AN INVENTORY MODEL WITH TWO SUPPLIERS UNDER YIELD UNCERTAINTY

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By
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September 2001

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Master of Science.

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Abstract

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In this study, an inventory model with one retailer and two suppliers is considered for a single item. Different from most of the models in inventory literature, we do not make the assumption that we receive all the quantity that we ordered. It is assumed that a random fraction of the lot size is actually delivered by the suppliers. Hence, the model is constructed under yield uncertainty for both binomial yield and stochastically proportional yield model. The demand rate is constant, and backordering is allowed. The objective is to minimize the long-run average cost and find the near optimal values for the decision variables; order quantities and reorder point. Furthermore, the regions where diversification among suppliers is beneficial are investigated. The results are generalized to "M" suppliers (M>2) and solution method is proposed. Finally, experimental study is carried out for the two-suppliers problem.

Keywords: Random yield, two suppliers

Özet

BİRDEN FAZLA TEDARİKÇİNİN BULUNDUĞU ORTAMDA RASSAL VERİMLİ ENVANTER MODELİ

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Bu çalışmada bir perakendecinin ve iki tedarikçinin bulunduğu bir envanter modeli bir çeşit ürün için kurulmuştur. Envanter literatüründeki bir çok modelden farkı olarak, sipariş mikatarının tamamının tedarikçiler tarafından teslim edildiği varsayımı yapılmamıştır. Verilen siparişin tesadüfi bir miktarının gerçekte sağlandığı varsayılmıştır. Bu yüzden, model binom dağılımlı ve rassal orantılı olmak üzere iki farklı rassal verim modeli gözönüne alınarak kurulmuştur. Talep hızı sabittir ve geri ısmarlamaya izin verilmiştir. Amaç uzun dönemde ortalama maliyet fonksiyonunu enazlamak ve karar değişkenlerinin (yeniden ısmarlama noktası ve sipariş miktarları) değerlerini bulmaktır. Hangi parametre setlerinde toplam siparişin iki tedarikçi arasında paylaştırılmasının karlı olacağı incelenmiştir. Sonuçlar ikiden fazla ("M" sayıda)tedarikçi için genelleştirilmiştir ve çözüm yolları önerilmiştir. Son olarak iki tedarikçinin bulunduğu problem için sayısal analiz yapılmıştır.

Anahtar sözcükler: Rassal verim, iki tedarikçi

to my parents, and Defne

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Chapter 1

Introduction and Literature Review

1.1 Introduction

Management of the inventories that a firm keeps is very crucial for the firm to operate profitably, from both economical and physical perspectives. Inventory keeping costs constitute a significant portion of the total operating costs for companies. Keeping excess inventory may result in unnecessary holding costs including the opportunity costs. On the other side, if there is not enough inventory on-hand, stockouts occur and the demand occurring at that time period is either totally lost, or partially lost or fully backordered. But, in all three cases, the firm incurs shortage costs. Not satisfying the demand instantaneously results in loss of goodwill due to customer dissatisfaction. Hence, shortage costs do not only affect the present, they also affect the future sales of the company. Furthermore, one cannot keep as much inventory as he wants due to the capacity constraints of the warehouses. Therefore, the decision makers are to take into account the physical limitations of the problem.

Complexity of the inventory management problems depends on the structure of the problems. Randomness in lead time, demand, procurement/production make the problems harder to solve. Besides, as the number of products and suppliers increase, it becomes much harder to find analytical solutions. The nature of the items also play an important role in the complexity of the problem. For problems involving continuously deteriorating items, items that have fixed or random shelf lives, different models need to be constructed. The objective functions are usually the expected total cost or the expected profit.

Costs incurred in inventory problems can be classified into the following categories: replenishment, inventory carrying, backordering, system control and inspection costs. Almost all of the previous research in inventory theory assumes a fixed cost for placing an order, which is independent of the lot size. The replenishment may be instantaneous or we may face a positive lead time. In addition to the fixed ordering cost, we incur purchasing/production costs, which are mostly linear in the number of items purchased/produced. The holding costs include opportunity costs related with the cost of capital, taxes, warehouse operation costs, insurance, and finally deterioration costs. Most of the researchers assume that holding costs are directly proportional to the average inventory level. Shortage costs occur due to the unsatisfied demand when the system is out of stock. They are in the form of backordering or lost sales costs both of which cause loss of goodwill and customer dissatisfaction. System control costs may include the costs of reviewing the inventory level in a continuous or periodic fashion, acquiring data, computational costs, and inspection costs.

Mostly it is assumed that the suppliers do provide all the amount ordered by the retailers. Very few papers consider the unreliability of the suppliers. But, in some cases, the suppliers may provide only a fraction of the quantity ordered. In this case, the decision makers have to make their ordering decisions under uncertainty because they do not know the amount they will actually receive. According to the review by Yano and Lee [23], five different ways are proposed to model yield uncertainty. The first one assumes that producing a good unit is a Bernoulli process, so the number of good units has a binomial distribution. Second one is the stochastically proportional yield model in which a random fraction of the order quantity is actually received by the retailer in which the distribution of the fraction is independent of the batch size. The third is similar to the second one except for the fact that the distribution of the fraction changes with the batch size. In the fourth way, the output quantity turns out to be minimum of the input quantity and the realized capacity. Finally, the fifth approach involves specifying for each possible batch size the probability that each possible output quantity will occur. The second way of modeling yield uncertainty is the one that has been most extensively used in the literature.

The number of the suppliers is another factor complicating the analysis of the inventory management problems. Although there is a trend in reducing the number of suppliers due to the long-term contracts with the retailers, this is not always the case when we have unreliable suppliers. In order to reduce uncertainty on the amount that is actually received, retailers tend to order from more than one supplier. Not only do they reduce uncertainty, but also their purchasing costs may go down due to the competition between the suppliers to get a large share in the market. We consider a setting with two suppliers.

1.2 Literature Survey

One of the earliest studies on random yield is done by Wei [20] where a random fraction "p" of the lot is defective and "p" has a known probability distribution. He compares the results obtained by the model that ignores random yield with his model. Both constant demand and random demand (single period) cases are analyzed. He also discussed the effect of inspection policy on the average inventory level and adopted the assumption of 100 % inspection on receipt of order.

In the study by Gerchak, Vickson, and Parlar [11] a periodic review production model with stochastically proportional yield and uncertain demand has been analyzed assuming full backlogging, no set-up cost, unit production cost proportional to the realized yield, and a salvage value for each item unsold. They first analyze the single period and prove that the expected profit function is concave in initial stock and lot size. The optimal policy is characterized by a critical level above which no order will be placed. Furthermore it is observed that when an order is given in the case the initial stock is below the critical value, the expected yield generally does not equal the difference between the order point and available stock. The variable representing the random yield and demand are assumed to be independent and identically distributed over the periods in multiple period problem. Then, 2-period problem is formulated and the profit over two periods is also proven to be convex. The critical level for the first period in a 2-period problem turns out to be larger than the one in single period problem in case an expression of parameters is satisfied. Finally the 2-period formulation is generalized to "n" periods using a DP approach, concavity of the profit is shown and a finite critical value for each period is obtained.The structure of the optimal policy for finite horizon problem turns out to be myopic (it is not easy to tell how much to order at the beginning) which makes the multiple-period case hard to solve explicitly.

Similar to the previously mentioned model, Henig and Gerchak [14] discussed a periodic review model assuming general holding/shortage and production costs in the presence of random yield which is of stochastically proportional yield. It proves that the expected cost per period is convex given that the production, holding/shortage costs are convex in initial stock level and the order quantity. In multiple period problem again the DP formulation is constructed where unsatisfied demand is fully backordered. Under the same assumptions about the cost terms, the objective function is shown to be convex and critical values for each period above which no order is given are obtained. Additionally, the infinite horizon problem is analyzed. The existence of the limit of the expected cost function solving the infinite horizon equivalent of multi-period problem when the number of periods goes to infinity is proven under some assumptions which are sufficient to ensure monotonicity and boundedness. Lastly, they explore some generalizations of production costs depending on the level of inputs (order quantities) as well as the realized yield in addition to the existence of a set-up cost.

Ciarallo, Akella, and Morton [7] discuss a periodic review production planning model with uncertain capacity and uncertain demand. They assume that demand

and capacity for each period are independent and identically distributed random variables (hence stationary over the planning horizon), the holding/shortage costs are linear and stationary also. Actual output in this particular problem amounts to the minimum of the planned production and the uncertain capacity. This may be considered as a random yield problem where there is a probability mass for receiving all of the quantity ordered. Both single period and multiple period problems are analyzed. For single period case, the objective function is shown to be nonconvex but unimodular and it is observed that randomized capacity has no effect on the optimal order policy which is identical to the classical newsboy problem. The cost function is also not convex in multiple period problem but can be shown to be quasi-convex and to have a unique minimum. Optimal policy is found to be of order-up-to type for the multiple period and infinite horizon problem exhibits the same functional form for the cost with the single period problem.

In a study by Wang and Gerchak [21], the variable capacity problem above is extended to a setting with random yield. Again the random capacity, demand, and yield variables are independent and identically distributed and unsatisfied demand is fully backordered. In this case, the actual production is again the minimum of variable capacity and planned production quantity, but the yield of any executed quantity is random. Hence, the actual quantity of usable items is a random fraction of the executed quantity. The random yield is of stochastically proportional yield type. The productions cost is assumed to be proportional to the executed production. Stochastic dynamic programming is used to analyze finite horizon problem and optimal policy at each period is characterized by a single critical level where the objective function is shown to be quasi-convex. In the single period analysis, the reorder point turns out to be unaffected by the distribution of random capacity but depends on the yield rate.Thus, for the single period, optimal policy will be exactly the same as when there is no capacity randomness. They also explore the infinite horizon problem and show that there exists a limit for the objective function and that limit is convex. If the cost function is differentiable, then the reorder point and planned production quantity converge to their limiting values.

In a recent paper by Gurnani, Akella, and Lehoczky [13], an assembly system where the final product which is assembled using two components faces random demand in a single period setting. Suppliers provide random fractions of the order quantities (multiplicative yield) for the two components. An analytically complex exact cost function which is to be minimized is obtained and a modified cost function is introduced so as to determine the combined ordering and production decisions. Conditions under which the difference in the costs is bounded are provided and as a result of the numerical study, it is observed that the percent difference between exact and approximate cost is just 7.7 % in the worst case. It is assumed that shortages are allowed. The performance of the optimal policy is compared with two heuristic policies. In heuristic I, target level is determined for each component type separately without considering the effect of randomness in the supply of the other type, but still ordering and production decisions are made simultaneously. In the second heuristic, ordering and production decisions are made separately. Finally, they consider the case where there is a "joint supplier" from which both components can be ordered in addition to the individual suppliers and derived the conditions under which diversification pays.

Similar to the above study, Gerchak, Wang, and Yano [12] consider an assembly system in a single period setting with stochastically proportional type of yield. Two different models are discussed. The first one assumes components with identical yield distributions and costs, random demand, salvage values, and imperfect assembly stage (where detection of some components' imperfections is only possible after they are assembled). That is, different from the assumptions made by almost all previous researches on assembly systems with random yields, an "assembly yield" problem exists in the first model. Hence, a two stage decision problem is solved where the decision maker selects the lot size for the components and given the number of usable components, quantity to assemble. The results are simplified for a single stage setting where assembly is perfect. In the second model, a single stage system with non-identical component yield and costs is explored. The random fractions for components are not necessarily independent for this case. Concavity of the profit function is proven for the model with zero salvage values, and for the model with two components with independent yields and non-zero salvage values.

Basu and Mukerjee [5] discuss a single period model with random demand and yield, allowing shortages. The random fraction has a known distribution with mean being equal to the order size. For exponentially distributed yield, the optimal order quantity maximizing the profit comes out to be a function of demand distribution. They show that an estimator of maximin order quantity converges in distribution to an appropriate normal law when the sample size characterizing the demand function increases. A similar model is analyzed by Ehrhardt and Taube [8] where yield is of stochastically proportional type. Optimal order quantity minimizing expected cost is a generalization of the standard newsboy problem for the case with no setup cost. In the case of positive setup cost, optimal policy is the random yield analogue of optimal (s,S) policies. It is also found that, simple heuristics that account for the expected value of the replenishment quantity, but not its variability give good results for both uniformly and negative binomially distributed demands.

Anupindi and Akella [1] consider single period and multiple period problems with two suppliers, assuming full backlogging, random and continuous demand. They discuss three different models and lead time becomes a random variable in two of these models. In the first model for supply process, each supplier supplies all of the order quantity with zero lead time with a positive probability and delivers the order quantity next period if there is no delivery in this period. In model II, a random fraction of the order quantity is supplied and the portion that is not delivered is cancelled, which is equivalent to a pure random yield problem. Model III is the same as Model II except that the remaining quantity is to be delivered in the next period. So, uncertain lead times are observed in models I and III. The sum of ordering, holding, and penalty costs are minimized and the optimal policy in a particular period turns out to be characterized by three regions and two critical numbers. That is, it is optimal to order nothing when the on-hand inventory level is larger than u^n $(x > u^n)$, use only one supplier when $v^n < x < u^n$, and order from both suppliers when $x < v^n$. For models II and III, they demonstrate that the order quantities for the suppliers with equal-costs follow a ratio rule (similar to the one obtained by Gerchak and Parlar (1990))when demand is exponential and the supply process is either normal or gamma.

In a study by Baker and Ehrhardt [2] a periodic review model involving random demand, stochastically proportional yield is constructed where backordering is allowed. Rather than performing mathematical analysis, they use simulation to compare the results of the heuristics they propose with the best known (s,S) policy. The logic of the heuristic is to account for the mean of the amount of outstanding orders so that the expected value of the order size matches the deterministic-replenishment order size.

Mazzola, McCoy, and Wagner [16] consider a multi-period lot sizing problem where the production yield is variable according to a binomial probability distribution and demand over the planning horizon is deterministic and dynamic. It is assumed that the lead time is less than one period so that all production in a particular period can be used to satisfy the demand in that period. A setup cost is incurred each time an order is placed, finite production and storage capacities exist, all defective units are discarded with no salvage value, and all stockouts are backordered. A dynamic programming formulation solving to optimality is constructed for the problem and some heuristics are developed. In order to provide a basis for the heuristics, the continuous time version of the original problem is considered where demand is constant, lead time is zero and yield follows a binomial distribution. Using renewal/reward theorem, long-run average cost function is obtained and the optimal values for the quantity to be ordered and the reorder point (less than zero) are obtained. To solve the original problem six heuristics based on the EOQ model solutions are proposed. The first two use (s,S) and (s,Q) decision rules where Q and s are found by using Q^* and i^* found previously. The other heuristics are Wagner Whitin and Silver Meal based solutions of perfect yield version of the original problem multiplied by the reciprocal of the parameter "p" of the binomial distribution. Modified versions of the last two heuristics are also provided and they produce near optimal lot sizing policies for problems with stationary and time varying demands.

Bitran and Dasu [6] consider a multi-item system where the demand is deterministic and dynamic, backordering is allowed, lead time and set-up cost are zero, and higher grade products can be substituted for a lower grade one. The yield is of multiplicative type where "np_i" units of item "i" is actually produced when "n" units are produced (sum of the p_i's is assumed to be less than or equal to one). Two approximation procedures solving finite horizon problems are considered to study the infinite horizon problem for which determining the optimal solution is computationally intractable.

Wang and Gerchak [22] consider a batch production system with due dates allowing backorders. The yield of each batch is random (stochastically proportional) and the production lead time which is independent of the batch size is longer than the time interval between starting consecutive batches. The general model is formulated, but a simplified one which is easier to analyze is constructed where lead time is equal to one period, costs are linear, and production capacity is very large. The optimal policy (minimizing the cost) for the simplified model is characterized by a single critical level (but not order-up-to type) where a new input batch is started if and only if the size of work-in-process batch is less than that critical level.

In a paper published by Gerchak, Tripathy, and Wang [10], a production system with random yields is analyzed in a single period setting where shortages are allowed and demand is random. Higher and lower grade items are produced where the demand for lower grade items can be met by higher grade ones. Hence, the yield is two-fold here: total yield of usable products and the portions of each grade products are uncertain. The profit function is proven to be jointly concave and optimality conditions are driven in the analysis. Another contribution of this study is the possibility of using this solution as a basis for a heuristic approach to the multi-grade problem. Parlar and Perry [17] discuss a (Q, r, T) inventory policy for deterministic and random yields when future supply is uncertain. The lead time is assumed to be zero when the system is ON, that is, the supplier is available. When an order placement is necessary, the state of the supplier can be identified at a fixed cost k_0. There are three decision variables to be optimized which are the reorder point, order quantity, and T, the time to wait before the next order is placed if the first one was made during the OFF state (when the supplier is unavailable). The supplier's availability process is modeled as a two state continuous time Markov Chain consisting of ON and OFF periods for which the durations are assumed to be exponentially distributed. EOQ type model is constructed, there are no planned shortages since reorder point is larger than zero. But all demands occurring when when the system is out of stock are backordered. A fixed cost per unit backordered and a variable cost per unit linear in the length of time for which backorders continue are used. Also, T is supposed to be the same regardless of the inventory level. In addition to the deterministic yield, they also analyze the problem when the amount delivered is random where the yield is a "general function" of the quantity ordered. Expected cost in a cycle is found by conditioning on the state of CTMC when inventory level reaches the reorder point.

Bar-Lev, Parlar, and Perry [3] consider an EOQ model with inventory level-dependent demand rate and random yield which is of stochastically proportional type. Replenishment is instantaneous and no backorders are allowed (the reorder point is taken to be equal to zero). Using level crossing theory, an analysis of the stationary distribution of the inventory level is provided and the long-run average cost function is minimized. Three special cases are considered: standard EOQ model, EOQ model with random yield, and EOQ model modified to incorporate inventory level dependent demand rate. Explicit formulas for the expected cycle length, stationary distribution of the inventory level are given for the general case where the demand rate is a power function of the inventory level ($\lambda(x) = ax^b$ for $a > 0$ and $0 < b < 1$) and yield rate is a beta random variable.

In a study by Zhang and Gerchak [24] a model where a random proportion of units are defective is explored. The environment they use is that of classical EOQ model with no backlogging. The defective items can be identified through costly inspections where inspection costs are assumed to be linear. Two different models are analyzed. In the first one, the only penalty for uninspected defectives is financial in the first one; and defective units cannot be used and must be replaced by non-defective ones. Two levels of uncertainty exist in this particular problem: the percentage of defectives in a lot and the number of defectives in the inspected sub-lot. For a given defective percentage for the entire lot, number of defective items in a sample is a random variable having hypergeometric distribution. Therefore, both the quantity to be ordered and the fraction to inspect have to be optimized. Expected cost function per cycle is obtained. Due to the complexity in the structure of the objective function, the joint determination of f (fraction) and Q is difficult. Hence, some approximations are made in order to obtain explicit expressions. They provide a solution procedure (exhaustive search) to find optimal Q given the optimal value of "f". They also discuss the model with replacement of defective items for the immediate replacement case. They report that the optimal inspection fraction is either zero or one in most applications.

Gerchak and Parlar [9] consider an EOQ model with no backordering, zero lead time, and stochastically proportional yield (non-negative and unbounded random variable) for one supplier. They analyze a model where the decision maker has the option to play with the variation in the yield. They discuss two models where the mean value for the yield variable is fixed but the variance (σ^2) can be changed. In the first model, the cost associated with decreasing the standard deviation is incurred at each order regardless of its size, replacing the commonly fixed setup cost. The variable cost per item which is independent of variance (σ^2) is not included in the analysis. The cost rate function is obtained which is proven to be convex in Q and σ separately, and convex at the unique solution of the necessary conditions for some particular (power) form of cost of changing σ. It is shown that Q is decreasing in σ and optimal yield variability is attained when the relative rate of change in the ordering cost $c(\sigma)$ equals the relative rate of change in the second moment of Y_Q/Q. In the second model, the cost of changing yield variability is per unit ordered and assumed to be convex. We have K (additional) in this case. They also discuss where diversification between two suppliers is beneficial. An ordering cost K is incurred each time an order is given and two sources are assumed to charge the same price which is not a realistic assumption. They obtain explicit expressions for Q_1^*, Q_2^* and find a relation between the optimal values of Q_1 and Q_2. They also analyze the conditions for which it is profitable to order from both suppliers and find that diversification does not pay if $K \geq K_1 + K_2$(K_i is the ordering cost when supplier i is used only). Lastly, the optimal number of identical sources having identical yield distributions and pricing policies is found.

Parlar and Wang [18] extended the above model assuming that the suppliers charge different prices per unit and holding costs incurred for items purchased from the two suppliers also differ. The amount paid (purchasing cost) depends on the amount received, not the amount ordered (pay for output) in their model. In an EOQ model with no backlogging, the long-run average cost function is shown to be convex for a wide range of parameter values. They again find conditions where diversification is advantageous. Additionally, a single period problem in which demand is a random variable is also analyzed. It is assumed that there is a salvage value for unsold items at the end of period. Concavity of the expected profit function is shown. It is shown that it is impossible to obtain closed form solutions for the optimal order quantities. By the help of the concavity of the objective function, an approximate solution requiring the solution of a system of two linear equations and the performance of the approximation is measured. It is observed that the model produces reasonably low errors.

An inventory model where raw material supply and demand for finished goods are random is considered by Bassok and Akella [4]. There is a limited production capacity and backordering is allowed in their model. The distribution of the random fraction depends on the order quantity. That is, if the order quantity is between $\underline{b_i}$ and $\overline{b_i}$, then the density distribution of random yield is $g_i(.)$, where arrival process of raw material can be in one of "n" states. The optimal solution is the one with the minimum cost among "n" different problems. They also consider multi-item extension of the same model.

To summarize, stochastically proportional yield model is used extensively in the literature. Both pay-for-input and pay-for-output models are considered.

Periodic review models are more often used compared to continuous review models. (s,S) type policies are shown to be optimal for most of these models. A table (1.1) including the most relevant studies is given at the end of this chapter.

1.3 Motivation

In this thesis, we discuss an inventory problem under continuous review where the demand rate is assumed to be a constant. The problem is analyzed under an EOQ setting. The purchasing costs (c_1, c_2) for the products are different for the two suppliers For the purchasing cost, we preferred the "pay-for-input" model, where you pay for the amount that you order, not the amount that is actually received. The analysis can be simply modified for the pay-for-output type purchasing policy also, by just adjusting the selling prices for the two suppliers (multiply them by the expected values of the random fractions). The ordering cost K is same regardless of which supplier(s) is(are) used and the holding costs per unit per time are also assumed to be equal for both suppliers (the analysis can be easily extended for different ordering costs when just one supplier is used).

We incur the same holding cost for both suppliers' products, since if we had assumed different holding costs per unit per time, the analysis would be much more difficult in terms of finding average inventory level. The average inventory level would depend on the time when each item is sold. But, model with different holding costs can be handled by solving two different problems for the suppliers by assuming a constant demand rate $\frac{D}{2}$ for each supplier, as in the paper of Parlar and Wang [18]. Different from the model by Parlar and Wang [18], shortages are allowed since it is profitable to take advantage of backordering if there is not a significant difference between backlogging and holding costs. The shortage and holding costs that we used in numerical study allows us to backorder the unsatisfied demand (full backlogging is assumed). Also, the replenishment is instantaneous (lead time is negligible). The control policy used is as follows: When the inventory on-hand hits the reorder point, the retailer orders from the suppliers.

One needs to decide how much to order from both suppliers and when to order. As a result, there is an additional decision variable that is not considered mostly in the literature, the reorder point. Naturally, if the perfect yield case is considered, the problem turns out to be very simple; just order from the supplier offering less selling price. But, when random yield is present for the suppliers, the decision is not that simple because you are to make your decisions under uncertainty. In our model, we consider two models with different types of random yield, binomial yield and stochastically proportional yield. In the first one, each item produced can be either good or bad with some fixed probability. The probability of producing a good item is different for the two suppliers. We expect to observe a higher probability of producing a good unit for the product with higher selling price. Consequently, the number of good units in a lot is binomially distributed. This type of modeling is appropriate for the firms producing goods which have tight quality constraints, leading to a significant fraction of the lot size to be considered as defective items. Mazzola, McCoy, and Wagner [16] considers this type of yield uncertainty assuming continuity throughout their paper. Different from their model, we obtain the exact cost function taking into account the fact that there is a positive probability of not increasing inventory level to a positive value when the orders arrive. Also, two suppliers with different yield levels and selling prices per unit compete to get the market share in our setting where they had only one supplier. We obtain a simple analytical formula showing where diversification is advantageous that provides important managerial implications especially for the suppliers side in terms of the market share.

In the second yield model, we assume that a random fraction, independent of the lot size, of the quantity ordered is received. The suppliers are assumed to have known yield distributions which are independent for each supplier. This type of yield model is appropriate when the capacity of the supplier is random due to stoppages, strikes, machine breakdowns, etc. In addition to the variability in the production capacity, the supplier may face random demand from more than one retailer. In this case, it has to allocate its random capacity to each retailer. So, the uncertainty in the yield is two-fold here: variable capacity and proportion of the capacity allocated to a particular retailer.

It is observed that the behavior of the inventory level starts repeating itself at the beginning of each cycle (time interval between the arrival of consecutive orders). Hence, the exact long-run average cost function is obtained using renewal/reward theorem. Since, there is a probability of not increasing the inventory level to a positive value, determination of the optimal values of decision variables analytically, using first order conditions, is very hard. For that reason, an algorithm to obtain optimal values is proposed. The probability of not increasing the inventory level to a positive value is positive, is assumed to be equal to a constant at each iteration. The algorithm proceeds till the convergence in the optimal values is attained. The convexity of the cost function is proven for some particular combinations of parameter values and the regions where diversification among suppliers pays are determined.

The rest of the thesis is organized as follows: In Chapter 2 the assumptions, parameters, decision variables, and the optimal policy are introduced. Chapters 3 and 4 focus on deriving the optimal values and analytical properties of the expected total cost rate of the model for binomial yield and stochastically proportional yield, respectively. In Chapter 5, we present numerical results over a wide range of parameter settings for the two random yield types separately. Also, we measure the performance of the algorithm proposed, by comparing the results that the algorithm provides to the real optimal values. Finally, in Chapter 6, we conclude the study by summarizing our findings, and identifying possible future research venues.

Reference	Yield Structure	# of Supp.	Demand	Backorder		Review Policy		Lead Time		Purchasing Policy	
				Yes	No	Cont.	Per.	0	>0	PI	PO
Anupindi and Akella [1]	spt.	2	Random	√			√	√	√		√
Bar-Lev, Parlar, and Perry [3]	spt.	1	Inv. dependent rate		√	√		√			
Bassok and Akella [4]	spt.(depends on Q)	1	Random	√			√	√			√
Bitran and Dasu [6]	spt.	1	Deterministic	√			√	√		√	
Ciarallo, Akella, and Morton [7]	min(var. capacity,Q)	1	Random	√			√	√			
Ehrhardt and Taube [8]	spt.	1	Random	√			√	√			√
Gerchak and Vickson and Parlar [11]	spt.	1	Random	√			√	√			√
Gurnani and Akella [13]	spt.	1	Random	√			√	√			√
Mazzola, McCoy, and Wagner [16]	Binomial	1	Deterministic	√			√		√	√	
Parlar and Perry [17]	spt.	1	Deterministic and Random	√		√		√	√		
Parlar and Wang [18]	spt.	2	Constant		√	√		√			√
Shih [20]	spt.	1	Constant and Random	√		√	√	√			
Wang and Gerchak [21]	u*min(var. capacity,Q)	1	Random	√			√	√		√	
Zhang and Gerchak [24]	spt.	1	Constant		√	√		√		√	

Table 1.1: Summary of literature review where "spt" stands for stochastically proportional to, "u" is random fraction, "PI" is pay-for-input, and "PO" is pay-for-output

Chapter 2

The Model

We consider an inventory system where the manager has the option of ordering from two different suppliers facing random yield. That is, they supply a random fraction of the quantity ordered. In this work, two types of random yield, binomial yield and stochastically proportional yield, are considered.At the end of this chapter, the notation used throughout the analysis is given in Table A.1 in the Appendix. The following assumptions are made in the model:

• The purchasing prices are different for each supplier,

• A fixed ordering cost K is incurred when an order is placed regardless of which supplier(s) used (the model can easily be extended to the one where this cost is differentiated between suppliers),

• Backlogging is allowed,

• Replenishment is instantaneous (lead time is zero),

• Same holding cost is incurred for the items,

• Demand rate is constant,

• The yield distributions are independent from each other for the suppliers and they are stationary, i.e. the parameters of the distributions do not change over time,

• The good items produced (after taking the yield into consideration) by each supplier are of the same quality,

• The system is reviewed continuously.

Decisions as to when and how much to order are given at some predetermined points in time defined by the reorder point. After the order arrives the process starts repeating itself. The cycle is defined as the time between these regeneration points. Therefore it is appropriate to use reward/renewal theorem for this problem. Using the reward/renewal theorem, the expected cost rate (cost per unit time), which is the expected total cost divided by the expected cycle time is found by constructing an EOQ type model.

The decision variables and parameters of the model are the following:

Decision variables:

Q_1 : quantity ordered from supplier 1

Q_2 : quantity ordered from supplier 2

i : reorder point that triggers the placement of an order ($i < 0$)

Parameters:

c_H : holding cost per unit per time

c_S : shortage cost per unit per time

K : ordering cost

c_1 : purchasing cost of an item from supplier 1

c_2 : purchasing cost of an item from supplier 2

D : constant demand rate

In this chapter, the control policy, expected holding, backordering, procurement cost figures and cycle time expressions are given. There are three decision variables in the model. The reorder point, which is the level that triggers the orders, is the first decision variable. Other decision variables, Q_j for $j = 1, 2$ are the quantities ordered from each supplier.

Cycle: The time between the arrival of the consecutive orders is defined as a cycle as illustrated in Figure 1.

Control Policy: An order is placed for both suppliers (Q_1, Q_2) when inventory level hits i ($i < 0$).

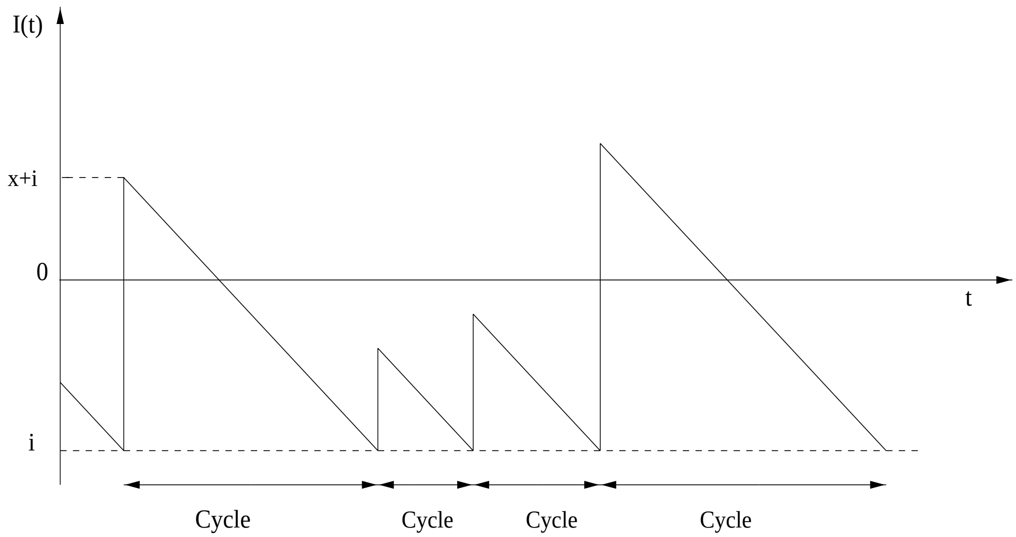

Figure 2.1: Behavior of the inventory level with constant demand rate

2.1 The Objective Function

To find out the cost rate, we first need to compute the expected ordering and procurement, holding, backordering costs per cycle. The quantities that are actually received from the suppliers are defined as follows:

X^1 : amount actually received from supplier 1

X^2 : amount actually received from supplier 2

X : total amount actually received $(X^1 + X^2)$

Since the amount actually received is random, it is not certain that the inventory level increases to a positive value after the arrival of the order. Therefore, we may face cycles where we incur holding cost and where the holding cost is zero. That is, the inventory level may be greater than zero (all backorders cleared) or inventory level may be negative after the shipment is received.

P_i : probability that the amount received is smaller than $'' - i''$:

$P_i = P(X < -i) = P(X^1 + X^2 < -i)$

Each time an order is placed, there is a Bernoulli trial taking place. The inventory level either increases to a positive level with probability $1 - P_i$ or stays negative with probability P_i.

2.1.1 Computation of Expected Cycle Cost

Each time an order is placed, the procurement cost incurred is as follows:

$$E[\text{Procurement Cost}] = K + c_1Q_1 + c_2Q_2 \tag{2.1}$$

We incur holding costs when the inventory level is above zero. To find the expected holding cost expression, we need to define a new random variable (conditional on the amount that is actually received). Inventory level becomes positive only when the amount received from the suppliers is greater than the magnitude of the reorder level. Hence, we need to consider x as if x is greater than $-i$. Also for the backordering cost, a similar reasoning works. Backordering cost expression incurred during cycles in which inventory level is always negative, we need to define another random variable (again conditional on the amount that is actually received), since we assume that the suppliers provide less than we expect such that the inventory level is not enough to clear all the backorders and to have excess inventory. Similarly, we need to consider x as if x is less than $-i$ for this case. Therefore, the conditional random variables and their expectations should be used in the analysis.

The expected holding cost per cycle can be found by computing the area (above x-axis) under the inventory level curve in Figure 2.1. Therefore, the expected holding cost per cycle is found as follows (where HC denotes holding cost):

$$\begin{aligned}
\text{If } X < -i, \text{ then } HC &= 0 \\
\text{If } X > -i, \text{ then } HC &= E[\frac{c_H}{2D}((XI(X > -i)) + i)^2] \\
&= \frac{c_H}{2D}(E[(XI(X > -i))^2|Q_1, Q_2] + i^2 \\
&+ 2iE[XI(X > -i)|Q_1, Q_2])
\end{aligned}$$

Then taking expectation over X yields:

$$\begin{aligned}
E[HC] &= 0P_i + (1 - P_i)\frac{c_H}{2D}(E[(XI(X > -i))^2|Q_1, Q_2] \\
&+ 2iE[XI(X > -i)|Q_1, Q_2] + i^2)
\end{aligned} \tag{2.2}$$

where,

$$E[X|X > -i] = \frac{E[XI(X > -i)]}{P(x > -i)} = \frac{E[XI(X > -i)]}{(1 - P_i)}$$

Note that I is the indicator function, where $I(X > -i) = 1$ if $X > -i$ and zero otherwise.

Expected backordering cost is also found with the same method used in deriving the holding cost expression (BC denotes backordering cost):

$$\begin{aligned} \text{If } X > -i, \text{ then } BC &= \frac{c_S i^2}{2D} \\ \text{If } X < -i, \text{ then } BC &= \frac{-c_S}{2D} E[(XI(X < -i))^2 + 2iXI(X < -i)] \end{aligned}$$

Then taking expectation over X yields:

$$E[BC] = \frac{c_S i^2}{2D}(1 - P_i) + P_i(\frac{-c_S}{2D} E[(XI(X < -i))^2 + 2iXI(X < -i)]) \quad (2.3)$$

where,

$$E[X|X < -i] = \frac{E[XI(X < -i)]}{P(x < -i)} = \frac{E[XI(X < -i)]}{P_i}$$

Consequently, the expected total cost per cycle will be as follows:

$$\begin{aligned} E[\text{TC}] &= K + c_1 Q_1 + c_2 Q_2 + (\frac{c_H + c_s}{2D}) i^2 (1 - P_i) \\ &+ \frac{c_H}{2D}(E[(XI(X > -i))^2] + 2iE[XI(X > -i)])(1 - P_i) \\ &- \frac{c_S P_i}{2D}(E[(XI(X < -i))^2] + 2iE[XI(X < -i)]) \quad (2.4) \end{aligned}$$

Since, $E[XI(X > -i)]$ can be written in terms of $E[x]$ and $E[XI(X < -i)]$, we can get rid of the term $E[XI(X > -i)]$ in the total cost per cycle expression. The following identities are used for this purpose:

$$E[X^2|X > -i] = \frac{E[X^2 I(X > -i)]}{(1 - P_i)} \text{ and}$$

$$E[X^2|X < -i] = \frac{E[X^2 I(X < -i)]}{P_i} \text{ and}$$

$$E[X^2] = E[X^2 I(X < -i)] + E[X^2 I(X > -i)]$$

Then the sum of the expected holding and backordering cost is rewritten as follows:

$$\begin{aligned} E[HC]+E[BC] &= (\frac{c_H+c_S}{2D})i^2(1-P_i) \\ &+ (1-P_i)\frac{c_H}{2D}[\frac{E[X^2I(X>-i)]}{(1-P_i)}+2i\frac{E[XI(X>-i)]}{(1-P_i)}] \\ &- P_i\frac{c_S}{2D}[\frac{E[X^2I(X<-i)]}{P_i}-2i\frac{E[XI(X<-i)]}{P_i}] \end{aligned}$$

$$\begin{aligned} E[HC]+E[BC] &= (\frac{c_H+c_s}{2D})i^2(1-P_i)+\frac{c_H}{2D}[E[X^2]+2iE[X]] \\ &- \frac{c_H+c_S}{2D}[E[X^2I(X<-i)]+2iE[XI(X<-i)]] \end{aligned}$$

As a result, the new form of the expected total cost per cycle (excluding the term $E[XI(X>-i)]$) is as follows:

$$\begin{aligned} E[\text{TC}] &= K+c_1Q_1+c_2Q_2+(\frac{c_H+c_s}{2D})i^2(1-P_i) \\ &+ \frac{c_H(E[X^2]+2iE[X])}{2D} \\ &- \frac{(c_H+c_S)[E[X^2I(X<-i)]+2iE[XI(X<-i)]]}{2D} \end{aligned} \tag{2.5}$$

2.1.2 Expected Cycle Time

After finding the expected total cost per cycle, the expected cycle time must be also found. The expected cycle time is found by conditioning on the amount that is actually received (T denotes cycle time):

$$\begin{aligned} \text{If } X>-i, \text{ then } T &= \frac{E[XI(X>-i)]}{D} \\ \text{If } X<-i, \text{ then } T &= \frac{E[XI(X<-i)]}{D} \end{aligned}$$

Taking expectation over X (treating P_i as a constant given i,Q_1, and Q_2) yields:

$$\begin{aligned} E[T] &= P_i(\frac{E[XI(X<-i)]}{DP_i})+(1-P_i)\frac{E[XI(X>-i)]}{D(1-P_i)} \\ &= \frac{E[XI(X<-i)]+E[XI(X>-i)]}{D}=\frac{E[X]}{D} \end{aligned} \tag{2.6}$$

2.1.3 Expected Cost Rate

Cost rate, which is the total cost per cycle divided by the cycle time, is the function that is to be minimized. Starting at the regeneration points, the process shows the same behavior. Cycle times and costs per cycles are independent and identically distributed. Hence, the long-run average cost is just the expected cost incurred during a cycle divided by the expected cycle length (see Ross [19], page 318). Then,

$$\text{Cost Rate} = \text{CR} = \lim_{t\to\infty} \frac{TotalCost}{t} = \frac{E[\text{Total Cost Per Cycle}]}{E[\text{Cycle Length}]}$$

$$CR = \frac{E[TC]}{E[T]} = \frac{K + c_1Q_1 + c_2Q_2 + E[HC] + E[BC]}{\frac{E[X]}{D}} \tag{2.7}$$

2.1.4 Approximate Expected Cost Rate

To find the minimum cost rate, we need to construct the first order conditions. As it is observed, P_i, $E[XI(X < -i)]$, and $E[X^2I(X < -i)]$ are dependent on the decision variables. The partial derivatives of these expressions with respect to Q_1, Q_2, and i are very complex. The structure of the above expressions do not allow us to find expressions involving just one decision variable,i.e. the decision variables cannot be separated from each other using first order conditions. As a result, it seems impossible to come up with closed form formulas giving the optimal values for the decision variables. Thus, we define a new cost rate called approximate expected cost rate. In this new cost rate function, the cycle time is still the same (since we do not have P_i, $E[XI(X < -i)]$, and $E[X^2I(X < -i)]$ in the cycle time expression), but the total cost is modified. For the approximate cost rate function, we assume that P_i, $E[XI(X < -i)]$, and $E[X^2I(X < -i)]$ do not depend on the decision variables. So, they are taken to be as constants in this new cost rate. The following notation is used:

$$P_i = \alpha \tag{2.8}$$

$$E[XI(X < -i)] = m_1 \tag{2.9}$$

$$E[X^2I(X < -i)] = m_2 \tag{2.10}$$

Let the new expected total cost per cycle be TC^a, then:

$$
\begin{aligned}
E[TC^a] &= K + c_1Q_1 + c_2Q_2 + (\frac{c_H + c_s}{2D})i^2(1-\alpha) \\
&+ \frac{c_H(E[X^2] + 2iE[X])}{2D} \\
&- \frac{(c_H + c_S)(m_2 + 2im_1)}{2D}
\end{aligned} \tag{2.11}
$$

Then, the approximate expected cost rate is the following:

$$
CR^a = \frac{E[TC^a]}{E[T]} \tag{2.12}
$$

2.2 An Iterative Solution Procedure

Giving the optimal procurement decision requires the minimization of the expected cost rate function with respect to three decision variables i, Q_1, and Q_2. In the cost rate expression we have the P_i, $E[XI(X < -i)]$, and $E[(XI(X < -i))^2]$ terms, which are also functions of the decision variables above. In order to find the expressions above, we need to use the sum of two different random variables both of which are independent and identically distributed. So, P_i, $E[XI(X < -i)]$, and $E[(XI(X < -i))^2]$ are found by using conditional probabilities and expectations. In the following part, we condition on X^2 assuming that X^2 (which is the actual amount received from supplier 2) is equal to a value y where $y \in [0, Q_2]$. As a result, the following are obtained by conditioning on X^2:

$$
\begin{aligned}
P_i = P(X^1 + X^2 < -i) &= E_{X^2}[P(X^1 < -i - y|X^2 = y)] \\
E[XI(X < -i)] &= E_{X^2}[E[XI(X^1 < -i - y)]|X^2 = y)] \\
E[X^2I(X < -i)] &= E_{X^2}[E[X^2I(X^1 < -i - y)]|X^2 = y)]
\end{aligned} \tag{2.13}
$$

An algorithm which uses an iterative solution procedure that will be discussed in detail below, is proposed (first order conditions of approximate expected cost rate function is used for this algorithm). Throughout the algorithm, it is assumed that the dependence of P_i, $E[XI(X < -i)]$, and $E[(XI(X < -i))^2]$ on decision variables is small enough to neglect the partial derivatives with respect to $Q_1, Q_2,$ and$i$. In other words, these expressions are treated as constants in the algorithmic solution procedure. We expect the algorithm to work properly for the realizations where the change in these expressions due to changes in the values of the decision variables are small enough. But, especially for $E[XI(X < -i)]$ and $E[X^2 I(X < -i)]$, when they take larger values, our approximation may not always work well.

The stopping point is the point where the convergence in long-run average cost is attained (however,it is not guaranteed that we will obtain convergence), and the following is the algorithm that is used:

2.2.1 Algorithm

1. Find the exact cost rate (CR) assuming $\alpha = 0$, $m_1 = 0$, and $m_2 = 0$.
2. Setup the first order conditions for CR^a

2.1 Find the optimal values for i, Q_1, Q_2 for CR^a

2.2 Assign $Q_1^0 = Q_1, Q_2^0 = Q_2$ and $i^0 = i$

2.3 Using Q_1^0, Q_2^0 and i^0 compute P_i, $E[XI(X < -i)]$, and $E[(XI(X < -i))^2]$ via equation 2.13.

2.4 Assign $\alpha^{new} = P_i$, $m_1^{new} = E[XI(X < -i)]$, and $m_2^{new} = E[(XI(X < -i))^2]$.

3. Find the new exact cost rate with the new values of α, m_1, and m_2

3.1 Setup the first order conditions for the new approximate cost rate

3.2 Find the new optimal values for i, Q_1, Q_2 for CR^a

3.3 Assign $Q_1^{new} = Q_1, Q_2^{new} = Q_2$ and $i^{new} = i$

3.4 Compute P_i, $E[XI(X < -i)]$, and $E[(XI(X < -i))^2]$ using $Q_1^{new} = Q_1, Q_2^{new} = Q_2$ and $i^{new} = i$

3.5 Repeat Step 2.4

4. Compute $|CR^{new} - CR^{old}|$

4.1 If the value found in step 4 is smaller than a predetermined constant ϵ, go to the next step

4.1.1 Assign $Q_1^* = Q_1^{new}, Q_2^* = Q_2^{new}$, and $i^* = i^{new}$ and stop.

4.2 Else, go to step 3.

Chapter 3

Binomial Yield

For this type of yield model, the following notation is used:

p_1 : probability of producing a good unit for supplier 1

p_2 : probability of producing a good unit for supplier 2

Therefore, for the case where the random yield is assumed to have binomial distribution, each unit is supplied instantaneously with a probability of p_j by supplier j ($j = 1, 2$) and with a probability of $1 - p_j$ the unit does not reach the customers. Therefore the number of units that are supplied have the following binomial distribution:

$$P(x^j = k|Q_j) = \begin{pmatrix} Q_j \\ k \end{pmatrix} (p_j)^k (1 - p_j)^{Q_j - k}$$

Hence, the expected amount actually received from the two suppliers and the second moment of the same quantity comes out to be the following:

$$E[x^1 + x^2|Q_1, Q_2] = E[x|Q_1, Q_2] = p_1 Q_1 + p_2 Q_2$$

$$\begin{aligned} E[(x^1 + x^2)^2|Q_1, Q_2] &= E[(x)^2|Q_1, Q_2] = p_1(1 - p_1)Q_1 + p_2(1 - p_2)Q_2 \\ &+ (p_1 Q_1 + p_2 Q_2)^2 \end{aligned}$$

Note that the quantities here are discrete but we are making a continuity assumption throughout the analysis.

The expected total cost per cycle and cycle time become the following for this particular yield model:

$$\begin{aligned} E[\text{TC}] &= K + c_1Q_1 + c_2Q_2 + (\frac{c_H + c_s}{2D})i^2(1 - P_i) \\ &+ \frac{c_H(p_1(1-p_1)Q_1 + p_2(1-p_2)Q_2 + (p_1Q_1 + p_2Q_2)^2 + 2i(p_1Q_1 + p_2Q_2))}{2D} \\ &- \frac{(c_H + c_S)[E[X^2I(X < -i)] + 2iE[XI(X < -i)]]}{2D} \end{aligned} \quad (3.1)$$

$$E[T] = \frac{E[x]}{D} = \frac{p_1Q_1 + p_2Q_2}{D} \quad (3.2)$$

At the very beginning of the iterative solution procedure,P_i, $E[XI(X < -i)]$, and $E[X^2I(X < -i)]$ are assigned zero and corresponding optimal values of decision variables are computed. Then, using Q_1^*, Q_2^*, and i^* new values of the expressions above are found as follows for this yield model:

$$\begin{aligned} P_i &= P(x^1 + x^2 < -i) = E_{x^2}[P(x^1 < -i - k | x^2 = k)] \\ &= \sum_{k=0}^{Q_2}[\sum_{x^1=0}^{-i-k} \begin{pmatrix} Q_1 \\ x^1 \end{pmatrix} (p_1)^{x^1}(1-p_1)^{Q_1 - x^1}] \begin{pmatrix} Q_2 \\ k \end{pmatrix} (p_2)^k(1-p_2)^{Q_2-k} \end{aligned}$$

$$\begin{aligned} E &\quad [XI(X < -i)] = E_{x^2}[E[XI(x^1 < -i - k)] | x^2 = k)] \\ &= \sum_{k=0}^{Q_2}[\sum_{x^1=0}^{-i-k} (x^1 + x^2) \begin{pmatrix} Q_1 \\ x^1 \end{pmatrix} (p_1)^{x^1}(1-p_1)^{Q_1 - x^1}] \begin{pmatrix} Q_2 \\ k \end{pmatrix} (p_2)^k(1-p_2)^{Q_2-k} \end{aligned}$$

$$\begin{aligned} E &\quad [X^2I(X < -i)] = E_{x^2}[E[X^2I(x^1 < -i - k)] | x^2 = k)] \\ &= \sum_{k=0}^{Q_2}[\sum_{x^1=0}^{-i-k} (x^1 + x^2)^2 \begin{pmatrix} Q_1 \\ x^1 \end{pmatrix} (p_1)^{x^1}(1-p_1)^{Q_1 - x^1}] \begin{pmatrix} Q_2 \\ k \end{pmatrix} (p_2)^k(1-p_2)^{Q_2-k} \end{aligned}$$

3.1 Analytical Properties of Approximate Objective Function

In order to use the first order conditions to find the optimal values of decision variables, the long-run average cost function must be convex, since the problem is a minimization problem. Firstly, we need show that the objective function is strictly convex either for the whole space or just for some particular parameter sets.When the function is convex, the optimal values occur at the points where the first partial derivatives are equal to zero. We are also guaranteed that these values are global optimums.

In the following part, we are going to analyze the analytical properties of the cost rate with respect to each decision variable given the values of the other two decision variables(recall that this analysis is done under the assumption that P_i, $E[XI(X<-i)]$, and $E[(XI(X<-i))^2]$ are constant with respect to the decision variables). The second order partial derivatives will be found for this purpose. For the function to be convex, the sign of the second order derivative must be positive. Firstly, the convexity of the function with respect to the reorder level for given values of Q_1 and Q_2 is investigated :

Lemma 3.1: *The cost rate function is convex with respect to the reorder point "i" for given Q_1 and Q_2.*

Proof:

$$\frac{\partial^2 CR}{\partial i^2} = \frac{(\frac{\partial^2 E[TC]}{\partial i^2}E[T] - E[TC]\frac{\partial^2 E[T]}{\partial i^2})E[T] - (\frac{\partial E[TC]}{\partial i}E[T] - E[TC]\frac{\partial E[T]}{\partial i})2\frac{\partial E[T]}{\partial i}}{(E[T])^3}$$

Since the first and second order partial derivatives of the expected cycle time with respect to the reorder point are both equal to zero, the second order derivative of the expected cost rate reduces to the following:

$$\frac{\partial^2 CR}{\partial i^2} = ((E[T])^2)\frac{\partial^2 E[TC]}{\partial i^2} = \frac{c_H + c_S}{D}$$

The expression above is always positive, so Lemma 3.1 is proven. □

Lemma 3.2: *Approximate cost rate (CR^a) is convex with respect to Q_1 for Q_2, and i, and with respect to Q_2 for Q_1, and i iff the following inequalities hold, respectively:*

$$\begin{aligned}(c_H \quad &+ \quad c_S)p_1 i(\frac{i(1-\alpha)}{2} - m_1) - (\frac{c_H + c_S}{2})p_1 m_2 \\ &+ \quad Q_2[D(c_2 p_1 - c_1 p_2) + \frac{c_H}{2}p_1 p_2(p_1 - p_2)] + KDp_1 > 0\end{aligned}$$

$$\begin{aligned}(c_H &+ c_S)p_2 i(\frac{i(1-\alpha)}{2} - m_1) - (\frac{c_H + c_S}{2})p_2 m_2 \\ &+ Q_1[D(c_1 p_2 - c_2 p_1) + \frac{c_H}{2} p_1 p_2 (p_2 - p_1)] + KDp_2 > 0\end{aligned}$$

Proof:

The second order partial derivative with respect to Q_1 (similar for Q_2) is the following:

$$\frac{\partial^2 CR^a}{\partial Q_1^2} = \frac{(\frac{\partial^2 E[TC^a]}{\partial Q_1^2}E[T] - E[TC^a]\frac{\partial^2 E[T]}{\partial Q_1^2})E[T] - (\frac{\partial E[TC^a]}{\partial Q_1}E[T] - E[TC^a]\frac{\partial E[T]}{\partial Q_1})2\frac{\partial E[T]}{\partial Q_1}}{(E[T])^3}$$

Since the expected cycle time ($E[T]$) is always positive, the sign of the second order derivative depends on the following expression:

$$(\frac{\partial^2 E[TC^a]}{\partial Q_1^2}E[T] - E[TC^a]\frac{\partial^2 E[T]}{\partial Q_1^2})E[T] - (\frac{\partial E[TC^a]}{\partial Q_1}E[T] - E[TC^a]\frac{\partial E[T]}{\partial Q_1})2\frac{\partial E[T]}{\partial Q_1}$$

We have the same expressions for Q_2 except that Q_1's are replaced by Q_2.

- *For* Q_1*:*

$$\frac{\partial E[T]}{\partial Q_1} = \frac{p_1}{D}, \text{ and } \frac{\partial^2 E[T]}{\partial Q_1^2} = 0$$

$$\begin{aligned}\frac{\partial E[TC^a]}{\partial Q_1} &= c_1 + \frac{c_H}{2D}(p_1(1-p_1) + 2p_1(p_1 Q_1 + p_2 Q_2) + 2ip_1) \\ &\Rightarrow \frac{\partial^2 E[TC^a]}{\partial Q_1^2} = \frac{c_H p_1^2}{D}\end{aligned}$$

After some algebraic simplifications, the expression indicating the sign of the second order derivative turns out to be the following one:

$$\begin{aligned}(c_H &+ c_S)p_1 i(\frac{i(1-\alpha)}{2} - m_1) - (\frac{c_H + c_S}{2})p_1 m_2 \\ &+ Q_2[D(c_2 p_1 - c_1 p_2) + \frac{c_H}{2} p_1 p_2 (p_1 - p_2)] + KDp_1\end{aligned}$$

- *For* Q_2*:*

$$\frac{\partial E[T]}{\partial Q_2} = \frac{p_2}{D}, \text{ and } \frac{\partial^2 E[T]}{\partial Q_2^2} = 0$$

$$\frac{\partial E[TC^a]}{\partial Q_2} = c_2 + \frac{c_H}{2D}(p_2(1-p_2) + 2p_2(p_1Q_1 + p_2Q_2) + 2ip_2)$$
$$\Rightarrow \frac{\partial^2 E[TC^a]}{\partial Q_2^2} = \frac{c_H p_2^2}{D}$$

Again the expression indicating the sign of the second order derivative reduces to the following one for Q_2:

$$(c_H + c_S)p_2 i(\frac{i(1-\alpha)}{2} - m_1) - (\frac{c_H + c_S}{2})p_2 m_2 + Q_1[D(c_1p_2 - c_2p_1) + \frac{c_H}{2}p_1p_2(p_2 - p_1)] + KDp_2$$

As a result Lemma 3.2 is proven. □

Proposition 3.1: *The condition that* $Q_2[D(c_2p_1 - c_1p_2) + \frac{c_H}{2}p_1p_2(p_1 - p_2)] - (\frac{c_H+c_S}{2})p_1m_2 \geq 0$, *and* $Q_1[D(c_1p_2 - c_2p_1) + \frac{c_H}{2}p_1p_2(p_2 - p_1)] - (\frac{c_H+c_S}{2})p_2m_2 \geq 0$ *is a sufficient but not a necessary condition for the approximate cost rate function* (CR^a) *to be convex with respect to* Q_1 *given the values for* i *and* Q_2, *and with respect to* Q_2 *for given values of* Q_1, i, *respectively.*

Proof:

The first two terms in the expression indicating the sign of the second order partial derivative are always positive. $i(\frac{i(1-\alpha)}{2} - m_1)$ is also positive since i is less than zero and m_1 is always greater than 0. As a result if the last two terms are positive we are guaranteed that the cost rate function is convex and Proposition 3.1 is proven. □

3.2 Optimization

For the regions where the approximate expected cost rate (CR^a) is convex, the approximate first order partial derivatives are taken to find the near optimal values of decision variables. Then, the first order conditions are used to find the relations among i, Q_1, and Q_2. The first order partial derivatives are as follows:

Approximate F.O.C. for "i":

$$\text{Since} \qquad \frac{\partial E[T]}{\partial i} = 0 \Rightarrow$$

$$\frac{\partial CR^a}{\partial i^*} = 0 \Rightarrow E[T]\frac{\partial E[TC^a]}{\partial i^*} = E[TC^a]\frac{\partial E[T]}{\partial i^*} \Rightarrow \frac{\partial E[TC^a]}{\partial i^*} = 0$$

$$\Rightarrow \quad (\frac{c_H + c_S}{D})i^*(1-\alpha) + \frac{c_H(p_1Q_1 + p_2Q_2) - (c_H + c_S)m_1}{D} = 0$$

$$\Rightarrow \quad i^* = \frac{(c_H + c_S)m_1 - c_H(p_1Q_1 + p_2Q_2)}{(c_H + c_S)(1-\alpha)} \tag{3.3}$$

Approximate F.O.C. for "Q_1":

$$\begin{aligned}\frac{\partial CR^a}{\partial Q_1^*} &= 0 \Rightarrow (p_1Q_1^* + p_2Q_2)[c_1 + \frac{c_H(p_1(1-p_1) + 2(p_1Q_1^* + p_2Q_2)p_1 + 2ip_1)}{2D}] \\ &= E[TC^a]p_1 \Rightarrow\end{aligned}$$

$$\begin{aligned}(\quad & p_1Q_1^* + p_2Q_2)[Dc_1 + \frac{c_Hp_1(1-p_1)}{2}] = p_1[D(K + c_1Q_1^* + c_2Q_2) \\ + \quad & (\frac{c_H + c_S}{2})(1-\alpha)i^2 + \frac{c_H(p_1(1-p_1)Q_1^* + p_2(1-p_2)Q_2 - (p_1Q_1^* + p_2Q_2)^2)}{2} \\ - \quad & \frac{(c_H + c_S)(m_2 + 2im_1)}{2}]\end{aligned} \tag{3.4}$$

Approximate F.O.C. for "Q_2":

$$\begin{aligned}\frac{\partial CR^a}{\partial Q_2^*} &= 0 \Rightarrow (p_1Q_1 + p_2Q_2^*)[c_2 + \frac{c_H(p_2(1-p_2) + 2(p_1Q_1 + p_2Q_2^*)p_2 + 2ip_2)}{2D}] \\ &= E[TC^a]p_2 \Rightarrow\end{aligned}$$

$$\begin{aligned}(\quad & p_1Q_1 + p_2Q_2^*)[Dc_2 + \frac{c_Hp_2(1-p_2)}{2}] = p_2[D(K + c_1Q_1 + c_2Q_2^*) \\ + \quad & (\frac{c_H + c_S}{2})(1-\alpha)i^2 + \frac{c_H(p_1(1-p_1)Q_1 + p_2(1-p_2)Q_2^* - (p_1Q_1 + p_2Q_2^*)^2)}{2} \\ - \quad & \frac{(c_H + c_S)(m_2 + 2im_1)}{2}]\end{aligned} \tag{3.5}$$

3.2.1 A Marginal Analysis

The relation between i^*, Q_1^*, and Q_2^* is found from (3.3) and the relation between Q_1^*, Q_2^* is obtained by equating (3.4) and (3.5) as follows:

$$i^* = \frac{(c_H + c_S)m_1 - c_H(p_1 Q_1^* + p_2 Q_2^*)}{(c_H + c_S)(1-\alpha)} \tag{3.6}$$

Equating (3.4) and (3.5) yields:

$$p_2[Dc_1 + \frac{c_H p_1(1-p_1)}{2}] = p_1[Dc_2 + \frac{c_H p_2(1-p_2)}{2}]$$

$$\Rightarrow c_2 p_1 - c_1 p_2 + \frac{c_H}{2D} p_1 p_2 (p_1 - p_2) = 0 \tag{3.7}$$

It is an interesting result that the equation above involves only parameters. So, there are two cases to be considered. First case is the one where we have the optimal solution at a point where the first derivative is equal to zero (i.e. equation (3.7) holds). For the second case (where equation (3.7) does not hold), the optimal value occurs at the boundaries. Note that, when you divide the expression above by $p_1 p_2$, it is observed that the decision as to which supplier should be used is given by comparing the effective unit selling prices and unit holding cost. This aspect will be discussed in more detail at the end of this section. Also note that, the selection of the supplier does not depend on the unit shortage cost per time, since the expected amount to receive is the same leading to the fact that the reorder point is the same regardless of which supplier you order from. The analytical reasoning of the above explanation will be given later. Below, both cases are discussed in detail:

Lemma 3.3: *If $c_2 p_1 - c_1 p_2 + \frac{c_H}{2D} p_1 p_2 (p_1 - p_2) = 0$ holds, then the optimal values of Q_1 and Q_2 for CR^a will be any pair (Q_1, Q_2) satisfying the following equation:*

Let $Q_G^* = p_1 Q_1^* + p_2 Q_2^*$ be defined as the expected amount of good units

$$(p_1 Q_1^* + p_2 Q_2^*) = Q_G^* = \sqrt{\frac{(c_H + c_S)(\frac{(m_1)^2}{1-\alpha} + m_2) - 2KD}{c_H(\frac{(c_H+c_S)\alpha - c_S}{(c_H+c_S)(1-\alpha)})}} \tag{3.8}$$

Proof:

Using Equation (3.4) and writing i^* in terms of Q_1^* and Q_2^* yields the following equation when (3.7) holds:

$$\begin{aligned}(\quad & c_H + c_S)\frac{(\frac{(m_1)^2}{1-\alpha} + m_2)}{2} - KD \\ = \quad & \frac{c_H((c_H + c_S)\alpha - c_S)((p_1 Q_1^* + p_2 Q_2^*)^2)}{2(c_H + c_S)(1-\alpha)}\end{aligned}$$

It is inferred that, the sum of expected amount to receive from both suppliers should be equal to a constant value (recall that α, m_1, and m_2 are considered as constants in the algorithm). Also it follows that any pair (Q_1, Q_2) satisfying equation (3.8) is a solution to the problem for which the proof is discussed below:

Keeping $(p_1 Q_1^* + p_2 Q_2^*)$ the same, when we increase Q_1 and decrease Q_2 accordingly, the expected cycle time will remain the same. Only the approximate expected total cost (TC^a) may be affected from this substitution from Q_2 to Q_1. So, the change in approximate expected total cost (TC^a) will reflect the change in the approximate cost rate (CR^a). Below, the change in TC^a is investigated:

Suppose we increase Q_1 by Δ, that is, $Q_1 \to Q_1 + \Delta$,then since $p_1 Q_1 + p_2 Q_2$ should remain the same,the new value of Q_2 becomes, $Q_2 \to Q_2 - \frac{p_1}{p_2}\Delta$

Let the new approximate expected total cost per cycle be $E^{new}[TC^a]$. We are going to look at the difference in the approximate expected total cost per cycle, $\Delta E[TC^a]$, which is equal to $E^{new}[TC^a] - E[TC^a]$:

$$\begin{aligned}\Delta E[TC^a] &= c_1\Delta - c_2\frac{p_1}{p_2}\Delta + \Delta\frac{c_H}{2D}p_1(p_2 - p_1) \\ &= \Delta[c_1 p_2 - c_2 p_1 + \frac{c_H}{2D}p_1 p_2(p_2 - p_1)] = 0\end{aligned}$$

Consequently, for Case 1, increasing the amount of Q_1 or Q_2 while keeping $p_1 Q_1 + p_2 Q_2$ constant will not change the approximate expected cost rate (CR^a). Hence, any pair satisfying equation (3.8) will give the solution to our problem. $\square$.

Lemma 3.4: *If* $c_2 p_1 - c_1 p_2 + \frac{c_H}{2D}p_1 p_2(p_1 - p_2) > 0$, *then use Supplier 1 only,* *otherwise use just Supplier 2.*

Proof:

When the equation obtained from the first order conditions does not hold, we have the following situation(obtained from the partial derivatives):
$Q_2^* p_2 = -Q_1^* p_1$, which is not possible for positive values of the order quantities. Hence,the optimal should occur at the boundaries. So, either Q_1 or Q_2 is equal to zero. Two cases need to be considered:
Since the minimum will occur at the boundaries, we are going to look at the cost rates when $Q_1 = 0$ and $Q_2 = 0$.

Case 1: When $Q_2 = 0$, we have the following:

$$\begin{aligned} E[TC^a] &= K + c_1 Q_1 + (\frac{c_H + c_s}{2D})(1-\alpha)i^2 \\ &+ \frac{c_H(p_1(1-p_1)Q_1 + (p_1 Q_1)^2 + 2ip_1 Q_1)}{2D} \\ &- \frac{(c_H + c_S)[m_2 + 2im_1]}{2D} \end{aligned}$$

$$E[T] = \frac{p_1 Q_1}{D}$$

Let the cost rate be CR_1^a in this realization. Now, we need to find the optimal value of Q_1 minimizing cost rate 1 (CR_1^a):

$$\frac{\partial CR_1^a}{\partial Q_1^*} = 0 \Rightarrow \frac{\partial E[TC^a]}{\partial Q_1^*} E[T] = E[TC^a] \frac{\partial E[T]}{\partial Q_1^*}$$

$$\begin{aligned} KD &- \frac{(c_H + c_S)}{2}(\frac{(m_1)^2}{1-\alpha} + m_2) \\ &= p_1^2 (Q_1^*)^2 \frac{c_H}{2}(\frac{c_S - (c_H + c_S)\alpha}{(c_H + c_S)(1-\alpha)}) \Rightarrow \end{aligned}$$

$$p_1 Q_1^* = Q_G^* \tag{3.9}$$

Case 2: When $Q_1 = 0$, we have the following:

$$\begin{aligned} E[TC^a] &= K + c_2 Q_2 + (\frac{c_H + c_s}{2D})(1-\alpha)i^2 \\ &+ \frac{c_H(p_2(1-p_2)Q_2 + (p_2Q_2)^2 + 2ip_2Q_2)}{2D} \\ &- \frac{(c_H + c_S)[m_2 + 2im_1]}{2D} \end{aligned}$$

$$E[T] = \frac{p_2 Q_2}{D}$$

Let the cost rate be CR_2^a in this realization. Now, we need to find the optimal value of Q_2 minimizing cost rate 2 (CR_2^a):

$$\frac{\partial CR_2^a}{\partial Q_2^*} = 0 \Rightarrow \frac{\partial E[TC^a]}{\partial Q_2^*} E[T] = E[TC^a] \frac{\partial E[T]}{\partial Q_2^*}$$

$$\begin{aligned} KD &- \frac{(c_H + c_S)}{2}(\frac{(m_1)^2}{1-\alpha} + m_2) \\ &= p_2^2 (Q_2^*)^2 \frac{c_H}{2}(\frac{c_S - (c_H + c_S)\alpha}{(c_H + c_S)(1-\alpha)}) \Rightarrow \end{aligned}$$

$$p_2 Q_2^* = Q_G^* \tag{3.10}$$

It is observed from equations (3.9) and (3.10) that the values $p_1 Q_1^*$ and $p_2 Q_2^*$ are equal. So, the same rule applies in Case 2 as in Case 1, that is, the amount that we expect to receive is equal to a constant value. Since we are going to use only one supplier, we need to look at the difference of the cost rates and choose the one with the minimum cost rate at the optimal values of Q_1 and Q_2. Since $p_1 Q_1^*$ and $p_2 Q_2^*$ are equal, it is possible to compare the expected total costs instead of the cost rates. Following notation is used:

TC_1^a=Expected total cost when $Q_2 = 0$
TC_2^a=Expected total cost when $Q_1 = 0$

In the following part the identity $Q_1^* = \frac{p_2}{p_1} Q_2^*$ is used. We need to look at the difference of the expected total costs at Q_1^* and Q_2^* in order to decide whether supplier 1 or 2 must be used. After some simplifications, the difference of expected total costs per cycle turns out to be:

$$\begin{aligned} TC_1^a - TC_2^a &= c_1 Q_1 - c_2 Q_2 + \frac{c_H}{2D} p_1 Q_1 (p_2 - p_1) \Rightarrow \\ TC_1^a - TC_2^a &= Q_2 [c_1 \frac{p_2}{p_1} - c_2 + \frac{c_H}{2D} p_2 (p_2 - p_1)] \Rightarrow \\ TC_1^a - TC_2^a &= \frac{Q_2}{p_1} [c_1 p_2 - c_2 p_1 + \frac{c_H}{2D} p_1 p_2 (p_2 - p_1)] \end{aligned}$$

As observed, when $c_2 p_1 - c_1 p_2 + \frac{c_H}{2D} p_1 p_2 (p_1 - p_2) > 0$, TC_1^a is less than TC_2^a. Therefore it would be less costly to use supplier 1. On the other hand, if $c_2 p_1 - c_1 p_2 + \frac{c_H}{2D} p_1 p_2 (p_1 - p_2) < 0$, TC_2^a is less than TC_1^a. Therefore it would be less costly to use supplier 2.So, Lemma 3.4 is proven. □

Using the Lemmas proved above, the following theorem is constructed:

Theorem 3.1:

i) If $c_2 p_1 - c_1 p_2 + \frac{c_H}{2D} p_1 p_2 (p_1 - p_2) = 0$, *any pair* (Q_1^*, Q_2^*) *satisfying the following equation is an optimal solution for* CR^a*:*

$$(p_1 Q_1^* + p_2 Q_2^*) = \sqrt{\frac{(c_H + c_S)(\frac{(m_1)^2}{1-\alpha} + m_2) - 2KD}{c_H(\frac{(c_H + c_S)\alpha - c_S}{(c_H + c_S)(1-\alpha)})}} = Q_G^*$$

and the optimal value of reorder point is given as:

$$i^* = \frac{m_1}{(1-\alpha)} - \frac{c_H Q_G^*}{(c_H + c_S)(1-\alpha)}$$

ii) If $c_2 p_1 - c_1 p_2 + \frac{c_H}{2D} p_1 p_2 (p_1 - p_2) > 0$, *order from supplier 1 only, and the optimal values for* CR^a *are as follows:*

$$\begin{aligned} Q_2^* &= 0 \\ Q_1^* &= \frac{Q_G^*}{p_1} \\ i^* &= \frac{m_1}{(1-\alpha)} - \frac{c_H Q_G^*}{(c_H + c_S)(1-\alpha)} \end{aligned}$$

iii) If $c_2p_1 - c_1p_2 + \frac{c_H}{2D}p_1p_2(p_1 - p_2) < 0$, *order from supplier 2 only, and the optimal values for* CR^a *are as follows:*

$$
\begin{aligned}
Q_1^* &= 0 \\
Q_2^* &= \frac{Q_G^*}{p_2} \\
i^* &= \frac{m_1}{(1-\alpha)} - \frac{c_H Q_G^*}{(c_H + c_S)(1-\alpha)}
\end{aligned}
$$

Proof:

Proofs are given separately in the lemmas above. □

3.2.2 Initial Solution of the Algorithm

In the analysis done up to this point, we assumed that α, m_1, and m_2 are constants. But, we first need to assign values to the expressions above. Therefore, we started with assigning 0 to these values. Plugging 0 in place of these expressions, into the formulas giving the relation between the optimal decision variables and the values of i^*, Q_1^*, and Q_2^* for CR^a yields the following:

$$
i^* = (\frac{-c_H}{c_H + c_S})(p_1Q_1^* + p_2Q_2^*) \tag{3.11}
$$

The equation of parameters $c_2p_1 - c_1p_2 + \frac{c_H}{2D}p_1p_2(p_1 - p_2)$ that is obtained before is also valid for the case where $\alpha = m_1 = m_2 = 0$. As a result, the following is constructed by just using 0 instead of P_i, $E[XI(X < -i)]$, and $E[X^2I(X < -i)]$:

Corollary 3.1:

i) If $c_2p_1 - c_1p_2 + \frac{c_H}{2D}p_1p_2(p_1 - p_2) = 0$, *any pair* (Q_1^*, Q_2^*) *satisfying the following equation is an optimal solution for* CR^a *when* $P_i = E[XI(X < -i)] = E[X^2I(X < -i)] = 0$:

$$
\begin{aligned}
(p_1Q_1^* + p_2Q_2^*) &= \sqrt{\frac{2KD(c_S + c_H)}{c_Sc_H}} \\
i^* &= -\sqrt{\frac{2KDc_H}{(c_S + c_H)c_S}}
\end{aligned}
$$

ii) If $c_2p_1 - c_1p_2 + \frac{c_H}{2D}p_1p_2(p_1 - p_2) > 0$*, order from supplier 1 only, and the optimal values for* CR^a *when* $P_i = E[XI(X < -i)] = E[X^2I(X < -i)] = 0$ *are as follows:*

$$
\begin{aligned}
Q_2^* &= 0 \\
Q_1^* &= \sqrt{\frac{2KD(c_H + c_S)}{c_H c_S p_1^2}} \\
i^* &= -\sqrt{\frac{2KDc_H}{(c_H + c_S)c_S}}
\end{aligned}
$$

iii) If $c_2p_1 - c_1p_2 + \frac{c_H}{2D}p_1p_2(p_1 - p_2) < 0$*, order from supplier 2 only, and the optimal values for* CR^a *when* $P_i = E[XI(X < -i)] = E[X^2I(X < -i)] = 0$ *are as follows:*

$$
\begin{aligned}
Q_1^* &= 0 \\
Q_2^* &= \sqrt{\frac{2KD(c_H + c_S)}{c_H c_S p_2^2}} \\
i^* &= -\sqrt{\frac{2KDc_H}{(c_H + c_S)c_S}}
\end{aligned}
$$

The first values of the optimal decision variables for CR^a will be the ones just given above. Using these values P_i, $E[XI(X < -i)]$, and $E[X^2I(X < -i)]$ are computed. Then, Theorem 3.1 is used to find the new values of i^*, Q_1^*, and Q_2^*. This process continues until convergence in total cost is attained. The physical implication of the theorem is discussed below:

Dividing the expression (involving parameters in Theorem 3.1) by p_1p_2 yields $\frac{c_2}{p_2} - \frac{c_1}{p_1} - \frac{c_h}{2D}(p_2 - p_1)$.The sum of the first two items can be regarded as the additional purchasing cost of moving one item from Q_1 to Q_2. $\frac{c_h}{2D}(p_2 - p_1)$ can be thought of as the additional expected holding cost of moving one item from Q_1 to Q_2. When the sum of the first two items is greater than the last item, intuitively, it is more profitable to order from supplier 1 (as Theorem 3.1 suggests) in this case, because the purchasing cost contributes to the total cost more than the holding cost in general.

3.3 Generalization to "M" Suppliers

In this setting, more than two suppliers produce the same item of the same quality. However, the selling prices and the probability of producing a good item are different for each supplier. The retailer takes advantage of reducing uncertainty on the amount actually received by order-splitting among suppliers. The following is the expected total cost per cycle and expected cycle time for this particular case:

$$\begin{aligned} E[\text{Total Cost}] &= K + c_1Q_1 + \ldots + c_MQ_M + (\frac{c_H + c_s}{2D})i^2(1 - P_i) \\ &+ \frac{c_H(E[X^2|Q_1, \ldots, Q_M] + 2iE[X|Q_1, \ldots, Q_M])}{2D} \\ &- \frac{(c_H + c_S)[E[X^2 I(X < -i)] + 2iE[XI(X < -i)]]}{2D} \end{aligned}$$

$$E[T] = \frac{E[X|Q_1, \ldots, Q_M]}{D}$$

where the expected value and the second moment of the amount actually received are as follows:

$$E[x^1 + \ldots + x^M|Q_1, \ldots, Q_M] = E[x|Q_1, \ldots, Q_M] = p_1Q_1 + p_2Q_2 + \ldots + p_MQ_M$$

$$\begin{aligned} E[(x)^2|Q_1, \ldots, Q_M] &= p_1(1 - p_1)Q_1 + p_2(1 - p_2)Q_2 + \ldots + p_M(1 - p_M)Q_M \\ &+ (p_1Q_1 + p_2Q_2 + \ldots + p_MQ_M)^2 \end{aligned}$$

Again, the long-run average cost will be minimized. Following are the approximate first order conditions for the reorder point and the order quantities:

For i:

$$\frac{\partial CR^a}{\partial i^*} = 0 \Rightarrow E[T]\frac{\partial E[TC^a]}{\partial i^*} = E[TC^a]\frac{\partial E[T]}{\partial i^*} \Rightarrow \frac{\partial E[TC^a]}{\partial i^*} = 0$$

$$\Rightarrow (\frac{c_H + c_S}{D})i^*(1 - \alpha) + \frac{c_H(p_1Q_1 + \ldots + p_MQ_M) - (c_H + c_S)m_1}{D} = 0$$

$$\Rightarrow i^* = \frac{(c_H + c_S)m_1 - c_H(p_1Q_1 + \ldots + p_MQ_M)}{(c_H + c_S)(1 - \alpha)} \tag{3.12}$$

For Q_1:

$$
\begin{aligned}
(& \; p_1Q_1^* + \ldots + p_MQ_M)(Dc_1 + \frac{c_H}{2}p_1(1-p_1)) = p_1[KD + Dc_1Q_1^* + \ldots + Dc_MQ_M \\
+ & \; (\frac{c_H + c_S}{2})(1-\alpha)i^2 + \frac{c_H}{2}(p_1(1-p_1)Q_1^* + \ldots + p_M(1-p_M)Q_M \\
- & \; (p_1Q_1^* + \ldots + p_MQ_M)^2)] - (c_H + c_S)(\frac{m_2}{2} + im_1)
\end{aligned}
$$

The first order conditions for $Q_2, \ldots, Q_M$ are also taken. Equating the first order conditions, we obtain equations of parameters below, that was also the case for the setting with two suppliers.

$$
\begin{aligned}
p_1(Dc_2 + \frac{c_H}{2}p_2(1-p_2)) &= p_2(Dc_1 + \frac{c_H}{2}p_1(1-p_1)) \\
p_1(Dc_3 + \frac{c_H}{2}p_3(1-p_3)) &= p_3(Dc_1 + \frac{c_H}{2}p_1(1-p_1)) \\
. \quad . & \quad\quad . \\
. \quad . & \quad\quad . \\
p_1(Dc_M + \frac{c_H}{2}p_M(1-p_M)) &= p_M(Dc_1 + \frac{c_H}{2}p_1(1-p_1))
\end{aligned}
$$

If "M" equations above hold simultaneously, we obtain the following result using the first order condition for Q_1 for CR^a:

$$
p_1Q_1^* + \ldots + p_MQ_M^* = \sqrt{\frac{(c_H + c_S)(\frac{(m_1)^2}{1-\alpha} + m_2) - 2KD}{c_H(\frac{(c_H+c_S)\alpha - c_S}{(c_H+c_S)(1-\alpha)})}}
$$

Note that the expression giving the expected amount to receive $(p_1Q_1^* + \ldots + p_MQ_M^*)$ is the same with the one found for the problem with two suppliers. Similar to the argument in the 2-suppliers problem, any pair $(Q_1^*, \ldots, Q_M^*)$ satisfying the equation above is a solution when "M" equations of parameters hold simultaneously. Decision maker will then decide how to split among suppliers considering the contracts with the suppliers. After finding the optimal values for decision variables, the value of P_i is computed accordingly and the algorithm continues.

When the equations of parameters do not hold, the optimal values occur at the boundaries. Order splitting is not profitable in this case, so the decision maker needs to decide which supplier to use. For choosing the right supplier, s/he needs to compare the suppliers employing the following method which was also used in 2-suppliers case:

Use supplier j if:

$$c_j p_k - c_k p_j + \frac{c_H}{2D} p_k p_j (p_k - p_j) > 0 \quad \text{for} \quad \forall k \neq j. \tag{3.13}$$

One has to make $M-1$ such comparisons to give the final decision, and the optimal order quantity for CR^a will be the following:

$$\begin{aligned} Q_j^* &= \sqrt{\frac{(c_H + c_S)(\frac{(m_1)^2}{1-\alpha} + m_2) - 2KD}{c_H(\frac{(c_H+c_S)\alpha - c_S}{(c_H+c_S)(1-\alpha)})(p_j)^2}} \\ Q_k^* &= 0 \text{ for } k \neq j \\ i^* &= \frac{m_1}{(1-\alpha)} - \sqrt{\frac{c_H((c_H + c_S)(m_2 + \frac{(m_1)^2}{(1-\alpha)}) - 2KD)}{(c_H + c_S)(1-\alpha)((c_H + c_S)\alpha - c_S)}} \end{aligned}$$

We may face a situation in which the equations of parameters hold for a subset of the M suppliers. Suppose that the subset is denoted by S where cardinality of S is l where the whole set is denoted by E. If we have:

$$c_j p_k - c_k p_j + \frac{c_H}{2D} p_k p_j (p_k - p_j) = 0 \text{ for } \forall k, j \in S.$$

Then the order quantities will be any combination $(Q_{S_1}^*, \ldots, Q_{S_l}^*)$ satisfying the following:

$$\begin{aligned} (p_{S_1} Q_{S_1}^* + \ldots + p_{S_l} Q_{S_l}^*) &= \sqrt{\frac{(c_H + c_S)(\frac{(m_1)^2}{1-\alpha} + m_2) - 2KD}{c_H(\frac{(c_H+c_S)\alpha - c_S}{(c_H+c_S)(1-\alpha)})}} \\ Q_j^* &= 0 \text{ for } \forall j \in E - S \end{aligned}$$

The order quantity given above is splitted among the suppliers in any way the decision maker chooses.

Chapter 4

Stochastically Proportional Yield

In the "stochastically proportional to yield" case, a random fraction,which is between zero and one, of the order quantity is received. The two suppliers have different distributions for this random fraction and the amounts received from the two suppliers are independent from each other. In this case in addition to quality problems, the suppliers may face random demand from their customers and the capacity of the suppliers are random. The means and the variances are different from each other for two suppliers. The following are the additional notation that will be used in this chapter:

u	:	random fraction for supplier 1 with density function $f(u)$
μ_1	:	mean of the random variable u
σ_1^2	:	variance of the random variable u
v	:	random fraction for supplier 2 with density function $f(v)$
μ_2	:	mean of the random variable v
σ_2^2	:	variance of the random variable v

The amount that is actually received is modeled as follows:

$x^1 = uQ_1$ and $x^2 = vQ_2$

Hence, the expected amount actually received from the two suppliers and the second moment of the same quantity comes out to be the following:

$$E[x^1 + x^2 | Q_1, Q_2] = E[x | Q_1, Q_2] = \mu_1 Q_1 + \mu_2 Q_2$$

$$E[(x^1 + x^2)^2 | Q_1, Q_2] = E[(x)^2 | Q_1, Q_2] = \sigma_1^2 Q_1^2 + \sigma_2^2 Q_2^2 + (\mu_1 Q_1 + \mu_2 Q_2)^2$$

The expected total cost per cycle and cycle time become the following for this particular yield model:

$$\begin{aligned} E[\text{Total Cost}] &= K + c_1 Q_1 + c_2 Q_2 + (\frac{c_H + c_s}{2D})(1 - P_i) i^2 \\ &+ \frac{c_H(\sigma_1^2 Q_1^2 + \sigma_2^2 Q_2^2 + (\mu_1 Q_1 + \mu_2 Q_2)^2 + 2i(\mu_1 Q_1 + \mu_2 Q_2))}{2D} \\ &- \frac{(c_H + c_S)[E[X^2 I(X < -i)] + 2iE[XI(X < -i)]]}{2D} \end{aligned} \quad (4.1)$$

$$E[T] = \frac{E[x]}{D} = \frac{\mu_1 Q_1 + \mu_2 Q_2}{D} \quad (4.2)$$

Using the following formulas new values for P_i, $E[XI(X < -i)]$, and $E[X^2 I(X < -i)]$ are computed:

- If $\frac{-i}{Q_2} \le 1$ and $\frac{-i}{Q_1} \le 1$, then

$$P_i = \int_{v=0}^{\frac{-i}{Q_2}} \int_{u=0}^{\frac{-i - vQ_2}{Q_1}} f(u) g(v) du dv$$

- If $\frac{-i}{Q_2} \le 1$ and $\frac{-i}{Q_1} > 1$, then

$$P_i = \int_{v=0}^{\frac{-i-Q_1}{Q_2}} \int_{u=0}^{1} f(u) g(v) du dv + \int_{v=\frac{-i-Q_1}{Q_2}}^{\frac{-i}{Q_2}} \int_{u=0}^{\frac{-i-vQ_2}{Q_1}} f(u) g(v) du dv$$

- If $\frac{-i}{Q_2} > 1$ and $\frac{-i}{Q_1} \le 1$, then

$$P_i = \int_{v=0}^{1} \int_{u=0}^{\frac{-i-vQ_2}{Q_1}} f(u) g(v) du dv$$

- If $\frac{-i}{Q_2} > 1$ and $\frac{-i}{Q_1} > 1$, then

$$P_i = \int_{v=0}^{\frac{-i-Q_1}{Q_2}} \int_{u=0}^{1} f(u) g(v) du dv + \int_{v=\frac{-i-Q_1}{Q_2}}^{1} \int_{u=0}^{\frac{-i-vQ_2}{Q_1}} f(u) g(v) du dv$$

The expected value and the second moment of the random variable X^- are computed by the same method defined above. We just replace $f(u)$ by $(uQ_1 + vQ_2) f(u)$ to find $E[XI(X < -i)]$, and $f(u)$ by $(uQ_1 + vQ_2)^2 f(u)$ to find $E[X^2 I(X < -i)]$.

4.1 Analytical Properties of the Approximate Objective Function

In order to use the first order conditions to find the optimal values of decision variables, the long-run average cost function must be convex, since the problem is a minimization problem. Therefore, firstly we need to show that the objective function is strictly convex either for the whole space or just for some particular parameter sets.

When the function is convex, the optimal values occur at the points where the first partial derivatives are equal to zero. We are also guaranteed that these values are global optimums.

In the following part, we are going to analyze the analytical properties of the cost rate with respect to each decision variable given the values for the other two decision variables(just as in the binomial case, this analysis is done under the assumption that P_i, $E[XI(X < -i)]$, and $E[(XI(X < -i))^2]$ are constant in the iterative solution procedure). The second order partial derivatives will be found for this purpose. For the function to be convex, the sign of the second order derivative must be positive.Firstly, the convexity of the function with respect to the reorder level for given values of Q_1 and Q_2 is investigated :

For i**:**

$$\frac{\partial^2 CR}{\partial i^2} = \frac{(\frac{\partial^2 E[TC]}{\partial i^2}E[T] - E[TC]\frac{\partial^2 E[T]}{\partial i^2})E[T] - (\frac{\partial E[TC]}{\partial i}E[T] - E[TC]\frac{\partial E[T]}{\partial i})2\frac{\partial E[T]}{\partial i}}{(E[T])^3}$$

Lemma 4.1: *The cost rate function is convex with respect to the reorder point "i" given the values of the order quantities.*

Proof:

Since the first and second order partial derivatives of the expected cycle time with respect to the reorder point are both equal to zero, the second order derivative reduces to the following:

$$\frac{\partial^2 CR}{\partial i^2} = ((E[T])^2)\frac{\partial^2 E[TC]}{\partial i^2} = \frac{c_H + c_S}{D}$$

The expression above is always positive, so Lemma 4.1 is proven. □

Lemma 4.2: *Approximate cost rate (CR^a) is convex with respect to Q_1 given Q_2, and i, and with respect to Q_2 given Q_1, and i iff the following inequalities hold, respectively:*

$$\begin{aligned} 2KD(\mu_1)^2 &+ (c_H + c_S)(\mu_1)^2 i((1-\alpha)i - 2m_1) + c_H Q_2^2(\sigma_1^2(\mu_2)^2 + \sigma_2^2(\mu_1)^2) \\ &+ 2D\mu_1 Q_2(c_2\mu_1 - c_1\mu_2) - (c_H + c_S)(\mu_1)^2 m_2 > 0 \end{aligned}$$

$$\begin{aligned} 2KD(\mu_2)^2 &+ (c_H + c_S)(\mu_2)^2 i((1-\alpha)i - 2m_1) + c_H Q_1^2(\sigma_1^2(\mu_2)^2 + \sigma_2^2(\mu_1)^2) \\ &+ 2D\mu_2 Q_1(c_1\mu_2 - c_2\mu_1) - (c_H + c_S)(\mu_2)^2 m_2 > 0 \end{aligned}$$

Proof:

The second order partial derivative with respect to Q_1 (similar for Q_2) is the following:

$$\frac{\partial^2 CR^a}{\partial Q_1^2} = \frac{(\frac{\partial^2 E[TC^a]}{\partial Q_1^2}E[T] - E[TC^a]\frac{\partial^2 E[T]}{\partial Q_1^2})E[T] - (\frac{\partial E[TC^a]}{\partial Q_1}E[T] - E[TC^a]\frac{\partial E[T]}{\partial Q_1})2\frac{\partial E[T]}{\partial Q_1}}{(E[T])^3}$$

Since the expected cycle time ($E[T]$) is always positive, the sign of the second order derivative depends on the following expression:

$$(\frac{\partial^2 E[TC^a]}{\partial Q_1^2}E[T] - E[TC^a]\frac{\partial^2 E[T]}{\partial Q_1^2})E[T] - (\frac{\partial E[TC^a]}{\partial Q_1}E[T] - E[TC^a]\frac{\partial E[T]}{\partial Q_1})2\frac{\partial E[T]}{\partial Q_1}$$

We have the same expressions for Q_2 except that Q_1's are replaced by Q_2.

- *For Q_1:*

$$\frac{\partial E[T]}{\partial Q_1} = \frac{\mu_1}{D}, \text{ and } \frac{\partial^2 E[T]}{\partial Q_1^2} = 0$$

$$\begin{aligned} \frac{\partial E[TC^a]}{\partial Q_1} &= c_1 + \frac{c_H}{2D}(2\sigma_1^2 Q_1 + 2\mu_1(\mu_1 Q_1 + \mu_2 Q_2) + 2i\mu_1) \\ &\Rightarrow \frac{\partial^2 E[TC^a]}{\partial Q_1^2} = \frac{c_H(\sigma_1^2 + (\mu_1)^2)}{D} \end{aligned}$$

After some algebraic simplifications, the expression indicating the sign of the second order derivative turns out to be the following one:

$$\begin{aligned} 2KD(\mu_1)^2 &+ (c_H + c_S)(\mu_1)^2 i((1-\alpha)i - 2m_1) + c_H Q_2^2(\sigma_1^2(\mu_2)^2 + \sigma_2^2(\mu_1)^2) \\ &+ 2D\mu_1 Q_2(c_2\mu_1 - c_1\mu_2) - (c_H + c_S)(\mu_1)^2 m_2 \end{aligned}$$

- *For Q_2:*

$$\frac{\partial E[T]}{\partial Q_2} = \frac{\mu_2}{D}, \text{ and } \frac{\partial^2 E[T]}{\partial Q_2^2} = 0$$

$$\begin{aligned} \frac{\partial E[TC^a]}{\partial Q_2} &= c_2 + \frac{c_H}{2D}(2\sigma_2^2 Q_2 + 2\mu_2(\mu_1 Q_1 + \mu_2 Q_2) + 2i\mu_2) \\ &\Rightarrow \frac{\partial^2 E[TC^a]}{\partial Q_2^2} = \frac{c_H(\sigma_2^2 + (\mu_2)^2)}{D} \end{aligned}$$

Again the expression indicating the sign of the second order derivative reduces to the following one for Q_2:

$$\begin{aligned} 2KD(\mu_2)^2 &+ (c_H + c_S)(\mu_2)^2 i((1-\alpha)i - 2m_1) + c_H Q_1^2(\sigma_1^2(\mu_2)^2 + \sigma_2^2(\mu_1)^2) \\ &+ 2D\mu_2 Q_1(c_1\mu_2 - c_2\mu_1) - (c_H + c_S)(\mu_2)^2 m_2 \end{aligned}$$

Consequently, Lemma 4.2 is proven. □.

Proposition 4.1: *The condition that $2DQ_2(c_2\mu_1 - c_1\mu_2) - (c_H + c_S)\mu_1 m_2 \geq 0$, and $2DQ_1(c_2\mu_1 - c_1\mu_2) - (c_H + c_S)\mu_2 m_2 \geq 0$ is a sufficient but not necessary condition for the approximate cost rate function (CR^a)to be convex with respect to Q_1 given the values for i and Q_2, and with respect to Q_2 for given values of Q_1, i.*

Proof:

The first three terms in the expression indicating the sign of the second order partial derivative are always positive since i is less than 0, and $m_1 > 0$, and dividing the last two terms by μ_1 (by μ_2 for the condition for Q_2) gives the conditions above. As a result Proposition 4.1 is proven. □

4.2 Optimization

For the regions in which the approximate cost rate is convex, the first partial derivatives are taken to find the optimal values of decision variables. Then, the first order conditions are used to find the relations among i, Q_1, and Q_2. The partial derivatives are taken below:

Approximate F.O.C. for "i":

$$\text{Since} \qquad \frac{\partial E[T]}{\partial i} = 0$$

$$\frac{\partial CR^a}{\partial i^*} = 0 \Rightarrow E[T]\frac{\partial E[TC^a]}{\partial i^*} = E[TC^a]\frac{\partial E[T]}{\partial i^*} \Rightarrow \frac{\partial E[TC^a]}{\partial i^*} = 0$$

$$\Rightarrow \quad (\frac{c_H + c_S}{D})(1-\alpha)i^* + \frac{c_H(\mu_1 Q_1 + \mu_2 Q_2) - (c_H + c_S)m_1}{D} = 0$$

$$\Rightarrow \quad i^* = \frac{(c_H + c_S)m_1 - c_H(\mu_1 Q_1 + \mu_2 Q_2)}{(c_H + c_S)(1-\alpha)} \tag{4.3}$$

Approximate F.O.C. for "Q_1":

$$\frac{\partial CR^a}{\partial Q_1^*} = 0 \quad \Rightarrow \quad (\mu_1 Q_1^* + \mu_2 Q_2)[c_1 + \frac{c_H(2\sigma_1^2 Q_1^* + 2(\mu_1 Q_1^* + \mu_2 Q_2)\mu_1 + 2i\mu_1)}{2D}] = E[TC^a]\mu_1 \Rightarrow$$

$$\begin{aligned}(\mu_1 Q_1^* + \mu_2 Q_2)[Dc_1 + c_H\sigma_1^2 Q_1^*] &= \mu_1[D(K + c_1 Q_1^* + c_2 Q_2) + (\frac{c_H + c_S}{2})(1-\alpha)i^2 \\ &+ \frac{c_H(\sigma_1^2(Q_1^*)^2 + \sigma_2^2(Q_2)^2 - (\mu_1 Q_1^* + \mu_2 Q_2)^2)}{2} \\ &- \frac{(c_H + c_S)(m_2 + 2im_1)}{2}]\end{aligned} \tag{4.4}$$

Approximate F.O.C. for "Q_2":

$$\frac{\partial CR^a}{\partial Q_2^*} = 0 \quad \Rightarrow \quad (\mu_1 Q_1 + \mu_2 Q_2^*)[c_2 + \frac{c_H(2\sigma_2^2 Q_2^* + 2(\mu_1 Q_1 + \mu_2 Q_2^*)\mu_2 + 2i\mu_2)}{2D}]$$

$$= \quad E[TC^a]\mu_1 \Rightarrow$$

$$\begin{aligned}(\mu_1 Q_1 + \mu_2 Q_2^*)[Dc_2 + c_H\sigma_2^2 Q_2^*] &= \mu_2[(K + c_1Q_1 + c_2Q_2^*) + (\frac{c_H + c_S}{2})(1-\alpha)i^2 \\ &+ \frac{c_H(\sigma_1^2(Q_1)^2 + \sigma_2^2(Q_2^*)^2 - (\mu_1 Q_1 + \mu_2 Q_2^*)^2)}{2} \\ &- \frac{(c_H + c_S)(m_2 + 2im_1)}{2}]\end{aligned} \tag{4.5}$$

Using the first order conditions for Q_1 and Q_2, equations (4.4) and (4.5) we obtain the following equality:

$$(c_2 + \frac{c_H}{2D}2\sigma_2^2 Q_2^*)\mu_1 = (c_1 + \frac{c_H}{2D}2\sigma_1^2 Q_1^*)\mu_2 \Rightarrow$$

$$Q_1^* = \frac{\frac{(c_2\mu_1 - c_1\mu_2)D}{c_H} + \mu_1\sigma_2^2 Q_2^*}{\mu_2\sigma_1^2} \tag{4.6}$$

As a result, we are able to find a linear relationship between the optimal values of Q_1^* and Q_2^*.

Let A and B be two constants where,

$$\begin{aligned} A &= \frac{(c_2\mu_1 - c_1\mu_2)D}{c_H\mu_2\sigma_1^2} \\ B &= \frac{\mu_1\sigma_2^2}{\mu_2\sigma_1^2}\end{aligned}$$

Then, the relation between Q_1^* and Q_2^* becomes:

$$\begin{aligned} Q_1^* &= A + BQ_2^* \text{ and from equation (4.3) we have} \\ i^* &= \frac{(c_H + c_S)m_1 - c_H(\mu_1 A + (\mu_1 B + \mu_2)Q_2^*)}{(c_H + c_S)(1-\alpha)}\end{aligned}$$

Since all three variables can be represented in terms of Q_2^*, we are able to find an equation in only one variable. Using the relations above and equation (4.5), a quadratic equation (for which a real solution may or may not exist) involving only Q_2^*,$a(Q_2^*)^2 + bQ_2^* + c = 0$, is obtained , where;

$$
\begin{aligned}
a &= \frac{c_H(\mu_1 B + \mu_2)}{2}[B\sigma_1^2 + \mu_1(\mu_1 B + \mu_2)\frac{(c_S - (c_H + c_S)\alpha)}{(c_H + c_S)(1-\alpha)}] \\
b &= c_H\mu_1 A[B(\sigma_1^2 + (\mu_1)^2) + \mu_1\mu_2 - \frac{c_H\mu_1(\mu_1 B + \mu_2)}{(c_H + c_S)(1-\alpha)}] \\
c &= \frac{c_H\mu_1 A^2}{2}[\frac{(\mu_1^2)(c_S - (c_H + c_S)\alpha)}{(c_H + c_S)(1-\alpha)} + \sigma_1^2] + H \text{ where,} \\
H &= \frac{(c_H + c_S)\mu_1}{2}[m_2 + \frac{(m_1)^2}{(1-\alpha)}] - K\mu_1 D
\end{aligned}
$$

Solving this quadratic equation will give the optimal value for Q_2 (when $\Delta > 0$). But it is not guaranteed that $\Delta > 0$. When $\Delta < 0$, it means that the quadratic equation has no real solution and the optimal values occur at the boundaries. Also, for the cases where $\Delta > 0$ but one of Q_1 or Q_2 is smaller than zero (infeasible solution for the problem), the optimal solution is at the boundaries. Therefore, a similar analysis carried out for the binomial case will be done for this case also. After some algebraic simplifications, the value of Δ comes out to be the following:

$$
\begin{aligned}
\Delta &= [2H(\mu_1 B + \mu_2) + DA(c_2\mu_1 - c_1\mu_2)\mu_1] * \\
&[\quad c_H\mu_1((\frac{(c_H + c_S)\alpha - c_S}{(c_H + c_S)(1-\alpha)})(\mu_1 B + \mu_2) - \frac{\sigma_2^2}{\mu_2})]
\end{aligned}
$$

All cases considered, the following theorem is constructed:

Theorem 4.1

i) Optimal values of Q_1, Q_2, and i for CR^a are:

$$
\begin{aligned}
Q_2^* &= \frac{-b + \sqrt{\Delta}}{2a} \\
Q_1^* &= A + BQ_2^* \\
i^* &= \frac{(c_H + c_S)m_1 - c_H(\mu_1 A + (\mu_1 B + \mu_2)Q_2^*)}{(c_H + c_S)(1-\alpha)}
\end{aligned}
$$

iff the following conditions hold;

$$
\begin{aligned}
\Delta &= [2H(\mu_1 B + \mu_2) + DA(c_2\mu_1 - c_1\mu_2)\mu_1] * \\
&[\quad c_H\mu_1((\frac{(c_H + c_S)\alpha - c_S}{(c_H + c_S)(1-\alpha)})(\mu_1 B + \mu_2) - \frac{\sigma_2^2}{\mu_2})] > 0
\end{aligned}
$$

$$\text{and} \quad min(Q_1^*, Q_2^*) > 0$$

ii) Else if, $\Delta < 0$, *or* $min(Q_1, Q_2) < 0$ *the optimal value for* CR^a *occurs at the boundaries and only one of the suppliers is used. Let* CR_1^a, *and* CR_2^a *be the cost rates when only supplier 1, 2 is used respectively. Hence,*

- *If* $CR_1^a|_{Q_1^*, Q_2^*=0} - CR_2^a|_{Q_1^*=0, Q_2^*} < 0$, *optimal values will be:*

$$\begin{aligned}
Q_1^* &= \sqrt{\frac{2KD - (c_H + c_S)(m_2 + \frac{(m_1)^2}{(1-\alpha)})}{c_H(\sigma_1^2 + (\mu_1)^2 - \frac{c_H(\mu_1)^2}{(1-\alpha)(c_H+c_s)})}} \\
Q_2^* &= 0 \\
i^* &= \frac{(c_H + c_S)m_1 - c_H\mu_1 Q_1^*}{(c_H + c_S)(1-\alpha)}
\end{aligned}$$

- *Else if,* $CR_1^a|_{Q_1^*, Q_2^*=0} - CR_2^a|_{Q_1^*=0, Q_2^*} > 0$, *then optimal values will be:*

$$\begin{aligned}
Q_1^* &= 0 \\
Q_2^* &= \sqrt{\frac{2KD - (c_H + c_S)(m_2 + \frac{(m_1)^2}{(1-\alpha)})}{c_H(\sigma_2^2 + (\mu_2)^2 - \frac{c_H(\mu_2)^2}{(1-\alpha)(c_H+c_s)})}} \\
i^* &= \frac{(c_H + c_S)m_1 - c_H\mu_2 Q_2^*}{(c_H + c_S)(1-\alpha)}
\end{aligned}$$

Proof:

Case 1: $\Delta > 0$, **and** $min(Q_1, Q_2) > 0$**:**

The solution of the quadratic equation $a(Q_2^*)^2 + bQ_2^* + c$ derived from equation (4.5) gives the optimal value for Q_2 for CR^a. Then using the linear relation between Q_1^* and Q_2^* ($Q_1^* = A + BQ_2^*$) optimal value for Q_1 is found. Finally, from equation (4.3), the optimal value for the reorder point is computed using the optimal values of the order quantities previously found.

Case 2: $\Delta < 0$ **or** $min(Q_1, Q_2) < 0$**:**

In this case the quadratic equation has no solution, that is, there is not a real value for Q_2^* satisfying this particular equation. From the relation between Q_1^* and Q_2^*, no real value of Q_1^* is available. Consequently, the optimal value will occur at the boundaries. Since the minimum will occur at the boundaries, we are going to look at the cost rates when $Q_1 = 0$ and $Q_2 = 0$.

- When $Q_2 = 0$, we have the following:

$$\begin{aligned} E[TC^a] &= K + c_1 Q_1 + (\frac{c_H + c_s}{2D})(1-\alpha)i^2 \\ &+ \frac{c_H(\sigma_1^2 Q_1^2 + (\mu_1 Q_1)^2 + 2i\mu_1 Q_1)}{2D} \\ &- \frac{(c_H + c_S)[m_2 + 2im_1]}{2D} \end{aligned} \tag{4.7}$$

$$E[T] = \frac{E[x]}{D} = \frac{\mu_1 Q_1}{D} \tag{4.8}$$

Let the cost rate be CR_1^a in this realization. Now, we need to find the optimal value of Q_1 minimizing cost rate 1 (CR_1^a):

$$\frac{\partial CR_1^a}{\partial Q_1^*} = 0 \Rightarrow \frac{\partial E[TC^a]}{\partial Q_1^*} E[T] = E[TC^a] \frac{\partial E[T]}{\partial Q_1^*} \Rightarrow$$

$$\begin{aligned} [\quad & \frac{c_H \mu_1}{2}(\sigma_1^2 + (\mu_1)^2) - \frac{(c_H)^2(\mu_1)^3}{2(1-\alpha)(c_H + c_S)}](Q_1^*)^2 - KD\mu_1 \\ = \quad & -\frac{(c_H + c_S)\mu_1}{2}(m_2 + \frac{(m_1)^2}{(1-\alpha)}) \Rightarrow \end{aligned}$$

$$Q_1^* = \sqrt{\frac{2KD - (c_H + c_S)(m_2 + \frac{(m_1)^2}{(1-\alpha)})}{c_H(\sigma_1^2 + (\mu_1)^2 - \frac{c_H(\mu_1)^2}{(1-\alpha)(c_H + c_s)})}} \tag{4.9}$$

i^* is found from equation (4.3).

- When $Q_1 = 0$, we have the following:

$$E[TC^a] = K + c_2 Q_2 + (\frac{c_H + c_s}{2D})(1-\alpha)i^2 + \frac{c_H(\sigma_2^2 Q_2^2 + (\mu_2 Q_2)^2 + 2i\mu_2 Q_2)}{2D} - \frac{(c_H + c_S)[m_2 + 2im_1]}{2D} \tag{4.10}$$

$$E[T] = \frac{E[x]}{D} = \frac{\mu_2 Q_2}{D} \tag{4.11}$$

Let the cost rate be CR_2^a in this setting. Now, we need to find the optimal value of Q_2 minimizing cost rate 2 (CR_2^a):

$$\frac{\partial CR_2^a}{\partial Q_2^*} = 0 \Rightarrow \frac{\partial E[TC^a]}{\partial Q_2^*} E[T] = E[TC^a] \frac{\partial E[T]}{\partial Q_2^*} \Rightarrow$$

$$[\quad \frac{c_H \mu_2}{2}(\sigma_2^2 + (\mu_2)^2) - \frac{(c_H)^2(\mu_2)^3}{2(1-\alpha)(c_H + c_S)}](Q_2^*)^2 - KD\mu_2 = -\frac{(c_H + c_S)\mu_2}{2}(m_2 + \frac{(m_1)^2}{(1-\alpha)}) \Rightarrow$$

$$Q_2^* = \sqrt{\frac{2KD - (c_H + c_S)(m_2 + \frac{(m_1)^2}{(1-\alpha)})}{c_H(\sigma_2^2 + (\mu_2)^2 - \frac{c_H(\mu_2)^2}{(1-\alpha)(c_H + c_s)})}} \tag{4.12}$$

Again i^* is found from equation (4.3).
Then, $CR_1^a - CR_2^a$ evaluated at the optimal values of Q_1, Q_2, which is a function of parameters (recall that α, m_1, and m_2) is computed. If $CR_1^a(CR_2^a)$ is smaller than $CR_2^a(CR_1^a)$ it is less costly to order from supplier 1(2) only. □.

4.2.1 Initial Solution of the Algorithm

In the analysis done up to this point, we assumed that α, m_1, and m_2 are assumed to be constants, so that the first order partial derivatives of them with respect to

Q_1, Q_2, and i are zero. According to the solution procedure proposed in Chapter 2, we first need to assign values to the expressions above. Therefore, we started with assigning 0 to these values. Substituting 0 in place of these expressions yields the following:

$$i^* = (\frac{-c_H}{c_H + c_S})(\mu_1 Q_1^* + \mu_2 Q_2^*) \tag{4.13}$$

The relation between optimal order quantities is preserved since it is not dependent on P_i, or $E[XI(X < -i)]$, or $E[X^2 I(X < -i)]$. Thus, we use the quadratic equation for the first step of the algorithm also. Therefore, for the first step of the algorithm the following corollary is constructed:

Corollary 4.1

i) Optimal values of Q_1, Q_2, *and* i *for* CR^a *when* $P_i = E[XI(X < -i)] = E[X^2 I(X < -i)] = 0$ *are:*

$$\begin{aligned} Q_2^* &= \frac{-b + \sqrt{\Delta_0}}{2a} \\ Q_1^* &= A + BQ_2^* \\ i^* &= \frac{-c_H(\mu_1 A + (\mu_1 B + \mu_2)Q_2^*)}{(c_H + c_S)} \end{aligned}$$

where,

$$\Delta_0 = [c_H \mu_1 (\frac{c_s}{(c_H + c_S)} + \frac{\sigma_2^2}{\mu_2})][2K\mu_1 D(\mu_1 B + \mu_2) - DA(c_2\mu_1 - c_1\mu_2)]$$

iff the following conditions hold;

$$2K\mu_1 D(\mu_1 B + \mu_2) - DA(c_2\mu_1 - c_1\mu_2) > 0 \ \text{ and } \ min(Q_1^*, Q_2^*) > 0$$

ii) Else if, $\Delta < 0$, *or* $min(Q_1, Q_2) < 0$ *the optimal value for* CR^a *occurs at the boundaries and only one of the suppliers is used. Let* CR_1^a, *and* CR_2^a *be the cost rates when only supplier 1, 2 is used respectively. Hence,*

- *If* $CR_1^a|_{Q_1^*} - CR_2^a|_{Q_2^*} < 0$, *optimal values (for* $P_i = E[XI(X < -i)] = E[X^2 I(X < -i)] = 0$*) will be:*

$$Q_1^* = \sqrt{\frac{2KD}{c_H[\frac{c_S \mu_1^2}{c_H + c_S} + \sigma_1^2]}}$$

$$Q_2^* = 0$$
$$i^* = \frac{-c_H\mu_1 Q_1^*}{(c_H + c_S)}$$

• *Else if,* $CR_1^a|_{Q_1^*} - CR_2^a|_{Q_2^*} > 0$*, then optimal values for* ($P_i = E[XI(X < -i)] = E[X^2 I(X < -i)] = 0$) *will be:*

$$Q_1^* = 0$$
$$Q_2^* = \sqrt{\frac{2KD}{c_H[\frac{c_S\mu_2^2}{c_H+c_S} + \sigma_2^2]}}$$
$$i^* = \frac{-c_H\mu_2 Q_2^*}{(c_H + c_S)}$$

Proof:

The discriminant of the quadratic equation was:

$$\Delta = [2H(\mu_1 B + \mu_2) + DA(c_2\mu_1 - c_1\mu_2)\mu_1]$$
$$[\quad c_H\mu_1((\frac{(c_H + c_S)P_i - c_S}{(c_H + c_S)(1 - P_i)})(\mu_1 B + \mu_2) - \frac{\sigma_2^2}{\mu_2})]$$

When $P_i, E[XI(X < -i)], E[X^2 I(X < -i)]$ are all equal to zero, we have:

$$H = -K\mu_1 D$$
$$\Delta_0 = [c_H\mu_1(\frac{c_s}{(c_H + c_S)} + \frac{\sigma_2^2}{\mu_2})][2K\mu_1 D(\mu_1 B + \mu_2) - DA(c_2\mu_1 - c_1\mu_2)]$$

Δ_0 being greater than zero (provided that $min(Q_1, Q_2) > 0$) proves that the optimal occurs where first order conditions hold. The first term in square brackets in Δ_0 is always positive so that the sign of Δ_0 depends on the sign of the second term,$2K\mu_1 D(\mu_1 B + \mu_2) - DA(c_2\mu_1 - c_1\mu_2)$. If the second term is greater than zero we are guaranteed that the quadratic equation has real roots. Otherwise, the optimal occurs at the boundaries and assigning 0 to $P_i, E[XI(X < -i)], E[X^2 I(X < -i)]$ yields:

$$Q_1^* = \sqrt{\frac{2KD}{c_H[\frac{c_S\mu_1^2}{c_H+c_S} + \sigma_1^2]}} \text{ when supplier 1 is used only}$$
$$Q_2^* = \sqrt{\frac{2KD}{c_H[\frac{c_S\mu_2^2}{c_H+c_S} + \sigma_2^2]}} \text{ when supplier 2 is used only}$$

Corollary 4.1 is proven. □

4.3 Generalization to "M" Suppliers:

In this setting, there are "M" suppliers producing the same unit, and the yield distribution of each supplier is independent from others. The following are the expected total cost per cycle and expected cycle time for this particular case:

$$\begin{aligned} E[\text{Total Cost}] &= K + c_1 Q_1 + \ldots + c_M Q_M + (\frac{c_H + c_s}{2D})(1 - P_i) i^2 \\ &+ \frac{c_H (E[X^2 | Q_1, \ldots, Q_M] + 2iE[X | Q_1, \ldots, Q_M])}{2D} \\ &- \frac{(c_H + c_S)[E[X^2 I(X < -i)] + 2iE[XI(X < -i)]]}{2D} \end{aligned}$$

$$E[T] = \frac{E[X|Q_1, \ldots, Q_M]}{D}$$

where the expected amount to receive and the second moment are as follows:

$$E[x|Q_1, \ldots, Q_M] = \mu_1 Q_1 + \ldots + \mu_M Q_M$$

$$E[x^2|Q_1, \ldots, Q_M] = \sigma_1^2 Q_1^2 + \ldots + \sigma_M^2 Q_M^2 + (\mu_1 Q_1 + \ldots + \mu_M Q_M)^2$$

The approximate cost rate is to be minimized. Following are the first order conditions for the reorder point and the order quantities:

For i:

$$\frac{\partial CR^a}{\partial i^*} = 0 \Rightarrow E[T] \frac{\partial E[TC^a]}{\partial i^*} = E[TC^a] \frac{\partial E[T]}{\partial i^*} \Rightarrow \frac{\partial E[TC^a]}{\partial i^*} = 0$$

$$\begin{aligned} &\Rightarrow \quad (\frac{c_H + c_S}{D})(1 - \alpha) i^* + \frac{c_H(\mu_1 Q_1 + \ldots + \mu_M Q_M) - (c_H + c_S) m_1}{D} = 0 \\ &\Rightarrow \quad i^* = \frac{(c_H + c_S) m_1 - c_H(\mu_1 Q_1 + \ldots + \mu_M Q_M)}{(c_H + c_S)(1 - \alpha)} \end{aligned} \tag{4.14}$$

For Q_1**:**

$$\begin{aligned} (\quad & \mu_1 Q_1^* + \ldots + \mu_M Q_M)(Dc_1 + c_H \sigma_1^2 Q_1) = \mu_1 [KD + Dc_1 Q_1^* + \ldots + Dc_M Q_M \\ + \quad & (\frac{c_H + c_S}{2})(1 - \alpha) i^2 + \frac{c_H}{2}(\sigma_1^2 (Q_1^*)^2 + \ldots + \sigma_M^2 (Q_M)^2 - (\mu_1 Q_1^* + \ldots + \mu_M Q_M)^2)] \\ - \quad & (c_H + c_S)(\frac{m_2}{2} + i m_1) \end{aligned}$$

Taking first order partial derivatives for the other decision variables $(Q_2 \ldots, Q_M)$ and equating them yields:

$$
\begin{aligned}
\mu_1(Dc_2 + c_H\sigma_2^2 Q_2^*) &= \mu_2(Dc_1 + c_F\sigma_1^2 Q_1^*) \Rightarrow Q_1^* = A_2 + B_2 Q_2^* \\
\mu_1(Dc_3 + c_H\sigma_3^2 Q_3^*) &= \mu_3(Dc_1 + c_F\sigma_1^2 Q_1^*) \Rightarrow Q_1^* = A_3 + B_3 Q_3^* \\
\ldots \quad &. \quad \ldots\ldots\ldots \quad . \\
\ldots \quad &. \quad \ldots\ldots\ldots \quad . \\
\mu_1(Dc_M + c_H\sigma_M^2 Q_M^*) &= \mu_M(Dc_1 + c_F\sigma_1^2 Q_1^*) \Rightarrow Q_1^* = A_M + B_M Q_M^*
\end{aligned}
$$

where,

$$
\begin{aligned}
A_j &= \frac{D(c_j\mu_1 - c_1\mu_j)}{c_H\mu_j\sigma_1^2} \\
B_j &= \frac{\mu_1\sigma_j^2}{\mu_j\sigma_1^2}
\end{aligned}
$$

Similar to the 2-supplier case, we are able to express all decision variables in terms of Q_1^* using first order conditions. Therefore, a quadratic equation involving only Q_1^* can be found. If the discriminant of this quadratic equation is positive, Q_1^* will be the solution of this equation, and the optimal values of the order quantities will be computed from the relations given above (given that they are positive). Otherwise, if the equation has no real (and positive) solution, the optimal solution is at the boundaries. In this case, some of the order quantities will be equal to zero. To find the optimal value of the cost rate, all subsets of the set 1,2,...,M need to be considered. The quadratic equation for each possible subset need to be found, and the subset with the minimum cost rate value should be chosen. As a result:

$Q_j = 0$ for $j \in S$ where S is a subset of $1, 2, \ldots, M$ and $Q_k > 0$ for $k \neq j$ for which values are found from the quadratic equations. All subsets "S" will be analyzed in this manner, and the one with the minimum cost will be selected as the best solution.

Chapter 5

Numerical Analysis

In order to investigate the behavior of the optimal values of decision variables and the cost rate with respect to the cost parameters and the distribution parameters a numerical study is carried out in this chapter. This study is done for both binomial yield and stochastically proportional yield. For each case, we first present the results for different *parameter* settings and then the results under different *distribution* parameter sets. The results are obtained by using the algorithm defined in Chapter 2 using the software package Matlab. Additionally, the regions showing where diversification among suppliers pays or where using only one supplier is beneficial are provided. Finally, the comparison of the results obtained by the algorithm that we proposed and the optimal values are given in a table for different cost and yield structures.

Throughout the analysis the demand rate is taken to be equal to one (D=1). The experimental set-up for both cases, is given in Table 5.1.

K	200, 400, 600
c_H	5, 10, 20, 30, 40
c_S	50, 60, 80

Table 5.1: Experimental Design # 1

5.1 Binomial Yield Case

In Tables 5.3 to 5.6, the effect of the change in K, c_H, c_S on decision variables and cost rate is investigated. The following observations are made for the purchasing costs and *distribution* parameters given in Table 5.2. We used the values in Table 5.2 in order to see the effect of different selling prices and distribution parameters.

Table #	c_1	c_2	p_1	p_2
Table 5.3	96	120	0.6	0.8
Table 5.4	96	120	0.75	0.9
Table 5.5	80	120	0.6	0.8
table 5.6	108	120	0.75	0.9

Table 5.2: Experimental Design # 2

- As K increases and c_H, c_S are held constant, the retailer orders more in order to place orders less frequently and consequently minimize the procurement cost in the long-run. The magnitude of the reorder level gets larger in order to balance increasing inventory holding cost.

- As c_H increases only, the quantities ordered decrease to incur less inventory holding cost. The absolute value of reorder level, $|i|$ also increases to take advantage of backordering which is now relatively inexpensive when c_H is increased. When c_S increases (while K, c_H do not change), i becomes closer to zero to allow as little shortages as possible. Also, since one would incur extra holding cost when the reorder level is closer to zero and the total quantity ordered is the same, the retailer orders less from the suppliers as the unit shortage cost per unit time increases.

- The cost rate increases when K, c_H, and c_S increases as expected. Also, the values of reorder point are almost the same (at least for two decimal places), since the expected amount to receive is almost equal when cost parameters c_H, c_S are same (do not change with c_1, c_2, p_1, p_2), complying with the theoretical findings.

In Tables 5.3 and 5.4, the ratio $\frac{c_1}{c_2}$ is unchanged, but the probabilities of

		$c_S = 50$				$c_S = 60$				$c_S = 80$			
K	c_H	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
200	5	0.00	11.73	-0.85	193.140	0.00	11.64	-0.72	193.467	0.00	11.52	-0.54	193.886
	10	0.00	8.66	-1.15	208.735	0.00	8.54	-0.98	209.554	0.00	8.39	-0.75	210.628
	20	0.00	6.61	-1.51	227.589	0.00	6.45	-1.29	229.458	0.00	6.25	-1.00	231.999
	30	0.00	5.77	-1.73	239.565	0.00	5.59	-1.49	242.420	0.00	5.35	-1.17	246.412
	40	0.00	5.30	-1.89	248.216	0.00	5.10	-1.63	251.938	0.00	4.84	-1.29	257.205
400	5	0.00	16.58	-1.21	210.802	0.00	16.46	-1.01	211.264	0.00	16.30	-0.77	211.857
	10	0.00	12.25	-1.63	232.650	0.00	12.08	-1.38	233.808	0.00	11.86	-1.05	235.327
	20	0.00	9.35	-2.14	258.904	0.00	9.13	-1.83	261.544	0.00	8.84	-1.41	265.137
	30	0.00	8.16	-2.45	275.472	0.00	7.91	-2.11	279.487	0.00	7.57	-1.65	285.114
	40	0.00	7.50	-2.67	287.311	0.00	7.22	-2.31	292.555	0.00	6.85	-1.83	300.046
600	5	0.00	20.31	-1.48	224.355	0.00	20.16	-1.24	224.921	0.00	19.96	-0.94	225.647
	10	0.00	15.00	-2.00	251.000	0.00	14.79	-1.69	252.419	0.00	14.52	-1.29	254.280
	20	0.00	11.46	-2.62	282.931	0.00	11.18	-2.24	286.164	0.00	10.83	-1.73	290.564
	30	0.00	10.00	-3.00	303.000	0.00	9.68	-2.58	307.918	0.00	9.27	-2.02	314.808
	40	0.00	9.19	-3.27	317.295	0.00	8.84	-2.83	323.698	0.00	8.39	-2.24	332.884

Table 5.3: Results with $c_1 = 96, c_2 = 120, p_1 = 0.6, p_2 = 0.8$

		$c_S = 50$				$c_S = 60$				$c_S = 80$			
K	c_H	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
200	5	12.51	0.00	-0.85	171.265	12.41	0.00	-0.72	171.592	12.29	0.00	-0.54	172.011
	10	9.24	0.00	-1.15	186.985	9.11	0.00	-0.98	187.804	8.94	0.00	-0.75	188.878
	20	7.06	0.00	-1.51	206.090	6.89	0.00	-1.29	207.954	6.67	0.00	-1.00	210.498
	30	6.16	0.00	-1.73	218.326	5.96	0.00	-1.49	221.135	5.71	0.00	-1.17	225.147
	40	5.66	0.00	-1.89	227.122	5.44	0.00	-1.63	230.876	5.16	0.00	-1.29	236.235
400	5	17.69	0.00	-1.21	188.927	17.55	0.00	-1.01	189.389	17.38	0.00	-0.77	189.982
	10	13.06	0.00	-1.63	210.900	12.88	0.00	-1.38	212.058	12.65	0.00	-1.05	213.577
	20	9.98	0.00	-2.14	237.404	9.74	0.00	-1.83	240.044	9.43	0.00	-1.41	243.637
	30	8.71	0.00	-2.45	254.214	8.43	0.00	-2.11	258.237	8.07	0.00	-1.65	263.864
	40	8.00	0.00	-2.67	266.312	7.70	0.00	-2.31	271.533	7.30	0.00	-1.83	279.048
600	5	21.66	0.00	-1.48	202.480	21.50	0.00	-1.24	203.046	21.29	0.00	-0.94	203.772
	10	16.00	0.00	-2.00	229.250	15.78	0.00	-1.69	230.669	15.49	0.00	-1.29	232.530
	20	12.22	0.00	-2.62	261.431	11.93	0.00	-2.24	264.664	11.55	0.00	-1.73	269.064
	30	10.67	0.00	-3.00	281.747	10.33	0.00	-2.58	286.668	9.89	0.00	-2.02	293.557
	40	9.80	0.00	-3.27	296.280	9.43	0.00	-2.83	302.697	8.94	0.00	-2.24	311.878

Table 5.4: Results with $c_1 = 96, c_2 = 120, p_1 = 0.75, p_2 = 0.9$

		$c_S = 50$				$c_S = 60$				$c_S = 80$			
K	c_H	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
200	5	15.63	0.00	-0.85	176.973	15.52	0.00	-0.72	177.300	15.37	0.00	-0.54	177.719
	10	11.55	0.00	-1.15	193.068	11.39	0.00	-0.98	193.887	11.18	0.00	-0.75	194.962
	20	8.82	0.00	-1.51	212.903	8.61	0.00	-1.29	214.779	8.33	0.00	-1.00	217.327
	30	7.70	0.00	-1.73	225.813	7.45	0.00	-1.49	228.698	7.14	0.00	-1.17	232.718
	40	7.07	0.00	-1.89	235.409	6.80	0.00	-1.63	238.998	6.45	0.00	-1.29	244.459
400	5	22.11	0.00	-1.21	194.636	21.94	0.00	-1.01	195.098	21.73	0.00	-0.77	195.691
	10	16.33	0.00	-1.63	216.983	16.10	0.00	-1.38	218.141	15.81	0.00	-1.05	219.661
	20	12.47	0.00	-2.14	244.235	12.17	0.00	-1.83	246.876	11.79	0.00	-1.41	250.469
	30	10.89	0.00	-2.45	261.771	10.54	0.00	-2.11	265.806	10.09	0.00	-1.65	271.440
	40	10.00	0.00	-2.67	274.593	9.62	0.00	-2.31	279.815	9.13	0.00	-1.83	287.357
600	5	27.08	0.00	-1.48	208.188	26.87	0.00	-1.24	208.754	26.61	0.00	-0.94	209.480
	10	20.00	0.00	-2.00	235.333	19.72	0.00	-1.69	236.752	19.36	0.00	-1.29	238.613
	20	15.28	0.00	-2.62	268.263	14.91	0.00	-2.24	271.497	14.43	0.00	-1.73	275.897
	30	13.33	0.00	-3.00	289.325	12.91	0.00	-2.58	294.243	12.36	0.00	-2.02	301.138
	40	12.25	0.00	-3.27	304.592	11.79	0.00	-2.83	310.995	11.18	0.00	-2.24	320.205

Table 5.5: Results with $c_1 = 80, c_2 = 120, p_1 = 0.6, p_2 = 0.8$

		$c_S = 50$				$c_S = 60$				$c_S = 80$			
K	c_H	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
200	5	0.00	10.42	-0.85	176.223	0.00	10.34	-0.72	176.550	0.00	10.24	-0.54	176.969
	10	0.00	7.70	-1.15	191.568	0.00	7.59	-0.98	192.387	0.00	7.45	-0.75	193.462
	20	0.00	5.88	-1.51	209.925	0.00	5.74	-1.29	211.793	0.00	5.56	-1.00	214.333
	30	0.00	5.13	-1.73	221.434	0.00	4.97	-1.49	224.265	0.00	4.76	-1.17	228.250
	40	0.00	4.71	-1.89	229.581	0.00	4.54	-1.63	233.292	0.00	4.30	-1.29	238.606
400	5	0.00	14.74	-1.21	193.886	0.00	14.63	-1.01	194.348	0.00	14.49	-0.77	194.941
	10	0.00	10.89	-1.63	215.483	0.00	10.73	-1.38	216.641	0.00	10.54	-1.05	218.161
	20	0.00	8.31	-2.14	241.238	0.00	8.11	-1.83	243.878	0.00	7.86	-1.41	247.470
	30	0.00	7.26	-2.45	257.308	0.00	7.03	-2.11	261.324	0.00	6.73	-1.65	266.949
	40	0.00	6.67	-2.67	268.661	0.00	6.42	-2.31	273.896	0.00	6.09	-1.83	281.392
600	5	0.00	18.05	-1.48	207.438	0.00	17.92	-1.24	208.004	0.00	17.74	-0.94	208.730
	10	0.00	13.33	-2.00	233.833	0.00	13.15	-1.69	235.252	0.00	12.91	-1.29	237.113
	20	0.00	10.18	-2.62	265.264	0.00	9.94	-2.24	268.497	0.00	9.62	-1.73	272.897
	30	0.00	8.89	-3.00	284.833	0.00	8.61	-2.58	289.753	0.00	8.24	-2.02	296.641
	40	0.00	8.16	-3.27	298.632	0.00	7.86	-2.83	305.038	0.00	7.45	-2.24	314.219

Table 5.6: Results with $c_1 = 108, c_2 = 120, p_1 = 0.75, p_2 = 0.9$

producing a good item are increased for each supplier. The variance for supplier 1 ($\sigma_1^2 = p_1(1-p_1)$) is higher than the one for supplier 2 ($\sigma_2^2 = p_2(1-p_2)$) and p_1 is smaller than p_2 in both tables. That is, the second supplier is more reliable. However, in Table 5.4, Q_1 is always positive whereas $Q_2 = 0$, since c_1 is low enough to force the retailer to order from the first supplier. Also, $\frac{p_1}{p_2}$ is now 0.83 approximately while it is 0.75 in Table 5.4 (which is higher than $\frac{c_1}{c_2} = 0.8$), therefore two suppliers are closer to each other in terms of their reliability.
When Tables 5.3 and 5.5 are compared, we see that both suppliers are equal in terms of reliability. But, in Table 5.5, the item produced by the first supplier is relatively inexpensive as compared to c_1 in Table 5.3(i.e. $\frac{c_1}{c_2}$ is lower in Table 5.5). Hence, the retailer switches to the first supplier (again in this case $\frac{c_1}{c_2} = 0.66$ is less than $\frac{p_1}{p_2} = 0.75$).

For Tables 5.4 and 5.6, we observe that as c_1 gets large enough when the other parameters are unchanged, retailer orders from second supplier only. In Table 5.6, we also observe that $\frac{c_1}{c_2} = 0.9$ is higher than $\frac{p_1}{p_2} = 0.83$.
When we analyze Tables 5.3 to 5.6, we see that, the retailer chooses the supplier to order from according to the equation of parameters derived in Chapter 3. But, from the numerical results, for the range of unit holding cost per time from 5 to 40 and for D=1, the selection criterion seems to be simpler. In all tables, it is observed that, when $\frac{c_1}{c_2} < \frac{p_1}{p_2}$ first supplier is used, and when $\frac{c_1}{c_2} > \frac{p_1}{p_2}$ second supplier is used. These ratios can be rewritten as $\frac{c_1}{p_1} < \frac{c_2}{p_2}$ and $\frac{c_1}{p_1} > \frac{c_2}{p_2}$. Since, $\frac{c_j}{p_j}$ can be regarded as the effective selling price of a *good* item produced by supplier "j", it is intuitively reasonable to order from the less expensive one. Consequently, when the unit holding cost per unit time is not so large, the selection among suppliers can be done by comparing the effective prices for each supplier.

In Table 5.7, as we move downwards, p_1 gets larger which results in higher mean and lower variance for supplier 1. That is, first supplier becomes more reliable and retailer uses supplier 1 more often. As we keep p_j the same and increase p_i, it is observed that the amount ordered decreases for the cases where supplier i is used, since level of uncertainty involved in the amount delivered is less. Obviously, the amount ordered Q_i is independent of p_j. As we increase p_2

keeping p_1 the same, the retailer switches to the second supplier in some cases where probabilities are more close to each other. This move from supplier 1 to 2 is again determined by the "approximate selection criterion" (done by comparing effective selling prices) since holding cost is not that high to have a significant effect on the equation of parameters ($c_2p_1 - c_1p_2 + \frac{c_H}{2D}p_1p_2(p_1 - p_2) = 0$). Clearly, retailer orders more often from supplier 2 when p_1 is low enough, since $\frac{c_1}{c_2} = 0.8$ which makes first supplier advantageous. Lastly, reorder level is not affected by the change in p_1, p_2 (since expected amount to receive, $p_1Q_1 + p_2Q_2$ is the same)and the cost rate is an increasing function of p_1 or p_2.

In Table 5.8, c_1 is increased from 96 to 120, and c_H is decreased to 20 from 30. Since $\frac{c_1}{c_2} = 0.9$ now, the retailer orders from supplier 2 in 8 cases whereas supplier 2 is used 4 times in Table 5.7. Q_1, Q_2, and i and cost rate show the same behavior as in Table 5.7. For the cases where supplier 1 is used only, we observe that the amount ordered (Q_1 or Q_2) are higher since the holding cost is lower even though c_1 is increased. Again, the approximate selection criteria works in Table 5.8. Note that, in Tables 5.7-5.9, the value of the reorder point stays the same, since it depends on the expected amount to receive (p_jQ_j), so on K, D, c_H, c_S.

We choose smaller values for p_1 and p_2 in Table 5.9 in order to see the effect of the increase in variance when the mean gets higher. In Table 5.9, the probabilities are increased in such a way that the variance also increases as the mean increases, using the structure of binomial density function (that is, for $p + p' < 1$ where $p < p'$, then $p(1-p) < p'(1-p')$.

So, as we move downwards, the mean is larger but the variance is also larger. But as the results in Table 5.9 suggest, the amounts ordered are again decreasing functions of p_1 and p_2. As a result, we conclude that the effect of the increase in variance has a negligible effect in terms of the change in Q_1 or Q_2 when the mean is higher. Obviously, the amounts ordered increases significantly when p_1 and p_2 are taken to be equal to values smaller than 0.5.

p_1	p_2	Q_1^*	Q_2^*	i^*	CR
0.60	0.60	12.172	0.000	-2.739	302.9175
	0.70	12.172	0.000	-2.739	302.9175
	0.75	0.000	9.737	-2.739	300.6751
	0.80	0.000	9.129	-2.739	289.9294
	0.90	0.000	8.114	-2.739	271.7639
0.70	0.60	10.433	0.000	-2.739	278.5666
	0.70	10.433	0.000	-2.739	278.5666
	0.75	10.433	0.000	-2.739	278.5666
	0.80	10.433	0.000	-2.739	278.5666
	0.90	0.000	8.114	-2.739	271.7639
0.75	0.60	9.737	0.000	-2.739	268.6751
	0.70	9.737	0.000	-2.739	268.6751
	0.75	9.737	0.000	-2.739	268.6751
	0.80	9.737	0.000	-2.739	268.6751
	0.90	9.737	0.000	-2.739	268.6751
0.80	0.60	9.129	0.000	-2.739	259.9294
	0.70	9.129	0.000	-2.739	259.9294
	0.75	9.129	0.000	-2.739	259.9294
	0.80	9.129	0.000	-2.739	259.9294
	0.90	9.129	0.000	-2.739	259.9294
0.90	0.60	8.114	0.000	-2.739	245.0972
	0.70	8.114	0.000	-2.739	245.0972
	0.75	8.114	0.000	-2.739	245.0972
	0.80	8.114	0.000	-2.739	245.0972
	0.90	8.114	0.000	-2.739	245.0972

Table 5.7: Results with $K = 500, c_1 = 96, c_2 = 120, c_H = 30, c_S = 50$

p_1	p_2	Q_1^*	Q_2^*	i^*	CR
0.60	0.60	13.944	0.000	-2.390	303.5209
	0.70	0.000	11.952	-2.390	293.9507
	0.75	0.000	11.155	-2.390	282.0227
	0.80	0.000	10.458	-2.390	271.5228
	0.90	0.000	9.296	-2.390	253.8562
0.70	0.60	11.952	0.000	-2.390	276.8078
	0.70	11.952	0.000	-2.390	276.8078
	0.75	11.952	0.000	-2.390	276.8078
	0.80	0.000	10.458	-2.390	271.5228
	0.90	0.000	9.296	-2.390	253.8562
0.75	0.60	11.155	0.000	-2.390	266.0227
	0.70	11.155	0.000	-2.390	266.0227
	0.75	11.155	0.000	-2.390	266.0227
	0.80	11.155	0.000	-2.390	266.0227
	0.90	0.000	9.296	-2.390	253.8562
0.80	0.60	10.458	0.000	-2.390	256.5228
	0.70	10.458	0.000	-2.390	256.5228
	0.75	10.458	0.000	-2.390	256.5228
	0.80	10.458	0.000	-2.390	256.5228
	0.90	0.000	9.296	-2.390	253.8562
0.90	0.60	9.296	0.000	-2.390	240.5229
	0.70	9.296	0.000	-2.390	240.5229
	0.75	9.296	0.000	-2.390	240.5229
	0.80	9.296	0.000	-2.390	240.5229
	0.90	9.296	0.000	-2.390	240.5229

Table 5.8: Results with $K = 500, c_1 = 108, c_2 = 120, c_H = 20, c_S = 50$

p_1	p_2	Q_1^*	Q_2^*	i^*	CR
0.25	0.25	33.466	0.000	-2.390	559.0115
	0.30	0.000	27.889	-2.390	526.5124
	0.35	0.000	23.905	-2.390	468.8715
	0.40	0.000	20.917	-2.390	425.5161
	0.45	0.000	18.592	-2.390	391.6851
0.30	0.25	27.889	0.000	-2.390	486.5124
	0.30	27.889	0.000	-2.390	486.5124
	0.35	0.000	23.905	-2.390	468.8715
	0.40	0.000	20.917	-2.390	425.5161
	0.45	0.000	18.592	-2.390	391.6851
0.35	0.25	23.905	0.000	-2.390	434.5858
	0.30	23.905	0.000	-2.390	434.5858
	0.35	23.905	0.000	-2.390	434.5858
	0.40	0.000	20.917	-2.390	425.5161
	0.45	0.000	18.592	-2.390	391.6851
0.40	0.25	20.917	0.000	-2.390	395.5161
	0.30	20.917	0.000	-2.390	395.5161
	0.35	20.917	0.000	-2.390	395.5161
	0.40	20.917	0.000	-2.390	395.5161
	0.45	0.000	18.592	-2.390	391.6851
0.45	0.25	18.592	0.000	-2.390	365.0184
	0.30	18.592	0.000	-2.390	365.0184
	0.35	18.592	0.000	-2.390	365.0184
	0.40	18.592	0.000	-2.390	365.0184
	0.45	18.592	0.000	-2.390	365.0184

Table 5.9: Results with $K = 500, c_1 = 108, c_2 = 120, c_H = 20, c_S = 50$

5.1.1 Diversification Among Suppliers

Note that, only one supplier is used in Tables 5.3 to 5.6. This is due to the fact that diversification among suppliers is beneficial for very limited number of cases (for cases where the equation of parameters $c_2p_1 - c_1p_2 + \frac{c_H}{2D}p_1p_2(p_1 - p_2) = 0$ holds). Dividing the equation of parameters by p_1p_2 and rewriting it yields: $\frac{c_2}{p_2} - \frac{c_1}{p_1} = \frac{c_H}{2D}(p_2 - p_1) =$. If the left hand side (difference of effective selling prices) is greater than right hand side (marginal holding cost of moving one item from supplier 1 to 2), retailer orders from Supplier 1 only. The retailer orders from both suppliers when the marginal holding cost of moving one item from supplier 1 to 2 is equal to additional purchasing cost of moving again one item from supplier 1 to 2, as explained in Chapter 3. Even in this case, the retailer could order from each supplier in the way s/he desires. That is, s/he can order from just one supplier despite the equation of parameters mentioned above holds, since the expected cost rate does not change when Q_1, or Q_2 changes (as proved in Chapter 3, we just need to keep $p_1Q_1 + p_2Q_2$ constant and any feasible pair (Q_1, Q_2) can be a solution). Moreover, as the numerical results suggest, the selection among suppliers can be done by the so called "approximate selection criterion" explained before when c_H is not significantly high. In the following part, the regions showing where ordering from supplier 1 or 2 is less costly are given for some given values of c_H, D, c_1, and c_2:

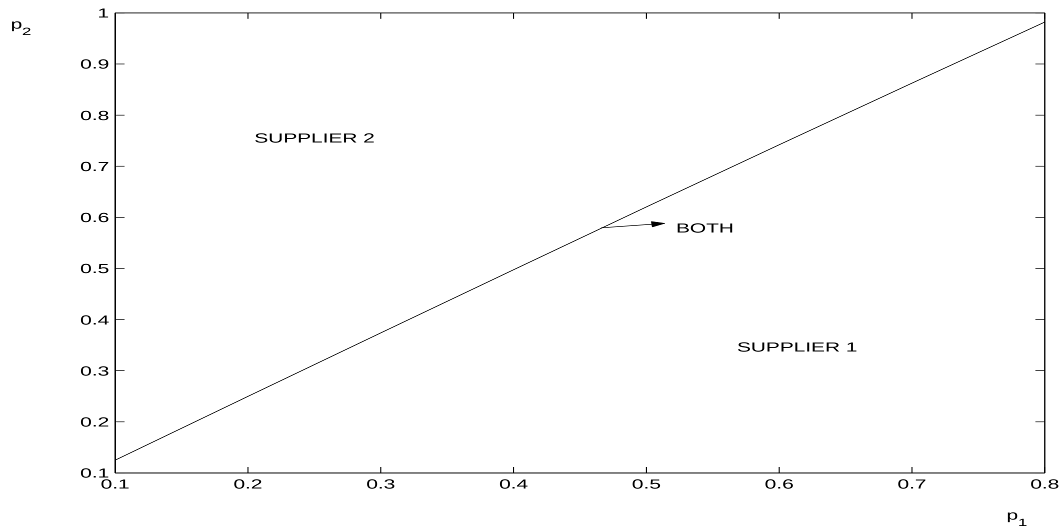

Figure 5.1: Where to use supplier 1 or 2 for $c_H = 20, D = 1, c_1 = 80, c_2 = 100$

5.2 Stochastically Proportional Yield Case

For this type of yield model, the random fractions for each is supplier (u, v) are assumed to have beta distributions for numerical analysis. The parameters defining the distributions are α_1, β_1 and α_2, β_2 for suppliers 1 and 2,respectively. Recall that the mean and the variance of a beta random variable are defined as:

$$\begin{aligned} \mu_j &= \frac{\alpha_j}{\alpha_j + \beta_j} \\ \sigma_j^2 &= \frac{\alpha_j \beta_j}{(\alpha_j + \beta_j)^2(\alpha_j + \beta_j + 1)} \end{aligned}$$

In almost all of the parameter sets, we considered the cases where $1 < \beta < \alpha$ where the distribution is said to be negatively skewed (skewed to left) for this α, β values (i.e. higher probability of receiving a high proportion of the amount ordered). The shape of the beta distribution is given below: (see Larson [15], page 207):

In Tables 5.11 to 5.14 the effect of the cost parameters on decision variables and objective function is investigated with different selling prices per item and distribution parameters for the suppliers, given in Table 5.10 (where the values used for K, c_H, and c_S are given in Table 5.1). For Tables 5.11, 5.12, 5.13, 5.14,

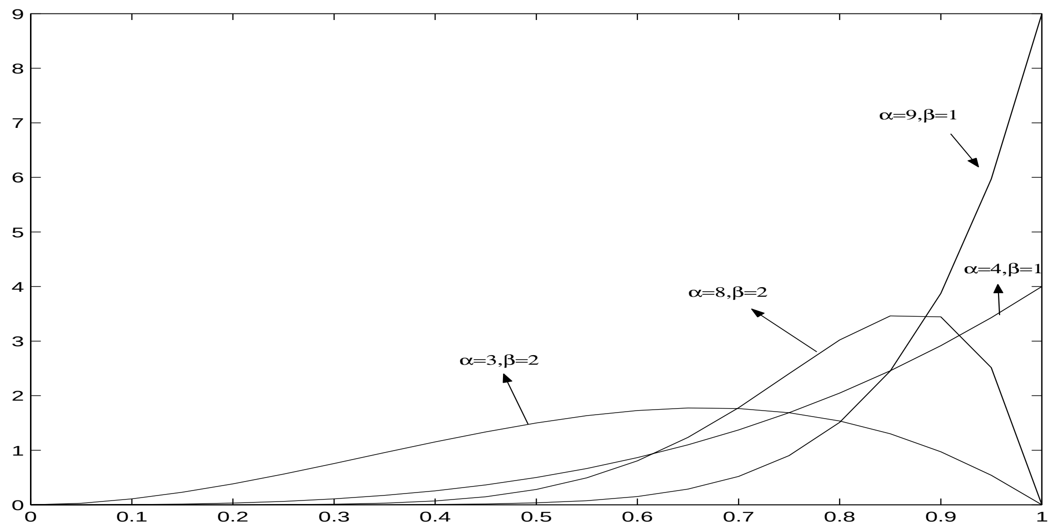

Figure 5.2: Beta density function for different $\alpha(a)$, and $\beta(b)$ values

the following observations are made:

Table #	c_1	c_2	α_1	α_2	β_1	β_2
Table 5.11	120	135	3	4	2	1
Table 5.12	120	135	8	9	2	1
Table 5.13	90	120	3	4	2	1
table 5.14	90	120	8	9	2	1

Table 5.10: Experimental Design # 3

- As the holding cost per unit per time increases, the values of Q_1^*, Q_2^*, and the cost rate go down. The reorder level decreases, that is, the absolute value of i increases. The results are also intuitive since the retailer tends to order less not to incur too much holding cost, and wants to take advantage of backordering which is less expensive as the holding cost increases.

- As the fixed ordering cost increases, Q_1^*, Q_2^*, $|i^*|$ increase since it is more profitable to place orders less frequently. The cost rate again increases as K increases.

• The reorder point approaches zero when the cost of shortage per unit per time goes up as expected (when backordering is more expensive, number of backorders less is expected to go down). The magnitudes of the ordering quantities also go down. If the retailer keeps ordering the same amount, s/he would incur extra holding cost since the reorder level is closer to zero than before as c_H increases. Hence, to decrease the inventory keeping cost, the retailer tends to order less. The cost rate again increases with the increase in c_H.

		$c_S = 50$				$c_S = 60$				$c_S = 80$			
K	c_H	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
200	5	4.19	8.39	-0.84	193.345	4.16	8.33	-0.70	193.666	4.12	8.25	-0.53	194.079
	10	3.09	6.19	-1.13	208.775	3.05	6.10	-0.96	209.580	3.00	6.00	-0.73	210.636
	20	2.36	4.71	-1.48	227.180	2.30	4.60	-1.27	229.009	2.23	4.46	-0.98	231.501
	30	2.05	4.10	-1.69	238.676	1.99	3.98	-1.46	241.453	1.91	3.81	-1.14	245.346
	40	1.88	3.76	-1.84	246.814	1.81	3.62	-1.59	250.421	1.72	3.44	-1.26	255.600
400	5	5.93	11.86	-1.19	211.299	5.89	11.78	-1.00	211.754	5.83	11.67	-0.75	212.337
	10	4.37	8.75	-1.60	233.121	4.32	8.63	-1.36	234.259	4.24	8.48	-1.04	235.753
	20	3.33	6.66	-2.09	259.148	3.25	6.51	-1.79	261.736	3.15	6.31	-1.39	265.260
	30	2.90	5.80	-2.39	275.407	2.81	5.62	-2.06	279.333	2.70	5.39	-1.62	284.840
	40	2.66	5.31	-2.60	286.915	2.56	5.12	-2.25	292.017	2.43	4.87	-1.79	299.341
600	5	7.27	14.53	-1.45	225.076	7.21	14.42	-1.22	225.633	7.14	14.29	-0.92	226.347
	10	5.36	10.72	-1.96	251.802	5.29	10.57	-1.66	253.196	5.19	10.39	-1.27	255.025
	20	4.08	8.16	-2.56	283.679	3.99	7.97	-2.19	286.848	3.86	7.73	-1.70	291.164
	30	3.55	7.10	-2.93	303.592	3.44	6.89	-2.53	308.400	3.30	6.61	-1.98	315.144
	40	3.25	6.51	-3.18	317.686	3.14	6.27	-2.76	323.935	2.98	5.96	-2.19	332.905

Table 5.11: Results with $c_1 = 90, c_2 = 120, \mu_1 = 0.6, \mu_2 = 0.8, \frac{\sigma_1^2}{\sigma_2^2} = 1.5$

• In Tables 5.11 and 5.14 retailer always uses both suppliers since the discriminant of the quadratic equation is always positive and positive real root is available for Q_2^*. When we compare Tables 5.11 and 5.14, we see that the relation between Q_1^* and Q_2^* is pretty much the same since $A = 0$ and B's($B = \frac{\mu_1 \sigma_2^2}{\mu_2 \sigma_1^2}$) are almost the same. But since the selling prices (c_1, c_2) are higher, the order quantities are lower in Table 5.14.

•*Comparison of Tables 5.11 & 5.12:*

When we analyze Table 5.12, we see that first supplier is used all the time. In Table 5.12, μ_1 is less than μ_2, and σ_1^2 is higher than σ_2^2. Hence, the uncertainty in the amount that is actually delivered is higher for the first supplier. However,

		$c_S = 50$				$c_S = 60$				$c_S = 80$			
K	c_H	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
	5	11.58	0.00	-0.84	155.670	11.50	0.00	-0.71	155.993	11.39	0.00	-0.54	156.407
	10	8.54	0.00	-1.14	171.017	8.43	0.00	-0.96	171.825	8.28	0.00	-0.74	172.886
200	20	6.51	0.00	-1.49	189.286	6.36	0.00	-1.27	191.125	6.16	0.00	-0.99	193.628
	30	5.67	0.00	-1.70	200.663	5.50	0.00	-1.47	203.454	5.27	0.00	-1.15	207.368
	40	5.20	0.00	-1.85	208.689	5.01	0.00	-1.60	212.317	4.76	0.00	-1.27	217.525
	5	16.38	0.00	-1.19	173.551	16.26	0.00	-1.00	174.008	16.10	0.00	-0.76	174.594
	10	12.08	0.00	-1.61	195.256	11.92	0.00	-1.36	196.399	11.71	0.00	-1.04	197.899
400	20	9.21	0.00	-2.10	221.092	8.99	0.00	-1.80	223.692	8.72	0.00	-1.39	227.233
	30	8.02	0.00	-2.41	237.181	7.77	0.00	-2.07	241.129	7.45	0.00	-1.63	246.664
	40	7.35	0.00	-2.61	248.531	7.08	0.00	-2.27	253.663	6.73	0.00	-1.80	261.028
	5	20.06	0.00	-1.46	187.272	19.91	0.00	-1.23	187.831	19.72	0.00	-0.93	188.549
	10	14.80	0.00	-1.97	213.854	14.60	0.00	-1.67	215.254	14.34	0.00	-1.27	217.092
600	20	11.28	0.00	-2.58	245.497	11.01	0.00	-2.20	248.682	10.67	0.00	-1.71	253.019
	30	9.82	0.00	-2.95	265.202	9.52	0.00	-2.54	270.038	9.13	0.00	-1.99	276.817
	40	9.00	0.00	-3.20	279.104	8.68	0.00	-2.78	285.389	8.25	0.00	-2.20	294.409

Table 5.12: Results with $c_1 = 90, c_2 = 120, \mu_1 = 0.8, \mu_2 = 0.9, \frac{\sigma_1^2}{\sigma_2^2} = 1.77$

		$c_S = 50$				$c_S = 60$				$c_S = 80$			
K	c_H	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
	5	0.00	11.47	-0.83	212.356	0.00	11.38	-0.70	212.676	0.00	11.28	-0.53	213.086
	10	0.00	8.45	-1.13	227.911	0.00	8.34	-0.95	228.710	0.00	8.20	-0.73	229.760
200	20	0.00	6.43	-1.47	246.514	0.00	6.28	-1.26	248.331	0.00	6.09	-0.97	250.806
	30	0.00	5.59	-1.68	258.179	0.00	5.42	-1.45	260.938	0.00	5.21	-1.14	264.806
	40	0.00	5.11	-1.82	266.464	0.00	4.93	-1.58	270.052	0.00	4.70	-1.25	275.200
	5	0.00	16.22	-1.18	230.419	0.00	16.10	-0.99	230.871	0.00	15.95	-0.75	231.451
	10	0.00	11.95	-1.59	252.416	0.00	11.79	-1.35	253.547	0.00	11.59	-1.03	255.031
400	20	0.00	9.09	-2.08	278.725	0.00	8.89	-1.78	281.294	0.00	8.62	-1.38	284.795
	30	0.00	7.91	-2.37	295.222	0.00	7.67	-2.05	299.124	0.00	7.36	-1.61	304.594
	40	0.00	7.23	-2.57	306.938	0.00	6.98	-2.23	312.013	0.00	6.64	-1.77	319.293
	5	0.00	19.86	-1.44	244.278	0.00	19.72	-1.21	244.832	0.00	19.53	-0.92	245.542
	10	0.00	14.64	-1.95	271.219	0.00	14.44	-1.65	272.604	0.00	14.19	-1.26	274.422
600	20	0.00	11.14	-2.55	303.441	0.00	10.88	-2.18	306.588	0.00	10.55	-1.69	310.876
	30	0.00	9.68	-2.90	323.646	0.00	9.39	-2.50	328.425	0.00	9.02	-1.97	335.124
	40	0.00	8.86	-3.15	337.995	0.00	8.55	-2.74	344.211	0.00	8.13	-2.17	353.126

Table 5.13: Results with $c_1 = 120, c_2 = 135, \mu_1 = 0.6, \mu_2 = 0.8, \frac{\sigma_1^2}{\sigma_2^2} = 1.5$

		$c_S = 50$				$c_S = 60$				$c_S = 80$			
K	c_H	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
200	5	3.59	7.19	-0.85	192.804	3.57	7.13	-0.71	193.129	3.53	7.07	-0.54	193.547
	10	2.65	5.31	-1.15	207.977	2.62	5.23	-0.97	208.792	2.57	5.14	-0.74	209.863
	20	2.03	4.05	-1.50	225.962	1.98	3.95	-1.29	227.820	1.91	3.83	-1.00	230.349
	30	1.77	3.53	-1.72	237.086	1.71	3.42	-1.48	239.911	1.64	3.28	-1.16	243.868
	40	1.62	3.24	-1.87	244.872	1.56	3.12	-1.62	248.549	1.48	2.96	-1.28	253.820
400	5	5.08	10.17	-1.20	210.534	5.04	10.09	-1.01	210.994	5.00	9.99	-0.76	211.585
	10	3.75	7.51	-1.63	231.992	3.70	7.40	-1.37	233.145	3.63	7.27	-1.05	234.658
	20	2.86	5.73	-2.13	257.427	2.80	5.59	-1.82	260.054	2.71	5.42	-1.41	263.630
	30	2.50	5.00	-2.44	273.158	2.42	4.84	-2.10	277.153	2.32	4.64	-1.64	282.749
	40	2.29	4.59	-2.65	284.170	2.21	4.42	-2.30	289.369	2.10	4.19	-1.82	296.823
600	5	6.23	12.45	-1.47	224.138	6.18	12.36	-1.24	224.702	6.12	12.24	-0.94	225.426
	10	4.60	9.19	-1.99	250.419	4.53	9.06	-1.68	251.831	4.45	8.90	-1.29	253.685
	20	3.51	7.02	-2.61	281.570	3.42	6.85	-2.23	284.788	3.32	6.63	-1.72	289.168
	30	3.06	6.12	-2.98	300.837	2.96	5.93	-2.57	305.730	2.84	5.68	-2.01	312.584
	40	2.81	5.62	-3.25	314.324	2.70	5.41	-2.81	320.692	2.57	5.13	-2.22	329.821

Table 5.14: Results with $c_1 = 120, c_2 = 135, \mu_1 = 0.8, \mu_2 = 0.9, \frac{\sigma_1^2}{\sigma_2^2} = 1.77$

numerical results show that retailer orders just from the first supplier,since c_1 is low enough with respect to c_2 that suppresses the negative effect of low level of reliability.

When Tables 5.11 and 5.12 are compared, it is observed that the retailer orders more from supplier 1(i.e. Q_1(Table 5.12) $>$ Q_1(Table 5.11)). Despite the fact that c_1 stays the same, the mean (μ_1) is higher and the variance (σ_1^2) is lower (i.e. the level of uncertainty for supplier is lower), the retailer orders more from supplier 1 as compared to the values of Q_1 in Table 5.11(where there is a tendency to order less when the mean is higher and variance is lower, intuitively). The reason why $Q_1's$ are higher in Table 5.12 is that the retailer needs to keep the lot size at a level which is enough to meet the demand in the long-run (i.e. the quantity ordered from supplier 2 is moved to supplier 1).

•*Comparison of Tables 5.11 & 5.13:*

In Table 5.13, the retailer orders from supplier 2 only. Both suppliers have the same distribution parameters as in Table 5.11, but supplier 1 is relatively expensive in Table 5.13. Even though the unit price c_1 is still less than c_2, the effect of lower mean and higher variance forces the retailer to order from supplier 2 only. The values of Q_2 also increases due to reasons explained above (the desire to meet the total demand).

•*Comparison of Tables 5.13 & 5.14:*

It is observed that supplier 1 is also used because the prices are the same for both tables but the mean is higher and the variance is lower for supplier 1.

After we investigate the effect of cost parameters on the values of decision variables and the cost rate under different yield distribution parameters and selling price schemes, the effect of means and variances of yield distributions under some particular cost parameters and selling prices is explored. In Tables 5.15, 5.16, and 5.17 the cost structures are the same. In these tables, μ_2 increases and σ_2^2 decreases as we move rightwards. So, as we improve the process capability of supplier 2 (move to right in the table), the retailer starts ordering more from supplier 2 for the cases where both suppliers are used. Obviously, we observe lower values for Q_1. The absolute value of the reorder point gets higher since uncertainty is lower. Additionally, since the process capability stays the same for supplier 1 and gets better for supplier 2 as we move rightwards, the total amount ordered $(Q_1 + Q_2)$ decreases. In Table 5.15, we observe cases where supplier 1 is used only. For these cases, the quantity ordered (Q_1) is independent of the distribution parameters for supplier 2 (α_2, β_2) complying with the theoretical findings in Chapter 4. As we move downwards, the effect of the change in α_1, β_1 on the values of decision variables is observed. When the mean gets higher and variance gets lower (for supplier 1), we see that the retailer starts ordering from supplier 1 only. In this case, the total amount ordered decreases. But when μ_1 is kept constant and σ_1^2 is lowered, total amount ordered increases for this particular parameter set. Therefore, we are not guaranteed to order less when the variance is lower and the mean is the same contrary to the intuition that you always order less when the variance decreases.

Within Tables 5.16 and 5.17, we observe similar behavior for the values of the decision variables. The mean μ_2 is kept the same but the variance σ_2^2 is decreased further when Tables 5.15,5.16 and Tables 5.16, 5.17 are compared. It is observed from numerical results that the retailer orders more from supplier 2 and less from the first supplier, since the second supplier became more reliable for the cases when both suppliers are used.

		$\beta_2 = 1$											
		$\alpha_2 = 3$				$\alpha_2 = 4$				$\alpha_2 = 5$			
β_1	α_1	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
1	2	6.26	3.77	-2.62	296.887	2.67	6.68	-2.67	290.310	0.04	8.54	-2.68	284.020
	3	9.26	0.00	-2.60	277.295	9.26	0.00	-2.60	277.295	7.80	1.44	-2.64	277.097
	4	8.84	0.00	-2.65	266.400	8.84	0.00	-2.65	266.400	8.84	0.00	-2.65	266.400
2	4	8.71	1.63	-2.63	294.117	3.95	5.65	-2.68	289.783	0.06	8.52	-2.68	284.020
	6	9.46	0.00	-2.66	274.259	9.46	0.00	-2.66	274.259	9.46	0.00	-2.66	274.259
	8	8.97	0.00	-2.69	264.398	8.97	0.00	-2.69	264.398	8.97	0.00	-2.69	264.398
3	6	10.32	0.21	-2.64	292.292	4.89	4.89	-2.69	289.398	0.08	8.51	-2.68	284.020
	9	9.54	0.00	-2.68	273.044	9.54	0.00	-2.68	273.044	9.54	0.00	-2.68	273.044
	12	9.02	0.00	-2.71	263.632	9.02	0.00	-2.71	263.632	9.02	0.00	-2.71	263.632

Table 5.15: Results with $K = 500, c_1 = 100, c_2 = 120, c_H = 30, c_S = 50$

		$\beta_2 = 2$											
		$\alpha_2 = 6$				$\alpha_2 = 8$				$\alpha_2 = 10$			
β_1	α_1	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
1	2	5.51	4.53	-2.65	296.291	1.66	7.61	-2.70	289.021	0.00	8.66	-2.71	282.606
	3	9.26	0.00	-2.60	277.295	9.26	0.00	-2.60	277.295	7.54	1.69	-2.65	277.054
	4	8.84	0.00	-2.65	266.400	8.84	0.00	-2.65	266.400	8.84	0.00	-2.65	266.400
2	4	8.22	2.10	-2.65	293.999	2.61	6.83	-2.70	288.806	0.00	8.66	-2.71	282.606
	6	9.46	0.00	-2.66	274.259	9.46	0.00	-2.66	274.259	9.46	0.00	-2.66	274.259
	8	8.97	0.00	-2.69	264.398	8.97	0.00	-2.69	264.398	8.97	0.00	-2.69	264.398
3	6	10.24	0.29	-2.64	292.290	3.38	6.20	-2.71	288.632	0.00	8.66	-2.71	282.606
	9	9.54	0.00	-2.68	273.044	9.54	0.00	-2.68	273.044	9.54	0.00	-2.68	273.044
	12	9.02	0.00	-2.71	263.632	9.02	0.00	-2.71	263.632	9.02	0.00	-2.71	263.632

Table 5.16: Results with $K = 500, c_1 = 100, c_2 = 120, c_H = 30, c_S = 50$

		$\beta_2 = 3$											
		$\alpha_2 = 9$				$\alpha_2 = 12$				$\alpha_2 = 15$			
β_1	α_1	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
1	2	5.13	4.91	-2.66	295.990	1.20	8.03	-2.71	288.444	0.00	8.69	-2.72	282.079
	3	9.26	0.00	-2.60	277.295	9.26	0.00	-2.60	277.295	7.43	1.80	-2.65	277.034
	4	8.84	0.00	-2.65	266.400	8.84	0.00	-2.65	266.400	8.84	0.00	-2.65	266.400
2	4	7.95	2.36	-2.65	293.933	1.95	7.41	-2.71	288.328	0.00	8.69	-2.72	282.079
	6	9.46	0.00	-2.66	274.259	9.46	0.00	-2.66	274.259	9.46	0.00	-2.66	274.259
	8	8.97	0.00	-2.69	264.398	8.97	0.00	-2.69	264.398	8.97	0.00	-2.69	264.398
3	6	10.19	0.33	-2.64	292.289	2.58	6.89	-2.71	288.229	0.00	8.69	-2.72	282.079
	9	9.54	0.00	-2.68	273.044	9.54	0.00	-2.68	273.044	9.54	0.00	-2.68	273.044
	12	9.02	0.00	-2.71	263.632	9.02	0.00	-2.71	263.632	9.02	0.00	-2.71	263.632

Table 5.17: Results with $K = 500, c_1 = 100, c_2 = 120, c_H = 30, c_S = 50$

In Tables 5.18, 5.19, and 5.20 a similar analysis is done under a different cost structure. In this setting, selling prices c_1, c_2 are more close to each other. Since, the item provided by supplier 1 is relatively expensive (with respect to the one in Tables 5.15, 5.16, and 5.17) in this setting, retailer orders more from the second supplier than before. Similarly, when just supplier 2 is used, value of Q_2 is independent of α_1, β_1 (which must be the case according to closed form formulas derived in Chapter 4). Also, the reduction in the variance (in σ_1^2) keeping μ_1 the same leads to an increase in Q_1 (same rule applies for Q_2 also).

		$\beta_2 = 1$											
		$\alpha_2 = 3$				$\alpha_2 = 4$				$\alpha_2 = 5$			
β_1	α_1	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
1	2	2.27	8.78	-2.31	284.442	0.00	10.17	-2.32	272.955	0.00	9.84	-2.34	265.889
	3	10.67	0.00	-2.29	271.620	6.26	4.38	-2.34	270.055	1.43	8.60	-2.35	265.758
	4	10.17	0.00	-2.32	260.455	10.17	0.00	-2.32	260.455	9.92	0.24	-2.33	260.453
2	4	3.18	8.02	-2.32	284.232	0.00	10.17	-2.32	272.955	0.00	9.84	-2.34	265.889
	6	10.88	0.00	-2.33	269.249	8.63	2.19	-2.35	268.959	2.08	8.03	-2.36	265.698
	8	10.30	0.00	-2.35	258.909	10.30	0.00	-2.35	258.909	10.30	0.00	-2.35	258.909
3	6	3.78	7.52	-2.33	284.092	0.00	10.17	-2.32	272.955	0.00	9.84	-2.34	265.889
	9	10.96	0.00	-2.35	268.316	10.10	0.83	-2.35	268.280	2.53	7.64	-2.36	265.657
	12	10.35	0.00	-2.36	258.323	10.35	0.00	-2.36	258.323	10.35	0.00	-2.36	258.323

Table 5.18: Results with $K = 500, c_1 = 110, c_2 = 120, c_H = 20, c_S = 50$

		$\beta_2 = 2$											
		$\alpha_2 = 6$				$\alpha_2 = 8$				$\alpha_2 = 10$			
β_1	α_1	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
1	2	0.49	10.46	-2.33	282.561	0.00	10.30	-2.35	271.409	0.00	9.93	-2.37	264.803
	3	10.67	0.00	-2.29	271.620	5.28	5.35	-2.36	269.709	0.00	9.93	-2.37	264.803
	4	10.17	0.00	-2.32	260.455	10.17	0.00	-2.32	260.455	9.87	0.30	-2.33	260.453
2	4	0.73	10.26	-2.34	282.551	0.00	10.30	-2.35	271.409	0.00	9.93	-2.37	264.803
	6	10.88	0.00	-2.33	269.249	7.90	2.90	-2.36	268.866	0.00	9.93	-2.37	264.803
	8	10.30	0.00	-2.35	258.909	10.30	0.00	-2.35	258.909	10.30	0.00	-2.35	258.909
3	6	0.92	10.10	-2.34	282.543	0.00	10.30	-2.35	271.409	0.00	9.93	-2.37	264.803
	9	10.96	0.00	-2.35	268.316	9.76	1.16	-2.36	268.266	0.00	9.93	-2.37	264.803
	12	10.35	0.00	-2.36	258.323	10.35	0.00	-2.36	258.323	10.35	0.00	-2.36	258.323

Table 5.19: Results with $K = 500, c_1 = 110, c_2 = 120, c_H = 20, c_S = 50$

		$\beta_2 = 3$											
		$\alpha_2 = 9$				$\alpha_2 = 12$				$\alpha_2 = 15$			
β_1	α_1	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR	Q_1^*	Q_2^*	i^*	CR
1	2	0.00	10.96	-2.35	281.649	0.00	10.35	-2.36	270.823	0.00	9.97	-2.37	264.400
	3	10.67	0.00	-2.29	271.620	4.79	5.84	-2.36	269.537	0.00	9.97	-2.37	264.400
	4	10.17	0.00	-2.32	260.455	10.17	0.00	-2.32	260.455	9.84	0.33	-2.33	260.453
2	4	0.00	10.96	-2.35	281.649	0.00	10.35	-2.36	270.823	0.00	9.97	-2.37	264.400
	6	10.88	0.00	-2.33	269.249	7.49	3.30	-2.36	268.813	0.00	9.97	-2.37	264.400
	8	10.30	0.00	-2.35	258.909	10.30	0.00	-2.35	258.909	10.30	0.00	-2.35	258.909
3	6	0.00	10.96	-2.35	281.649	0.00	10.35	-2.36	270.823	0.00	9.97	-2.37	264.400
	9	10.96	0.00	-2.35	268.316	9.55	1.36	-2.36	268.258	0.00	9.97	-2.37	264.400
	12	10.35	0.00	-2.36	258.323	10.35	0.00	-2.36	258.323	10.35	0.00	-2.36	258.323

Table 5.20: Results with $K = 500, c_1 = 110, c_2 = 120, c_H = 20, c_S = 50$

5.2.1 Diversification Among Suppliers

In the following part, by changing the unit selling prices (in Figures 5.3 to 5.5) and then the distribution parameters (in Figures 5.6 and 5.7), we analyzed where the retailer orders from both suppliers and where he/she orders from one supplier only. We observe a stepwise behavior for the regions where it is better to use both suppliers due to numerical search increments of selling prices. A smoother region would have appeared if we had taken the increments smaller.

- In Figure 5.3, the means μ_1 and μ_2 are the same but the variances are different. The ratio of the variances is $\frac{\sigma_1^2}{\sigma_2^2} = \frac{11}{6}$. Hence, level of uncertainty involved with supplier 2 is lower. As it is observed, both suppliers are used mostly when the selling prices c_1, c_2 are close to each other. Out of 961 cases, supplier 1 is used 378 times (%39), supplier 2 is used 435 times (%45), and both suppliers are used 148 times (%16). Even the means are the same and σ_1^2 is almost two times σ_2^2, numerical results show that the effect of variance is not that much significant since the percentages mentioned above are close to each other. In Figure 5.4, the ratio of the means and variances are $\frac{\mu_1}{\mu_2} = 1.33$ and $\frac{\sigma_1^2}{\sigma_2^2} = 1.22$. As the results suggest, supplier 1 dominates in this case. Out of 961 cases, supplier 1 is used 816 times (%84), supplier 2 is used 45 times (%5), and both suppliers are used 100 times (%11). Since both μ_1 and σ_1^2 are higher than that of supplier 2, it can be assumed that they are close in terms of the level of uncertainty. But again, it is verified from numerical results that the difference in the mean suppresses the difference in the variances. In Figure 5.5, the ratio $\frac{\sigma_1^2}{\sigma_2^2} = 1.96$ which is close to the ratio of variances in Figure 5.3. But, different from Figure 5.3, the means are not the same and $\frac{\mu_1}{\mu_2} = 1.07$. In this figure, out of 961 cases, supplier 1 is used 455 times (%47), supplier 2 is used 286 times (%30), and both suppliers are used 220 times (%23). Obviously, these percentage figures are just for the cases analyzed in this study. These percentages depend on the zone (interval of the parameters) investigated. Comparing the results obtained from Figures 5.3 and 5.5, we see that a small change in the ratio of means keeping the ratio of variances almost the same results in a significant change in the retailers decisions as to from which supplier to order. Even though the first supplier has a higher variance (almost twice), retailer orders from supplier 1 approximately in half of the cases due to μ_1 being slightly greater than μ_2. Consequently, Figures 5.3 to 5.5 suggest that the means are more effective in shaping the decision of the retailer than the variances.

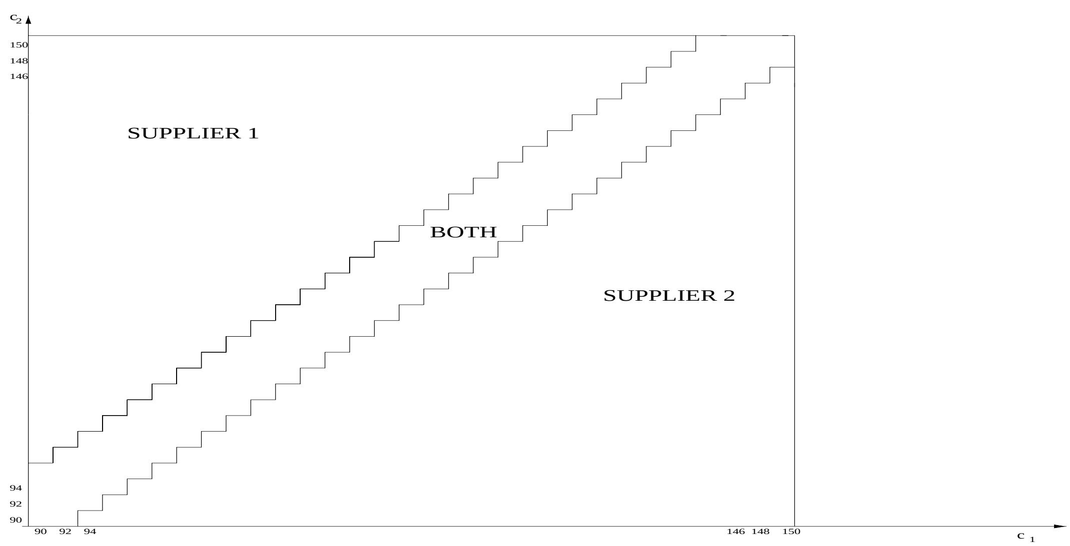

Figure 5.3: Where to use supplier 1 or 2 for $K = 500, c_H = 30, c_S = 50, \alpha_1 = 4, \alpha_2 = 8, \beta_1 = 1, \beta_2 = 2$

• In Figure 5.6, ratio of selling prices is $\frac{c_1}{c_2} = 0.8$, and μ_1, μ_2 is increased by 0.05 in intervals $[0.65, 0.95]$ and $[0.55, 0.85]$, respectively (whereas the variances are decreasing). This figure is similar to the previous figures in the sense that the

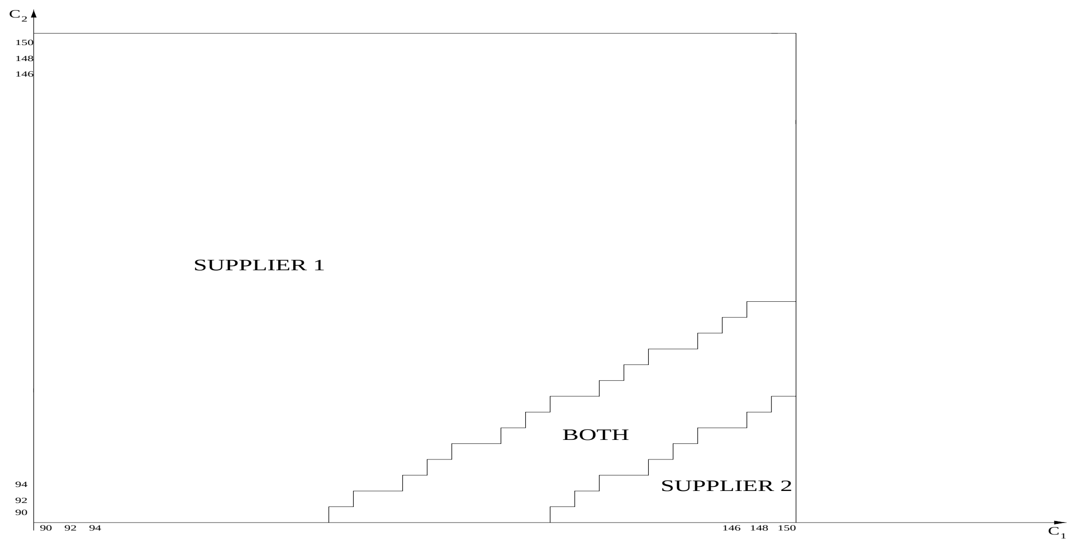

Figure 5.4: Where to use supplier 1 or 2 for $K = 500, c_H = 30, c_S = 50, \alpha_1 = 4, \alpha_2 = 6, \beta_1 = 1, \beta_2 = 4$

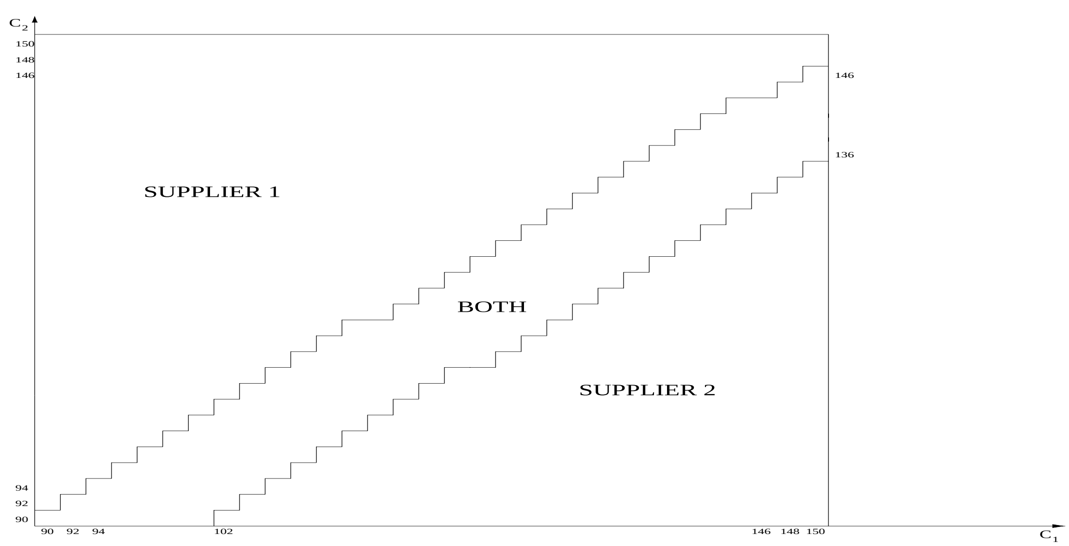

Figure 5.5: Where to use supplier 1 or 2 for $K = 500, c_H = 30, c_S = 50, \alpha_1 = 6, \alpha_2 = 4, \beta_1 = 2, \beta_2 = 1$

regions where retailer orders from only one supplier is separated by the region where both suppliers are used. In Figure 5.7, the only difference from Figure 5.6 is c_2 which is increased to 90 from 80.Due to the effect of this price change, the percentage of the times supplier 1 is used increased to %57 from %31, percentage of the times supplier 2 is used decreased to %39 from %61, and finally there is a small change in the percentage of the times both suppliers are used (a decrease from %4 to %2).

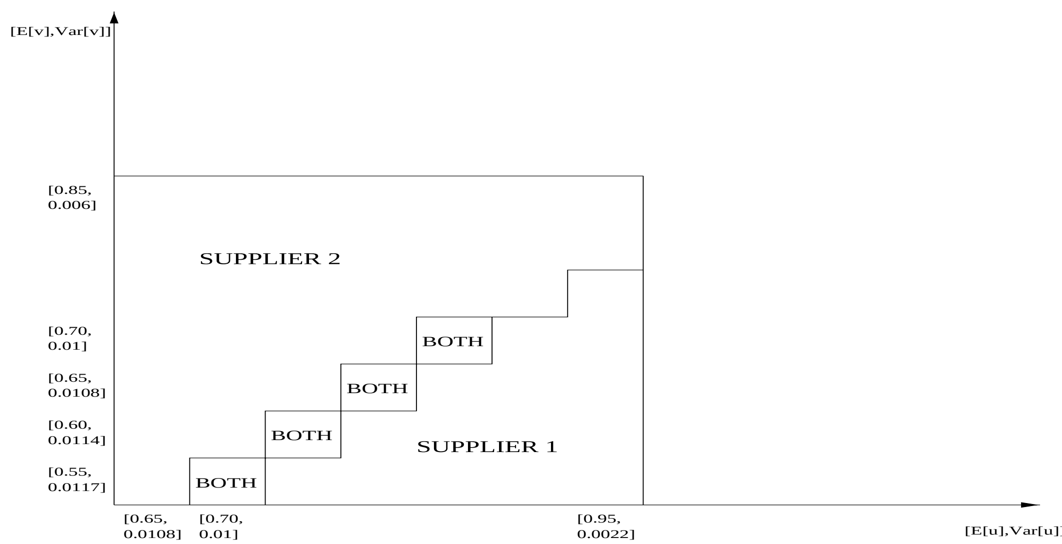

Figure 5.6: Where to use supplier 1 or 2 for $K = 500, c_H = 30, c_S = 50, c_1 = 100, c_2 = 80$

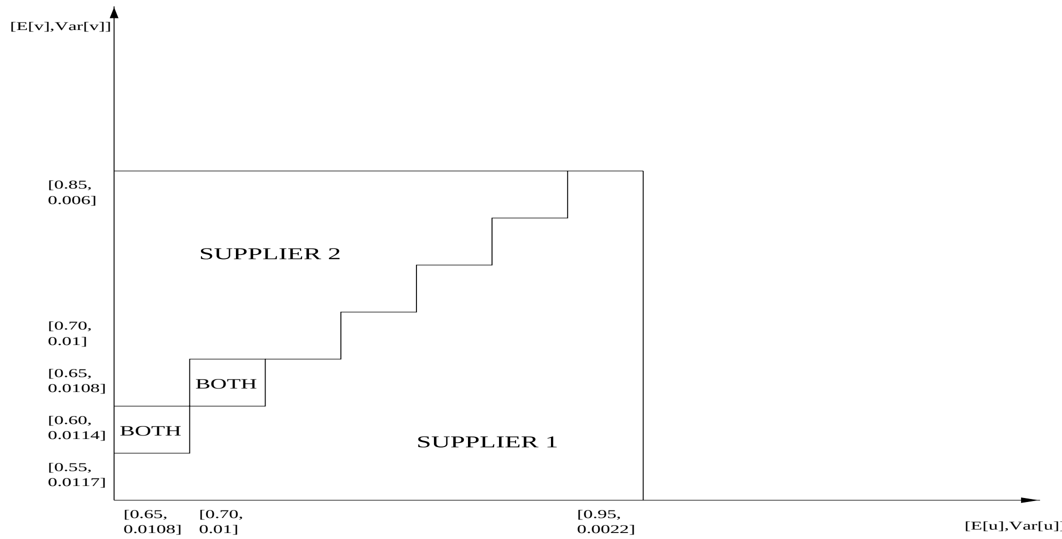

Figure 5.7: Where to use supplier 1 or 2 for $K = 500, c_H = 30, c_S = 50, c_1 = 100, c_2 = 90$

5.2.2 Performance of the Algorithm

In Table 5.22, the results obtained from the algorithmic solution procedure are compared with the optimal ones found by Matlab for the parameter sets given in Table 5.21. The optimization procedure used by Matlab itself is an unconstrained one that does not take into account the non negativity of the order quantities. Therefore, it does not always give the optimal values for all parameter sets. For the parameter sets given in Table 5.21, optimal solutions found by Matlab's minimization function are all feasible. Also, the computation time for the algorithm proposed is significantly less than the computation time for Matlab's unconstrained optimization function. The probabilities P_i are also given in the Table. As it is observed, P_i values are very much close to zero in most of the cases. But, for the cases where holding cost is relatively higher and the means of the random fractions are lower, P_i values start increasing. From, the numerical study, we have seen that taking $P_i = 0$ is a very good approximation, since the incorporation of P_i into the analysis does not make much difference in terms of the cost rate. But, of course, in order to find the exact cost rate, $P_i^\prime s$ should be computed. Additionally, difference between the optimal cost rate and the cost rate obtained by algorithmic solution is very low. Thus, this algorithm is a very practical tool to find near optimal solutions in a very short time.

K	D	c_H	c_S	c_1	c_2	$\frac{\mu_1}{\mu_2}$	$\frac{\sigma_1^2}{\sigma_2^2}$	Experiment #
500	1	20	30	80	120	0.75	1.5	1
500	1	20	40	80	120	0.75	1.5	2
500	1	20	50	80	120	0.75	1.5	3
500	1	10	30	80	110	0.75	1.5	4
500	1	15	20	80	120	0.75	1.5	5
500	1	30	20	80	120	0.75	1.5	6
500	1	50	20	80	120	0.75	1.5	7
500	1	70	20	80	120	0.75	1.5	8
500	1	100	20	80	120	0.75	1.5	9
500	1	30	50	100	120	0.88	1.48	10
500	1	30	50	100	120	0.825	2.08	11
500	1	30	50	100	120	0.88	0.85	12
500	1	30	50	100	120	0.825	2.18	13
500	1	20	50	110	120	0.88	1.48	14
500	1	10	30	100	110	0.94	1.41	15
500	1	20	50	100	120	0.76	0.68	16
500	1	50	30	135	150	0.8	0.48	17

Table 5.21: Experimental Design # 4

	Algorithm					Optimal				
Experiment #	Q_1	Q_2	i	P_i	CR	Q_1^*	Q_2^*	i^*	CR*	%Δ in CR
1	12.95	0.90	-3.39	0.030	252.4375	13.61	0.33	-3.41	252.4155	0.0087
2	12.76	0.52	-2.69	0.019	257.9852	13.18	0.16	-2.70	257.9750	0.0039
3	12.64	0.29	-2.23	0.014	261.7807	12.95	0.02	-2.24	261.070	0.0027
4	9.66	6.82	-2.81	0.000	224.1950	9.66	6.82	-2.81	224.1950	0
5	16.47	0.00	-4.23	0.054	234.3285	16.50	0.00	-4.30	234.3214	0.0031
6	9.93	3.21	-5.11	0.059	255.4081	10.59	2.65	-5.15	255.3709	0.015
7	7.18	4.37	-5.57	0.094	268.4309	7.74	3.93	-5.66	268.3674	0.024
8	5.97	4.80	-5.77	0.134	275.8104	6.54	4.38	-5.92	275.6800	0.047
9	5.03	5.05	-5.88	0.185	282.7369	5.61	4.70	-6.14	282.4248	0.11
10	6.26	3.76	-2.62	0.005	296.8871	6.27	3.76	-2.62	296.8871	0
11	2.67	6.68	-2.67	0.001	290.3097	2.67	6.68	-2.67	290.3097	0
12	8.70	1.62	-2.63	0.004	294.1169	8.72	1.60	-2.63	294.1169	0
13	2.61	6.83	-2.70	0.000	288.8064	2.61	6.83	-2.70	288.8063	0
14	2.27	8.78	-2.31	0.002	284.4417	2.26	8.78	-2.31	284.4417	0
15	10.90	3.85	-2.81	0.000	223.2593	10.91	3.86	-2.82	223.2593	0
16	5.19	16.67	-1.95	0.098	513.1126	4.91	16.87	-2.03	513.0784	0.0066
17	5.45	7.76	-3.79	0.212	475.7064	4.02	8.91	-4.20	474.8081	0.19

Table 5.22: Comparison of the algorithmic results and optimal values

Chapter 6

Conclusion

In this thesis, we discuss an inventory model with two suppliers without making the general assumption that all ordered units are received. Instead, we consider a system where the suppliers deliver a random fraction of the quantity ordered. This type of a system may appear especially in firms producing electronic products having very tight quality constraints in real life. We consider real life examples such that the defective units are detected through inspection performed by the suppliers, and consider pay-for-input, that is, pay for the amount that you order not the amount that you receive. Our model could easily be extended to the one considering pay-for-output (which was the case in the study by Parlar and Wang [18]), by just adjusting the selling prices of the items. We assume a constant demand, same holding cost for each supplier's product, a fixed ordering cost that does not depend on which supplier is used, and allow backordering. The lead time is assumed to be zero. In order to analyze the effect of random yield on the decision variables and the expected cost rate, we discuss two different yield models: binomial yield and stochastically proportional yield. It is assumed that density distributions for each supplier's random fractions are known. These distributions are independent from each other and stationary. The policy used is of (Q, r) type. That is, you order the amount Q when the inventory level hits the reorder point r.

The regions where the expected cost rate function is convex are obtained.

As the theoretical findings suggest, we are able to find which supplier to use by the simple and closed form expressions. The optimal values of the decision variables and cost rate are also found by simple formulas. It is shown that the order quantity for supplier "j" is independent of the distribution parameters of the other supplier, given that supplier "j" is used only. The cost rate function for the case of "M" suppliers ($M > 2$) is obtained and solution method is proposed.

A detailed numerical analysis is done in order to observe the effect of the cost figures and the parameters of the distributions on the order quantities, reorder level, and expected cost rate. The expected cost rate increases as we increase K, c_H, and c_S. Quantity ordered decreases as c_H, c_S increase and increases as K increases. The magnitude of the reorder level increases as we increase K, c_H and decreases as we increase c_S. All of these findings are intuitive. When we analyze the effect of the mean and variances of the random yield distributions, we observe that as the reliability of supplier j increases (i.e. mean is higher and the variance is lower), the retailer starts ordering more from supplier j when both suppliers are used. When the retailer orders from just one supplier, s/he orders less from that supplier as reliability increases. A lower level of uncertainty also leads to an increase in the magnitude of the reorder level, since P_i is less when randomness is reduced. Moreover, the effect of a change in μ (when σ^2 is kept constant) suppresses the effect of a change in σ^2 (when μ is kept constant). That is, the values of decision variables and cost rate are much more sensitive to changes in mean than variance. Another result obtained from numerical study shows that the total amount ordered is pretty much the same to meet the total demand in the long-run regardless of which supplier(s) is used.

For the cases where just one supplier is used, the amount ordered from supplier j is independent of the distribution parameters of the other supplier, complying with the theoretical findings. An extensive numerical study is done to construct the regions showing which supplier is used for which parameter set. As a result, for both yield models, ordering from just one supplier is optimal most of the time which forces the other supplier to improve its process capability or to offer less prices. Especially, for the binomial yield case, the retailer always orders from one supplier, since both suppliers are used only when the equation of parameters ($c_2p_1 - c_1p_2 + \frac{c_H}{2D}p_1p_2(p_1 - p_2) = 0$) is equal to zero. Even when we have the equality, the retailer can still order from one supplier as it is proved in Chapter 3. Also, the decision as to which supplier should be used can be made by comparing the effective selling prices ($\frac{c_1}{p_1}, \frac{c_2}{p_2}$) for the binomial case, as the numerical results suggest. For the stochastically proportional yield case, it is optimal to order from both suppliers for a limited number of cases, but more than that of binomial yield case. For this case, the effect of the change in variances is not that much significant in the decision process (to choose the right supplier(s)). But, a small change in the ratio of means (when σ^2 is the same), results in a significant change in the retailer's decision as to from which supplier to order.

The comparison of the results of the algorithm with the optimal ones (only feasible solutions that are found by Matlab function "fminsearch" solving unconstrained minimization problems are given) for the stochastically proportional yield case shows that the algorithm works very well for a wide range of parameter sets. Furthermore, we also conclude that the results with $P_i = 0$ are very close to the ones with positive $P_i's$. That is, the algorithm converges very fast.

Consequently, the main contribution of this study is providing simple, and practical closed form expressions for inventory managers, that would help them make decisions as to which supplier should be used for a particular parameter set for the problems involving two suppliers. For the binomial yield case, it is concluded that using only one supplier is better almost always. Also, the selection between suppliers could be done by just comparing effective selling prices as the numerical results suggest. For the stochastically proportional yield case, the selection criterion is slightly harder than that of binomial yield case. But, we are still able to choose which supplier(s) should be used by looking for a feasible solution to the quadratic equation obtained. The *n-supplier version* is also analyzed except for the numerical part. The exact cost expressions are obtained considering the effect of P_i, that was missing in the literature. Also, for the stochastically proportional yield case, this is the only model that allows backordering with two suppliers. This thesis can be further extended to analyze models involving positive lead time (fixed or random), random demand. Also, the effect of different inspection policies (the supplier provides all the amount ordered, but the retailer does the inspection business) on the model. The models allowing substitution between products of two suppliers can also be analyzed. Lastly, game theoretical models can be constructed using the results of this study where the retailer is the "follower" and the suppliers are leader, since the market share is very sensitive to changes in cost parameters and the process capability of the suppliers. By modeling this problem, the suppliers could be able to optimize their profits by improving the quality of their products or employing some discount policies.

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Appendix

Notation	
Q_1	Quantity ordered from supplier 1
Q_2	Quantity ordered from supplier 2
i	Reorder point that triggers the placement of an order ($i < 0$)
X^1	Amount actually received from supplier 1
X^2	Amount actually received from supplier 2
X	Total amount actually received ($X^1 + X^2$)
c_H	Holding cost per unit per time
c_S	Shortage cost per unit per time
K	Ordering cost
c_1	Purchasing cost of an item from supplier 1
c_2	Purchasing cost of an item from supplier 2
D	Constant demand rate
P_i	Probability that the amount received is smaller than $'' - i''$
p_1	Probability of producing a good unit for supplier 1 (binomial yield)
p_2	Probability of producing a good unit for supplier 2 (binomial yield)
u	Random fraction for supplier 1 with density function $f(u)$
μ_1	Mean of the random variable u
σ_1^2	Variance of the random variable u
v	Random fraction for supplier 2 with density function $f(v)$
μ_2	Mean of the random variable v
σ_2^2	Variance of the random variable v

Table A.1: Notation