigvee_{g \in H} \langle g\rangle$. □

So, it is enough to compute $\eta(\langle g\rangle)$ for all $g \in G$ in order to compute the lattice of periods. $\Gamma(G, X)$ is generated by taking the arbitrary joins of the elements from the set $\{\eta(\langle g\rangle) : g \in G\}$.

Corollary 2.2.9. *The poset $\Gamma(G, X)$ is a lattice.*

Proof. By Proposition 2.2.6, $\Gamma(G, X)$ has a well defined join. Since η is an order preserving map $\eta(\{1\})$ is the minimum element of $\Gamma(G, X)$. We conclude the proof by Lemma 2.1.9. □

Definition 2.2.10. Let P and Q be two posets and $\varphi : P \to Q$ be an order preserving map. The kernel of φ is the partition of P where a and b are in the same block if and only if $\varphi(a) = \varphi(b)$. It is denoted by $\ker \varphi$.

Recall that, for a poset P and two elements a and b in it, the interval $[a, b]$ is defined as

$$[a, b] = \{c \in P : a \leq c \leq b\}.$$

Definition 2.2.11. Given a poset P, a partition π of P is called an *interval partition* if $a \sim_\pi b$ implies that $a \sim_\pi c \sim_\pi b$ for each $c \in [a, b]$.

Definition 2.2.12. Given a poset P, a partition π of P is called a *normal partition* if for any two blocks A_i and A_j, the following two conditions together imply that $i = j$.

(i) There exist $x \in A_i$ and $y \in A_j$ with $x \leq y$,

(ii) There exist $z \in A_i$ and $t \in A_j$ with $z \geq t$.

Recall that a poset map is an order preserving map between two posets.

Lemma 2.2.13. *A partition of a poset P is a normal partition if and only if it is the kernel of some poset map $\varphi : P \to Q$.*

Proof. Let π be a partition of P which is equal to $\ker \varphi$ where $\varphi : P \to Q$ is a poset map and let A_i, A_j be two blocks of π. Assume that there are elements $x, z \in A_i$ and $y, t \in A_j$ such that $x \leq y$ and $z \geq t$. Then, $f(x) \leq f(y)$ and $f(z) \geq f(t)$. But since $f(x) = f(z)$ and $f(y) = f(t)$, all the elements must be in the same block, i.e., $i = j$. Hence, π is a normal partition.

Now assume that π is a normal partition. For each block A_i in π create an element q_i. If there are two elemets $x \in A_i$ and $y \in A_j$ such that $x \leq y$ then let $q_i \leq q_j$. Let Q be the poset with the elements q_i and with this ordering. Let $\varphi : P \to Q$ be the map sending an element in A_i to q_i. Then, $\ker \varphi$ is the same partition as π. □

Definition 2.2.14. Let $\pi = A_1| \ldots |A_r$ be a normal partition of P. The *quotient poset* of π is the poset whose elements are $A_1, \ldots, A_r$ and $A_i \leq A_j$ if and only if $x \leq y$ in P for some $x \in A_i$ and $y \in A_j$. This poset is denoted by P/π.

Since η is an order preserving map (poset map), $P/\ker \eta$ is well defined as a poset. We need this construction for the proof of next proposition.

Remark 2.2.15. In [2], the quotient poset P/π is defined when π is an interval partition. However, when π is not a normal partition, the quotient poset may not be well defined. For instance let P be the poset with the set of elements $\{a, b, c, d\}$ and with the relations $a \leq b$ and $c \leq d$. Then $\pi = ad|bc$ is an interval partition but P/π is not well defined.

Recall that two posets P and Q are said to be isomorphic, denoted by $P \cong Q$, if there is a bijective map $f : P \to Q$ such that f and f^{-1} are order preserving.

Proposition 2.2.16. *The lattice of periods $\Gamma(G, X)$ is isomorphic to the quotient lattice $L(G)/\ker\eta$.*

Proof. Let $\ker\eta = A_1|\ldots|A_m$ and $\eta(H) = \pi_i$ for $H \in A_i$. So, the elements $A_1, \ldots, A_m$ constitute the lattice $L(G)/\ker\eta$ and the elements $\pi_1, \ldots, \pi_m$ constitute the lattice $\Gamma(G, X)$.

Since the map η is order preserving, if $A_i \leq A_j$, it is clear that $\pi_i \leq \pi_j$. Now let $\pi_i \leq \pi_j$ for some $i \neq j$. Take $H \in A_i$ and $K \in A_j$. Then $H \vee K \in A_j$ by Proposition 2.2.6 and hence $A_i \leq A_j$ since $H \leq H \vee K$. Hence we are done. □

Proposition 2.2.17. *Let G be a finite group acting on the finite sets X and Y and let η_1 and η_2 be the corresponding maps respectively. If $\ker\eta_1 = \ker\eta_2$ then*

$$\Gamma(G, X) \cong \Gamma(G, Y).$$

Proof. Let the defining maps for $\Gamma(G, X)$ and $\Gamma(G, Y)$ be the maps

$$\eta_1 : L(G) \to \Pi(X)$$

$$\eta_2 : L(G) \to \Pi(Y).$$

By Proposition 2.2.16, we have $\Gamma(G, X) \cong L(G)/\ker\eta_1$ and $\Gamma(G, Y) \cong L(G)/\ker\eta_2$. Combining the two isomorphisms we get the desired result. □

For a finite group G and a G-set X, the set of G fixed points of X is denoted by X^G and the set of G orbits of X is denoted by X/G. For a group element g, the notation X^g is used to denote the g fixed points of X, i.e., $X^g = \{\alpha \in X : g\alpha = \alpha\}$

Lemma 2.2.18. *Let G be a group and X and Y be two G-sets. Then, the following are equivalent:*

(i) $|X^H| = |Y^H|$ for each cyclic subgroup H of G.

(ii) $|X/H| = |Y/H|$ for each subgroup H of G.

(iii) $|X/H| = |Y/H|$ *for each cyclic subgroup* H *of* G.

(iv) The complex representations $\mathbb{C}X$ *and* $\mathbb{C}Y$ *are isomorphic.*

Proof. We will first show that the first three statements are equivalent. Then we will show that (i) implies (iv) and finally we will show that (iv) implies (ii).

$(i) \Rightarrow (ii)$ We have

$$\begin{aligned} |X/H| &= \frac{1}{|H|} \sum_{g \in H} |X^g| \\ &= \frac{1}{|H|} \sum_{g \in H} |Y^g| \\ &= |Y/H| \end{aligned}$$

where the first and third equalities are due to Cauchy-Frobenius Theorem [4]. The second equality is followed by (i).

$(ii) \Rightarrow (iii)$ This is obvious.

$(iii) \Rightarrow (i)$ We have $|X^{\langle 1 \rangle}| = |Y^{\langle 1 \rangle}|$. Assume by induction for all the proper subgroups $\langle h \rangle$ of $\langle g \rangle$, $|X^{\langle h \rangle}| = |Y^{\langle h \rangle}|$. If the number of $\langle g \rangle$ orbits of X and Y are equal then

$$\frac{1}{|\langle g \rangle|} \sum_{h \in \langle g \rangle} |X^h| = \frac{1}{|\langle g \rangle|} \sum_{h \in \langle g \rangle} |Y^h|$$

by Cauchy-Frobenius Theorem. Then,

$$\sum_{h \in \langle g \rangle} |X^h| = \sum_{h \in \langle g \rangle} |Y^h|.$$

The last equation and the induction hypothesis together imply that $|X^{\langle g \rangle}| = |Y^{\langle g \rangle}|$.

$(i) \Rightarrow (iv)$ Let χ_1 be the character of complex representation $\mathbb{C}X$ and χ_2 be the character of complex representation $\mathbb{C}Y$. Assume that $|X^g| = |Y^g|$ for every $g \in G$. In order to show that $\mathbb{C}X$ and $\mathbb{C}Y$ are isomorphic representations it is enough to show the characters χ_1 and χ_2 are equal. But for any $g \in G$, $\chi_1(g)$ is the number of g fixed points of X and $\chi_2(g)$ is the number of g fixed points of Y. Since $|X^g| = |Y^g|$ for every group element g, the characters χ_1 and χ_2 are equal.

(iv) $\Rightarrow$ *(ii)* Now assume that the complex representations $\mathbb{C}X$ and $\mathbb{C}Y$ are isomorphic. Then $\dim_{\mathbb{C}}(\mathbb{C}X)^H = \dim_{\mathbb{C}}(\mathbb{C}Y)^H$ for every subgroup H of G where $(\mathbb{C}X)^H$ denotes the H fixed points of $\mathbb{C}X$ and $(\mathbb{C}Y)^H$ denotes the H fixed points of $\mathbb{C}Y$. But since $\dim_{\mathbb{C}}(\mathbb{C}X)^H = |X/H|$ and $\dim_{\mathbb{C}}(\mathbb{C}Y)^H = |Y/H|$ for any subgroup H we conclude that $|X/H| = |Y/H|$ for every subgroup H of G. $\square$

Proposition 2.2.19. *Let $\eta_1 : L(G) \to \Pi(X)$ and $\eta_2 : L(G) \to \Pi(Y)$ be the usual maps where G is a finite group and X, Y are finite G-sets. If $\ker \eta_1 \neq \ker \eta_2$ then there exist subgroups $H_1 \geq H_2$ such that one of the following statements holds,*

$$(i)\ \eta_1(H_1) = \eta_1(H_2) \text{ but } \eta_2(H_1) \neq \eta_2(H_2)$$

$$(ii)\ \eta_1(H_1) \neq \eta_1(H_2) \text{ but } \eta_2(H_1) = \eta_2(H_2)$$

Proof. Suppose $\ker \eta_1 \neq \ker \eta_2$. Then, there exist subgroups H and K of G such that $\eta_1(H) = \eta_1(K)$ but $\eta_2(H) \neq \eta_2(K)$, or vice versa. Assume WLOG, $\eta_1(H) = \eta_1(K)$ but $\eta_2(H) \neq \eta_2(K)$. Then, $\eta_1(H \vee K) = \eta_1(H) = \eta_1(K)$ but at least one of $\eta_2(H)$ and $\eta_2(K)$ is not equal to $\eta_2(H \vee K)$, say $\eta_2(H)$. Letting $H \vee K = H_1$ and $H = H_2$ completes the proof. $\square$

Theorem 2.2.20 (Thm 3.2, Doran [2])**.** *Let G be a finite group acting on finite sets X and Y. If the complex permutation representations of X and Y are isomorphic, then*

$$\Gamma(G, X) \cong \Gamma(G, Y).$$

Proof. If the complex representations $\mathbb{C}X$ and $\mathbb{C}Y$ are isomorphic then the number of H orbits of X and the number of H orbits of Y are equal by Lemma 2.2.18, for any subgroup H of G. Let $\eta_1 : L(G) \to \Pi(X)$ and $\eta_2 : L(G) \to \Pi(Y)$ be the usual maps. If we show that $\ker \eta_1 = \ker \eta_2$ then we are done by Proposition 2.2.17.

Assume that $H_1 \geq H_2$. Since H_2 is a subgroup of H_1, any H_2 orbit of a G-set is included in an H_1 orbit. On the other hand, the images of H_1 and H_2 under η_1 are same if and only if H_1 and H_2 orbits of X are same. Combining these two facts we conclude that the images $\eta_1(H_1)$ and $\eta_1(H_2)$ are same if and only if the number of H_1 orbits and H_2 orbits are equal. Similarly, the images $\eta_2(H_1)$ and $\eta_2(H_2)$ are same if and only if the number of H_1 and H_2 orbits of Y are equal. But then, by the contrapositive of Proposition 2.2.19, the kernels $\ker \eta_1$ and $\ker \eta_2$ are same. □

Definition 2.2.21. Let G be finite a group and ϕ be a complex representation of G. The support of ϕ is the set of irreducible representations (up to isomorphism) appearing in ϕ, i.e., if $\phi \cong a_1\phi_1 \oplus \cdots \oplus a_k\phi_k$ where $a_1, \dots, a_k$ are positive integers and $\phi_1, \dots, \phi_k$ are pairwise nonisomorphic irreducible representations then the set $\{\phi_1, \dots, \phi_k\}$ is the support of ϕ.

Actually, the next theorem says that it is enough to look at the support of the representation to determine the lattice of periods of a group action.

Theorem 2.2.22 (Thm 5.2, Doran [2])**.** *Let G be a finite group and X, Y be two finite G-sets. Let χ_1 and χ_2 be the characters of complex representations $\mathbb{C}X$ and $\mathbb{C}Y$, respectively. If the supports of χ_1 and χ_2 are same, then $\Gamma(G, X) \cong \Gamma(G, Y)$.*

Proof. Let $\eta_1 : L(G) \to \Pi(X)$ and $\eta_2 : L(G) \to \Pi(Y)$ be the defining maps. By Proposition 2.2.17 it is enough to show that $\ker \eta_1 = \ker \eta_2$. Let $H_1 \geq H_2$. Then, as in the proof of Theorem 2.2.20, the images of H_1 and H_2 under η_1 are the same if and only if the number of H_1 orbits of X is equal to the number of H_2 orbits of X. But the number of H_1 orbits is equal to $\dim_{\mathbb{C}}(\mathbb{C}X)^{H_1}$ and the number of H_2 orbits is equal to $\dim_{\mathbb{C}}(\mathbb{C}X)^{H_2}$. Hence, $\eta_1(H_1) = \eta_1(H_2)$ if and only if $\dim_{\mathbb{C}}(\mathbb{C}X)^{H_1} = \dim_{\mathbb{C}}(\mathbb{C}X)^{H_2}$.

Let

$$\chi_1 = a_1\psi_1 + \cdots + a_n\psi_n$$
$$\chi_2 = b_1\psi_1 + \cdots + b_n\psi_n$$

where $a_i, b_i \in \mathbb{Z}^+$ and ψ_i's are irreducible characters $(i = 1, \dots, n)$. Then,

$$\mathbb{C}X \cong a_1V_1 \oplus \cdots \oplus a_nV_n$$

where V_i is an irreducible $\mathbb{C}G$ submodule of $\mathbb{C}X$ whose character is ψ_i, $(i = 1, \dots, n)$. Similarly,

$$\mathbb{C}Y \cong b_1V_1 \oplus \cdots \oplus b_nV_n.$$

First note that since $H_1 \geq H_2$, the H_1 fixed points of any $\mathbb{C}G$-module M is contained in the H_2 fixed points of M. We have $\dim_{\mathbb{C}}(\mathbb{C}X)^{H_1} = \dim_{\mathbb{C}}(\mathbb{C}X)^{H_2}$ if and only if

$$\dim_{\mathbb{C}}(a_1V_1 \oplus \cdots \oplus a_nV_n)^{H_1} = \dim_{\mathbb{C}}(a_1V_1 \oplus \cdots \oplus a_nV_n)^{H_2}.$$

The last equality holds if and only if $\dim_{\mathbb{C}}(V_i)^{H_1} = \dim_{\mathbb{C}}(V_i)^{H_2}$ for $i = 1, 2, \ldots, n$ since $\dim_{\mathbb{C}}(V_i)^{H_1} \leq \dim_{\mathbb{C}}(V_i)^{H_2}$ for all i. Thus, $\eta_1(H_1) = \eta_1(H_2)$ if and only if $\dim_{\mathbb{C}}(V_i)^{H_1} = \dim_{\mathbb{C}}(V_i)^{H_2}$ for $i = 1, 2, \ldots, n$. Similarly, one can show that $\eta_2(H_1) = \eta_2(H_2)$ if and only if $\dim_{\mathbb{C}}(V_i)^{H_1} = \dim_{\mathbb{C}}(V_i)^{H_2}$ for $i = 1, 2, \ldots, n$. Thus, $\eta_1(H_1) = \eta_1(H_2)$ if and only if $\eta_2(H_1) = \eta_2(H_2)$. We conclude that the kernels are the same by Proposition 2.2.19. □

We end this section with a proposition from [2].

Proposition 2.2.23 (Proposition 4.4, Doran [2]). *Let H_1 and H_2 be conjugate subgroups of G. Then $\ker H_1 = \ker H_2$ where $\ker H_i = \ker(\eta_i)$ and $\eta_i : L(G) \to \Pi(G/H_i)$ is the defining map for lattice of periods on the group G on the set G/H_i for $i = 1, 2$.*

2.3 Constructing the Lattice of Periods

In this section, we will determine the possible lattice of periods for a given finite group G. All we need is the kernels of defining maps $\eta : L(G) \to \Pi(X)$ where X runs over the representatives of conjugacy classes of transitive G-sets.

By Proposition 2.2.17, constructing the possible lattices of periods for a group action is equivalent to determining the possible partitions of the subgroup lattice $L(G)$ which can arise as the kernel of a map $\eta : L(G) \to \Pi(S)$.

Definition 2.3.1. Let G be a group and X, Y be two G-sets. X and Y are said to be G-isomorphic if there exists a bijective function $\varphi : X \to Y$ such that $\varphi(g \cdot x) = g \cdot \varphi(x)$ for any $g \in G$ and $x \in X$.

Let G be a finite group and X be a transitive G-set. There exists a subgroup H of G such that X is G-isomorphic to G/H. Indeed, for any $\alpha \in X$ one can take the subgroup G_α for H where G_α denotes the stabilizer of α.

If X and Y be two disjoint G-sets one can extend the action of G to $X \amalg Y$ by

$$g \cdot a = \begin{cases} g \cdot_{(\text{action on X})} a, & \text{if } a \in X, \\ g \cdot_{(\text{action on Y})} a, & \text{if } a \in Y, \end{cases}$$

for $a \in X \amalg Y$.

Theorem 2.3.2. *Let X and Y be two G-sets. Let $\eta_1 : L(G) \to \Pi(X)$ and $\eta_2 : L(G) \to \Pi(Y)$ be the defining maps and $\ker \eta_1$ and $\ker \eta_2$ be the kernels of these maps respectively. Then,*

$$\Gamma(G, X \amalg Y) \cong L(G)/(\ker \eta_1 \wedge \ker \eta_2)$$

Proof. The defining map for $\Gamma(G, X \amalg Y)$ is

$$\begin{aligned} \eta :& L(G) \to \Pi(X \amalg Y) \\ & H \mapsto \eta_1(H)|\eta_2(H) \end{aligned}$$

It is enough to show that $\ker \eta = \ker \eta_1 \wedge \ker \eta_2$.

If two subgroups H_1 and H_2 are in the same block in $\ker \eta$ then obviously they are in the same block in $\ker \eta_1$ and $\ker \eta_2$ and hence in $\ker \eta_1 \wedge \ker \eta_2$. Thus, $\ker \eta \subseteq \ker \eta_1 \wedge \ker \eta_2$.

Conversely, if H_1 and H_2 are in the same block in $\ker \eta_1 \wedge \ker \eta_2$, then $\eta_1(H_1) = \eta_1(H_2)$ and $\eta_2(H_1) = \eta_2(H_2)$. But, then $\eta(H_1) = \eta(H_2)$. So, they are in the same block in $\ker \eta$. Thus, $\ker \eta \supseteq \ker \eta_1 \wedge \ker \eta_2$. Hence, $\ker \eta = \ker \eta_1 \wedge \ker \eta_2$. $\square$

For a subgroup H of G, let $\ker H$ denote the kernel of the map $\varphi : L(G) \to \Pi(G/H)$. Since any G-set X is a union of transitive sets, it can be written as $X = G/H_1 \amalg \cdots \amalg G/H_n$. Thus, above theorem provides a method for obtaining all possible lattice of periods for a given group G. The method can be described as follows: Calculate $\ker H$ for each subgroup H of G and take all the possible meets of these kernels. For any combination of subgroups $H_1, \cdots, H_n$, the quotient poset $L(G)/(\ker H_1 \wedge \cdots \wedge \ker H_n)$ gives a lattice of periods for group G and G-set $X \cong G/H_1 \amalg \cdots \amalg G/H_n$ and conversely any lattice of periods for group G can be obtained by this way.

Corollary 2.3.3. *If the G-set X contains an orbit isomorphic to $G/\{1\}$, then $\Gamma(G, X) \cong L(G)$.*

Proof. This follows from the observation that the meet of $\ker\{1\}$ with $\ker H$ for any $H \leq G$ gives $\ker\{1\}$. □

Proposition 2.3.4. *Let G be a finite group and H be a subgroup of it. If there is a normal subgroup N of G such that $N \leq H \leq G$ then,*

$$\Gamma(G, G/H) \cong \Gamma(G/N, G/N\big/H/N).$$

Proof. Let $\eta_1 : L(G) \to \Pi(G/H)$ and $\eta_2 : L(G/N) \to \Pi(G/N\big/H/N)$ be the defining maps. First let's show that $\eta_1(M) = \eta_1(MN)$ for a subgroup M of G. It is obvious that $\eta_1(MN) \geq \eta_1(M)$. Let aH be a coset of H in G for some arbitrary $a \in G$. It is enough to show that the block of aH in $\eta_1(M)$ contains the block of aH in $\eta_1(MN)$. The block of aH in $\eta_1(MN)$ is $\{mnaH : m \in M, n \in N\}$ and the block of aH in $\eta_1(M)$ is $\{maH : m \in M\}$. If we show that $mnaH = maH$ for any $n \in N$ we are done. The last equality holds if and only if $a^{-1}m^{-1}mna = a^{-1}na \in H$. But the last expression holds since $a^{-1}na \in N \leq H$. So, for any partition π in $\Gamma(G, G/H)$, there is a subgroup $N \leq K \leq G$ such that $\eta_1(K) = \pi$. So, the map $r : \Omega \to \Gamma(G, G/H)$ defined by $r(K) = \eta_1(K)$ is a surjective map where $\Omega = \{K : N \leq K \leq G\}$.

Now, let $\phi : \Gamma(G, G/H) \to \Gamma(G/N, G/N\big/H/N)$ be such that

$$\eta_1(K) \mapsto \eta_2(K/N)$$

for $N \leq K \leq G$.

Assume that $\eta_1(K_1) = \eta_1(K_2)$ for $N \leq K_1, K_2 \leq G$. This is possible iff

$$aH \sim_{\eta_1(K_1)} bH \iff aH \sim_{\eta_1(K_2)} bH \text{ for any } a, b \in G.$$

This is equivalent to saying that

$$\exists k_1 \in K_1 \text{ s.t. } aH = k_1 bH \iff \exists k_2 \in K_2 \text{ s.t. } aH = k_2 bH.$$

Since $N \leq H$ is a normal subgroup of G, the last expression is equivalent to

$$\exists k_1 \in K_1 \text{ s.t. } a(H/N) = k_1 b(H/N) \iff \exists k_2 \in K_2 \text{ s.t. } a(H/N) = k_2 b(H/N)$$

and this equivalent to the expression

$$a(H/N) \sim_{\eta_2(K_1/N)} b(H/N) \iff a(H/N) \sim_{\eta_2(K_2/N)} b(H/N) \text{ for any } a, b \in G.$$

Thus the map ϕ is well defined and injective. It is obvious that it is surjective and order preserving. It is also obvious that the inverse map ϕ^{-1} is order preserving. Hence, ϕ is an order preserving bijective map such that its inverse is order preserving. Therefore, it is an isomorphism between two lattices. □

Corollary 2.3.5. *Let G be a finite group and X is a transitive G-set which is isomorphic to G/N as a G-set where N is a normal subgroup of G. Then, $\Gamma(G, X)$ is isomorphic to $L(G/N)$.*

Proof. First apply Proposition 2.3.4 with $H = N$ and then Corollary 2.3.3. □

Chapter 3

Topology of the Lattice of Periods

In the first part of this chapter we give the necessary definitions for poset topology. Then we state some general results about the topology of lattice of periods.

3.1 Poset Topology

In order to be able to talk about the topology of a poset, we will associate a simplicial complex to a given poset. In this way it will be clear what is meant by 'topology of a poset'. First, we need some definitions.

Definition 3.1.1. Let X and Y be two spaces and $f, g : X \to Y$ be two continuous maps between X and Y. The maps f and g are said to be homotopic, denoted by $f \simeq g$, if there is a continuous map $H : X \times I \to Y$ such that

(i) $H(x, 0) = f(x)$ for all $x \in X$ and

(ii) $H(x, 1) = g(x)$ for all $x \in X$.

The map H is called homotopy.

Suppose X and Y are two spaces and $f : X \to Y$ is a map. The map f is called homotopy equivalence if there is a map $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. In this case, X and Y are said to be homotopy equivalent. The map g is called the homotopy inverse of f. A space Z is called contractible if it is homotopy equivalent to a point.

The notation $\simeq$ is used both to denote the homotopic maps and homotopy equivalent spaces.

The topology of the space $\mathbb{R}^{\mathbb{N}}$ has the basis elements $U_1 \times U_2 \times \ldots$ where the U_i's are open sets of $\mathbb{R}$ and $U_i = \mathbb{R}$ except for a finite number of i. A set of points $\{p_0, p_1, \ldots, p_n\}$ in $\mathbb{R}^{\mathbb{N}}$ is said to be geometrically independent if the equations

$$\sum_{i=0}^{n} t_i p_i = 0 \text{ and} \sum_{i=0}^{n} t_i = 0$$

together imply that $t_i = 0$ for $i = 0, 1, \ldots, n$.

An n-simplex is defined as the convex hull of $n+1$ geometrically independent points. Technically, if $V = \{p_0, \ldots, p_n\}$ is a geometrically independent set in $\mathbb{R}^{\mathbb{N}}$, the n-simplex σ with vertex set V is the set of points x in $\mathbb{R}^{\mathbb{N}}$ such that $x = \sum_{i=0}^{n} t_i p_i$ where $\sum_{i=0}^{n} t_i = 1$ for nonnegative $t_0, \cdots, t_n$.

If an n-simplex σ is a convex hull of points $\{p_0, p_1, \ldots, p_n\}$, then these points are called the vertices of σ. The set of vertices is denoted by $V(\sigma)$. The dimension of σ is n. In general the dimension of any simplex is one less than the number of vertices of that simplex. Any simplex with vertex set S where $S \subseteq V(\sigma)$ is called a face of σ. A simplex σ can be topologized by the subspace topology, i.e., a subset τ of σ is open in σ if and only if $\tau = U \cap \sigma$ for some open set U of $\mathbb{R}^{\mathbb{N}}$.

Definition 3.1.2. A simplicial complex Δ in $\mathbb{R}^{\mathbb{N}}$ is a collection of simplices satisfying:

1. If σ is a simplex in Δ, then so is any face of it.

2. Intersection of two simplices in Δ is a face of both simplices.

A simplicial complex Δ can be topologized as follows: A subset X of Δ is closed if and only if $X \cap \sigma$ is closed in σ for any simplex σ of Δ. This topologized space is called the polytope of Δ and sometimes denoted by $|\Delta|$. We will not distinguish Δ and $|\Delta|$.

It is possible to define a simplicial complex in an abstract way which makes it an appealing object for mathematicians working in combinatorics.

Definition 3.1.3. An abstract simplicial complex Δ on vertex set V is a subset of $\mathcal{P}(V) - \{\emptyset\}$ such that:

1. $\{v\} \in \Delta$ for any $v \in V$
2. If $A \in \Delta$ then $B \in \Delta$ for any $B \subseteq A$.

An element of a simplicial complex Δ is called a face or a simplex of Δ. A maximal face, i.e., a face which is not included in any other face, is called a facet. It suffices to know the facets of a simplicial complex to know the simplicial complex. The dimension of Δ is the maximum dimension among all the dimensions of its faces, or equivalently its facets. Any simplicial complex Δ' which is a subset of Δ is called a subcomplex of Δ.

A simplicial map between two simplicial complexes Δ and Δ' is a continuous map $f : \Delta \to \Delta'$ sending a simplex of Δ to a simplex of Δ', i.e., if $\sigma \in \Delta$ then $f(\sigma) \in \Delta'$.

It is possible to define an abstract simplicial complex from a geometric one uniquely. Similarly, it is possible to construct a geometric simplicial complex from an abstract simplicial complex S. Although this construction is not unique, it is unique up to homeomorphism. Such a complex is called a geometric realization of S. This gives us the ability to talk about the topology of an abstract simplicial complex without any confusion. In the remaining of this chapter and next chapter we will work with abstract simplicial complexes. By abuse of terminology, we will mean a simplicial complex by 'complex'.

Definition 3.1.4. Let Δ be a simplicial complex with vertex set V and F be the set of facets of Δ. If w is a point which is not in $V(\Delta)$ then the cone on Δ with vertex w is defined as the complex with facet set $\{\{w\} \cup V(\sigma) : \sigma \in F\}$. It is denoted as $w * \Delta$. It is clear that Δ is a subcomplex of $w * \Delta$ and it is called the base of the cone $w * \Delta$.

A poset P can be viewed as a topological object by associating a simplicial complex $\Delta(P)$ to P. The elements of the poset constitute the vertices of the simplicial complex $\Delta(P)$ and any chain of length n is considered as an n-simplex with the corresponding vertices in $\Delta(P)$. Naturally, by 'the topology of P' we mean the topology of $\Delta(P)$. Any topological aspect of P such as contractibility is indeed the topological aspect of $\Delta(P)$. If L is a finite lattice then it has a greatest element $\widehat{1}$ and a least element $\widehat{0}$. Every maximal chain in such a lattice contains the elements $\widehat{0}$ and $\widehat{1}$. This is equivalent to saying that every facet in $\Delta(L)$ contains the vertices $\widehat{0}$ and $\widehat{1}$. So, $\Delta(L)$ is a cone on $\Delta(L - \{\widehat{0}\})$ with vertex $\widehat{0}$ and it is a cone on $\Delta(L - \{\widehat{1}\})$ with vertex $\widehat{1}$. It is well known that a cone on a complex with vertex w is contractible to w. Hence $\Delta(L)$ is contractible for any finite lattice L.

Recall that a poset map $f : P \to Q$ between two posets P and Q is an order preserving map, i.e., $x \leq_P y$ implies that $f(x) \leq_Q f(y)$.

Proposition 3.1.5. *Any poset map $f : P \to Q$ induces a simplicial map $|f| : \Delta(P) \to \Delta(Q)$.*

A poset P is called conically contractible if there is a poset map $f : P \to P$ such that $p \leq f(p) \geq p_0$ for all $p \in P$ and for some $p_0 \in P$. A lattice L is conically contractible since $f(p) = p \vee p_0$ for any $p_0 \in L$ is a poset map from L to itself satisfying the above condition. Similarly, a poset P with a least (greatest) element is conically contractible since the function $f : P \to P$ which sends $p \in P \mapsto sup\{p, p_0\}$ where p_0 denotes the least (greatest) element of P is well defined and satisfies the above condition. A conically contractible poset is contractible. This is an easy consequence of the following proposition.

Proposition 3.1.6 (Homotopy Property, Quillen [5])**.** *If $f, g : X \to Y$ are poset maps such that $f(x) \leq g(x)$ for every $x \in X$ then $|f|$ and $|g|$ are homotopic maps.*

Proof. If X and Y are posets then $X \times Y$ is a poset with $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$. Similarly, if Δ_1 and Δ_2 are simplicial complexes, then

$\Delta_1 \times \Delta_2$ is a simplicial complex with faces $\{(\sigma_1, \sigma_2) : \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}$. There is a homeomorphism between $\Delta(X \times Y)$ and $\Delta(X) \times \Delta(Y)$ induced by the maps $|pr_1|$ and $|pr_2|$. We have a map $F : X \times \{0, 1\} \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. This induces a homotopy $|F| : \Delta(X) \times I \to \Delta(Y)$ such that $|F|(x, 0) = |f|(x)$ and $|F|(x, 1) = |g|(x)$ for all $x \in \Delta(X)$.

Definition 3.1.7. For a poset P we define $P_{\leq x}$ as the set $\{y \in P : y \leq x\}$. $P_{<x}$, $P_{>x}$ and $P_{\geq x}$ are defined similarly. If f is a poset map from P to Q, then $f^{-1}(Q_{\leq q}) = \{p \in P : f(p) \leq q\}$. This is clearly a subposet of P. The set $f^{-1}(Q_{\geq q})$ is defined similarly.

The next proposition, known as Quillen Fiber Lemma, is a very important result and tool in this subject [5].

Proposition 3.1.8 (Quillen Fiber Lemma)**.** *Let P and Q be posets and let $\phi : P \to Q$ be a poset map. Suppose $\Delta(f^{-1}(Q_{\leq q}))$ is contractible for each $q \in Q$. Then, $\Delta(P) \simeq \Delta(Q)$.*

In [1], some generalizations of this lemma are given.

Theorem 3.1.9. *Let P be a p-group which is not elementary abelian and $L(P)$ be the subgroup lattice of P. Then the poset $L(P) - \{\widehat{0}, \widehat{1}\}$ is contractible where $\widehat{0}$ denotes the trivial subgroup of P and $\widehat{1}$ denotes the group P itself.*

Proof. Let $L_0(G)$ denote $L(G) - \{\widehat{0}, \widehat{1}\}$ for any group G. Now, let P be a group as above. Then $\Phi(P) \neq \{1\}$. Since the Frattini group is a normal subgroup, $\Phi(P)H$ is a subgroup of G for any subgroup H of P. Let $\psi : L_0(P) \to L_0(P)$ be such that $\psi : H \mapsto \Phi(P)H$. It is clear that $H \leq \psi(H) \geq \Phi(P)$. Then, the poset $L_0(P)$ is conically contractible and hence it is contractible. □

3.2 A Homotopy Equivalence for the Lattice of Periods

In this section, we will give some general results about the topology of lattice of periods. We have seen in the first chapter that, $\ker \eta$ gives a partition of $L(G)$

where G is a finite group and η is the defining map for the lattice of periods. We will denote the block of trivial subgroup of G by S_0 and the block of G by S_1 in $\ker \eta$. The set S will denote the union $S_0 \cup S_1$.

Theorem 3.2.1. *$\Delta(\Gamma_0(G, X))$ is homotopy equivalent to $\Delta(L(G) - S)$.*

Proof. Let ϕ be the restriction of η to $L(G) - S$. Then the image of ϕ is $\Gamma_0(G, X) = \Gamma(G, X) - \{\widehat{0}, \widehat{1}\}$ where $\widehat{0}$ denotes the image of trivial subgroup and $\widehat{1}$ denotes the image of G under η. Let $Q = \Gamma_0(G, X)$. If we show that $\phi^{-1}(Q_{\leq q})$ is contractible for each $q \in Q$ then we are done by Quillen Fiber Lemma. In order to show $\phi^{-1}(Q_{\leq q})$ is contractible, it is enough to prove that $\phi^{-1}(Q_{\leq q})$ has a greatest element. Assume it does not have such an element. Then there must be two distinct maximal elements, H and K in this set. On the other hand by Proposition 2.2.6, $\phi(H \vee K) = \phi(H) \vee \phi(K) \leq q$. So, we will have $H \vee K \in \phi^{-1}(Q_{\leq q})$. This contradicts the maximality of H and K. $\square$

Definition 3.2.2. Given a poset P with a greatest element $\widehat{1}$ and least element $\widehat{0}$, the poset $P - \{\widehat{0}, \widehat{1}\}$ is denoted by P_0.

Recall that, for a lattice L, the meet sublattice L^* is defined as follows:

$L^* = \{\bigwedge I : I \text{ is a subset of coatoms in } L\}$,

Lemma 3.2.3 (Lemma 2.2, Shareshian [8])**.** *Let L be a finite lattice and P be a subposet of L which contains $L^* \cup \{\widehat{0}\}$.*
(i) If $\widehat{0} \in L^*$ then $\Delta(P_0) \simeq \Delta(L_0^*)$.
(ii) Otherwise $\Delta(P_0)$ is contractible.

Proof. Define the poset Q such that,

$$Q = \begin{cases} L_0^*, & \widehat{0} \in L^*; \\ L^* - \{\widehat{1}\}, & \text{otherwise.} \end{cases}$$

Let $i : Q \to P_0$ be the inclusion map. For $x \in P_0$ define x^* to be the meet of all coatoms greater than x. Clearly, the preimage of i restricted to the elements greater than or equal to x in P_0 is equal to $Q_{\geq x^*}$ which is contractible since it has a least element. We conclude by Quillen Fiber Lemma that $\Delta(Q) \simeq \Delta(P_0)$. Thus the first part is proved. But if $\widehat{0} \notin L^*$ then L^* contains a least element and so does Q. This finishes the proof of second part. $\square$

Corollary 3.2.4. *Let L be a finite lattice. If $\widehat{0} \notin L^*$, then L_0 is contractible. Otherwise L_0 is homotopy eqivalent to L_0^*.*

Proof. Take $P = L$ in the Lemma 3.2.3. $\square$

In the next chapter we will examine the topology of lattice of periods in some special cases and Lemma 3.2.3 will be the main tool in the proofs.

Chapter 4

Calculations

In this chapter we will identify the topology of lattices of periods in some special cases. We will consider transitive G-sets where G belongs to one of the following family of groups: D_{2^n}, D_{2p^n}, SD_{2^n}, Q_{2^n} where p is an odd prime and D_{2n} denotes the dihedral group of order $2n$, SD_{2^n} denotes the semi-dihedral group of order 2^n, and Q_{2^n} denotes the quaternion group of order 2^n.

4.1 The Dihedral Group of Order 2^n

The presentation of dihedral group D_{2^n} is given by:

$$D_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, bab = a^{-1} \rangle.$$

Lemma 4.1.1. *Let $G = D_{2^n}$ for $n > 1$. Then G has 3 maximal subgroups which are $H_1 = \langle a \rangle$, $H_2 = \langle a^2, b \rangle$ and $H_3 = \langle a^2, ab \rangle$.*

Proof. Any maximal subgroup of a p-group has index p. So, all the maximal subgroups of D_{2^n} has order 2^{n-1}. Clearly, H_1, H_2 and H_3 are subgroups of order 2^{n-1}. Indeed, a has order 2^{n-1} in G and hence $H_1 = \langle a \rangle$ has order 2^{n-1}. The element a^2 has order 2^{n-2} and $H_2 \supsetneqq \langle a^2 \rangle$, so H_2 has order 2^{n-1}. Similarly, H_3 has order 2^{n-1}. Now, let H be a subgroup of index 2. It contains a^2 for otherwise

H, aH and a^2H would be distinct left cosets of H which is not possible since an index 2 subgroup has two left cosets. If $a \in H$, then $H = H_1$. Otherwise, either $b \in H$ or $ab \in H$. The former case corresponds to $H = H_2$ and the latter corresponds to $H = H_3$. □

The group element $a^i b$ has order 2 for $i = 0, 1, \ldots$. Moreover, $a^i b a^k a^i b = a^{-k}$ and in particular $a^i b a^2 a^i b = a^{-2}$. Hence, the above lemma tells us that the dihedral group of order 2^n ($n \geq 2$) has a cyclic maximal subgroup and two other maximal subgroups which are dihedral of order 2^{n-1}.

Corollary 4.1.2. *Any subgroup of dihedral group of order 2^n is either a cyclic group or a dihedral group.*

Proof. It becomes apparent if we apply Lemma 4.1.1 repeatedly. □

Corollary 4.1.3. *If H is a noncyclic subgroup of D_{2^n} with index 2^k, then $H = \langle a^{2^k}, a^i b\rangle$ for some $i \in \{0, 1, \ldots, 2^k - 1\}$.*

Proof. The claim holds for maximal dihedral subgroups by Lemma 4.1.1. Assume it holds for dihedral subgroups of index 2^j. Let H be a dihedral subgroup of index 2^{j+1}. Then it is a subgroup of a subgroup K where K has index 2^j. By induction hypothesis $K = \langle a^{2^j}, a^i b\rangle$ for some $i \in \{0, 1, \ldots, 2^j - 1\}$. Since $|K : H| = 2$, $H = \langle a^{2^{j+1}}, a^l a^i b\rangle$ for some $l \in \{0, 2^j\}$ by Lemma 4.1.1. □

Corollary 4.1.4. *Let H be a subgroup of D_{2^n} with index 2^k. Then either H is cyclic generated by $a^{2^{k-1}}$ or H is a dihedral group generated by a^{2^k} and $a^i b$ for some $i = 0, 1, \ldots, 2^k - 1$.*

Proof. This is an immediate consequence of Corollary 4.1.2 and Corollary 4.1.3. □

The first three lines of the subgroup lattice of dihedral group of order 2^n ($n \geq 3$) is shown in Figure 4.1. In the second row we have three maximal subgroups; a cyclic subgroup and two dihedral subgroups. In the third row, we have a cyclic subgroup of order 2^{n-2} and four dihedral subgroups of the same order. In general, in row k (for $2 \leq k \leq n$) we have one cyclic subgroup of order 2^{n-k+1} and 2^{k-1} dihedral subgroups of the same order.

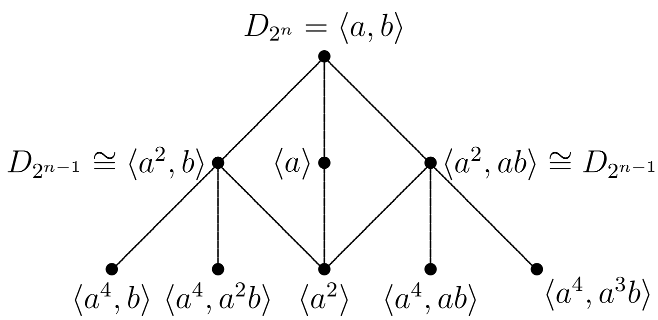

Figure 4.1: D_{2^n}

Lemma 4.1.5. *Let H and K be two proper subgroups of $G = D_{2^n}$ which are not maximal. Then, $HK \neq G$.*

Proof. It is enough to show that for two subgroups H and K of index 4, $HK \neq G$ since any subgroup of index greater than 4 is contained in a subgroup of index 4. Any subgroup of G of index 4 is either cyclic generated by a^2 or generated by the elements a^4 and a^ib for $i \in \{0, 1, 2, 3\}$. Hence, two subgroups of index 4 intersect at the subgroup $\langle a^4\rangle$. $|HK| = |H||K|/|H \cap K| = \frac{|G|/4.|G|/4}{|G|/8} = |G|/2 \neq |G|$. Hence $HK \neq G$. □

Proposition 4.1.6. *Let G be a group, H be a subgroup of G, and let G act on G/H in the usual way. If $\eta(K_1) \geq \eta(K_2)$ in $\Gamma(G, G/H)$ then $K_1H \supseteq K_2H$. In particular, if $\eta(K_1) = \eta(K_2)$ in $\Gamma(G, G/H)$ then $K_1H = K_2H$.*

Proof. This becomes clear with the following observation: For any subgroup K, the union of the left cosets of H which are in the same block with the coset H in partition $\eta(K)$ is equal to the product KH. □

Let $X = G/H$ be a transitive G-set where G is isomorphic to dihedral group of order 2^n. If $n = 1$, then the only possible lattices of periods of this group are the lattice with one element and the lattice with two elements. But, we are interested only in the poset where the greatest and least elements of the lattice are removed. In the case of D_2 this poset is the empty poset hence everything is trivial for this case. Now assume $n \geq 2$. We need two lemmas before stating one of the main theorems of this thesis.

Lemma 4.1.7. *Let H be a subgroup of $G = D_{2^n}$ such that $H = \langle a^{2^k}, a^ib\rangle$ where*

$k > 1$ and $i \in \{0, 1, \dots 2^k - 1\}$. Then the maximal elements of $\Gamma_0(G, G/H)$ are $\eta(\langle a^2, a^i b\rangle)$, $\eta(\langle a^4, a^{i+1}b\rangle)$ and $\eta(\langle a^4, a^{i+3}b\rangle)$.

Proof. The cosets of H are $H, aH, \dots, a^{2^k-1}H$. Let $\langle a\rangle = H_1$ and $\langle a^2, a^{i+1}b\rangle = H_2$. Since $H_1 \cdot H = G = H_2 \cdot H$ we have $\eta(H_1) = \eta(G) = \eta(H_2)$. On the other hand,

$$\begin{aligned} &G \neq \langle a^2, a^i b\rangle H = \langle a^2, a^i b\rangle \\ &G \neq \langle a^4, a^{i+1}b\rangle H \text{ (by Lemma 4.1.5)} \\ &G \neq \langle a^4, a^{i+3}b\rangle H \text{ (by Lemma 4.1.5)} \end{aligned}$$

Any proper subgroup of G other than H_1 and H_2 is a subgroup of at least one of the given subgroups. Since η is an order preserving map all the possible maximal elements of $\Gamma_0(G, G/H)$ are the corresponding images of these subgroups under the map η. Now,

$$\begin{aligned} &\eta(\langle a^2, a^i b\rangle) = H, a^2H, \dots, a^{2^k-2}H | aH, a^3H, \dots, a^{2^k-1}H \\ &\eta(\langle a^4, a^{i+1}b\rangle) = H, aH, \dots, a^{2^k-4}H, a^{2^k-3}H | a^2H, a^3H, \dots, a^{2^k-2}H, a^{2^k-1}H \\ &\eta(\langle a^4, a^{i+3}b\rangle) = H, a^3H, \dots, a^{2^k-4}H, a^{2^k-1}H | aH, a^2H, \dots, a^{2^k-3}H, a^{2^k-2}H \end{aligned}$$

Clearly, above three partitions are not comparable. Hence we are done. □

Lemma 4.1.8. *Let H be a subgroup of D_{2^n} which is generated by a^{2^k} for $k = 1, 2, \dots$. Then the maximal elements in the poset $\Gamma_0(G, G/H)$ are $\eta(\langle a\rangle)$, $\eta(\langle a^2, b\rangle)$, $\eta(\langle a^2, ab\rangle)$, i.e., the maximal elements of $\Gamma_0(G, G/H)$ are the images of maximal subgroups in D_{2^n}.*

Proof. Since H is a subgroup of the Frattini group $\Phi(D_{2^n}) = \langle a^2\rangle$, it is a subgroup of each of the maximal subgroups. Hence $HM = M \neq D_{2^n}$ when M is one of these subgroups which implies that the image of M under η is not equal to $\eta(D_{2^n})$. This fact guarantees that the images of maximal subgroups exist in the poset $\Gamma_0(G, G/H)$. Since η is order preserving, the image of any proper subgroup K is smaller than the image of the maximal subgroup containing K. Now it is enough to prove that any pair of these three elements are not comparable. But this is an immediate consequence of Proposition 4.1.6. □

Theorem 4.1.9. *Let H be a subgroup of $G = D_{2^n}$ and $X = G/H$.*
(i) If H has index 1 or 2, then $\Gamma_0(G, G/H)$ is empty.
(ii) If H has index 4, then $\Gamma_0(G, G/H)$ is homotopy equivalent to 3 points.
(iii) If H has index greater than 4, then $\Gamma_0(G, G/H)$ is contractible.

Proof. (i) If H has index 1 or 2, then lattice of periods has one or two elements respectively. When the least and greatest elements are removed, the remaining poset is empty in either case.

(ii) Suppose now that H has index 4. If $H = \langle a^2 \rangle = \Phi(G)$ then $\Gamma(G, G/H)$ is isomorphic to $L(G/H)$ and hence $\Gamma_0(G, G/H)$ is isomorphic to $L(G/H) - \{\widehat{0}, \widehat{1}\}$ where $\widehat{0}$ and $\widehat{1}$ denote the trivial subgroup and G/H itself. Since G/H is isomorphic to D_4, it follows that $\Gamma_0(G, G/H)$ has the homotopy type of 3 points. Let $H = \langle a^4, a^i b \rangle$ where $i \in \{0, 1, 2, 3\}$. Then the cosets of H are $\{H, aH, a^2H, a^3H\}$ and $\eta(G) = \eta(\langle a \rangle) = \eta(\langle a^2, a^{i+1}b \rangle) = H, aH, a^2H, a^3H$. We have

$$\eta(\langle a^2, a^i b \rangle) = H, a^2H | aH, a^3H,$$
$$\eta(\langle a^4, a^{i+1} b \rangle) = H, aH | a^2H, a^3H, \text{ and}$$
$$\eta(\langle a^4, a^{i+3} b \rangle) = H, a^3H | aH, a^2H.$$

Since η is order preserving, the coatoms of $\Gamma(G, G/H)$ are the above three partitions. The meet of any pair of these partitions is the partition $H|aH|a^2H|a^3H$, which is the least element in $\Gamma(G, G/H)$. So, the meet sublattice consists of five elements; the greatest element H, aH, a^2H, a^3H, the least element $H|aH|a^2H|a^3H$, and three atoms (or coatoms) appearing above. Since the least element of $\Gamma(G, G/H)$ is contained in the meet sublattice, the poset $\Gamma_0(G, G/H)$ is homotopy equivalent to 3 points by Lemma 3.2.3.

(iii) Now assume that H has index greater than 4 in G. If H is generated by a^{2^m} for some $m = 2, 3, \ldots$ then the maximal elements of $\Gamma_0(G, G/H)$ are exactly the images of maximal subgroups of D_{2^n} by Lemma 4.1.8. Since H is properly included in the Frattini subgroup $\Phi(D_{2^n})$, the images $\eta(\Phi(D_{2^n}))$ and $\eta(\langle 1 \rangle)$ are distinct elements of $\Gamma(G, G/H)$. The meet semilattice has the least element $\eta(\Phi(D_{2^n}))$ and hence does not contain the least element of $\Gamma(G, G/H)$. Hence, $\Gamma_0(G, G/H)$ is contractible by Corollary 3.2.4.

Now assume that $H = \langle a^{2^m}, a^i b\rangle$ for some $m = 3, 4, \ldots$ and $0 \leq i < 2^m$. The maximal elements of $\Gamma_0(G, G/H)$ are $\eta(\langle a^2, a^i b\rangle)$, $\eta(\langle a^4, a^{i+1}b\rangle)$ and $\eta(\langle a^4, a^{i+3}b\rangle)$ by Lemma 4.1.8. Since $\langle a^4\rangle$ is an index 8 subgroup, it is not included in H and hence $\langle a^4\rangle H \neq H$. So, $\eta(\langle a^4\rangle) \neq \eta(\langle 1\rangle)$. On the other hand $\eta(\langle a^4\rangle)$ is smaller than all these maximal elements. So, the meet of all maximal elements is equal to $\eta(\langle a^4\rangle)$. Hence the least element of the meet sublattice of $\Gamma(G, G/H)$ is different than $\eta(\{1\})$. Thus, $\Gamma_0(G, G/H)$ is contractible by Corollary 3.2.4. $\square$

4.2 The Dihedral Group of Order $2p^n$

The presentation of dihedral group D_{2p^n} is given by:

$$D_{2p^n} = \langle a, b : a^{p^n} = b^2 = 1, bab = a^{-1}\rangle.$$

Proposition 4.2.1. *The group D_{2p^n} has $p+1$ maximal subgroups, namely a cyclic subgroup of index 2 and p dihedral groups of index p.*

Proof. $C_{p^n} = \langle a\rangle$ is a normal subgroup of D_{2p^n}. Sylow's Theorem tells us that there is no other subgroup of D_{2p^n} of index 2. Assume H is a maximal subgroup of D_{2p^n} which is different from C_{p^n}. Let k be the least positive integer such that $a^k \in H$. It is clear that $k = p^l$ for some $l \in \{0, 1, \ldots, n\}$. Similarly, let i be the least nonnegative integer such that $a^i b \in H$ (there does exist such an element). This i is necessarily smaller than k. Thus k must be greater than 1 otherwise H would be the whole group D_{2p^n}. But if $k = p^l$ for $l > 1$ then by adding a^p to the generating set of H we obtain a larger subgroup which is not D_{2p^n}. This contradicts the maximality of H. So, $H = \langle a^p, a^i b\rangle$ for some $i = 0, 1, \ldots, p-1$. Clearly, different i's generate different subgroups. The elements a^p and $a^i b$ have orders p^{n-1} and 2 respectively. Moreover $a^i b a^p a^i b = a^{-p}$, so $\langle a^p, a^i b\rangle$ is a dihedral group of order $2p^{n-1}$. $\square$

Corollary 4.2.2. *All the subgroups of D_{2p^n} are either cyclic groups or dihedral groups.*

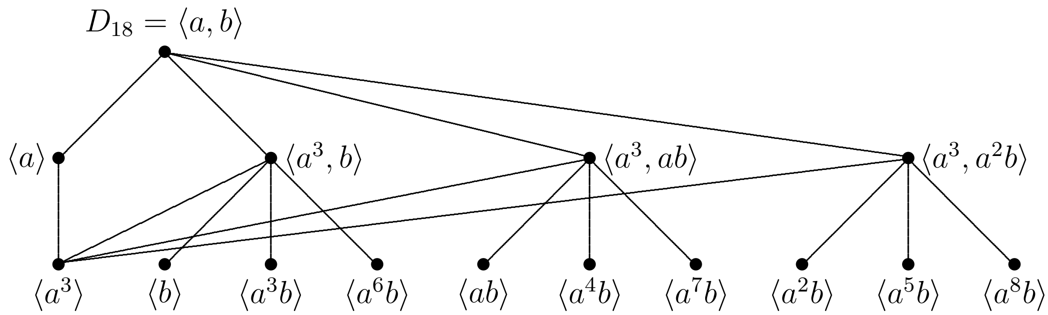

Figure 4.2: D_{18}

Proof. Any maximal subgroup of cyclic group is cyclic. Maximal subgroups of dihedral groups D_{2p^k} for any k are either cyclic of index 2 or dihedral of index p by Proposition 4.2.1. The desired result follows by induction. □

Proposition 4.2.3. *Let H be a proper subgroup of D_{2p^n} which is not maximal. Then there exists a maximal subgroup $\langle a^p, a^i b\rangle$ of index p which contains H where $i \in \{0, 1, \ldots, p-1\}$.*

Proof. If H is a dihedral subgroup then it is contained in a maximal dihedral subgroup. If H is not a dihedral subgroup, then it is cyclic generated by a^{p^k} for some $k \geq 1$. The subgroup H is included in a maximal dihedral group in this case too. □

The top part of the subgroup lattice of D_{18} is illustrated in Figure 4.2.

Lemma 4.2.4. *Let H_1 and H_2 be subgroups of D_{2p^n} where $H_1, H_2 \notin \{\langle a\rangle, D_{2p^n}\}$. Then, $H_1H_2 \neq D_{2p^n}$.*

Proof. By Proposition 4.2.3, it is enough to consider maximal subgroups $M_i = \langle a^p, a^i b\rangle, i = 0, 1, \ldots, p-1$. Since $M_i \cap M_j = \langle a^p\rangle$ we have $|M_i \cap M_j| = p^{n-1}$.

$$|M_iM_j| = \frac{|M_i| \cdot |M_j|}{|M_i \cap M_j|} = \frac{2p^{n-1}2p^{n-1}}{p^{n-1}} = 4p^{n-1} < 2p^n = |D_{2p^n}|.$$

□

Theorem 4.2.5. *Assume $G = D_{2p^n}$ where p is an odd prime and let G act on G/H for $H \leq G$.*

(i) If H has index 1 or index 2, then $\Gamma_0(G, G/H) = \emptyset$.

(ii) If H has index p, then $\Gamma_0(G, G/H)$ is homotopy equivalent to the disjoint union of p points.
(iii) If H has index $2p$, then $\Gamma_0(G, G/H)$ is homotopy equivalent to the disjoint union of $p+1$ points.
(iv) Otherwise, $\Gamma_0(G, G/H)$ is contractible.

Proof. (i) The first case which corresponds to $|G : H| = 1$ or $[G : H] = 2$ is trivial since in each of these cases, any element of $\Gamma(G, G/H)$ is either the least element or the greatest element. Hence the removal of the least and the greatest elements of $\Gamma(G, G/H)$ leaves the poset $\Gamma_0(G, G/H)$ empty.

(ii) D_{2p^n} has p subgroups with index p. These subgroups are $\{\langle a^p, a^i b\rangle : 0 \leq i < p\}$ which are conjugate to each other. Let $K_i = \langle a^p, a^i b\rangle$ for $i = 0, 1, \ldots, p-1$. Hence $H = K_m$ for some $m \in \{0, 1, \ldots, p-1\}$. Without loss of generality, we can assume $m = 0$ by Proposition 2.2.23. Since $\langle a\rangle$ has index 2, it is normal in D_{2p^n} and $\langle a\rangle H = G$ which means that $\eta(\langle a\rangle) = \eta(D_{2p^n})$. By Lemma 4.2.4, $K_i H \neq D_{2p^n}$ for $i \in \{0, 1, \ldots, p-1\}$. By Proposition 4.1.6, $\eta(K_i) \neq \eta(G)$ for $i \in \{0, 1, \ldots, p-1\}$ and $\eta(K_j) \neq \eta(K_l)$ for distinct $j, l \in \{0, 1, \ldots, p-1\}$. So, the images of these subgroups are maximal in $\Gamma_0(G, G/H)$. The maximal subgroups of D_{2p^n} are $\{\langle a\rangle\} \cup \{K_0, K_1, \ldots, K_{p-1}\}$ and any other proper subgroup of D_{2p^n} is contained in at least one of the subgroups in $\{\langle a^p, a^i b\rangle : 0 \leq i < p\}$ by Proposition 4.2.3. So, the only maximals in $\Gamma_0(G, G/H)$ are $\{\eta(K_i) : i \in \{0, 1, \ldots, p-1\}\}$. Let's now show that $\eta(K_i) \wedge \eta(K_j)$ does not lie in $\Gamma_0(G, G/H)$ for $i \neq j$. Assume otherwise, let $\eta(K_i) \wedge \eta(K_j) = \eta(K)$ for some $K \leq D_{2p^n}$ and $\eta(K) \neq \eta(\{1\})$. The cosets of H are $\{H, aH, \ldots, a^{p-1}H\}$. Assume $a^s H \sim_{\eta(K)} a^r H$ for $s \neq r$. Then $a^s H \sim_{\eta(K_i)} a^r H$ and $a^s H \sim_{\eta(K_j)} a^r H$.

$$\begin{aligned} a^s H \sim_{\eta(K_i)} a^r H &\iff k_i a^s H = a^r H \text{ for some } k_i \in K_i \\ &\iff a^{-r} k_i a^s \in H \text{ for some } k_i \in K_i \end{aligned}$$

$K_i = \langle a^p, a^i b\rangle = \{a^{px} : x \in \mathbb{N}\} \cup \{a^{py+i} b : y \in \mathbb{N}\}$. The element k_i can not be of the form a^{px} so $k_i = a^{py+i} b$ for some $y \in \mathbb{N}$. Then, the element $a^{-r} a^{py+i} b a^s = a^{py+i-s-r} b$ is in H if and only if $s + r = i \mod p$. So, $a^s H \sim_{\eta(K_i)} a^r H$ if and only if $s + r = i \mod p$.
Similarly, $a^s H \sim_{\eta(K_j)} a^r H$ if and only if $s + r = j \mod p$. Clearly, both of

these cannot be satisfied if $i \neq j$. Hence contradiction. So, $\eta(K_i) \wedge \eta(K_j) = \eta(\{1\})$ for any $i \neq j$. So, the meet sublattice of $\Gamma(G, G/H)$ consists of these maximals together with the greatest and the least elements of $\Gamma(G, G/H)$. Hence, by Lemma 3.2.3, $\Gamma_0(G, G/H)$ has the homotopy type of p distinct points.

(iii) If H has index $2p$, then $H = \langle a^p \rangle = \Phi(D_{2p^n})$. The lattice $\Gamma(G, G/\Phi(G))$ is isomorphic to $L(G/\Phi(G))$ by Corollary 2.3.5. If we take $G = D_{2p^n}$ and $H = \Phi(D_{2p^n})$ then, $\Gamma_0(D_{2p^n}, D_{2p^n}/\Phi(D_{2p^n})) \simeq L_0(D_{2p^n}/\Phi(D_{2p^n})) \simeq L_0(D_{2p})$ and $L_0(D_{2p})$ is homotopy equivalent to $p+1$ points.

(iv) Assume now H has index greater than $2p$. If $H \leq \langle a \rangle$ then the maximal elements of the poset $\Gamma_0(G, G/H)$ are $\eta(\langle a \rangle)$ and $\eta(K_i)$ for $i = 0, 1, \ldots, p-1$ where K_i is as defined above. But the meet of any pair is greater than $\eta(\Phi(G))$ since η is order preserving. Hence the least element of the meet sublattice of $\Gamma(G, G/H)$ is greater than or equal to $\eta(\Phi(G))$ which is strictly greater than $\eta(\{1\})$. Therefore, the meet sublattice does not contain the least element of $\Gamma(G, G/H)$. We conclude by Lemma 3.2.3 that $\Gamma_0(G, G/H)$ is contractible. If H is not a subgroup of $\langle a \rangle$ then the maximal elements of $\Gamma_0(G, G/H)$ are $\eta(K_i)$ for $i = 0, 1, \ldots, p-1$. Nevertheless, the same argument works well in this case also to show that $\Gamma_0(G, G/H)$ is conically contractible. $\square$

4.3 Semi-dihedral and Quaternion Groups

In this section we restrict our attention to semi-dihedral groups and quaternion groups. If G is one of these groups and X is a transitive G-set which is G-isomorphic to G/H for some subgroup H with index more than 2, then $\Gamma_0(G, G/H)$ is either homotopic to 3 points or it is contractible.

Lemma 4.3.1. *Let $H \leq SD_{2^n} = \langle x, y : x^{2^{n-1}} = y^2 = 1,\ yxy = x^{2^{n-2}-1} \rangle$. Assume that $x^k \in H$ where k is a positive integer and there is no natural number l less than k such that $x^l \in H$. Similarly, assume $x^i y \in H$ where i is a nonnegative integer and there is no natural j less than i such that $x^j y \in H$. Then, $H = \langle x^k, x^i y \rangle$ and $|H| = 2|\langle x^k \rangle|$.*

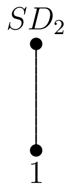

Figure 4.3: SD_2

Proof. Let $x^t \in H$ such that $t = kr + q$ with $0 \leq q < k$. Then, $x^{-kr}x^{kr+q} = x^q \in H$. By the minimality of k, q must be 0. We claim that if $x^t y \in H$ then, $t = kr + i$ for some convenient integer r.

Assume otherwise, let $x^t y \in H$ such that $k(r-1) + i < t < kr + i$. Then, $x^{kr+i}yx^t y = x^{kr+i}x^{(2^{n-2}-1)t} = x^{kr+i-t}x^{2^{n-2}t} \in H$. But then, $x^{kr+i-t} \in H$ since $x^{2^{n-2}t} \in H$ and $x^{-2^{n-2}t} \in H$. This contradicts the minimality of k since $0 < kr + i - t < k$. So, any element of H is either of the form x^{kr} or $x^{kj+i}y$ for integer r and j. This completes the first part of the proof.

It is clear that $|H| = 2|\langle x^k \rangle|$. $\square$

Let H be a subgroup of SD_{2^n} as in the previous lemma. Then i must be less than k, otherwise $x^{-k}x^i y = x^{i-k}y$ would be in H. Since the order of SD_{2^n} is a power of 2, k must be a power of 2. Let $k = 2^e$. If $e = n-1$ then i must be even since $x^i y x^i y = x^{2^{n-2}}$ for odd i which is contradicting the minimality of k. If $e < n-1$ then i can be anything less than k. So, there are k subgroups which contain x^k and do not contain x^l for $l < k$ for $k \neq 2^{n-1}$. These are $\{\langle x^k, x^i y\rangle : i < k\}$. There are 2^{n-2} subgroups with $k = 2^{n-1}$, which constitute the set $\{\langle x^i y\rangle : i \text{ is even and } i < 2^{n-1}\}$.

The subgroup lattices of SD_{2^n} for $n = 1$, $n = 2$, and $n = 3$ are given in Figures 4.3, 4.4, and 4.5 respectively.

For $n > 3$, first three lines of the subgroup lattice of SD_{2^n} is illustrated in Figure 4.6. The maximal subgroups of SD_{2^n} are the three subgroups in the middle line of the figure.

Lemma 4.3.2. *If H and K are two subgroups of SD_{2^n} which are both not maximal then $HK \neq SD_{2^n}$.*

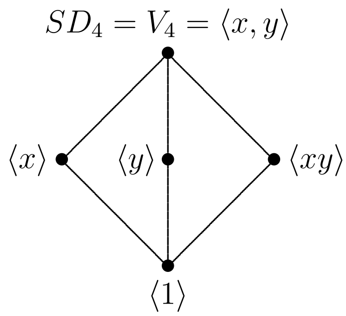

Figure 4.4: SD_4

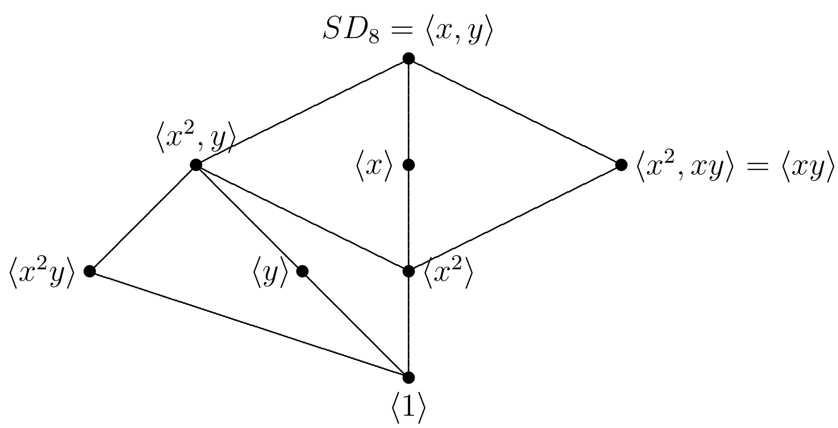

Figure 4.5: SD_8

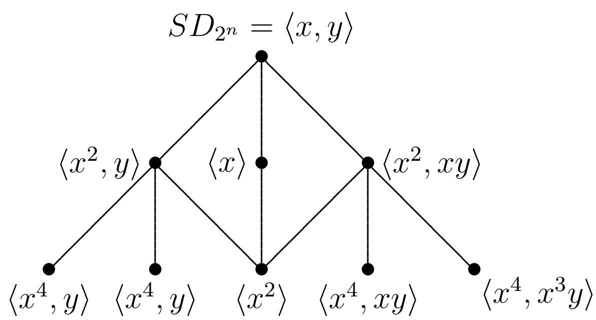

Figure 4.6: SD_{2^n}

Proof. It is enough to consider the subgroups of index 4 since any nonmaximal subgroup lies in an index 4 subgroup. But any two subgroups of index 4 intersect at $\frac{2^n}{8}$ elements generated by x^4. $|HK| = \frac{|H||K|}{|H\cap K|} = \frac{2^n}{2} = 2^{n-1} < 2^n$. So, HK can not be equal to SD_{2^n}.

Proposition 4.3.3. *Let G be a semi-dihedral group of order 2^n where $n > 3$ and $X \cong G/H$ is a transitive G-set for some subgroup H of G.*
(i) If H has index 1 or 2 then $\Gamma_0(G, G/H)$ is empty.
(ii) If H has index 4, then $\Gamma_0(G, G/H)$ has the homotopy type of 3 points.
(iii) Otherwise $\Gamma_0(G, G/H)$ is contractible.

Proof. There are three maximal subgroups of G: $M_1 = \langle x^2, y\rangle$, $M_2 = \langle x\rangle$ and $M_3 = \langle x^2, xy\rangle$. Let $H_1, \ldots, H_5$ be the index 4 subgroups of G from left to right respectively in Figure 4.6.

(i). This is obvious since in this case the corresponding lattice has one element or two elements. Removing the greatest and the least elements results in an empty poset.

(ii) If $H = H_3$ is the Frattini subgroup then $\Gamma(G, G/H)$ is isomorphic to $L(G/H) \cong D_4$. Hence $\Gamma_0(G, G/H)$ has the homotopy type of 3 points. Otherwise assume without loss of generality that $H = H_1$. The images of M_2 and M_3 under η are equal to $\eta(G)$ since $M_2H = G = M_3H$. Then the maximal elements of $\Gamma_0(G, G/H)$ are $\eta(M_1)$, $\eta(H_4)$ and $\eta(H_5)$. The cosets of H are H, xH, x^2H and x^3H. The maximal elements are: $\eta(M_1) = H, x^2H|xH, x^3H$; $\eta(H_4) = H, xH|x^2H, x^3H$; and $\eta(H_5) = H, x^3H|xH, x^2H$. Since the meet of any two maximal elements is $\eta(\{1\})$, the meet sublattice consists of five elements; two are the least and the greatest elements of $\Gamma(G,G/H)$ and the others are $\eta(M_1)$, $\eta(H_4)$ and $\eta(H_5)$. Hence, the poset $\Gamma_0(G,G/H)$ is homotopy equivalent to 3 points by Lemma 3.2.3.

(iii) If H has index greater than 4, then either it lies in the Frattini subgroup H_3 or it lies in exactly one of M_1 and M_3. In the former case, the maximal elements of $\Gamma_0(G,G/H)$ are the images of maximal subgroups of G under the map η. Since $\eta(M_1)\wedge\eta(M_2)\wedge\eta(M_3)=\eta(\Phi(G))\neq\eta(\{1\})$, the meet sublattice does not contain the least element of $\Gamma(G,G/H)$. Thus, the poset $\Gamma_0(G,G/H)$ contractible by Corollary 3.2.4. In the latter case, assume that H lies in M_1. Then the maximal elements of the poset $\Gamma_0(G,G/H)$ are $\eta(M_1)$, $\eta(H_4)$ and $\eta(H_5)$. All these maximal elements are greater than or equal to $\eta(\langle x^4\rangle)$ since the map η is order preserving. So, $\eta(M_1)\wedge\eta(H_4)\wedge\eta(H_5)\geq\eta(\langle x^4\rangle)$. Since $\langle x^4\rangle$ is not contained in H, the product $H\langle x^4\rangle\neq H$. This means that the coset H does not appear alone in $\eta(\langle x^4\rangle)$. Hence, $\eta(\langle x^4\rangle)\neq\eta(\{1\})$. Hence, $\eta(\{1\})$ does not appear in the meet sublattice. By Lemma 3.2.3, we conclude that $\Gamma_0(G,G/H)$ is contractible. If H does not lie in M_1 but lies in M_3, then we can replace M_1 with M_3, H_4 with H_1 and H_5 with H_2 in the above argument and get the same result. □

In the remaining part of this section we will consider quaternion groups.

Lemma 4.3.4. *Let* $H\leq Q_{2^n}=\langle x,y : x^{2^{n-1}}=1,\ x^{2^{n-2}}=y^2,\ yxy^{-1}=x^{-1}\rangle$. *Assume that* $x^k\in H$ *where* k *is positive and for* $0<l<k$, $x^l\notin H$. *Similarly, assume that* $x^iy\in H$ *where* i *is nonnegative and for* $0\leq j<i$, $x^jy\notin H$. *Then,* $H=\langle x^k,x^iy\rangle$ *and* $|H|=2|\langle x^k\rangle|$.

Proof. It is clear that $\langle x^k,x^iy\rangle\subseteq H$. In order to show the equality, it is enough to show that every element of H is either of the form x^{kr} for some integer r or it is of the form $x^{kr+i}y$ for some integer r.

Suppose that $x^t\in H$. Let $t=kr+q$ with $0\leq q<k$. Then, $x^{-kr}x^{kr+q}=x^q\in H$. So, $q=0$ by the minimality of k. If $x^iy\in H$ then $x^iyx^iy=y^2=x^{2^{n-2}}\in H$. It is clear that k is a power of two. So, $k=2^l$ for some $l=0,1,\ldots n-2$.

Assume $x^{kr+i+q}y\in H$ with $0\leq q<k$. Then, $x^{k(r+1)+i}yx^{kr+i+q}y=x^{k-q}y^2=x^{k-q}x^{2^{n-2}}\in H$ and $x^{k-q}x^{2^{n-2}}x^{2^{n-2}}=x^{k-q}\in H$ which implies $q=0$. So,

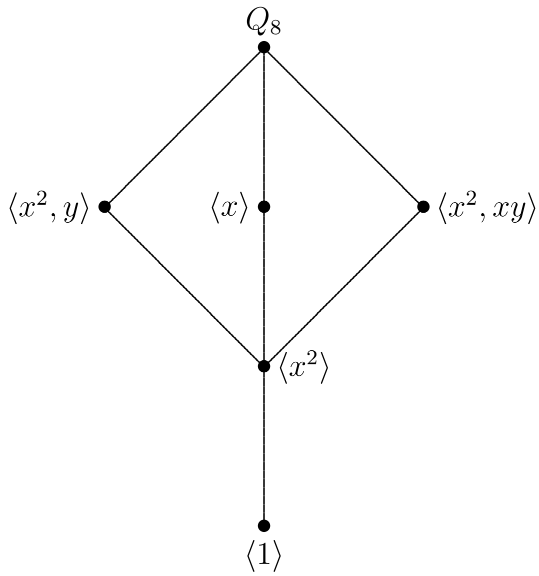

Figure 4.7: Q_8

$H = \langle x^k, x^i y\rangle = \{x^{ks} : s = 0, 1, \dots\} \cup \{x^{kj+i} : j = 0, 1, \dots\}$. It is clear that $|H| = 2|\langle x^k\rangle|$. $\square$

As in the case of semi-dihedral group i must be smaller than k in the above setting. Any nontrivial subgroup of Q_{2^n} contains $x^{2^{n-2}} = y^2$. So, if H is a nontrivial subgroup of Q_{2^n} containing $x^i y$ for some i, then k is a member of the set $\{2^e : e = 0, 1, \dots, n-2\}$. There are $2^t + 1$ subgroups with index 2^t for $t = 1, 2, \dots, n-2$. These are $\{\langle x^{2^t}, x^i y\rangle : 0 \leq i < 2^t\}$ and $\langle x^{2^{t-1}}\rangle$. There is only one subgroup with index 2^{n-1} which is generated by y^2. The quaternion group with two elements is isomorphic to C_2, indeed there is only one group up to isomorphism with two elements. The quaternion group with four elements is isomorphic to SD_4 which is isomorphic to V_4 and its subgroup lattice has the shape in Figure 4.4. The subgroup lattice of Q_8 is illustrated in Figure 4.7.

For $n > 3$, the shape of the subgroup lattice of Q_{2^n} is similar to the shape of the subgroup lattice of SD_{2^n}. The only difference occurs in subgroups of order 2. Quaternion group has a unique subgroup of order 2 but semi-dihedral group has $2^{n-2}+1$ subgroups of order 2. Therefore, the first three lines of the subgroup lattice of Q_{2^n} for $n > 3$ is exactly the same as the first three lines of lattice of SD_{2^n}. It is illustrated in Figure 4.8.

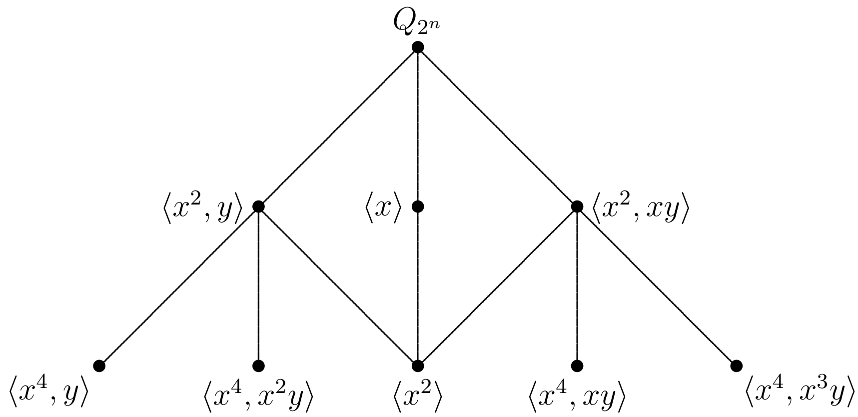

Figure 4.8: Q_{2^n}

Lemma 4.3.5. *Let H and K be two subgroups of Q_{2^n} which are both not maximal . Then, $HK \neq Q_{2^n}$.*

Proof. It is enough to consider index 4 subgroups of Q_{2^n}. All these subgroups include the element x^4. Consequently, any two of these groups intersect at the subgroup $\langle x^4\rangle$ which is an index 8 subgroup. So, if H and K are such two subgroups then $|HK| = \frac{|H||K|}{|H\cap K|} = 2^{n-1}$. Hence the product of these groups can not be equal to the group Q_{2^n}. □

Proposition 4.3.6. *For $n > 3$, let H be a subgroup of $G = Q_{2^n}$ and let X be a G-set which is G-isomorphic to G/H.*
(i) If H has index 1 or 2 then $\Gamma_0(G, G/H)$ is empty.
(ii) If H has index 4 then $\Gamma_0(G, G/H)$ is homotopic to 3 points.
(iii) Otherwise, $\Gamma_0(G, G/H)$ is contractible.

Proof. One can replace SD_{2^n} with Q_{2^n} in the proof of Theorem 4.3.3 and obtain the same results.

We can also argue as follows: There is a unique subgroup of order two of the quaternion group. This subgroup is generated by y^2 and it is a normal subgroup of G. If $H = \{1\}$, then $\Gamma(G, G/H)$ is isomorphic to the subgroup lattice $L(G)$ and hence $\Gamma_0(G, G/H)$ is contractible by Theorem 3.1.9. Otherwise, we can apply Theorem 2.3.4 with $N = \langle y^2\rangle$ since every non-trivial subgroup contains $\langle y^2\rangle$. The quotient $G/\langle y^2\rangle$ is isomorphic to the dihedral group of order 2^{n-1}. The group H/N is either a dihedral group or a cyclic group of order $|H|/2$. So, the problem is reduced to find the homotopy type of the lattice of periods generated by the dihedral group of order 2^{n-1} and a transitive set. Since the index of H/N in G/N is the same as the index of H in G, we conclude the proof by Theorem 4.1.9. $\square$

Remark 4.3.7. Also in the case of semi-dihedral group we could use Theorem 2.3.4 to reduce the problem to the dihedral case if the subgroup H has order greater than 2. This is because all the subgroups with order greater than 2 contains the central subgroup $C = \langle x^{2^{n-2}} \rangle$, so for a semi-dihedral group G we have $G/C \cong D_{2^{n-1}}$. But for the subgroups of order 2 we can not use the theorem unless the subgroup H is C itself.

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