ROBUST ENTANGLEMENT IN ATOMIC SYSTEMS

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

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Various models for generation of robust atomic entangled states and their implementation with current accessible technologies are proposed and worked out. Deterministic creation of long living Bell states with respect to metastable states in three-level Λ type systems is studied. Strong atom-field coupling drives atoms into a transient entangled state followed by an irreversible evolution towards a long-living maximally entangled state featuring robustness against dipole-allowed transitions. First, generation of pairwise atomic entanglement in cavities in ideal case is discussed, extension to multi-party entangled states is made. Observation of photons emitted from the system signals the generation of a Bell state.

The interaction of multi-level atoms with body-assisted electro-magnetic field in the presence of dispersing and absorbing media is studied and these results are applied to the description of a pair of Λ type atoms passing by a microsphere. Microspheres give rise to resonances of well defined height and width with easy access to strong and weak coupling regimes for atom-field interaction, thus enabling realization of the proposed scheme of "robust entanglement of three-level atoms". Even in realistic settings it is possible to obtain quite high amount of entanglement at spatially well separated distances.

Then we focus on steady state entanglement between atomic dipoles. It is shown that two dipoles in free space driven by a classical driving field become entangled in the steady state. The crucial point is that, this entanglement is irrespective of the initial state and may be preserved as long as the engineered system is kept intact.

Absorption effects in real cavities are studied, and an input-output relation is formulated in the presence of a source in the cavity. Extraction of non-classical

photon states from a cavity is investigated.

Keywords: Quantum Optics, Quantum Information Theory, EPR paradox, Entanglement, Cavity Quantum Electrodynamics, Quantum Open Systems, Decoherence, Quantum Noise .

ÖZET

ATOM SİSTEMLERİNDE KALICI DOLAŞIKLIK

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Kalıcı atomik dolanık durumları oluşturabilecek bir dizi teorik model ve bu modellerin bugünkü teknoloji ile gerçekleştirilebilmesi için değişik deneysel yöntemler önerilmiş ve irdelenmiştir. Üç seviyeli atomlarda en alt enerji seviyesine dipol geçisi mümkün olmadığı için ömrü uzun olan yarı-kararlı seviyeler kullanılarak, boş uzayda bile dolanıklığını koruyabilecek Bell durumlarının deterministik bir şekilde oluşturulabilmesi üzerinde duruldu. İkili atomik dolanık durumların kovukçuklarda ideal koşullarda oluşturulabilmesi üzerinde durulmuş ve çoklu dolanık durumlara genellenmiştir. Bu sistemlerin kendiliğinden yayımladıgı fotonlar dolanıklığın oluşumuna işaret eder.

Çok seviyeli atomların saçılım ve emilimin mevcut olduğu bir ortamda birbirleriyle ve elektromagnetik(EM) alanla etkileşimleri çalısılmış ve bu sonuçlar üç seviyeli atomlara uygulanmıştır. Önerilen "kalıcı dolanık durumların" sistemin mikroküreçik çınlaçlar kullanılarak gerçekleştirilebilmesi irdelenmiştir. Mikroküreçikler EM alanın genliği ve yüksekliği belirli rezonanslar göstermesine neden olur ve bu sayede etraftaki atom ve EM alan arasında geçiş frekansına bağlı olarak zayıf veya güçlü etkileşim rejimlerine ulaşılabilir. Gerçekçi durumlarda bile birbirlerinden yeterince uzak mesafede bulunan atomlar arasında yüksek oranda dolanıklığın oluşabilmesi mümkündür.

Bir sonraki aşamada atomik dipoller arasında durağan dolanık durumların oluşumu çalısılmıştır. Klasik bir EM dalga tarafından beslenen iki dipolun boş uzayda bile dolanık duruma geçebilecekleri ortaya çıkmaktadır. Bu dolanık durumların en çarpıcı özelliklerinden birisi ortaya cıkan durumun baslangıç durumundan bağımsız olması ve kurulan sistem korunduğu sürece dolanık durumun korunmasıdır.

Kovukçuklarda emilim etkileri, bu tür gerçekçi kovukçuklardan fotonik kuvantum durumların dışarı cıkartılması incelenmiş ve bu sistemler için içeri gelen ve dısari çıkan durumlar arasında ilişki kurulmuştur.

Anahtar sözcükler: Kuantum Optik, Kuantum Bilgi Kuramı, EPR paradoksu, Dolanıklık, Oyuk Kuvantum Elektrodinamiği, Açık Kuantum Sistemler, Kuantum Uyumsuzlaşma, Kuantum Gürültü .

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Chapter 1

Introduction

Quantum entanglement is one of the central themes making distinction between classical and quantum mechanics. On the other hand the interference phenomena or quantum superposition constitutes another distinctive behavior of quantum mechanics and is a well understood phenomena. For a long time, entanglement has been recognized as a curious phenomenon of no practical importance. However, with the advent of experimental techniques and *quantum information science*, entanglement and generation of robust entangled states has become a subject of intense research regarding its fundamental and technological implications.

Quantum superposition principle is the most intriguing feature of quantum mechanics, and rules the microscopic world. A quantum system may be in a superposition state of the eigenstates of an observable, i.e., it is likely to be found in different classical realities. Once the measurement is performed, only one of these possibilities is realized. When the superposition principle is applied to a composite system then the concept of *entanglement* arises. If the composite system is initially unentangled, it will be in a tensor product state of the eigenstates of observables corresponding to subsystems. However, once they are allowed to interact with each other then they may be in a superposition state of different tensor product states, namely an *entangled state*. Entangled states exist for composite quantum systems that can be decomposed into subsystems, whereas

for composite classical systems no such analogue exists. The very striking manifestation of quantum entanglement is the impossibility of local description of subsystems comprising the total entangled system, even in the absence of a physical interaction between the subsystems. This is in contrast with the *principle of locality* which asserts that space-like separated events are independent of each other. A measurement performed on one of the subsystems leads to an instantaneous global state reduction which implies strong non-local correlations between the measurement results performed on the subsystems even when the parties are spatially well separated and this serves as a very important test of quantum mechanics[1, 2]. In a composite system, the observables belonging to different parties commute (are compatible) and allow for correlation type measurements. However this type of correlation measurements cannot be realized for a single component system characterized by an indecomposable Hilbert space, since it is not possible to find a set of compatible observables. Bell inequalities, contrasting the presence of high amount of correlations in a non-local theory (Quantum theory), with that of a local theory (classical mechanics) were shown to be violated for polarization entangled photons, and this was a conclusive test of quantum mechanics against classical mechanis[2] (See Appendix-A for further details and references on entanglement).

An entangled state was first exemplified by Einstein, Podolsky, and Rosen[3]. Following the example of Bohm[4], consider an entangled state of two spin-1/2 particles,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle), \tag{1.1}$$

where $|\pm\rangle$ are the eigenstates of spin along z axis. The salient features of entanglement is present in this example (see Appendix-A). Before any measurement, neither of the particles is in a well defined state. However whenever a measurement is performed on one of the particles, the other spin points in the opposite direction. These correlations are basis (observable) independent, measurement results will always be anti-correlated, irrespective of the measurement axis and spatial separation. Note that quantum interference in a single Hilbert space may not exist for some observables.

Entanglement naturally may exist in many-body systems, however entanglement is of practical interest only when the subsystems are spatially well separated so that the subsystems can individually be addressed. In principle it is possible to entangle different degrees of freedom of a single particle, for instance the momentum and spin of a single particle are described by distinct Hilbert spaces, thus it is possible to obtain entanglement in the tensor product of these two Hilbert spaces. However this type of entangled states are not suitable for testing locality principle, and during the measurement process it might be quite difficult to address these Hilbert spaces individually. Further it is essential to have distant parties for quantum information protocols.

Entanglement became an important resource in *quantum information science* enabling the realization of some data processing and communication tasks which would be regarded difficult or even impossible with classical reasoning such as quantum teleportation[5], cryptography[6], dense coding[7], distributed computational tasks[8], improvement of performance in some competitive games[9]. On the other hand entanglement provides an unprecedented increase in precision of frequency standarts[10, 11], and lithography[12] which would otherwise be impossible. Further, entanglement is of fundamental importance in *quantum computation*[13]. Entanglement should be present at some stage to achieve exponential speed up compared to classical computers, and information must be encoded in entangled states for error corrections. On the other hand, algebraic properties of entangled states are still not very well understood, especially for higher dimensional systems and it is also of interest regarding its mathematical structure[14, 15, 16].

Generation of controlled spatially well separated entangled states were first achieved using *polarization entangled photon states*, which are produced by strongly pumping a non-linear crystal[2, 17], and these were used to realize quantum key distribution, and teleportation[17]. *Cavity QED* techniques were also widely used in order to produce atom-atom , atom-photon entanglement. Rydberg atoms in high Q superconducting microwave cavities are strongly coupled to microwave radiation and EPR atom pairs were generated[18, 19, 20]. In cavity QED setting atoms may also provide a strong non-linearity for photons thus a

potentiality for photon-photon entanglement [21]. Efforts to realize experimentally the elements of quantum computation using *trapped atomic ions* have been stimulated by a proposal by Cirac and Zoller[22]. Ions confined in a linear radiofrequency(Paul) trap are cooled and form a spatial array. The motional mode can act as a data bus to transfer information between ions by mapping spin-qubit state of a particular ion onto the selected motion qubit with a laser beam focused onto that ion. In this manner universal quantum gates and thus entanglement of ions can be realized[23, 24, 25, 26, 22]. Direct manipulation and detection of nuclear spin states using radiofrequency Electro magnetic(EM) waves is a well-developed field known as nuclear magnetic resonance (NMR). NMR based elementary logic gates were proposed[27] and realized, even the Shor's factoring algorithm was implemented to factor the number 15 in a liquid state NMR quantum computer[28] (for an extensive bibliography see [29]).

Other methods, mostly theoretical at the moment, rely on using quantum correlated light field interacting with distant atoms, thus transferring entanglement of photons to the atoms[30, 31, 32, 33, 34, 35, 36] and conditional creation of entanglement realized by appropriate measurements[37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47] which usually make use of entanglement swapping[48] and the technique of reservoir engineering in a cascaded cavity QED setting[49].

For the realization of quantum information protocols, entangled states of long enough lifetime to allow for the necessary operations are needed and subsystems should be spatially well separated so that each subsystem can separately be addressed. Some quantum information and communication protocols, such as quantum teleportation and key distribution, could practically be useful only when the parties could be at any desired distance from each other[50]. The entangled state should be robust against the environmental noise, and in addition the physical nature of subsystems must still allow local operations, in particular measurements. If the entangled states are needed as a stationary component of some hardware then obviously photons are not good candidates since, they immediately leave the system or just disappear under any kind of measurement. As the components of hardware trapped atoms or solid state devices (see [51] and references therein) are the possible candidates and in such a system it is desirable

to have a deterministic scheme for the efficient generation of entanglement, so as to keep the size of system tractable.

In atomic systems it is possible to make the atoms interact with each other, thus enabling the qubit operations. This turns out to be a quite difficult task when it comes to interacting photons since nonlinear effects are very weak in nonlinear media. Thus photons are good candidates for communication purposes, and as the hardware components robust entangled states are needed.

Cavity QED is a domain of quantum optics which studies the behavior of Rydberg atoms confined in a limited region of space confined by metallic boundaries (see [52, 53, 54] and references therein). The modification of spectrum of the electromagnetic vacuum results in the modification of spontaneous emission rates of atoms which can be either inhibited or enhanced[55, 56]. This enabled the realization of previously predicted phenomena such as superradiance in atomic ensembles[57], and exchange of quanta between field and atoms, namely Rabi oscillations . Cavity QED serves as an entangling machine for atom-atom, atom-photon systems, and also serves for the generation of non-classical photon states such as Fock states or Schroedinger cat states. However this type of structures that modify the electromagnetic vacuum are not limited by metallic cavities. For instance microspheres, photonic band-gap materials may also give rise to a strong modification of the vacuum, therefore enable the enhancement or inhibition of spontaneous decay rates[58, 59, 60]. In particular for microspheres, it is possible to obtain high-Q resonators ($> 10^9$) at the optical frequencies at the ultimate level determined by intrinsic material absorption[61]. The study of various resonator like structures and the interaction of atoms with EM field in this media is an important issue.

In cavity QED systems another important issue is obtaining information from the cavity. The cavity modes should be coupled to the continuum of modes outside the cavity so as to gather information about the photonic and atomic states inside the cavity. Since the photon states extracted from the cavity are highly non-classical, e.g., Fock states, they are quite vulnerable to decoherence effects such as unavoidable spontaneous emission to the free modes, and absorption at

the walls as the photons are extracted out. Engineering these systems in order to reduce these effects is naturally quite important[18, 62, 63, 64].

Thesis is organized as follows: In chapter-2 creation of robust entangled states, using atoms and photons as the main physical objects is discussed. The use of three level atoms provides a deterministic scheme for the generation of spatially well separated maximally entangled states whose lifetime is determined by the lifetime of the metastable states of Λ type three level atoms. Conditional creation of maximally entangled state via the observation of spontaneously emitted photons is discussed. Finally generation of robust multi-party entangled states is studied which is a natural extension of the bipartite case. It is shown that maximally entangled GHZ[65] type states can be obtained if there is an even number of atoms and W[66] type states can be obtained if there is an odd number of atoms.

In chapter-3 the interaction of multi-level atoms with quantized EM field in the presence of dispersing-absorbing dielectric bodies is studied. First, quantization of EM field in dispersive-absorptive media is discussed then the master equation governing the atom-field system is obtained. Realization of the robust bipartite entanglement of three-level atoms in real physical settings is discussed, in particular for atoms passing by a dielectric microspheres is studied. It is shown that atoms may become entangled when they are spatially well separated. However in these real settings atoms will not be in a maximally entangled state, so the loss of entanglement will be under consideration. Also a proposal is made for the preparation of the initial state, i.e. the deposition of a single photonic excitation.

In chapter-4 it is shown that the environment can be engineered in order to stabilize entanglement. The stabilization of entanglement of two dipoles in free space with the help of classical driving field is discussed. In free space considerable amount of entanglement can be realized in Lamb-Dicke limit namely when the dipoles are close to each other.

In chapter-5 the absorption effects associated with the extraction of nonclassical photon states from a cavity will be studied. An input-output relation will

be formulated for a one dimensional cavity with absorptive walls. Dynamics of the intracavity field in the presence of a source and those of the field outside the cavity will be under consideration.

Chapter 2

Robust Entanglement of Three-Level Atoms

In this chapter we will discuss the generation of robust entangled states in bi-partite three level atomic systems and make an extension to multi-partite systems. Possible models that can be employed to describe these systems will be under consideration. In the next chapter a physical realization of this robust entangled state will be presented.

Introduction

During the last decade, the problem of engineered entanglement in atomic systems has attracted a great deal of interest (see [67, 18, 41, 23] and references therein). In particular, the atomic entangled states were successfully realized through the use of cavity QED [18] and the technique of ion traps [23]. At present, one of the most important problems under consideration is how to make a long-lived and easy-monitored atomic entangled state with existing experimental technique.

An interesting scheme has been proposed recently [68]. In this scheme, two

identical atoms are placed into a cavity tuned to resonance with one of the dipole-allowed transitions. Initially both atoms are prepared in the ground state, while the cavity field consists of a single photon. It is easy to show that the atom-field interaction leads in this case to a maximum atomic entangled state such that the single excitation is shared between the two atoms with equal probability. It was proposed in [68] to consider the absence of photon leakage from a non-ideal cavity as a signal that the atomic entangled state has been created. The scheme can also be generalized to the case of any even number of atoms $2n$, sharing n excitations.

In the schemes of Refs. [68] two-level atoms are used for generation of bipartite entanglement. The lifetime of the entanglement is defined by the specific time scale of the dipole-allowed radiative processes in atoms, which is usually quite short. Generally speaking, the lifetime of atomic entanglement is specified by the interaction of atoms with environment.

The interaction with environment can also be used to create a long-lived entanglement in atomic systems. For example, the initially non-entangled system may evolve to an entangled state connected with the atomic states that cannot be depopulated by radiative decay. In this case, the lifetime of the entangled state is specified by the considerably long nonradiative processes. Possible realization is provided by the use of three-level Λ-type process instead of the two-level scheme. The process is illustrated by Figure-2.1. Here the levels 1 and 3 are connected by the electric dipole transitions as well as the levels 2 and 3. In turn, the dipole transition between the levels 2 and 1 is forbidden because of the parity conservation [69]. The absorption of pumping photon by the transition $1 \leftrightarrow 3$ with further jump of the electron to the level 2 can be interpreted as a kind of Raman process in atomic system with emission of Stokes photon (see [70] and references therein). It is clear that the atom excited to the level 2 can change the state either by absorption of the Stokes photon resonant with respect to the transition $3 \leftrightarrow 2$ or through a nonradiative decay.

We assume now that the two identical Λ-type atoms in their ground state are placed inside a cavity of high quality with respect to the pumping photons which

are resonant with respect to the transition $1 \leftrightarrow 3$ and also that the Stokes photons created by the transition $3 \rightarrow 2$ either leave the cavity freely or are absorbed by the cavity walls where initially there exists single pump excitation. Then, the atom-field interaction may lead to creation of maximum entangled atomic state

$$\frac{1}{\sqrt{2}}(|2,1\rangle + |1,2\rangle), \tag{2.1}$$

whose lifetime is determined by the slow processes of nonradiative $2 \rightarrow 1$ decay. Let us stress that the monitoring of Stokes photons outside the cavity can be used to detect the atomic entangled state (2.1) in this case.

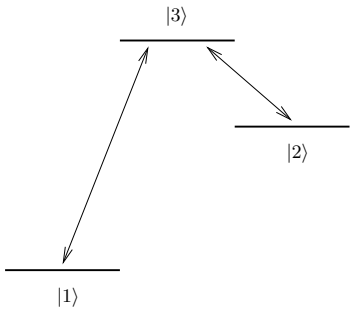


Figure 2.1: Scheme of the process and configuration of atomic levels and transitions.

The main objective of present chapter is to consider in details the evolution towards the long-lived atomic entangled state (2.1)[71, 72, 73, 74, 75].

In sections (2.1,2.2,2.3) we discuss the model Hamiltonians that can be used to describe the process under consideration and generation of robust entanglement in each model. viz, we discuss the following models,

- single cavity mode strongly coupled with $1 \leftrightarrow 3$ mode where the Stokes photons corresponding to $3 \leftrightarrow 2$ are allowed to escape from the cavity,
- two cavity modes strongly coupled with the two transitions, where Stokes photons are absorbed by the cavity walls.
- effective model in which the upper level (3rd atomic level) is not populated and adiabatically eliminated.

Then, in Sec. 2.1,2.2,2.3 we examine the irreversible dynamics, leading to the state (2.1) within the three models.

In section 2.4 the prescription for the generation of robust bipartite entanglement is extended to the multi atom case.

2.1 Cavity transparent to Stokes modes

Assume that a system of N identical three-level atoms with Λ-type transitions shown in Fig. 2.1 interacts with the cavity mode close to resonance with $1 \leftrightarrow 3$ transition and with the Stokes radiation that can leave the cavity freely. Then, we can choose the model Hamiltonian in the following form

$$H = H_0 + H_{int}, \tag{2.2}$$

$$H_0 = \omega_P a_P^+ a_P + \sum_k \omega_{Sk} a_{Sk}^+ a_{Sk} + \sum_f [\omega_{21} R_{22}(f) + \omega_{31} R_{33}(f)], \tag{2.3}$$

$$H_{int} = \sum_f \lambda_P R_{31}(f) a_P + \sum_{f,k} \lambda_{Sk} R_{32}(f) a_{Sk} + H.c. \tag{2.4}$$

Here a_P denotes the photon annihilation operator of the cavity mode with frequency ω_P, a_{Sk} is the annihilation operator of Stokes photon of the kth mode with frequency ω_{Sk}, and ω_{31}, ω_{21} are the energies of the corresponding atomic levels with respect to the ground level 1. The operator

$$R_{ij}(f) = |i_f\rangle\langle j_f|$$

describes the transition from level j to level i and index $\{f, (f = 1 \ldots N)\}$ labels the atoms. In Eq. 2.4, λ_P and λ_{Sk} are the coupling constants, specifying the dipole transitions $3 \leftrightarrow 1$ and $3 \leftrightarrow 2$, respectively. Summation over k in (2.4) implies that the Stokes photons do not feel the presence of the cavity walls. This summation involves the modes, corresponding to the natural line breadth near

$$\omega_S \equiv \omega_{32} = \omega_{31} - \omega_{21}. \tag{2.5}$$

Apart from the total electron occupation number, the Hamiltonian (2.2) has the two integrals of motion

$$N_P = a_P^+ a_P + \sum_f \{R_{22}(f) + R_{33}(f)\}$$
$$N_S = \sum_k a_{Sk}^+ a_{Sk} + \sum_f R_{22}(f). \tag{2.6}$$

2.1.1 Generation of robust bipartite entanglement

Consider the system of only two atoms. Assume that both atoms are prepared initially in the ground state $|1\rangle$, the cavity contains a single photon of frequency ω_P, and Stokes field is in the vacuum state. Then, because of the integrals of motion (2.6), the evolution of the system occurs in a single-excitation domain of the Hilbert space spanned by the vectors

$$\begin{cases} |\psi_1\rangle & = |1,1\rangle \otimes |1_P\rangle \otimes |0_S\rangle \\ |\psi_2^{(\pm)}\rangle & = \frac{1}{\sqrt{2}}(|1,3\rangle \pm |3,1\rangle) \otimes |0_P\rangle \otimes |0_S\rangle \\ |\psi_{3k}^{(\pm)}\rangle & = \frac{1}{\sqrt{2}}(|1,2\rangle \pm |2,1\rangle) \otimes |0_P\rangle \otimes |1_{Sk}\rangle \end{cases} \tag{2.7}$$

By construction, the four states (2.7) labelled by the superscripts $\pm$ manifest the maximum entanglement. It is easily seen that the action of operator (2.4) cannot transform the states

$$\{|\psi_1\rangle, |\psi_2^{(+)}\rangle, |\psi_{3k}^{(+)}\rangle\} \tag{2.8}$$

into the states

$$\{|\psi_2^{(-)}\rangle, |\psi_{3k}^{(-)}\rangle\} \tag{2.9}$$

and vice versa. Thus, the evolution of the system from the initial nonexcited state $|\psi_1\rangle$ takes place in the subspace spanned by only three vectors (2.8). Thus, the states (2.9) can be discarded.

Under the assumption that there are only two three-level Λ-type atoms in the cavity and that the system is initially prepared in the state $|\psi_1\rangle$ in (2.7), in

view of the results of previous section we should choose the time-dependent wave function as follows

$$|\Psi(t)\rangle = C_1|\psi_1\rangle + C_2|\psi_2\rangle + \sum_k C_{3k}|\psi_{3k}\rangle, \tag{2.10}$$

$$C_1(0) = 1, \quad C_2(0) = 0, \quad \forall k \quad C_{3k}(0) = 0, \tag{2.11}$$

using the reduced basis (2.8). Here we use the notations $|\psi_2\rangle \equiv |\psi_2^{(+)}\rangle$ and $|\psi_{3k}\rangle \equiv |\psi_{3k}^{(+)}\rangle$, for simplicity. The time-dependent Schrödinger equation with the Hamiltonian (2.3) and (2.4) then leads to the following set of equations for the coefficients in (2.10)

$$\begin{cases} i\dot{C}_1 &= \omega_P C_1 + \lambda_P\sqrt{2}C_2 \\ i\dot{C}_2 &= \omega_{31}C_2 + \lambda_P\sqrt{2}C_1 + \sum_k \lambda_{Sk}C_{3k} \\ i\dot{C}_{3k} &= (\omega_{21} + \omega_{Sk})C_{3k} + \lambda_{Sk}C_2. \end{cases} \tag{2.12}$$

To find solutions of (2.12), let us integrate out the last equation in (2.12) in the form

$$C_{3k}(t) = -i\lambda_{Sk}\int_0^t C_2(\tau)e^{i(\omega_{31}+\omega_{Sk})(\tau-t)}d\tau, \tag{2.13}$$

then the equation of motion for C_2 becomes,

$$i\dot{C}_2(t) = \omega_{31}C_2(t) + \sqrt{2}\lambda_P C_1(t) - i\sum_k \lambda_{Sk}^2 \int_0^t \mathtt{d}\tau e^{-i(\omega_{21}+\omega_{Sk})\tau}C_2(t-\tau). \tag{2.14}$$

Assuming exact resonance $\omega_P = \omega_{31}$, we introduce normal modes, $C_\pm = (C_1 \pm C_2)/\sqrt{2}$ for the equations of motion for C_1 and C_2 (2.12),(2.14)

$$i\dot{C}_\pm = (\omega_{31} \pm \sqrt{2}\lambda_P)C_\pm \mp \frac{i}{2}\sum_k \lambda_{Sk}^2 \int_0^t \mathtt{d}\tau e^{-i(\omega_{21}+\omega_{Sk})\tau}\big(C_+(t-\tau) - C_-(t-\tau)\big).$$

Now we can perform Markov approximation, assuming that time rate of change due to coupling with continuum of modes is slow, $C_\pm(t-\tau) \simeq C_\pm(t)e^{i(\omega_{31}\pm\sqrt{2}\lambda_P)\tau}$,

$$\begin{aligned} &i\dot{C}_\pm = (\omega_P \pm \sqrt{2}\lambda_P)C_\pm \\ &\mp \frac{i}{2}\sum_k \lambda_{Sk}^2 \int_0^t \mathtt{d}\tau\Big(e^{-i(\omega_{Sk}-\omega_{32}-\sqrt{2}\lambda_P)\tau}C_+(t) - e^{-i(\omega_{Sk}-\omega_{32}+\sqrt{2}\lambda_P)\tau}C_-(t)\Big). \end{aligned} \tag{2.15}$$

Assuming that the coupling constants λ_{Sk} are slowly varying as a function of frequency, the resulting frequency integral over ω_{Sk} would yield a sharply peaked function at time t, and its value will be irrespective of the value of time t. So one can take the limit $t \to \infty$ with the appropriate convergence factor,

$$\lim_{\delta\to 0}\lim_{t\to\infty}\int_0^t \mathtt{d}\tau e^{-i(\omega_{Sk}-\omega_{32}\pm\sqrt{2}\lambda_P-i\delta)\tau} = \\ -i\mathcal{P}\frac{1}{\omega_{Sk}-\omega_{32}\pm\sqrt{2}\lambda_P}+\pi\delta(\omega_{Sk}-\omega_{32}\pm\sqrt{2}\lambda_P), \tag{2.16}$$

$\mathcal{P}$ denoting the Principal part, which results in the equations of motion,

$$i\dot{C}_\pm = (\omega_P \pm \sqrt{2}\lambda_P)C_\pm \mp \frac{i}{2}\big(\frac{\Gamma_+}{2}C_+(t) - \frac{\Gamma_-}{2}C_-(t)\big) \tag{2.17}$$

where we have ignored the level shifts arising from the Principal part in (2.16). The spontaneous decay rates $\Gamma_\pm$ are given as follows,

$$\Gamma_\pm = 2\pi\sum_k \lambda_{Sk}^2\delta(\omega_{Sk}-\omega_{32}\mp\sqrt{2}\lambda_P), \tag{2.18}$$

which can be evaluated by converting the summation into an integral. For isotropic free space this factor turns out to be $\Gamma_\pm = (\omega_{32}\pm\sqrt{2}\lambda_P)^3 d_{32}^2/3\pi\epsilon_0 c^3$ where $d_{32} = -e\langle 3|r|2\rangle$ is the electric dipole moment, and $-e$ is the electron charge. Here we can assume that $\Gamma = \Gamma_+ \simeq \Gamma_-$, as long as $\omega_{32} \gg \lambda_P$ and the equations of motion for $C_1,\ C_2$ can be cast into the form

$$\dot{C}_1 = -i\omega_P C_1 - i\sqrt{2}\lambda_P C_2 \\ \dot{C}_2 = -i\omega_{31}C_2 - i\sqrt{2}\lambda_P C_1 - \frac{\Gamma}{2}C_2, \tag{2.19}$$

where Γ is the single atom decay rate, for $3 \to 2$ transition.

It follows from (2.10) that the probability to have the atomic entangled state (2.1) has the form

$$\sum_k |C_{3k}|^2 = 1 - |C_1(t)|^2 - |C_2(t)|^2. \tag{2.20}$$

Let us stress that, unlike the conventional Wigner-Weisskopf theory, Eqs. (2.12) describe a superposition of exponential decay and harmonic oscillations.

The latter are caused by the interaction between the $1 \leftrightarrow 2$ transitions and cavity field.

In the equations of motion 2.19 system features two distinct behavior depending on the two limiting cases, viz. $\lambda_P \gg \Gamma$ and $\Gamma \gg \lambda_P$.

For $\Gamma \gg \lambda$, the coefficients C_1 and C_2 have the form,

$$C_1(t) \approx \left[-\frac{2\lambda^2}{(\Gamma/2 - i\Delta_P)^2} e^{(-\Gamma/2 + i\Delta_P)t} + (1 + \frac{2\lambda^2}{(\Gamma/2 - i\Delta_P)^2}) e^{-\frac{2\lambda^2}{\Gamma/2 - i\Delta_P}t} \right] e^{-i\omega_P t}$$

$$C_2(t) \approx -\frac{\sqrt{2}\lambda}{i\Gamma/2 + \Delta_P} \left[e^{-\Gamma/2t} - e^{-(\frac{2\lambda^2}{\Gamma/2 - i\Delta_P} + i\Delta_P)t} \right] e^{-i\omega_{21}t} \tag{2.21}$$

to second order in $\lambda/(\Gamma - i\Delta_P)$. Here

$$\Delta_P = \omega_P - \omega_{31}$$

is the detuning factor for the pumping mode.

It is seen that Eq. (2.21) describes the damped oscillations of the coefficient $C_1(t)$ in (2.10). Thus, the probability (2.20) to get the robust entangled state tends to 1 as $t \to \infty$ (see Fig. 2.2). It is seen from Eq. 2.21, that the time τ required for persistent entanglement is typically,

$$\tau \sim \frac{\Gamma^2 + \Delta_P^2}{\lambda_P^2 \Gamma}. \tag{2.22}$$

The increase of detuning leads to a deceleration of evolution towards the persistent entangled state.

In case $\lambda_P \gg \Gamma$, the solution becomes,

$$C_1(t) \simeq e^{-i\frac{\omega_P + \omega_{31}}{2}t - \frac{\Gamma}{4}t} \big(\cos\Omega t - i\frac{\Delta_P}{2\Omega} \sin\Omega t \big)$$

$$C_2(t) \simeq e^{-i\frac{\omega_P + \omega_{31}}{2}t - \frac{\Gamma}{4}t} \big(\frac{\sqrt{2}\lambda_P}{\Omega} \sin\Omega t \big) \tag{2.23}$$

where $\Omega = \sqrt{2\lambda_P^2 + \Delta_P^2/4}$ is the Rabi frequency. The system exhibits damped Rabi oscillations. The time scale required for entanglement is $\tau \simeq 1/\Gamma$.

While the atomic system evolves to the maximum entangled state (2.1), the Stokes photon leaves the cavity. Thus, the observation of Stokes photon outside the cavity can be considered as a signal that the robust entangled state has been prepared.

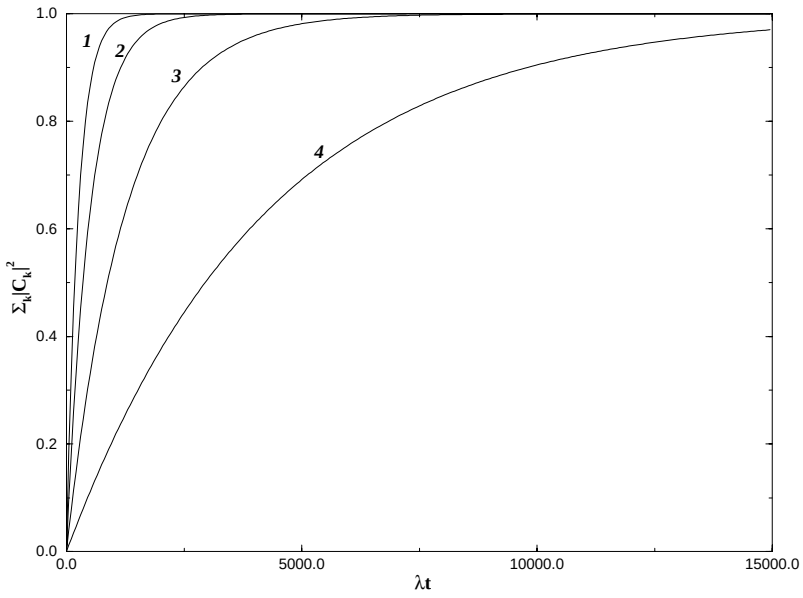


Figure 2.2: Time evolution of probability (2.20) to have the robust entanglement at $\lambda_P = 0.001\Gamma$ for $(1)\Delta_P = 0$; $(2)\Delta_P = \Gamma(3); \Delta_P = 2\Gamma; (4)\Delta_P = 4\Gamma$

2.2 Cavity with absorption of Stokes photons

The atomic entangled state (2.1) can also be realized when the Stokes mode is strongly damped in the cavity. For simplicity, we again assume no damping for the pumping mode. At the same time, the Stokes photons are supposed to be absorbed by the cavity walls.

The model Hamiltonian, describing the process under consideration, can be

chosen as follows

$$H = H_0 + H_{int} + H_{loss},$$
$$H_0 = \omega_P a_P^+ a_P + \omega_S a_S^+ a_S + \sum_f [\omega_{21} R_{22}(f) + \omega_{31} R_{33}(f)],$$
$$H_{int} = \sum_f [\lambda_P R_{31}(f) a_P + \lambda_S R_{32}(f) a_S] + H.c. \tag{2.24}$$
$$H_{loss} = \sum_q \eta_q (b_q^+ a_S + a_S^+ b_q) + \sum_q \Omega_q b_q^+ b_q. \tag{2.25}$$

H_{int} describes the interaction of three level atoms with the two modes of the cavity which are described by the photon annihilation operator a_P for the pump photons, and a_S for the Stokes photons. H_{loss} describes the cavity damping of Stokes modes. To take into account the cavity damping of Stokes photons, we consider an interaction with a "phonon reservoir" responsible for the absorption of photons by cavity walls, where b_q, b_q^+ are the Bose operators of phonons in the cavity walls[76].

We can now write the Master Equation, eliminating the phonon degrees of freedom (see Appendix-B,[77]),

$$\dot{\rho} = -i[H_0 + H_{int}, \rho] + \kappa\{2a_S \rho a_S^+ - a_S^+ a_S \rho - \rho a_S^+ a_S\}, \tag{2.26}$$

so that the contribution of (2.25) is taken into account effectively through the Liouville term. Here $1/\kappa$ is the lifetime of a Stokes photon in the cavity.

2.2.1 Generation of bipartite robust entanglement

In sec.2.2 we have obtained the Master equation(2.26) describing the situation when the cavity supports two modes and the Stokes photons are either absorbed by cavity walls or leak out of the cavity. Let us choose the same initial condition as in previous section, so that

$$\rho(0) = |\psi_1\rangle\langle\psi_1|, \tag{2.27}$$

subject to irreversible dynamics governed by the Master equation(2.26).

The Master equation(2.26) can be cast into the following form,

$$\begin{aligned}\dot{\rho} &= -i(H_{eff}\rho - \rho H_{eff}^{\dagger}) + 2\kappa a_S \rho a_S^{\dagger} \\ H_{eff} &= H_0 + H_{int} - i\kappa a_S^{\dagger} a_S,\end{aligned} \tag{2.28}$$

the solution of which can be expressed in the series form,

$$\begin{aligned}\rho(t) &= e^{\hat{S}(t-t_0)}\rho(t_0) \\ &+ \sum_{n=1}^{\infty} \int_{t_0}^{t} dt_n \int_{t_0}^{t_n} dt_{n-1} \ldots \int_{t_0}^{t_2} dt_1 e^{\hat{S}(t-t_n)} \hat{L} e^{\hat{S}(t_n - t_{n-1})} \ldots \hat{L} e^{\hat{S}(t_1 - t_0)} \rho(t_0),\end{aligned} \tag{2.29}$$

where the superoperators $\hat{S}$ and $\hat{L}$ are given as follows,

$$\begin{aligned}\hat{S}(\rho) &= -i(H_{eff}\rho - \rho H_{eff}) \\ \hat{L}(\rho) &= 2\kappa a_S \rho a_S^{\dagger}.\end{aligned}$$

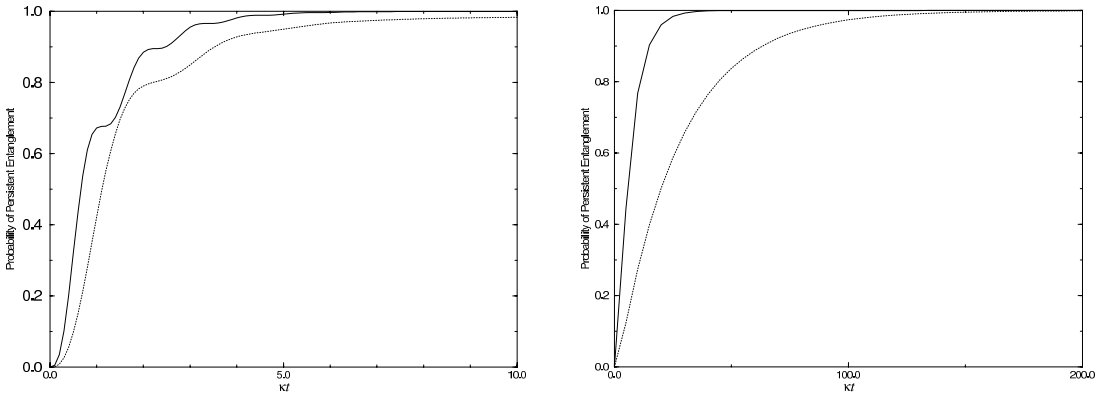
Since the initial state (2.27) contains only one excitation, the series (2.29) terminates at the second term,

$$\rho(t) = e^{\hat{S}(t-t_0)}\rho(t_0) + \int_{t_0}^{t} dt_1 e^{\hat{S}(t-t_1)} \hat{L} e^{\hat{S}(t_1 - t_0)} \rho(t_0). \tag{2.30}$$

It is seen that the system evolves to the robust atomic entangled state (2.1). The stairs-like structure is again caused by the competition between the transitions $3 \leftrightarrow 1$ and $3 \leftrightarrow 2$. Although such a behavior is an inherent property of the model under consideration, the stairs become more visible with decrease of κ (see the "dotted curves" in Fig. 2.3).

2.3 Effective Model

Consider the case when the cavity is a two-mode cavity which has support for pump and Stokes modes and the two transitions in three-level system are off-resonant with respect to these modes s.t. the transition $1 \leftrightarrow 2$ is an energy conserving process, i.e. $\omega_P = \omega_S + \omega_{21}$. This scheme is illustrated in Figure-2.4.



a) b)

Figure 2.3: Evolution to the persistent entangled state in the dynamics described by Eq. (2.26)(dotted curve) and in the effective model described by Eq. (2.37) (solid curve) for a)$\kappa = 0.1\lambda_P, \Delta_P = \Delta_S = 10\lambda_P, \lambda_S = \lambda_P$ b)$\kappa = \lambda_S = \lambda_P, \Delta_P = \Delta_S = 10\lambda_P$

In this situation, the 3rd level, if it is initially unpopulated, will not be populated and it can be adiabatically eliminated from the equations of motion.

For the moment disregarding the absorption of Stokes photons by cavity walls, from the Hamiltonian (2.24), $H = H_0 + H_{int}$, the Heisenberg equations of motion for the operators involving the 3rd atomic level are as follows,

$$\begin{aligned} i\dot{R}_{31} &= -\omega_{31}R_{31} - \lambda_S a_S^\dagger R_{21} + \lambda_P a_P^\dagger (R_{33} - R_{11}) \\ i\dot{R}_{32} &= -\omega_{32}R_{32} - \lambda_P a_P^\dagger R_{12} + \lambda_S a_S^\dagger (R_{33} - R_{22}) \\ i\dot{R}_{33} &= \lambda_P (R_{31}a_P - a_P^\dagger R_{13}) + \lambda_S (R_{32}a_S - a_S^\dagger R_{23}). \end{aligned} \tag{2.31}$$

The equations of motion can be integrated to yield,

$$\begin{aligned} R_{31}(t) =& R_{31}(0)e^{i\omega_{31}t} + i\int_0^t \mathrm{d}t' e^{i\omega_{31}(t-t')}\Big[\lambda_S a_S^\dagger(t')R_{21}(t') \\ &- \lambda_P a_P^\dagger(t')\big[R_{33}(t') - R_{11}(t')\big]\Big] \\ R_{32} =& R_{32}(0)e^{i\omega_{32}t} + i\int_0^t \mathrm{d}t' e^{i\omega_{32}(t-t')}\Big[\lambda_P a_P^\dagger(t')R_{12}(t') \\ &- \lambda_S a_S^\dagger(t')\big[R_{33}(t') - R_{22}(t'))\big]\Big] \end{aligned} \tag{2.32}$$

Now, when $\Delta = \omega_P - \omega_{31} = \omega_S - \omega_{32} \gg \lambda_P, \lambda_S$, we can make the following substitutions in the equations of motion (2.32),

$$\begin{aligned} a_S(t') &\simeq a_S(t)e^{i\omega_S(t-t')} \\ a_P(t') &\simeq a_P(t)e^{i\omega_P(t-t')}, \\ R_{33}(t') \simeq R_{33}(t),\ R_{22}(t') &\simeq R_{22}(t)\, R_{11}(t') \simeq R_{11}(t). \end{aligned} \tag{2.33}$$

The Heisenberg equations of motion (2.32) yield,

$$\begin{aligned} R_{31}(t) \simeq & R_{31}(0)e^{-i\omega_{31}t} \\ & - \frac{e^{-i\Delta t}-1}{\Delta}\Big[\lambda_S a_S^\dagger(t)R_{21}(t) - \lambda_P a_P^\dagger(t)\big[R_{33}(t) - R_{11}(t)\big]\Big] \end{aligned} \tag{2.34}$$

$$\begin{aligned} R_{32}(t) \simeq & R_{32}(0)e^{i\omega_{32}t} \\ & - \frac{e^{-i\Delta t}-1}{\Delta}\Big[\lambda_P a_P^\dagger(t)R_{12}(t) - \lambda_S a_S^\dagger(t)\big[R_{33}(t) - R_{22}(t)\big]\Big]. \end{aligned} \tag{2.35}$$

Substituting into (2.35) into (2.24) and discarding the fast oscillating terms we obtain the effective Hamiltonian in the subspace excluding the 3rd atomic level,

$$\begin{aligned} H_{eff} = \omega_{21}R_{22} + \omega_P a_P^\dagger a_P + \omega_S a_S^\dagger a_S + \tfrac{2\lambda_P\lambda_S}{\Delta}(a_S^\dagger a_P R_{21} + \texttt{H.C.}) \\ + \tfrac{2\lambda_P^2}{\Delta}a_P^\dagger a_P R_{11} + \tfrac{2\lambda_S^2}{\Delta}a_S^\dagger a_S R_{22}. \end{aligned} \tag{2.36}$$

Eq. (2.36) describes an effective two level system with Rabi frequency $\lambda_P\lambda_S/\Delta$, and the last two terms are Stark shifts, which can be ignored for small field populations.

In the adiabatic model the population of the 3rd atomic level, pump and the Stokes photons will remain small if these states are initially unpopulated. Here we should remark about another strategy of entanglement creation. The transition from the 1st atomic level to the 2nd atomic level will be a slow process, thus it might be possible to adjust the interaction time so that once the maximally entangled state (2.1) is obtained the interaction can be switched off. It is however also possible to achieve the same result using a single mode cavity if the 1st and the 3rd states both have the same energy. This single mode can address both transitions thus Rabi oscillations take place between the 1st and the 3rd atomic levels.

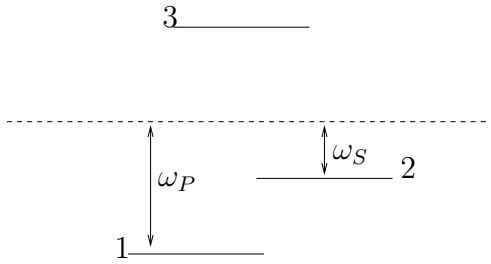


Figure 2.4: Effective Model: The 3rd level is adiabatically eliminated

Another possibility is to introduce an irreversible evolution to the 3rd level by assuming that the cavity is of low quality regarding the Stokes photons. Thus the system can be described by the following master equation similar to 2.26,

$$\dot{\rho} = -i[H_{eff}, \rho] + \kappa(2a_S\rho a_S^\dagger - a_S^\dagger a_S\rho - \rho a_S^\dagger a_S). \tag{2.37}$$

In Fig-2.3 a comparison can be made with the exact(2.26) and effective(2.37) models. It is seen that adiabatic model is unable to take short time behavior into account whereas it is more accurate for long time behavior.

2.4 Entanglement in the Multi Three-Level Atomic System

We are going to consider the case when the Stokes photons are allowed to escape from the cavity, initially all the atoms are in the ground state, and in the presence of pump photons which couple the 1st and 3rd levels. If there are N atoms in the ground state and n_P pump photons initially, in the final state n_P excitations in the 2nd state will be created, and these excitations will equally be distributed symmetrically over the N atoms,

$$|\Psi(t=0)\rangle = \bigotimes_{i=1}^{N} |1\rangle_i \otimes |n_P\rangle_P$$
$$\rightarrow \frac{1}{\sqrt{C_{n_P}(N)}} \sum_{\wp} \bigotimes_{i=1}^{n_P} |2\rangle_{\wp i} \bigotimes_{i=n_P+1}^{N} |1\rangle_{\wp i} \otimes |0\rangle_P,$$

where $\wp$ denotes all possible permutations over N atoms, and $C_{n_P}(N) = N!/n_P!(N-n_P)!$. In case $n_P \geq N$, all of the atoms will evolve to the second state, thus leading to an unentangled one. The Hamiltonian of the system in the interaction picture has the form,

$$\begin{aligned} H_0 &= \Delta_P a_P^\dagger a_P + g_P \Re_{31} a_P + g_P^* a_P^\dagger \Re_{13} \\ H_{int} &= \sum_k \Delta_k a_k^\dagger a_k + \sum_k g_k \Re_{32} a_k + g_k^* a_k^\dagger \Re_{23} \end{aligned} \tag{2.38}$$

where $\Re_{ij} = \sum_{f=1}^{N} R_{ij}(f)$ constitute the collective atomic operators. The Stokes modes make up the environment, and they lead to a spontaneous decay from the $2nd$ level to the $3rd$ level. Upon the elimination of Stokes modes, the Master equation for the reduced density matrix of atoms and pump photons, in a thermal environment is as follows,

$$\begin{aligned} \dot{\rho}(t) = -i[H_0, \rho(t)] &+ (\bar{n}+1)\frac{\Gamma}{2}(2\Re_{23}\rho(t)\Re_{32} - \rho(t)\Re_{32}\Re_{23} - \Re_{32}\Re_{23}\rho(t)) \\ &+ \bar{n}\frac{\Gamma}{2}(2\Re_{32}\rho(t)\Re_{23} - \rho(t)\Re_{23}\Re_{32} - \Re_{23}\Re_{32}\rho(t)), \end{aligned} \tag{2.39}$$

where Γ is the spontaneous decay rate for the $3 \rightarrow 2$ transition, and $\bar{n}$ is the average number of Stokes photons at the resonant frequency E_{32}. Consider for simplicity the case when the temperature is much smaller than the resonant energy E_{32}, so that the mean number of thermal photons $\bar{n} \sim 0$ and the Master equation reduces to

$$\dot{\rho}(t) = -i[H_0, \rho(t)] + \frac{\Gamma}{2}(2\Re_{23}\rho(t)\Re_{32} - \rho(t)\Re_{32}\Re_{23} - \Re_{32}\Re_{23}\rho(t)). \tag{2.40}$$

Initially all the atoms are in the ground state $|1\rangle$. Then due to coupling between the $1st$ and the $2nd$ levels mediated by the pump photons, an excitation in the $2nd$ level will appear. Assuming that the spontaneous decay rate Γ for $3 \rightarrow 2$ transition is much larger than the Rabi coupling constant g_P for $1 \leftrightarrow 3$ transition, the state with one excitation in the $3rd$ level will immediately decay to the $2nd$ state before any further Rabi oscillation $1 \leftrightarrow 3$ can take place. As a result, the

evolution will approximately take place in the subspace spanned by the vectors,

$$
\begin{aligned}
|\Psi_n\rangle &= \frac{1}{\sqrt{C_n(N)}} \sum_{\wp} \bigotimes_{i=1}^{n} |2\rangle_{\wp i} \bigotimes_{i=n+1}^{N} |1\rangle_{\wp i} \otimes |n_P - n\rangle_P \\
|\Phi_n\rangle &= \frac{1}{\sqrt{C_1(N-n)C_n(N)}} \sum_{\wp} \bigotimes_{i=1}^{n} |2\rangle_{\wp i} \bigotimes_{i=n+1}^{n+1} |3\rangle_{\wp i} \\
&\quad \bigotimes_{i=n+2}^{N} |1\rangle_{\wp i} \otimes |n_P - n - 1\rangle_P, \\
|\Phi'_n\rangle &= \frac{1}{\sqrt{C_2(N-n)C_n(N)}} \sum_{\wp} \bigotimes_{i=1}^{n} |2\rangle_{\wp i} \bigotimes_{i=n+1}^{n+2} |3\rangle_{\wp i} \\
&\quad \bigotimes_{i=n+3}^{N} |1\rangle_{\wp i} \otimes |n_P - n - 2\rangle_P,
\end{aligned} \tag{2.41}
$$

for $n = 0, 1, 2, \ldots, n_P$. First $|\Psi_n\rangle \rightarrow |\Phi_n\rangle$ transition takes place, followed by $|\Phi_n\rangle \rightarrow |\Phi'_n\rangle$ and $|\Phi_n\rangle \rightarrow |\Psi_{n+1}\rangle$ transitions, at a time scale of $t \sim 1/\Gamma$, the population of $|\Phi'_n\rangle$ to that of $|\Psi_{n+1}\rangle$ is of the order $g_P^2/\Gamma^2 \ll 1$. So we can confine ourselves to the subspace spanned by $\{|\Psi_n\rangle, |\Phi_n\rangle; n = 0, 1, 2, \ldots, n_P\}$.

The density matrix can approximately be expressed in the form,

$$
\rho \approx \sum_{n=0}^{n_P} a_n |\Psi_n\rangle\langle\Psi_n| + b_n |\Psi_n\rangle\langle\Phi_n| + b_n^* |\Phi_n\rangle\langle\Psi_n| + c_n |\Phi_n\rangle\langle\Phi_n|, \tag{2.42}
$$

from which the equations of motion for the coefficients, a_n, b_n, b_n^*, c_n are obtained, upon insertion into Eq.(2.40),

$$
\begin{aligned}
\dot{a}_n &= i\sqrt{(N-n)(n_P-n)}(g_P b_n - g_P^* b_n^*) + 2n\frac{\Gamma}{2} c_{n-1} \\
\dot{b}_n &= i\sqrt{(N-n)(n_P-n)}(g_P^* a_n - g_P^* c_n) - i\Delta b_n - (n+1)\frac{\Gamma}{2} b_n \\
\dot{b}_n^* &= -i\sqrt{(N-n)(n_P-n)}(g_P a_n - g_P c_n^*) + i\Delta b_n^* - (n+1)\frac{\Gamma}{2} b_n^* \\
\dot{c}_n &= -i\sqrt{(N-n)(n_P-n)}(g_P b_n - g_P^* b_n^*) - 2(n+1)\frac{\Gamma}{2} c_{n-1},
\end{aligned} \tag{2.43}
$$

keeping in mind that we are always projecting into the subspace spanned by $|\Psi_n\rangle, |\Phi_n\rangle,\ n = 0, 1, 2 \ldots$.

Given the initial condition $a_0(0) = 1$, we are going to assert that $\dot{a}_n/\Gamma \ll 1$, $\dot{c}_n/\Gamma \ll 1$, and accordingly solve the equations of motion(2.43), then the

assertions can be checked for consistency. b_n and b_n^* can be eliminated from the equations of motion,

$$\begin{aligned} b_n(t) &= i\sqrt{(N-n)(n_P-n)}g_P^*\int_0^t d\tau e^{-[(n+1)\Gamma/2+i\Delta]\tau}\left(a_n(t-\tau)-c_n(t-\tau)\right) \\ &\simeq i\sqrt{(N-n)(n_P-n)}\; g_P^*\frac{1}{(n+1)\Gamma/2+i\Delta}\left(a_n(t)-c_n(t)\right) \\ b_n^*(t) &\simeq -i\sqrt{(N-n)(n_P-n)}\; g_P\frac{1}{(n+1)\Gamma/2-i\Delta}\left(a_n(t)-c_n(t)\right), \end{aligned} \tag{2.44}$$

where it is assumed that $\dot{a}_n/\Gamma \ll 1, \dot{c}_n/\Gamma \ll 1$. Then the coupled equations for a_n, c_n are received,

$$\begin{aligned} \dot{a}_n &= -\gamma_n(a_n-c_n)+2n\frac{\Gamma}{2}c_{n-1} \\ \dot{c}_n &= \gamma_n(a_n-c_n)-2(n+1)\frac{\Gamma}{2}c_n, \\ \gamma_n &= 2(N-n)(n_P-n)(n+1)\frac{2\lambda^2\Gamma}{(n+1)^2\Gamma^2+4\Delta^2}, \end{aligned} \tag{2.45}$$

from which a_n and c_n can be obtained in terms of each other,

$$\begin{aligned} c_n(t) &= \gamma_n\int_0^t d\tau e^{-2(n+1)\Gamma\tau}\left(a_n(t-\tau)-c_n(t-\tau)\right) \\ &\simeq \frac{\gamma_n}{(n+1)\Gamma}\left(a_n(t)-c_n(t)\right) \\ &\simeq \frac{\gamma_n}{(n+1)\Gamma}a_n(t), \end{aligned} \tag{2.46}$$

thus obtaining the equations governing a_n's,

$$\dot{a}_n = -\gamma_n a_n+\gamma_{n-1}a_{n-1}. \tag{2.47}$$

The initial condition is $a_0(0)=1$ and all the other terms in the density matrix, are equal to zero, which lead to the solutions,

$$\begin{aligned} a_0(t) &= e^{-\gamma_0 t} \\ a_n(t) &= \gamma_{n-1}\int_0^t d\tau e^{-\gamma_n\tau}a_{n-1}(t-\tau), n=1,2,3,\ldots \end{aligned} \tag{2.48}$$

In general the solution for a_n and c_n's will be a linear sum of the terms of the form, $exp(-\gamma_i t)$, $i \le n$, which are in line with the assumption that $\dot{a}_n/\Gamma \ll$

$1, \dot{c}_n/\Gamma \ll 1$. When $n = \min(n_P, N)$, $\gamma_n = 0$, thus the final value is $n_f = min(n_P, N)$ and the system evolves to the state $|\Psi_{n_f}\rangle\langle\Psi_{n_f}|$, and remains in this state. The time dependence of a_{n_f} is,

$$a_{n_f}(t) = 1 - \sum_{i=0}^{n_f-1} e^{-\gamma_i t} \prod_{j\neq i}^{n_f-1} \frac{\gamma_j}{\gamma_j - \gamma_i}. \tag{2.49}$$

Thus the characteristic time scale needed in order to obtain the final state is $1/\gamma_{n_f-1}$, since γ_n is a monotonically decreasing sequence. A case of interest is the initial state for which, $N = 2m, n_P = m$ ($m = 1, 2, 3, \ldots$ which can produce maximally entangled states, $|\Psi_m\rangle\langle\Psi_m|$. For this case the characteristic time scale for obtaining entangled state is $\tau^{-1} = \gamma_{m-1} = 4m(m+1)\lambda^2\Gamma/(m^2\Gamma^2 + 4\Delta^2)$, for instance for vanishing detuning $\Delta = 0$, $\tau^{-1} = 4(1 + 1/m)\lambda^2/\Gamma$.

2.5 Summary and discussion

In this chapter, we have studied the quantum dynamics of a system of two three-level atoms in the Λ configuration interacting with two modes of quantized electromagnetic field in a cavity under the assumption that the Stokes-mode photons either leave the cavity freely or are damped rapidly. It is shown that in both cases the system evolves from the state when both atoms are in the ground state and cavity contains a pumping photon into the robust entangled state (2.1). The system is also studied within the adiabatic limit when both cavity modes are off-resonant with the dipole transitions. The lifetime of this final state is defined completely by the nonradiative processes and is therefore relatively long. The results that were obtained for a system of two atoms, are generalized to the case of big atomic clusters. In fact, it is shown that a certain robust entanglement can be obtained in a system with any even number $2N$ of three-level Λ-type atoms initially prepared in the ground state and interacting with N pumping photons.

In the case of a cavity transparent to the Stokes photons, the detection of Stokes photon signalizes the rise of atomic entanglement. Such a photon can be monitored outside the cavity.

Let us stress that the general models with the Hamiltonians (2.2, 2.25) , that take into account all three atomic levels, admit certain peculiarities in the evolution towards the robust entangled state caused by the competition of transitions $3 \leftrightarrow 1$ and $3 \leftrightarrow 2$. Moreover, the general model admits also a number of intermediate maximum entangled states ($|\psi_2\rangle$ and $|\psi_{3k}\rangle$ in Eq. (2.7)) that do not exist in the effective model. Lifetime of these entangled states are defined by the dipole radiative processes and are therefore too short.

One of the most important conditions of experimental realization of the robust entanglement discussed in this chapter is that the transitions $1 \leftrightarrow 3$ and $3 \leftrightarrow 2$, used for absorption of pumping photons and generation of Stokes photons, should have quite different frequencies. The considerable difference of frequencies ω_{31} and ω_{32} makes it possible to design a multi-mode cavity with high quality with respect to ω_{31}, permitting either leakage or strong absorption of Stokes photons. An important example is provided by the $3S \leftrightarrow 4P$ and $4P \leftrightarrow 4S$ transitions in sodium atom and similar transitions in other alkaline atoms (see Ref. [78]). These atoms are widely used in quantum optics, in particular in investigation of Bose-Einstein condensation [79]. Λ-type structures, obeying the condition $\omega_{31} \gg \omega_{32}$ can also be found in other atoms and molecules [78]. In particular, the cavities with necessary properties may be assembled using distributed Bragg reflectors (DBR) and double DBR structures to single out two different wavelengths [80].

The initial state of the system can be prepared in the same way as in Ref. [62]. The atoms can propagate through the cavity, using either the same opening or two different openings. The velocity of atoms should be chosen in a proper way so that the time they spend in the cavity $\tau \gg (\Gamma^2 + \Delta_P^2)/\lambda_P^2\Gamma$(2.22) or $\tau \gg 1/\Gamma$(2.23). All measurements aimed at the detection of atomic entanglement can be performed outside the cavity.

Chapter 3

Generation of Robust Entanglement in Dielectric Medium

In this chapter we are going to study the interaction of multi-level atoms with electromagnetic field in the presence of dispersing-absorbing dielectric bodies and make a realistic proposal for deterministic entanglement of two three-level atoms passing by a dispersing-absorbing dielectric microsphere. The preparation of the initial state and the possible sources of entanglement loss are discussed, and it is shown that entanglement might still be very close to its maximum value if the system is properly engineered.

Introduction

Photon exchange between two atoms is one of the simplest processes to entangle two atoms in a common electromagnetic field. The effect, which is very weak in free space, can be enhanced significantly when the atoms are in a cavity [18, 19, 20]. Usually attempts are made to minimize the effect of spontaneous emission. Quite counterintuitively, in certain situations one can take advantage

of the spontaneous emission for entanglement generation [81, 82, 83, 84]. Consider, for example, two two-level atoms located in free space with one of them being initially excited. This product state is a superposition of a symmetric (superradiant) state and an antisymmetric (subradiant) state. If the two atoms are separated by distances much smaller than the wavelength, the symmetric state decays must faster than the antisymmetric one, leaving the system in a mixture of the ground state and the entangled antisymmetric state.

The scheme also works at distances much larger than the wavelength, if a resonator-like equipment is used which sufficiently enhances the atom-field coupling, thereby ensuring that a photon emitted in the process of resonant photon exchange, which is mediated by real photon emission and absorption, is accessible to the two atoms. This condition can also be satisfied when the atoms pass by a dielectric microsphere at diametrically opposite positions [81]. If the distance of the atoms from the surface of the sphere becomes sufficiently small, then the excitation of surface-guided (SG) and whispering gallery (WG) waves can give rise to strong collective effects, which are necessarily required to generate substantial entanglement. Needless to say that nonspherical bodies can also be used to realize a noticeable mutual coupling of the atoms.

A drawback of the use of two-level-type atoms is that the entanglement is transient. In particular, when two atoms that have become entangled between each other near a body such as a microsphere move away from it (and from each other), then they undergo ordinary spontaneous emission (in free space), which destroys the quantum coherence. Preservation of the atomic entanglement over long distances between the atoms is therefore not possible in this way.

The contradicting effects of entanglement creation and destruction typical of two-level atoms can be combined in a more refined scheme involving two three-level atoms of Λ type (Fig. 2.1), where the two lower lying states $|1\rangle$ and $|2\rangle$, such as the ground state and a metastable state or two metastable states, represent the qubits that are desired to be entangled with each other [73]. Whereas the transition $|1\rangle \leftrightarrow |3\rangle$ is strongly coupled to the field, the transition $|2\rangle \leftrightarrow |3\rangle$ is only weakly coupled to the field. Each atom is initially in the state $|1\rangle$, while

the field is prepared in a single-photon state. Let us assume that due to Rabi oscillations the state $|3\rangle$ of one of the two atoms, we do not know which one, is populated. Irreversible decay to the state $|2\rangle$ is then accompanied with an entanglement transfer forming a (quasi-)stationary entangled state between the two atoms with respect to the states $|1\rangle$ and $|2\rangle$. Its lifetime is limited only by the lifetime of the metastable states, and the degree of entanglement achievable can approach 100% in principle. Moreover, the scheme is purely deterministic and realizable by means of current experimental techniques.

In fact, the model Hamiltonian used in Ref. [73] is based on a Dicke-type system and does not allow for atoms that are spatially well separated from each other, with the interatomic distance being much larger than the characteristic wavelengths. However, for many applications in quantum information processing or for testing Bell's inequalities, large interatomic distances and thus the possibility of individual manipulation of the atoms are necessary prerequisites. The aim of the present work is to close this loophole, by considering two spatially well separated Λ-type three-level atoms appropriately positioned with respect to macroscopic bodies, so that the two key ingredients – enhanced atom-field coupling and sharp field resonances can be realized. Note that the second ingredient is absent in the case of a super-lens geometry [85]. To illustrate the theory, we apply it to the case of the two atoms being near a realistic dielectric microsphere. The formalism used is based on the quantization of the macroscopic electromagnetic field and allows to take into account material dispersion and absorption in a quantum-mechanically consistent manner.

The chapter is organized as follows. In section-3.1 we outline the quantization of EM field in dispersing-absorbing medium(see [86]). In Sec. 3.2 the basic equations for describing the interaction of N multilevel atoms with the electromagnetic field in the presence of dispersing and absorbing macroscopic bodies are given. In Sec. 3.3 the theory is applied to the problem of formation of an entangled state between two Λ-type three-level atoms. Section 3.4 presents the results obtained for the case when the two atoms are at diametrically opposite positions outside a microsphere. Finally, a summary and some concluding remarks are given in Sec. 3.5[87].

3.1 Quantization of electromagnetic field in dispersing-absorbing medium

From Kramers-Kronnig relations it is evident that whenever the dielectric function, as a function of frequency, deviates from unity, inevitably has an imaginary part at some frequencies. From Maxwell equations one can easily see that imaginary part implies a dissipation of electro-magnetic fields. Whenever one tries to quantize EM fields in the presence of a material with a complex dielectric function then the field operators will be damped. The dissipation in quantum mechanics implies the existence of a noise, so in the Maxwell equations one should introduce polarization noise operators (noise magnetization as well if the permeability is a complex quantity) as the source terms (see [88, 89, 90, 91, 92] and [86] for a review).

In this section a microscopic derivation of Maxwell equations in dispersive-absorptive medium will be presented within the Drude-Lorentz model, though the resulting quantization scheme is not limited with the Drude-Lorentz model.

We are going to consider local harmonic oscillators under the action of *Markovian Langevin forces*[93], for which the Heisenberg equations of motion are as follows

$$m\ddot{q} + m\gamma\dot{q} + m\omega_0^2 q = F(t), \tag{3.1}$$

where *a priori* we make no assumption about the nature of Langevin forces, or their algebra. There are two physical constraints on the system: $\{q(t), p(t)\}$ should satisfy equal time commution relation, and under thermodynamic equilibrium the local oscillator should obey Bose statistics, which read as follows,

$$\begin{aligned} [q(t), p(t)] &= i\hbar, \\ \langle E \rangle &= (\bar{n}(\omega_0) + 1/2)\hbar\omega_0 \end{aligned} \tag{3.2}$$

where $\langle E \rangle$ denote the thermal average of the energy of the oscillator, and $\bar{n}(\omega) = (exp(\beta\hbar\omega) - 1)^{-1}$ denotes the Bose distribution function.

As time goes to infinity the transient motion will decay away, thus one is left with the inhomogenous solution of (3.1). The solution of (3.1) in frequency space yields,

$$\begin{aligned} q(\omega) &= \frac{F(\omega)}{m(\omega_0^2 - \omega^2 + im\gamma\omega)} \\ p(\omega) &= -i\omega q(\omega), \end{aligned} \tag{3.3}$$

where the Fourier transform into the frequency space is defined as follows,

$$q(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathtt{d}\omega q(\omega) e^{-i\omega t}, \tag{3.4}$$

and since $q(t)$ is a Hermitian quantity $q(\omega) = q^\dagger(-\omega)$, and so forth for the other observables $F(t), p(t)$.

From the equal time commutation relation (3.2) one can deduce the following fact,

$$\begin{aligned} [q(\omega), q^\dagger(\omega')] &= g(\omega)\delta(\omega - \omega') \\ [q(\omega), q(\omega')] &= 0 \quad \text{for} \quad \omega, \omega' \geq 0 \end{aligned} \tag{3.5}$$

where $g(\omega)$ depends on the system under consideration, but always has to satisfy the following condition

$$\int_0^\infty \mathtt{d}\omega \; \omega g(\omega) = \frac{\hbar\pi}{m}. \tag{3.6}$$

From thermodynamic equilibrium condition (3.2) follows

$$\begin{aligned} \int_0^\infty \mathtt{d}\omega \int_0^\infty \mathtt{d}\omega' (\omega_0^2 + \omega\omega') < q^\dagger(\omega) q(\omega') > e^{i(\omega - \omega')t} &= \frac{\bar{n}(\omega_0)\hbar\omega_0}{m} \\ \int_0^\infty \mathtt{d}\omega (\omega_0^2 + \omega^2) g(\omega) &= \frac{\hbar\omega_0}{m}. \end{aligned} \tag{3.7}$$

Now we can impose these conditions on the damped harmonic oscillator for the steady state solution (3.3). The condition (3.2), impose the following constraints on the dissipation rate γ and the Langevin force $\hat{F(t)}$,

$$\begin{aligned} [F(\omega), F^\dagger(\omega')] &= 2\hbar m\omega\gamma\delta(\omega - \omega') \\ \langle F^\dagger(\omega) F(\omega) \rangle &= 2\hbar m\omega\gamma\bar{n}(\omega)\delta(\omega - \omega'). \end{aligned} \tag{3.8}$$

with the assumption $\omega_0 \gg \gamma$.

Now we are going to extend the previous consideration to three dimensions and consider a medium with randomly distributed damped oscillators of charge e, each bound to a spatially fixed charge center at $\mathbf{q}'_\mu$ of charge $-e$. The oscillator is described by the phase space coordinates $\mathbf{q}_\mu, \mathbf{p}_\mu$, and obey the equations of motion,

$$m\ddot{\mathbf{q}}_\mu = -m\omega_0^2\mathbf{q}_\mu - im\gamma\dot{\mathbf{q}}_\mu + \mathbf{F}_\mu(t) + e\mathbf{E}(\mathbf{q}'_\mu, t) + e\dot{\mathbf{q}}_\mu \times \mathbf{B}(\mathbf{q}'_\mu, t) \tag{3.9}$$

where $\mathbf{E}(\mathbf{q}_\mu, t)$ and $\mathbf{B}(\mathbf{q}_\mu, t)$ are the electric and magnetic field vectors respectively. $\mathbf{F}_\mu$ is the three dimensional Langevin force acting on the μth oscillator and the force acting on two distinct oscillators are uncorrelated, i.e. $[\mathbf{F}_{\mu i}(\omega), \mathbf{F}_{\mu' j}(\omega')] = \delta_{\mu\mu'}\delta(\omega - \omega')\delta_{ij}$. We are going to ignore the magnetic force , namely the last term in (3.9) assuming that $1 \gg \dot{q}_\mu/c$. Then the solution of (3.9) yields,

$$\mathbf{q}_\mu(\omega) = \frac{\mathbf{F}_\mu(\omega) + e\mathbf{E}(\mathbf{q}'_\mu, \omega)}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}. \tag{3.10}$$

Now we can express the polarization of the medium as follows,

$$\begin{aligned}\mathbf{P}(\mathbf{r}, \omega) &= e\sum_\mu \mathbf{q}_\mu \delta(\mathbf{r} - \mathbf{q}'_\mu) \\ &= \mathbf{P}_F(\mathbf{r}, \omega) + \mathbf{P}_N(\mathbf{r}, \omega)\end{aligned} \tag{3.11}$$

where $P_F(\mathbf{r}, \omega)$ is the polarization of the medium, and $P_N(\mathbf{r}, \omega)$ is the noise induced polarization,

$$\begin{aligned}\mathbf{P}_F(\mathbf{r}, \omega) &= \epsilon_0\epsilon(\mathbf{r}, \omega)\mathbf{E}(\mathbf{r}, \omega) \\ \mathbf{P}_N(\mathbf{r}, \omega) &= e\sum_\mu \frac{\mathbf{F}_\mu(\omega)}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}\delta(\mathbf{r} - \mathbf{q}'_\mu).\end{aligned} \tag{3.12}$$

$\epsilon(\mathbf{r}, \omega)$ is the dielectric function in the Drude Lorentz model,

$$\epsilon(\mathbf{r}, \omega) = 1 + \frac{\omega_P^2}{\omega_0^2 - \omega^2 + i\omega\gamma}. \tag{3.13}$$

$\omega_P^2 = e^2 n(\mathbf{r})/m\epsilon_0$ is the plasma frequency. From (3.12) the commutation relation for the noise operators can be obtained as follows,

$$[P_{Ni}(\mathbf{r}, \omega), P^\dagger_{Nj}(\mathbf{r}', \omega')] = 2\hbar\epsilon_0\epsilon_I(\mathbf{r}, \omega)\delta(\omega - \omega')\delta(\mathbf{r} - \mathbf{r}')\delta_{ij} \quad for \quad \omega, \omega' \geq 0 \tag{3.14}$$

where it is assumed that oscillators are randomly distributed, so that a spatial averaging is performed. $\epsilon_I(\mathbf{r},\omega) = \mathtt{Im}\epsilon(\mathbf{r},\omega)$ is the imaginary part of the dielectric function. Then noise polarization can be expressed as follows,

$$\mathbf{P}_N(\mathbf{r},\omega) = i\sqrt{2\hbar\epsilon_0\epsilon_I}\mathbf{f}(\mathbf{r},\omega), \tag{3.15}$$

where the bosonic annihilation and creation operators $\mathbf{f}(\mathbf{r},\omega), \mathbf{f}^\dagger(\mathbf{r}',\omega')$are introduced,

$$[f_i(\mathbf{r},\omega), f_j^\dagger(\mathbf{r}',\omega')] = \delta(\omega-\omega')\delta_{ij}\delta(\mathbf{r}-\mathbf{r}'). \tag{3.16}$$

Noise induced currents, and charge operators can be inroduced,

$$\begin{aligned}\mathbf{j}_N(\mathbf{r},\omega) &= -i\omega\mathbf{P}_N(\mathbf{r},\omega)\\ \rho_N(\mathbf{r},\omega) &= -\boldsymbol{\nabla}\cdot\mathbf{P}_N(\mathbf{r},\omega).\end{aligned} \tag{3.17}$$

Then the Maxwell equations for the field amplitudes become,

$$\boldsymbol{\nabla}.\mathbf{B}(\mathbf{r},\omega) = 0 \tag{3.18}$$

$$\boldsymbol{\nabla}.\epsilon_0\epsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) = -\boldsymbol{\nabla}.\mathbf{P}_N(\mathbf{r},\omega) \tag{3.19}$$

$$\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r},\omega) = i\omega\mathbf{B}(\mathbf{r},\omega) \tag{3.20}$$

$$\boldsymbol{\nabla}\times\mathbf{B}(\mathbf{r},\omega) + i\omega\mu_0\epsilon_0\epsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) = \mu_0\mathbf{j}_N(\mathbf{r},\omega) \tag{3.21}$$

where the positive frequency part of the (noise) polarization and current reads,

$$\begin{aligned}\mathbf{P}_N(\mathbf{r},\omega) &= i\sqrt{2\hbar\epsilon_0\epsilon_i(\mathbf{r},\omega)}\mathbf{f}(\mathbf{r},\omega)\\ \mathbf{j}_N(\mathbf{r},\omega) &= -i\omega\mathbf{P}_N(\mathbf{r},\omega).\end{aligned} \tag{3.22}$$

The electric field can be calculated by the following partial differential equation,

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r},\omega) - \frac{\omega^2}{c^2}\epsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) = i\omega\mu_0\mathbf{j}_N(\mathbf{r},\omega). \tag{3.23}$$

3.2 Master equation

Consider N multilevel atoms at given positions $\mathbf{r}_A$ that interact with the electromagnetic field in the presence of some macroscopic, linear bodies, which are

allowed to be both dispersing and absorbing. In electric dipole approximation, the overall system can be described by the multipolar-coupling Hamiltonian [86],

$$\hat{H} = \int \mathrm{d}^3\mathbf{r} \int_0^\infty \mathrm{d}\omega\, \hbar\omega\, \hat{\mathbf{f}}^\dagger(\mathbf{r},\omega)\hat{\mathbf{f}}(\mathbf{r},\omega) + \sum_A \sum_m \hbar\omega_{Am}\hat{R}_{Amm} - \sum_A \int_0^\infty \mathrm{d}\omega \big[\hat{\mathbf{d}}_A \hat{\mathbf{E}}^+(\mathbf{r}_A,\omega) + \mathrm{H.c.}\big]. \tag{3.24}$$

Here, the bosonic fields $\hat{\mathbf{f}}(\mathbf{r},\omega)$ and $\hat{\mathbf{f}}^\dagger(\mathbf{r},\omega)$,

$$\big[\hat{f}_k(\mathbf{r},\omega), \hat{f}^\dagger_{k'}(\mathbf{r}',\omega')\big] = \delta_{kk'}\delta(\omega-\omega')\delta(\mathbf{r}-\mathbf{r}'), \tag{3.25}$$

are the canonically conjugated variables of the system, which consists of the electromagnetic field and the bodies (including the dissipative system responsible for absorption), the $\hat{R}_{Amn}$ are the atomic (flip) operators

$$\hat{R}_{Amn} = |m\rangle_{AA}\langle n|, \tag{3.26}$$

with $|m\rangle_A$ being the mth energy eigenstate of the Ath atom (of energy $\hbar\omega_{Am}$), and

$$\hat{\mathbf{d}}_A = \sum_{m,n} \mathbf{d}_{Amn}\hat{R}_{Amn} \tag{3.27}$$

are the electric dipole operators of the atoms ($\mathbf{d}_{Amn} = {}_A\langle m|\hat{\mathbf{d}}_A|n\rangle_A$). Further, the body-assisted electric field in the ω domain, $\hat{\mathbf{E}}^+(\mathbf{r},\omega)$, expressed in terms of the fundamental variables $\hat{\mathbf{f}}(\mathbf{r},\omega)$ reads

$$\hat{\mathbf{E}}^+(\mathbf{r},\omega) = \int \mathrm{d}^3\mathbf{r}'\, \tilde{\boldsymbol{G}}(\mathbf{r},\mathbf{r}',\omega)\hat{\mathbf{f}}(\mathbf{r}',\omega), \tag{3.28}$$

where

$$\tilde{\boldsymbol{G}}(\mathbf{r},\mathbf{r}',\omega) = i\sqrt{\frac{\hbar}{\pi\varepsilon_0}}\frac{\omega^2}{c^2}\sqrt{\mathrm{Im}\,\varepsilon(\mathbf{r}',\omega)}\,\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega) \tag{3.29}$$

with $\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega)$ being the classical Green tensor which satisfies the equation

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega) - \frac{\omega^2}{c^2}\varepsilon(\mathbf{r},\omega)\boldsymbol{G}(\mathbf{r},\mathbf{r}',\omega) = \boldsymbol{\delta}(\mathbf{r}-\mathbf{r}') \tag{3.30}$$

together with the boundary conditions at infinity [$\boldsymbol{\delta}(\mathbf{r}) = \mathbb{1}\delta(\mathbf{r}-\mathbf{r}')$, $\mathbb{1}$ is the the 3×3 unit matrix]. Throughout the chapter we restrict our attention to dielectric bodies, which are described by a spatially varying complex permittivity $\varepsilon(\mathbf{r},\omega) = \mathrm{Re}\,\varepsilon(\mathbf{r},\omega) + i\mathrm{Im}\,\varepsilon(\mathbf{r},\omega)$.

Next we assume that the macroscopic bodies, say, microspheres or photonic crystals, act as resonator-like structures such that the excitation spectrum of the body-assisted electromagnetic-field shows a resonance structure with the lines being well separated from each other. With regard to the atom–field coupling, we assume that a few atomic transitions can be strongly coupled to field resonances tuned to them, while all other transitions are weakly coupled to the field. Following Ref. [81], we decompose the body-assisted electromagnetic field into the part (denoted by $\int_0^{\prime\infty} \mathrm{d}\omega \ldots$) that can be strongly coupled to atomic transitions and the rest (denoted by $\int_0^{\prime\prime\infty} \mathrm{d}\omega \ldots$), which only gives rise to a weak atom–field coupling. The Heisenberg equation of motion for an arbitrary operator $\hat{O}$ that belongs to the system consisting of the atoms and the part of the body-assisted electromagnetic field that strongly interacts with the atoms can then be written in the form of

$$\begin{aligned}\dot{\hat{O}} = -\frac{i}{\hbar}\big[\hat{O},\hat{H}\big] &= -\frac{i}{\hbar}\big[\hat{O},\hat{H}_{\mathrm{S}}\big] \\ &+ \frac{i}{\hbar}\sum_A \int_0^{\prime\prime\infty} \mathrm{d}\omega \,\Big\{\big[\hat{O},\hat{\mathbf{d}}_A\big]\hat{\mathbf{E}}^{+}(\mathbf{r}_A,\omega) \\ &\qquad + \hat{\mathbf{E}}^{-}(\mathbf{r}_A,\omega)\big[\hat{O},\hat{\mathbf{d}}_A\big]\Big\}, \end{aligned} \tag{3.31}$$

where

$$\begin{aligned}\hat{H}_{\mathrm{S}} &= \int \mathrm{d}^3\mathbf{r} \int_0^{\prime\infty} \mathrm{d}\omega\, \hbar\omega\, \hat{\mathbf{f}}^{\dagger}(\mathbf{r},\omega)\hat{\mathbf{f}}(\mathbf{r},\omega) \\ &+ \sum_A \sum_m \hbar\omega_{Am}\hat{R}_{Amm} - \sum_A \int_0^{\prime\infty} \mathrm{d}\omega \big[\hat{\mathbf{d}}_A\hat{\mathbf{E}}^{+}(\mathbf{r}_A,\omega) + \mathrm{H.c.}\big]. \end{aligned} \tag{3.32}$$

To handle the weak atom–field interaction, i.e., the integral $\int_0^{\prime\prime\infty} \mathrm{d}\omega \ldots$ in Eq. (3.31), we first formally solve the Heisenberg equation of motion

$$\begin{aligned}\dot{\hat{\mathbf{f}}}(\mathbf{r},\omega) &= -\frac{i}{\hbar}\big[\hat{\mathbf{f}}(\mathbf{r},\omega),\hat{H}\big] \\ &= -i\omega\hat{\mathbf{f}}(\mathbf{r},\omega) + \frac{i}{\hbar}\sum_A \hat{\mathbf{d}}_A\tilde{\boldsymbol{G}}^{*}(\mathbf{r}_A,\mathbf{r},\omega), \end{aligned} \tag{3.33}$$

which yields

$$\hat{\mathbf{f}}(\mathbf{r},\omega,t) = \hat{\mathbf{f}}_{\mathrm{free}}(\mathbf{r},\omega,t) + \frac{i}{\hbar}\sum_A \int_0^t \mathrm{d}t'\, \hat{\mathbf{d}}_A(t')\,\tilde{\boldsymbol{G}}^{*}(\mathbf{r}_A,\mathbf{r},\omega)e^{-i\omega(t-t')}, \tag{3.34}$$

where $\hat{\mathbf{f}}_{\text{free}}(\mathbf{r},\omega,t)$ evolves freely,

$$\hat{\mathbf{f}}_{\text{free}}(\mathbf{r},\omega,t)=\hat{\mathbf{f}}_{\text{free}}(\mathbf{r},\omega,0)e^{-i\omega t}. \tag{3.35}$$

Inserting Eq. (3.34) into Eq. (3.28), we derive

$$\begin{aligned}\hat{\mathbf{E}}^{+}(\mathbf{r},\omega,t)&=\hat{\mathbf{E}}^{+}_{\text{free}}(\mathbf{r},\omega,t)\\&+\frac{i}{\pi\varepsilon_0}\frac{\omega^2}{c^2}\sum_A\int_0^t \mathrm{d}\tau\, e^{-i\omega\tau}\mathrm{Im}\,\boldsymbol{G}(\mathbf{r},\mathbf{r}_A,\omega)\,\hat{\mathbf{d}}_A(t-\tau),\end{aligned} \tag{3.36}$$

where $\hat{\mathbf{E}}^{+}_{\text{free}}(\mathbf{r},\omega,t)$ is defined according to Eq. (3.28) with $\hat{\mathbf{f}}_{\text{free}}(\mathbf{r},\omega,t)$ in place of $\hat{\mathbf{f}}(\mathbf{r},\omega,t)$.

Introducing slowly varying atomic operators

$$\hat{\tilde{R}}_{Amn}(t)=\hat{R}_{Amn}(t)e^{-i\tilde{\omega}_{Amn}t}, \tag{3.37}$$

we may write the electric dipole operator, Eq. (3.27), as

$$\hat{\mathbf{d}}_A(t)=\sum_{m,n}\mathbf{d}_{Amn}\hat{\tilde{R}}_{Amn}(t)e^{i\tilde{\omega}_{Amn}t}. \tag{3.38}$$

We now insert Eq. (3.36) together with Eq. (3.38) in the integral $\int_0^{\prime\prime\infty}\mathrm{d}\omega\ldots$ in Eq. (3.31), apply the Markov approximation to the slowly varying atomic variables. In the Markov limit, the time integral is performed, with the proper convergence factor,

$$\lim_{\delta\to 0,t\to\infty}\int_0^t \mathrm{d}\tau e^{-i(\omega-\omega_{Amn})\tau}=-i\mathcal{P}\frac{1}{\omega-\omega_{Amn}}+\pi\delta(\omega-\omega_{Amn}).$$

Now the positive frequency part of the electric field (3.36) becomes,

$$\begin{aligned}\hat{\mathbf{E}}^{+}(\mathbf{r},t)&=\int_0^\infty \mathrm{d}\omega\hat{\mathbf{E}}^{+}(\mathbf{r},\omega,t)=\hat{\mathbf{E}}^{+}_{\text{free}}(\mathbf{r},t)\\&+\frac{1}{\pi\varepsilon_0}\int_0^\infty \mathrm{d}\omega\frac{\omega^2}{c^2}\sum_A[\mathcal{P}\frac{1}{\omega-\omega_{Amn}}+i\pi\delta(\omega-\omega_{Amn})]\mathrm{Im}\boldsymbol{G}(\mathbf{r},\mathbf{r}_A,\omega)\mathbf{d}^{*}_{Amn}\hat{R}_{Anm}.\end{aligned} \tag{3.39}$$

We substitute (3.39) into the Heisenberg equations of motion (3.31) and take the expectation value with respect to the initial state. Assuming that the free field is initially in the vacuum state except for frequencies resonant with the atomic transitions, we derive (cf. App. A of Ref. [81])

$$\begin{aligned}\langle \dot{\hat{O}} \rangle = &-\frac{i}{\hbar}\langle \big[\hat{O}, \hat{\tilde{H}}_{\rm S}\big] \rangle \\ &+ i \sum_{A\neq A'} \sum_{m,n} \Big(\delta_{AA'}^{mn} \langle \big[\hat{O}, \hat{R}_{Amn}\big] \hat{R}_{A'nm} \rangle \\ &\quad + \delta_{AA'}^{mn\,*} \langle \hat{R}_{A'nm} \big[\hat{O}, \hat{R}_{Amn}\big] \rangle \Big) \\ &- \frac{1}{2} \sum_{A,A'} \sum_{m,n}{}' \Big(\Gamma_{AA'}^{mn} \langle \big[\hat{O}, \hat{R}_{Amn}\big] \hat{R}_{A'nm} \rangle \\ &\quad - \Gamma_{AA'}^{nm\,*} \langle \hat{R}_{A'nm} \big[\hat{O}, \hat{R}_{Amn}\big] \rangle \Big), \end{aligned} \tag{3.40}$$

where the primed sum $\sum'_{m,n}$ indicates that transitions that can strongly interact with the body-assisted electromagnetic field are excluded. In Eq. (3.40), $\hat{\tilde{H}}_{\rm S}$ is defined according to Eq. (3.32), with ω_{Am} being replaced by

$$\tilde{\omega}_{Am} = \omega_{Am} - \delta_{AA}^{m}, \tag{3.41}$$

where

$$\delta_{AA}^{m} = \sum_{n} \delta_{AA}^{mn}, \tag{3.42}$$

with δ_{AA}^{mn} being obtained from

$$\begin{aligned}\delta_{AA'}^{mn} = &\frac{1}{\hbar\pi\varepsilon_0 c^2}\, \mathcal{P} \int_0^\infty {\rm d}\omega \omega^2 \\ &\times \frac{\mathbf{d}_{Amn}\, {\rm Im}\, \boldsymbol{G}(\mathbf{r}_A, \mathbf{r}_{A'}, \omega)\, \mathbf{d}_{A'mn}^*}{\omega - \omega_{A'mn}} \end{aligned} \tag{3.43}$$

($\mathcal{P}$, principal part) for $A = A'$. For $A \neq A'$, the parameters $\delta_{AA'}^{mn}$ are the dipole–dipole coupling strengths between different atoms A and A'. Further, the decay rates $\Gamma_{AA'}^{mn}$ are defined according to

$$\begin{aligned}\Gamma_{AA'}^{mn} = &\frac{2\tilde{\omega}_{A'mn}^2}{\hbar\varepsilon_0 c^2}\, \Theta(\omega_{A'mn}) \\ &\times\, \mathbf{d}_{Amn}\, {\rm Im}\, \boldsymbol{G}(\mathbf{r}_A, \mathbf{r}_{A'}, \omega_{A'mn})\, \mathbf{d}_{A'mn}^* \end{aligned} \tag{3.44}$$

[$\Theta(x)$, unit step function].

Using the relationship

$$\begin{aligned}\langle \hat{O}(t) \rangle &= \mathrm{Tr}\big[\hat{\rho}(0)\hat{O}(t)\big] \\ &= \mathrm{Tr}\big[\hat{\rho}(t)\hat{O}(0)\big] = \mathrm{Tr}\big[\hat{\varrho}(t)\hat{O}(0)\big],\end{aligned} \tag{3.45}$$

where $\hat{\rho}$ is the density operator of the overall system, and $\hat{\varrho}$ is the (reduced) density operator of the system under consideration, and making use of the cyclic properties of the trace, from Eq. (3.39) we derive the following equation of motion for the system density operator in the Schrödinger picture:

$$\begin{aligned}\dot{\hat{\varrho}} = &-\frac{i}{\hbar}\big[\hat{\tilde{H}}_{\mathrm{S}}, \hat{\varrho}\big] + \Big[i \sum_{A,A'}{}' \sum_{m,n} \delta_{AA'}^{mn} (\hat{R}_{Amn}\hat{R}_{A'nm}\hat{\varrho} \\ &\qquad - \hat{R}_{A'nm}\hat{\varrho}\hat{R}_{Amn}) + \mathrm{H.c.}\Big] \\ &- \frac{1}{2}\sum_{A,A'}{}' \sum_{m,n} \Big[\Gamma_{AA'}^{mn} (\hat{R}_{Amn}\hat{R}_{A'nm}\hat{\varrho} \\ &\qquad - \hat{R}_{A'nm}\hat{\varrho}\hat{R}_{Amn}) + \mathrm{H.c.}\Big].\end{aligned} \tag{3.46}$$

Equation (3.46) is a generalization of the two-level-atom result in Ref. [81] to the case of multilevel atoms. In particular, if the conditions

$$\delta_{AA'}^{mn} = \delta_{A'A}^{mn\,*}, \tag{3.47}$$

$$\Gamma_{AA'}^{mn} = \Gamma_{A'A}^{mn\,*} \tag{3.48}$$

are fulfilled, which is the case when $\omega_{Amn} = \omega_{A'mn}$, for example this is always the case in particular for identical atoms, then the master equation (3.46) takes the somewhat simpler form of

$$\begin{aligned}\dot{\hat{\varrho}} = &-\frac{i}{\hbar}\Big[\hat{\tilde{H}}_{\mathrm{S}} + \hat{H}_{\mathrm{D}}, \hat{\varrho}\Big] \\ &- \frac{1}{2}\sum_{A,A'}{}' \sum_{m,n} \Gamma_{AA'}^{mn} (\hat{R}_{Amn}\hat{R}_{A'nm}\hat{\varrho} \\ &- 2\hat{R}_{A'nm}\hat{\varrho}\hat{R}_{Amn} + \hat{\varrho}\hat{R}_{Amn}\hat{R}_{A'nm}),\end{aligned} \tag{3.49}$$

where

$$\hat{H}_{\mathrm{D}} = -\sum_{A,A'}{}' \sum_{m>n} \hbar \Delta_{AA'}^{mn} \hat{R}_{Amn} \hat{R}_{A'nm} \tag{3.50}$$

describes the dipole-dipole interaction between the atoms, with $\Delta_{AA'}^{mn}$ being the dipole-dipole coupling strengths,

$$\Delta_{AA'}^{mn} = \delta_{AA'}^{mn} + \delta_{A'A}^{nm}. \tag{3.51}$$

which is symmetric with respect to the atoms i.e. $\Delta_{AA'}^{mn} = \Delta_{AA'}^{mn}$, only if the atoms are equivalently positioned,

$$\mathbf{d}_{Amn} \operatorname{Im} \boldsymbol{G}(\mathbf{r}_A, \mathbf{r}_{A'}, \tilde{\omega}_{A'mn})\, \mathbf{d}_{A'mn}^{*} = \mathbf{d}_{A'mn} \operatorname{Im} \boldsymbol{G}(\mathbf{r}'_A, \mathbf{r}_A, \tilde{\omega}_{Amn})\, \mathbf{d}_{A'mn}^{*}. \tag{3.52}$$

According to Eq. (3.49), the (undamped) system is governed by an effective Hamiltonian equal to $\hat{\tilde{H}}_{\mathrm{S}} + \hat{H}_{\mathrm{D}}$. Note that this is not true in general, but only under the conditions (3.47) and (3.48).

To construct the (formal) solution to the master equation (3.49), we first rewrite it in the form of

$$\dot{\hat{\varrho}} = \hat{L}\hat{\varrho} + \hat{S}\hat{\varrho}, \tag{3.53}$$

where $\hat{L}$ and $\hat{S}$ are superoperators which act on $\hat{\varrho}$ according to the rules

$$\hat{L}\hat{\varrho} \equiv -\frac{i}{\hbar}(\hat{\mathcal{H}}\hat{\varrho} - \hat{\varrho}\hat{\mathcal{H}}^{\dagger}), \tag{3.54}$$

$$\hat{S}\hat{\varrho} \equiv \sum_{A,A'} \sum_{m,n}{}' \Gamma_{AA'}^{mn} \hat{R}_{A'nm} \hat{\varrho} \hat{R}_{Amn}, \tag{3.55}$$

and the non-Hermitian "Hamiltonian" $\hat{\mathcal{H}}$ reads

$$\hat{\mathcal{H}} = \hat{\tilde{H}}_{\mathrm{S}} + \hat{H}_{\mathrm{D}} - \frac{i\hbar}{2} \sum_{A,A'} \sum_{m>n}{}' \Gamma_{AA'}^{mn} \hat{R}_{Amn} \hat{R}_{A'nm}. \tag{3.56}$$

From Eqs. (3.53)–(3.55) it then follows that

$$\hat{\varrho}(t) = e^{\hat{L}(t-t_0)} \hat{\varrho}(t_0) + \int_{t_0}^{t} \mathrm{d}t_1 e^{\hat{L}(t-t_1)} \hat{S} \hat{\varrho}(t_1). \tag{3.57}$$

By iteration, from Eq. (3.57) one readily finds

$$\hat{\varrho}(t) = \sum_{n=0}^{\infty} \hat{\varrho}^{(n)}(t), \tag{3.58}$$

where

$$\hat{\varrho}^{(0)}(t) = e^{\hat{L}(t-t_0)}\hat{\varrho}(t_0), \tag{3.59}$$

$$\hat{\varrho}^{(n)}(t) = \int_{t_0}^{t} \mathrm{d}t_n \int_{t_0}^{t_n} \mathrm{d}t_{n-1} \ldots \int_{t_0}^{t_2} \mathrm{d}t_1 e^{\hat{L}(t-t_n)} \times \hat{S} e^{\hat{L}(t_n-t_{n-1})} \ldots \hat{S} e^{\hat{L}(t_1-t_0)} \hat{\varrho}(t_0), \; n = 1, 2, 3 \ldots . \tag{3.60}$$

Although Eq. (3.58) is not a perturbative expansion, it can be helpful, as we shall see below, in finding the explicit solutions to the master equation.

3.3 Two three-level atoms of Λ type

3.3.1 Solution to the master equation

Let us specify the atomic system and consider two identical three-level atoms A and B of Λ type as sketched in Fig. 2.1.

We assume that the dipole-allowed transition $|1\rangle \leftrightarrow |3\rangle$ is tuned to a well pronounced body-induced electromagnetic field resonance, thereby giving rise to a strong dipole-allowed atom–field coupling. Further, the dipole-allowed transition $|2\rangle \leftrightarrow |3\rangle$ is assumed to be weakly coupled to the body-assisted electromagnetic field, and the transition between the states $|1\rangle$ and $|2\rangle$ is dipole-forbidden. Restricting our attention to two atoms at equivalent positions with respect to the macroscopic bodies, so that corresponding transition frequencies are equally shifted and the relations

$$\Delta_{AB}^{31} = \Delta_{BA}^{31}, \qquad \Delta_{AB}^{32} = \Delta_{BA}^{32}, \tag{3.61}$$

$$\Gamma_{AA}^{32} = \Gamma_{BB}^{32}, \qquad \Gamma_{AB}^{32} = \Gamma_{BA}^{32} \tag{3.62}$$

hold [cf. Eqs. (3.47) and (3.48)], we may apply the master equation in the form of Eq. (3.49) and its solution in the form of Eqs. (3.58)–(3.60), with Eqs. (3.55) and (3.56) being explicitly given by

$$\hat{S}\hat{\varrho} \equiv \sum_{A',A''=A,B} \Gamma_{A'A''}^{32} \hat{R}_{A''23} \hat{\varrho} \hat{R}_{A'32}, \tag{3.63}$$

and

$$\hat{\mathcal{H}} = \hat{\tilde{H}}_{\rm S} + \hat{H}_{\rm D} - \frac{i\hbar}{2}\sum_{A',A''}\Gamma^{32}_{A'A''}\hat{R}_{A'32}\hat{R}_{A''23}, \tag{3.64}$$

$$\begin{aligned}\hat{\tilde{H}}_{\rm S} = &\int {\rm d}^3\mathbf{r}\int_0^{\prime\infty}{\rm d}\omega\,\hbar\omega\,\hat{\mathbf{f}}^\dagger(\mathbf{r},\omega)\hat{\mathbf{f}}(\mathbf{r},\omega)\\ &+\sum_{A'=A,B}\sum_{m=1}^{3}\hbar\tilde{\omega}_{A'm}\hat{R}_{A'mm}\\ &-\sum_{A'=A,B}\int_0^{\prime\infty}{\rm d}\omega\big[\mathbf{d}_{A'31}\hat{R}_{A'31}\underline{\hat{\mathbf{E}}}(\mathbf{r}_{A'},\omega)+{\rm H.c.}\big],\end{aligned} \tag{3.65}$$

$$\hat{H}_{\rm D} = -(\hbar\Delta^{31}_{AB}\hat{R}_{A31}\hat{R}_{B13} + \Delta^{32}_{AB}\hat{R}_{A32}\hat{R}_{B23}) + {\rm H.c.}, \tag{3.66}$$

the rotating-wave approximation being made in Eq. (3.65).

To specify the initial condition at time t_0, let us assume that the two atoms are initially in the ground state $|1,1\rangle$ ($|i,j\rangle \equiv |i\rangle_A \otimes |j\rangle_B$, $i,j=1,2,3$) and the rest of the system is prepared in a state

$$|F\rangle = \int_0^{\prime\infty}{\rm d}\omega\int {\rm d}^3\mathbf{r}\,\mathbf{C}(\mathbf{r},\omega,t_0)\hat{\mathbf{f}}^\dagger(\mathbf{r},\omega)|\{0\}\rangle, \tag{3.67}$$

where $\mathbf{C}(\mathbf{r},\omega,t_0)$ as a function of ω is non-zero in a small interval around $\omega \simeq \tilde{\omega}_{A31} = \tilde{\omega}_{B31}$, and $|\{0\}\rangle$ is vacuum state with respect to this frequency interval. The initial density operator can then be given in the form of ($t_0 = 0$)

$$\hat{\varrho}(0) = |\Psi(0)\rangle\langle\Psi(0)|, \qquad |\Psi(0)\rangle = |1,1\rangle\otimes|F\rangle. \tag{3.68}$$

In order to determine the density operator at time t, we begin by calculating the first term of the series (3.58), viz.

$$\hat{\varrho}^{(0)}(t) = e^{\hat{L}t}\hat{\varrho}(0) = |\Psi(t)\rangle\langle\Psi(t)|, \tag{3.69}$$

where the (damped) state vector

$$|\Psi(t)\rangle = e^{-i\hat{\mathcal{H}}t/\hbar}|\Psi(0)\rangle \tag{3.70}$$

obviously obeys the equation

$$i\hbar\frac{{\rm d}|\Psi(t)\rangle}{{\rm d}t} = \hat{\mathcal{H}}|\Psi(t)\rangle. \tag{3.71}$$

Recalling the initial condition (3.68) and the form of $\hat{\mathcal{H}}$, Eqs. (3.64)–(3.66), we may expand $|\Psi(t)\rangle$ as

$$\begin{aligned}|\Psi(t)\rangle = & C_{31}(t)e^{-i(\tilde{\omega}_{A1}+\tilde{\omega}_{B3})t}|3,1\rangle\otimes|\{0\}\rangle \\ & + C_{13}(t)e^{-i(\tilde{\omega}_{A3}+\tilde{\omega}_{B1})t}|1,3\rangle\otimes|\{0\}\rangle \\ & + \int_0^{\prime\infty}\mathrm{d}\omega\int\mathrm{d}^3\mathbf{r}\,e^{-i(\tilde{\omega}_{A1}+\tilde{\omega}_{B1}+\omega)t} \\ & \times\ \mathbf{C}(\mathbf{r},\omega,t)\hat{\mathbf{f}}^\dagger(\mathbf{r},\omega)|\{0\}\rangle\otimes|1,1\rangle. \end{aligned} \tag{3.72}$$

We now substitute Eq. (3.72) into Eq. (3.71) and make use of Eqs. (3.64)–(3.66). Straightforward calculation yields the following system of differential equations for the expansion coefficients:

$$\dot{C}_{31} = -\tfrac{1}{2}\Gamma^{32}_{AA}C_{31} + i\Delta^{31}_{AB}C_{13} + \frac{i}{\hbar}\int_0^{\prime\infty}\mathrm{d}\omega\int\mathrm{d}^3\mathbf{r}\,\mathbf{d}_{A31}\tilde{\boldsymbol{G}}(\mathbf{r}_A,\mathbf{r},\omega)\mathbf{C}(\mathbf{r},\omega)e^{-i(\omega-\tilde{\omega}_{A31})t}, \tag{3.73}$$

$$\dot{C}_{13} = -\tfrac{1}{2}\Gamma^{32}_{BB}C_{13} + i\Delta^{31}_{AB}C_{31} + \frac{i}{\hbar}\int_0^{\prime\infty}\mathrm{d}\omega\int\mathrm{d}^3\mathbf{r}\,\mathbf{d}_{B31}\tilde{\boldsymbol{G}}(\mathbf{r}_B,\mathbf{r},\omega)\mathbf{C}(\mathbf{r},\omega)e^{-i(\omega-\tilde{\omega}_{B31})t}, \tag{3.74}$$

$$\dot{\mathbf{C}}(\mathbf{r},\omega) = \frac{i}{\hbar}e^{i(\omega-\tilde{\omega}_{A31})t}[\mathbf{d}^*_{A31}\tilde{\boldsymbol{G}}^*(\mathbf{r}_A,\mathbf{r},\omega)C_{31} + \mathbf{d}^*_{B31}\tilde{\boldsymbol{G}}^*(\mathbf{r}_B,\mathbf{r},\omega)C_{13}] \tag{3.75}$$

Recall that $\tilde{\omega}_{A31}=\tilde{\omega}_{B31}$. Inserting the formal solution to Eq. (3.75) in Eqs. (3.73) and (3.74), we derive, on making use of the properties of the Green tensor, the integro-differential equations

$$\begin{aligned}\dot{C}_{31} = & -\tfrac{1}{2}\Gamma^{32}_{AA}C_{31} + i\Delta^{31}_{AB}C_{13} \\ & + \int_0^t\mathrm{d}t'\,[K_{AA}(t-t')C_{31}(t') \\ & + K_{AB}(t-t')C_{13}(t')] + F_{31}(t), \end{aligned} \tag{3.76}$$

$$\begin{aligned}\dot{C}_{13} = & -\tfrac{1}{2}\Gamma^{32}_{BB}C_{13} + i\Delta^{31}_{AB}C_{31} \\ & + \int_0^t\mathrm{d}t'\,[K_{BB}(t-t')C_{13}(t') \\ & + K_{BA}(t-t')C_{31}(t')] + F_{13}(t), \end{aligned} \tag{3.77}$$

where the kernel function $K_{A'A''}(t)$ is defined by

$$K_{A'A''}(t) = -\frac{1}{\hbar\pi\varepsilon_0}\int_0^{\prime\infty} \mathrm{d}\omega\, \frac{\omega^2}{c^2}\, e^{-i(\omega-\tilde{\omega}_{A31})t} \times \mathbf{d}_{A'31}\, \mathrm{Im}\, \boldsymbol{G}(\mathbf{r}_{A'}, \mathbf{r}_{A''}, \omega)\, \mathbf{d}^*_{A''31} \tag{3.78}$$

$[A'(A'') = A, B]$, and the free-field driving terms F_{31} and F_{13} read

$$F_{31}(t) = \frac{i}{\hbar}\int_0^{\prime\infty} \mathrm{d}\omega \int \mathrm{d}^3\mathbf{r} \mathbf{d}_{A31}\tilde{\boldsymbol{G}}(\mathbf{r}_A, \mathbf{r}, \omega) \times \mathbf{C}(\mathbf{r}, \omega, 0) e^{-i(\omega-\tilde{\omega}_{A31})t}, \tag{3.79}$$

$$F_{13}(t) = \frac{i}{\hbar}\int_0^{\prime\infty} \mathrm{d}\omega \int \mathrm{d}^3\mathbf{r} \mathbf{d}_{B31}\tilde{\boldsymbol{G}}(\mathbf{r}_B, \mathbf{r}, \omega) \times \mathbf{C}(\mathbf{r}, \omega, 0) e^{-i(\omega-\tilde{\omega}_{B31})t}. \tag{3.80}$$

Note that for identical atoms at equivalent positions with respect to the macroscopic bodies

$$K_{AA}(t) = K_{BB}(t), \quad K_{AB}(t) = K_{BA}(t). \tag{3.81}$$

Instead of considering the probability amplitudes C_{31} and C_{13}, it is advantageous to introduce the probability amplitudes

$$C^{13}_{\pm} = 2^{-\frac{1}{2}}\left(C_{31} \pm C_{13}\right), \tag{3.82}$$

which are the expansion coefficients of $|\Psi\rangle$ with respect to the atomic basis

$$|\pm_{13}\rangle = 2^{-\frac{1}{2}}\left(|3,1\rangle \pm |1,3\rangle\right), \tag{3.83}$$

so that Eq. (3.72) takes the form of

$$\begin{aligned} |\Psi(t)\rangle = {} & C^{13}_{+}(t) e^{-i(\tilde{\omega}_{A1}+\tilde{\omega}_{B3})t}|+_{13}\rangle \otimes |\{0\}\rangle \\ & + C^{13}_{-}(t) e^{-i(\tilde{\omega}_{A1}+\tilde{\omega}_{B3})t}|-_{13}\rangle \otimes |\{0\}\rangle \\ & + \int_0^{\prime\infty} \mathrm{d}\omega \int \mathrm{d}^3\mathbf{r}\, e^{-i(\tilde{\omega}_{A1}+\tilde{\omega}_{B1}+\omega)t} \\ & \times \mathbf{C}(\mathbf{r}, \omega, t)\hat{\mathbf{f}}^{\dagger}(\mathbf{r}, \omega)|\{0\}\rangle \otimes |1,1\rangle. \end{aligned} \tag{3.84}$$

From Eqs. (3.76)–(3.82) it is not difficult to see that the differential equations for $C_{\pm}^{13}$ decouple

$$\dot{C}_{\pm}^{13} = \left(\pm i\Delta_{AB}^{31} - \tfrac{1}{2}\Gamma_{AA}^{32}\right) C_{\pm}^{13} + \int_0^t \mathrm{d}t'\, K_{\pm}(t-t') C_{\pm}^{13}(t') + F_{\pm}(t), \tag{3.85}$$

where

$$K_{\pm}(t) = K_{AA}(t) \pm K_{AB}(t), \tag{3.86}$$

$$F_{\pm}(t) = 2^{-1/2}[F_{31}(t) \pm F_{13}(t)]. \tag{3.87}$$

The field resonance strongly coupled to the atomic transition $|1\rangle \leftrightarrow |3\rangle$ can be typically modeled by a Lorentzian, with $\omega_{\mathrm{C}} \approx \tilde{\omega}_{A31}$ and $\Delta\omega_{\mathrm{C}}$ being the central frequency and the half width at half maximum, respectively. In this case, Eq. (3.78) can be approximated by

$$K_{A'A''}(t) = -\,\Gamma_{A'A''}^{31}\, e^{-i(\omega_{\mathrm{C}} - \tilde{\omega}_{A31})t} \times \frac{1}{2\pi} \int \mathrm{d}\omega\, \frac{\Delta\omega_{\mathrm{C}}^2 e^{-i(\omega-\omega_{\mathrm{C}})t}}{(\omega-\omega_{\mathrm{C}})^2 + \Delta\omega_{\mathrm{C}}^2}\,, \tag{3.88}$$

where $\Gamma_{A'A''}^{31}$ is defined according to Eq. (3.44), but with $\tilde{\omega}_{A31}$ being replaced by ω_{C},

$$\Gamma_{A'A''}^{31} = \frac{2\omega_{\mathrm{C}}^2}{\hbar\varepsilon_0 c^2}\, \mathbf{d}_{A'31}\, \mathrm{Im}\, \boldsymbol{G}(\mathbf{r}_{A'}, \mathbf{r}_{A''}, \omega_{\mathrm{C}})\, \mathbf{d}_{A''31}^{*}. \tag{3.89}$$

From Eq. (3.88) it then follows that ($t \geq 0$)

$$K_{A'A''}(t) = -\tfrac{1}{2}\Gamma_{A'A''}^{31} \Delta\omega_{\mathrm{C}}\, e^{-i(\Delta - i\Delta\omega_{\mathrm{C}})t} \tag{3.90}$$

($\Delta = \omega_{\mathrm{C}} - \tilde{\omega}_{A31}$). Using Eq. (3.90) and differentiating both sides of Eq. (3.85) with respect to time, we find that $C_{\pm}^{13}$ satisfies the second-order differential equation

$$\ddot{C}_{\pm}^{13} + a_{1\pm}\dot{C}_{\pm}^{13} + a_{2\pm} C_{\pm}^{13} = \dot{F}_{\pm}(t) + i(\Delta - i\Delta\omega_{\mathrm{C}}) F_{\pm}(t), \tag{3.91}$$

where

$$a_{1\pm} = i(\Delta \mp \Delta_{AB}^{31}) + \Delta\omega_{\mathrm{C}} + \tfrac{1}{2}\Gamma_{AA}^{32}, \tag{3.92}$$

$$a_{2\pm} = g_{\pm}^2 + (\Delta - i\Delta\omega_{\mathrm{C}})\left(\pm\Delta_{AB} + i\tfrac{1}{2}\Gamma_{AA}^{32}\right), \tag{3.93}$$

with

$$g_{\pm}^2 = \tfrac{1}{2}\Gamma_{\pm}^{31}\Delta\omega_{\rm C}, \quad \Gamma_{\pm}^{31} = \Gamma_{AA}^{31} \pm \Gamma_{AB}^{31}. \tag{3.94}$$

If $C_{\pm}^{13}(t)$ are known, then the probability amplitude $\mathbf{C}(\mathbf{r},\omega,t)$ can be obtained from Eq. (3.75) together with Eq. (3.82).

To calculate the terms $\hat{\varrho}^{(n)}(t)$ $(n>0)$, Eq. (3.60), of the series (3.58), we note that the action of the operator $\hat{S}$, Eq. (3.63), on $\hat{\varrho}^{(0)}(t) = |\Psi(t)\rangle\langle\Psi(t)|$ corresponds to atomic transitions $|3\rangle \to |2\rangle$. Thus, only the states $|1,3\rangle$ and $|3,1\rangle$, or equivalently $|\pm_{13}\rangle$, can contribute to $\hat{S}[|\Psi(t)\rangle\langle\Psi(t)|]$. It is not difficult to see that

$$\begin{aligned}&\hat{S}(|\pm_{13}\rangle\langle\pm_{13}|) = \\ &\Gamma_{AA}^{32}|\pm_{12}\rangle\langle\pm_{12}| \mp \tfrac{1}{2}\Gamma_{-}^{32}(|+_{12}\rangle\langle+_{12}| - |-_{12}\rangle\langle-_{12}|),\end{aligned} \tag{3.95}$$

$$\begin{aligned}&\hat{S}(|\pm_{13}\rangle\langle\mp_{13}|) = \\ &\Gamma_{AA}^{32}|\pm_{12}\rangle\langle\mp_{12}| - \tfrac{1}{2}\Gamma_{-}^{32}(|\pm_{12}\rangle\langle\mp_{12}| - |\mp_{12}\rangle\langle\pm_{12}|)\end{aligned} \tag{3.96}$$

$[\Gamma_{\pm}^{32} = \Gamma_{AA}^{32} \pm \Gamma_{AB}^{32},\ |\pm_{12}\rangle = 2^{-\frac{1}{2}}(|2,1\rangle \pm |1,2\rangle)]$. Combining Eqs. (3.84), (3.95), and (3.96), we derive

$$\begin{aligned}\hat{S}\hat{\varrho}^{(0)}(t) &= \hat{S}[|\Psi(t)\rangle\langle\Psi(t)|] \\ &= |\{0\}\rangle\langle\{0\}| \otimes \Big\{ \big(\tfrac{1}{2}\Gamma_{+}^{32}|C_{+}^{13}|^2 \\ &\quad + \tfrac{1}{2}\Gamma_{-}^{32}|C_{-}^{13}|^2\big)\,|+_{12}\rangle\langle+_{12}| + \big(\tfrac{1}{2}\Gamma_{+}^{32}|C_{-}^{13}|^2 \\ &\quad + \tfrac{1}{2}\Gamma_{-}^{32}|C_{+}^{13}|^2\big)\,|-_{12}\rangle\langle-_{12}| + \big[\big(\tfrac{1}{2}\Gamma_{+}^{32}C_{+}^{13}C_{-}^{13*} \\ &\quad + \tfrac{1}{2}\Gamma_{-}^{32}C_{+}^{13*}C_{-}^{13}\big)\,|+_{12}\rangle\langle-_{12}| + \text{H.c.}\big]\Big\},\end{aligned} \tag{3.97}$$

$$\hat{S}\hat{S}(|\Psi(t)\rangle\langle\Psi(t)|) = 0. \tag{3.98}$$

Recalling that $\hat{\mathcal{H}}$, Eqs. (3.64)–(3.66), acts on atomic states in the subspace spanned by $|\pm_{13}\rangle$, we see that

$$e^{\hat{L}(t-t_1)}\hat{S}(|\Psi(t_1)\rangle\langle\Psi(t_1)|) = \hat{S}(|\Psi(t_1)\rangle\langle\Psi(t_1)|), \tag{3.99}$$

leading to

$$\varrho^{(1)}(t) = \int_0^t \mathrm{d}t_1 \hat{S}(|\Psi(t_1)\rangle\langle\Psi(t_1)|) \tag{3.100}$$

[cf. Eq. (3.60)]. Further, Eqs. (3.98) and (3.99) imply that $\hat{\varrho}^{(n)} = 0$ if $n \geq 2$. Thus, the solution to the master equation reads

$$\hat{\varrho}(t) = |\Psi(t)\rangle\langle\Psi(t)| + \int_0^t \mathrm{d}t_1 \hat{S}[|\Psi(t_1)\rangle\langle\Psi(t_1)|] \tag{3.101}$$

together with Eqs. (3.84) and (3.97).

3.3.2 Stationary limit

Let us restrict our attention to the stationary limit $t \to \infty$. Since $F_{31}(t)$ and $F_{13}(t)$ approach zero as t tends to infinity, Eqs. (3.73) and (3.74) imply that

$$\lim_{t\to\infty} C_{\pm}^{13}(t) = 0. \tag{3.102}$$

Inserting Eq. (3.84) in Eq. (3.101), we derive

$$\hat{\varrho}(t \to \infty) = \hat{\varrho}_{\mathrm{at}} \otimes |\{0\}\rangle\langle\{0\}|, \tag{3.103}$$

$$\begin{aligned}\hat{\varrho}_{\mathrm{at}} = {} & \alpha_+ |+_{12}\rangle\langle+_{12}| + \alpha_- |-_{12}\rangle\langle-_{12}| \\ & + (\beta |+_{12}\rangle\langle-_{12}| + \mathrm{H.c.}) + (1-\alpha_+ - \alpha_-)|1,1\rangle\langle 1,1|,\end{aligned} \tag{3.104}$$

where

$$\alpha_\pm = \int_0^\infty \mathrm{d}t \left(\tfrac{1}{2}\Gamma_\pm^{32}|C_+^{13}|^2 + \tfrac{1}{2}\Gamma_\mp^{32}|C_-^{13}|^2\right), \tag{3.105}$$

$$\beta = \int_0^\infty \mathrm{d}t \left(\tfrac{1}{2}\Gamma_+^{32} C_+^{13} C_-^{13*} + \tfrac{1}{2}\Gamma_-^{32} C_+^{13*} C_-^{13}\right). \tag{3.106}$$

To determine the accessible entanglement of the two atoms, it may be instructive to study the concurrence of the atomic subsystem(See Appendix-B), which may be regarded as being a measure of entanglement [94]. For this purpose, we have to calculate the spin-flipped density operator

$$\hat{\tilde{\rho}}_{\mathrm{at}} = (\hat{\sigma}_{Ay} \otimes \hat{\sigma}_{By})\ \hat{\rho}_{\mathrm{at}}^*\ (\hat{\sigma}_{By} \otimes \hat{\sigma}_{Ay}), \tag{3.107}$$

where

$$\left(\sigma_{A(B)y}\right)_{mn} \mathrel{\widehat{=}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3.108}$$

$[m(n) = 1, 2]$, and to determine the two nonzero eigenvalues $\lambda_\pm$ of $\hat{\rho}_{\rm at}\hat{\tilde{\rho}}_{\rm at}$. A somewhat lengthy but straightforward calculation yields

$$\begin{aligned}\lambda_\pm = {} & \tfrac{1}{2}\left\{\alpha_+^2 + \alpha_-^2 - 2\left[(\mathrm{Re}\,\beta)^2 - (\mathrm{Im}\,\beta)^2\right]\right\} \\ & \pm \tfrac{1}{2}\sqrt{\left[(\alpha_+ + \alpha_-)^2 - 4(\mathrm{Re}\beta)^2\right]\left[(\alpha_+ - \alpha_-)^2 + 4(\mathrm{Im}\beta)^2\right]}\,,\end{aligned} \tag{3.109}$$

which then determine the concurrence

$$\mathcal{C} = \sqrt{\lambda_+} - \sqrt{\lambda_-}\,, \tag{3.110}$$

the value of which is in the interval $[0, 1]$. The nearer to 1 the value of $\mathcal{C}$ is, the higher is the degree of entanglement. Equations (3.109) and (3.110) reveal that a noticeably entangled state of the two atoms can be generated if

$$\alpha_+(\alpha_-) \gg \alpha_-(\alpha_+), |\beta|, \tag{3.111}$$

thus $\mathcal{C} \to \alpha_+(\alpha_-)$. Needless to say that the entanglement condition (3.111) is already expected from inspection of Eq. (3.104).

3.3.3 Different Coupling Regimes

Let us return to Eq. (3.91) and focus on the case where

$$\dot{F}_\pm(t) \simeq -i(\Delta - i\Delta\omega_{\rm C})F_\pm(t) \tag{3.112}$$

is valid, so that the term on the right-hand side in Eq. (3.91) can be omitted. Obviously, this is the case when initially the (Lorentzian) field resonance of mid-frequency $\omega_{\rm C}$ and width $\Delta\omega_{\rm C}$ is excited (for details, see Sec. 3.3.4). Under the initial conditions

$$C_\pm^{13}(0) = 0, \qquad \dot{C}_\pm^{13}(0) = F_\pm(0), \tag{3.113}$$

the solution to Eq. (3.91) can then be written in the form of

$$C_{\pm}^{13}(t) = \frac{F_{\pm}(0)}{q_{\pm}} e^{-a_{1\pm}t/2}(e^{q_{\pm}t/2} - e^{-q_{\pm}t/2}), \tag{3.114}$$

where

$$q_{\pm} = \sqrt{a_{1\pm}^2 - 4a_{2\pm}}\ . \tag{3.115}$$

Restricting again our attention to the stationary limit, we further assume, for simplicity, both the detuning Δ and the dipole-dipole coupling strength Δ_{AB}^{31} vanish, i.e., $\Delta = 0$ and $\Delta_{AB}^{31} = 0$. Since even under these conditions the explicit form of the expansion coefficients $\alpha_{\pm}$, Eq. (3.105), and β, Eq. (3.106), is rather involved, we renounce its presentation here but consider instead some instructive special cases.

From Eqs. (3.92) and (3.114) it is seen that the damping constant of $C_{\pm}^{13}$ is determined by the sum of the half width at half maximum of the field resonance strongly coupled to the transition $|3\rangle \leftrightarrow |1\rangle$ and the half width at half maximum of the transition $|3\rangle \rightarrow |2\rangle$, $\Delta\omega_{\rm C}$ and $\Gamma_{AA}^{32}/2$, respectively. Due to the finite $\Delta\omega_{\rm C}$ an atom tends to occupy the state $|1\rangle$, while the effect of the finite Γ_{AA}^{32} is that the atom prefers to occupy the state $|2\rangle$. We may therefore restrict ourselves to situations in which

$$\Gamma_{AA}^{32} \gg \Delta\omega_C. \tag{3.116}$$

To achieve noticeable entanglement, the interatomic coupling should be sufficiently strong, i.e., $|\Gamma_{AB}^{31}| \rightarrow \Gamma_{AA}^{31}$ and $|\Gamma_{AB}^{32}| \rightarrow \Gamma_{AA}^{32}$, equivalently,

$$\frac{\Gamma_{\pm}^{31}}{\Gamma_{\mp}^{31}} \gg 1, \quad \frac{\Gamma_{\pm(\mp)}^{32}}{\Gamma_{\mp(\pm)}^{32}} \gg 1. \tag{3.117}$$

Note that the first inequality is equivalent to $g_{\pm} \gg g_{\mp}$ [cf. Eq. (3.94)]. We now distinguish between the following three cases.

(a) $\quad g_{\pm} \gg \Gamma_{AA}^{32} \gg \Delta\omega_{\rm C} \gg g_{\mp}$

In this case, either the symmetric state $|+_{13}\rangle$ or the antisymmetric state $|-_{13}\rangle$ is strongly coupled to the medium-assisted electromagnetic field whereas the other

is weakly coupled. For the strongly and weakly-coupled states, respectively, Eq. (3.114) approximates to

$$C_{\pm}^{13}(t) = \frac{F_{\pm}(0)}{g_{\pm}}\, e^{-\Gamma_{AA}^{32}t/4} \sin(g_{\pm}t), \tag{3.118}$$

and

$$C_{\mp}^{13}(t) = \frac{2F_{\mp}(0)}{\Gamma_{AA}^{32}} \left[e^{-\Delta\omega_C t} - e^{-\Gamma_{AA}^{32}t/2}\right]. \tag{3.119}$$

It is seen that $C_{\pm}^{13}(t)$ undergoes damped Rabi oscillations of frequency $g_{\pm}$, while $C_{\mp}^{13}(t)$ undergoes a two-channel exponential decay. The steady-state density operator parameters $\alpha_{\pm}$, Eq. (3.105), and β, Eq. (3.106), approximate to

$$\alpha_{\pm} = \tfrac{1}{2}\Gamma_{\pm(\mp)}^{32} \frac{|F_{+(-)}(0)|^2}{g_{+(-)}^2 \Gamma_{AA}^{32}} + \Gamma_{\mp(\pm)}^{32} \frac{|F_{-(+)}(0)|^2}{(\Gamma_{AA}^{32})^2 \Delta\omega_C}, \tag{3.120}$$

$$\beta = \left[\Gamma_{+}^{32} F_{+}(0) F_{-}^{*}(0) + \Gamma_{-}^{32} F_{+}^{*}(0) F_{-}(0)\right] \frac{\Gamma_{AA}^{32}}{2g_{+(-)}^4} \tag{3.121}$$

for $g_{+(-)} \gg g_{-(+)}$.

(b) $\quad g_{\pm} \gg g_{\mp} \gg \Gamma_{AA}^{32} \gg \Delta\omega_C$

When both $g_{\pm}$ and $g_{\mp}$ dominate the other parameters, then the states $|+_{13}\rangle$ and $|-_{13}\rangle$ are both strongly coupled to the medium-assisted electromagnetic field, and Eq. (3.114) approximates to

$$C_{\pm}^{13}(t) = \frac{F_{\pm}(0)}{g_{\pm}}\, e^{-\Gamma_{AA}^{32}t/4} \sin(g_{\pm}t), \tag{3.122}$$

which is exactly analogous to Eq. (3.118). The steady-state density operator parameters $\alpha_{\pm}$ and β take the approximate form of

$$\alpha_{\pm} = \tfrac{1}{2}\Gamma_{\pm}^{32} \frac{|F_{+}(0)|^2}{g_{+}^2 \Gamma_{AA}^{32}} + \tfrac{1}{2}\Gamma_{\mp}^{32} \frac{|F_{-}(0)|^2}{g_{-}^2 \Gamma_{AA}^{32}} \tag{3.123}$$

and, for $g_{+(-)} \gg g_{-(+)}$,

$$\beta = \left[\Gamma_{+}^{32} F_{+}(0) F_{-}^{*}(0) + \Gamma_{-}^{32} F_{+}^{*}(0) F_{-}(0)\right] \frac{\Gamma_{AA}^{32}}{2g_{+(-)}^4}. \tag{3.124}$$

$$\text{(c)} \quad \Gamma_{AA}^{32} \gg g_{\pm} \gg g_{\mp},\, \Delta\omega_C$$

When the value of Γ_{AA}^{32} sufficiently exceeds the values of the other parameters, then from Eq. (3.114) it follows that

$$C_{\pm}^{13}(t) = \frac{2F_{\pm}(0)}{\Gamma_{AA}^{32}} \left[e^{-\Delta\omega_C t} - e^{-\Gamma_{AA}^{32} t/2} \right], \tag{3.125}$$

i.e., the behavior typical of weakly-coupled states is observed [cf. Eq. (3.119)]. In this approximation, the steady-state density operator parameters $\alpha_{\pm}$ and β read

$$\begin{aligned} \alpha_{\pm} = \Gamma_{\pm}^{32} & \frac{|F_{+}(0)|^2}{(\Gamma_{AA}^{32})^2(\Delta\omega_{\mathrm{C}} + 2g_{+}^2/\Gamma_{AA}^{32})} \\ & + \Gamma_{\mp}^{32} \frac{|F_{-}(0)|^2}{(\Gamma_{AA}^{32})^2(\Delta\omega_{\mathrm{C}} + 2g_{-}^2/\Gamma_{AA}^{32})} \end{aligned} \tag{3.126}$$

and, for $g_{+(-)} \gg g_{-(+)}$,

$$\beta = \frac{\Gamma_{+}^{32} F_{+}(0) F_{-}^{*}(0) + \Gamma_{-}^{32} F_{+}^{*}(0) F_{-}(0)}{(\Gamma_{AA}^{32})^2(\Delta\omega_{\mathrm{C}} + g_{+(-)}^2/\Gamma_{AA}^{32})}. \tag{3.127}$$

3.3.4 Preparation of the initial state

One possible way to initially prepare the medium-assisted electromagnetic field in the desired quantum state, is to use an additional atom, say atom D, such that $\tilde{\omega}_{D31} = \tilde{\omega}_{A31} = \tilde{\omega}_{B31} = \omega_C$. Let the transition $|1\rangle \leftrightarrow |3\rangle$ of atom D strongly interact with the medium-assisted electromagnetic field in the absence of atoms A and B. This can be achieved, for instance, by using atomic beams and letting atom D pass the equipment before atoms A and B pass it. When atom D initially prepared in the excited state $|3\rangle$ strongly interacts with the medium-assisted electromagnetic field initially prepared in the vacuum state, then an interaction time can be chosen after which the atomic excitation is transferred to the field.

The probability amplitude of finding, after some interaction time Δt, atom D (regarded as an effective two-level system) in the ground state and the $\hat{\mathbf{f}}(\mathbf{r}, \omega)$

field in a single-quantum state is [81]

$$\begin{aligned}&\mathbf{C}(\mathbf{r},\omega,t=0)\\&\quad=\frac{i}{\hbar}\int_{-\Delta t}^{0}\mathrm{d}t'\,\mathbf{d}_{D31}^{*}\tilde{\boldsymbol{G}}^{*}(\mathbf{r}_D,\mathbf{r},\omega)e^{i(\omega-\tilde{\omega}_{D31})t'}C_{U_D}(t'),\end{aligned}\tag{3.128}$$

where

$$C_{U_D}(t)=e^{-\Delta\omega_{\mathrm{C}}(t+\Delta t)/2}\cos[g_D(t+\Delta t)]\tag{3.129}$$

is the probability amplitude of finding the atom in the upper state. Here,

$$g_D=\sqrt{\Gamma_{DD}^{31}\Delta\omega_{\mathrm{C}}/2}\tag{3.130}$$

is the single-atom Rabi frequency, with Γ_{DD}^{31} being determined according to Eq. (3.89). Substitution of Eq. (3.128) into Eqs. (3.79) and (3.80) yields

$$F_{31}(t)=\int_{-\Delta t}^{0}\mathrm{d}t'\,K_{AD}(t-t')C_{U_D}(t'),\tag{3.131}$$

$$F_{13}(t)=\int_{-\Delta t}^{0}\mathrm{d}t'\,K_{BD}(t-t')C_{U_D}(t'),\tag{3.132}$$

where $K_{BD}(t)$ is defined according to Eq. (3.78). Note that $F_{\pm}(t)$, Eq. (3.87), calculated by using F_{31} and F_{13} given in Eqs. (3.131) and (3.132) fulfills Eq. (3.112). To calculate $F_{\pm}(0)$, we fix the interaction time Δt such that $C_{U_D}(0)=0$, thus

$$\Delta t=\frac{\pi}{2g_D}\,.\tag{3.133}$$

Combining Eq. (3.87) with Eqs. (3.129)–(3.133), we derive, on applying the Lorentz approximation according to Eq. (3.88),

$$F_{\pm}(0)=-\frac{1}{\sqrt{2}}\frac{g_{D\pm}^{2}}{g_D}\exp\left(-\Delta\omega_C\frac{\pi}{2g_D}\right),\tag{3.134}$$

where

$$g_{D\pm}=\sqrt{(\Gamma_{BD}^{31}\pm\Gamma_{AD}^{31})\Delta\omega_C/2}\,,\tag{3.135}$$

and Γ_{AD}^{31} and Γ_{BD}^{31} are defined according to Eq. (3.89).

In Eq. (3.134), the exponential factor characterizes the photon loss during the interaction time due to the finite width of the field resonance. Obviously, the better the strong-coupling condition $\Delta\omega_C\ll g_D$ is fulfilled, the less is the photon

loss. In particular, when atom A (or B) changes places with atom D and the orientations of the transition dipole moments of atoms A (or B) and D are the same, then from Eq. (3.134) it follows that ($\Delta\omega_C \ll g_D$)

$$F_{\pm}(0) \simeq -g_{\pm}, \quad F_{\mp}(0) \simeq -g_{\mp}^2/g_{\pm}. \tag{3.136}$$

It is worth noting that, as we will see in Sec. 3.4, the highest degree of entanglement can be achieved in case of equal positions of atoms D and A (or B).

3.4 Atomic entanglement near a dielectric microsphere

Let us apply the theory to two atoms near a dispersing and absorbing dielectric microsphere (of radius R) characterized by a Drude-Lorentz type permittivity

$$\varepsilon(\omega) = 1 + \frac{\omega_{\rm P}^2}{\omega_{\rm T}^2 - \omega^2 - i\omega\gamma} \tag{3.137}$$

($\omega_{\rm P}$, coupling constant; $\omega_{\rm T}$, transverse resonance frequency; γ, absorption parameter), which features a band gap in the region $\omega_{\rm T} < \omega < \omega_{\rm L} = \sqrt{\omega_{\rm T}^2 + \omega_{\rm P}^2}$, where $\mathrm{Re}\,\varepsilon(\omega) < 0$.

3.4.1 Two-atom coupling

Making use of the Green tensor for a dielectric sphere [95], one can show, on assuming radial dipole orientations, that Eq. (3.44) leads to

$$\Gamma_{A'A''} \equiv \Gamma_{A'A''}^{mn} = \tfrac{3}{2}\Gamma_0 \mathrm{Re} \sum_{l=1}^{\infty} \frac{l(l+1)(2l+1)}{(kr)^2}\, h_l^{(1)}(kr)$$
$$\times \left[j_l(kr) + B_l^N(\omega) h_l^{(1)}(kr) \right] P_l(\cos\theta) \tag{3.138}$$

[$\omega \equiv \tilde{\omega}_{A'}^{mn} = \tilde{\omega}_{A''}^{mn} > 0$; $k = \omega/c$; $r \equiv r_{A'} = r_{A''} (> R)$, radial position of the atoms]. Here, Γ_0 is the single-atom decay rate in free space, $j_l(z)$ and $h_l^{(1)}(z)$ are the spherical Bessel and Hankel functions, respectively, $P_l(x)$ is the Legendre function, θ is the

angle between the two transition dipole moments ($|\mathbf{d}_{A'mn}| = |\mathbf{d}_{A''mn}|$), and the scattering coefficients $B_l^N(\omega)$ read [95]

$$B_l^N(\omega) = -\frac{\varepsilon(\omega) j_l(z_2)[z_1 j_l(z_1)]' - j_l(z_1)[z_2 j_l(z_2)]'}{\varepsilon(\omega) j_l(z_2)[z_1 h_l^{(1)}(z_1)]' - h_l^{(1)}(z_1)[z_2 j_l(z_2)]'}\,, \tag{3.139}$$

where $z_i = k_i R$, $k_1 = k$, and $k_2 = \sqrt{\varepsilon(\omega)}\,\omega/c$. Note that radially oriented dipoles couple only to TM waves, whereas tangentially oriented dipoles couple to both TM and TE waves (for details, see, e.g., [96]). Needless to say that $\theta = 0$ in case of a single atom ($A' = A''$).

The complex roots of the denominator of the reflection coefficients $B_l^N(\omega)$ determine the positions and the widths of the sphere-assisted electromagnetic field resonances. When ω coincides with a resonance frequency, say ω_C, then the corresponding l term in Eq. (3.138) is the leading one, thus

$$\Gamma_{A'A''}^{mn} \simeq \tfrac{3}{2}\Gamma_0 \mathrm{Re}\Bigg\{ \frac{l(l+1)(2l+1)}{(kr)^2}\, h_l^{(1)}(kr) \times \Big[j_l(kr) + B_l^N(\omega) h_l^{(1)}(kr) \Big]\, P_l(\cos\theta) \Bigg\} \tag{3.140}$$

($\omega \simeq \omega_C$). Equation (3.140) implies that when the two atoms ($A' \neq A''$) are at diametrically opposite positions with respect to the sphere, i.e., $\theta = \pi$ and hence $P_l(\cos\theta) = (-1)^l$, then the interaction of the symmetric (antisymmetric) state with the sphere-assisted electromagnetic field is enhanced, while the antisymmetric (symmetric) state almost decouples [cf. Eq. (3.94)].

The dependence on θ of $\Gamma_{A'A''}$ ($A' \neq A''$) as given by Eq. (3.138) is illustrated in Fig. 3.1, where the atomic transition frequency ω is chosen to be close to a microsphere resonance frequency. From Figs. 3.2 and 3.3 it is clearly seen that the value of Γ_+ (Γ_-) can drastically exceed the value of Γ_- (Γ_+) when the two atoms approach the microsphere and the transition frequency equals a resonance frequency. Recall that Γ_+ (Γ_-) is a measure of the strength of coupling of the symmetric (antisymmetric) state to the sphere-assisted field. In particular, Fig. 3.3 reveals that there is an optimum distance – the distance at which the solid curve attains the minimum – for which the best contrast between Γ_+ and Γ_- can be realized. With increasing distance of the atoms from the sphere, the values of both Γ_+ and Γ_- tend to the free-space value Γ_0 as they should.

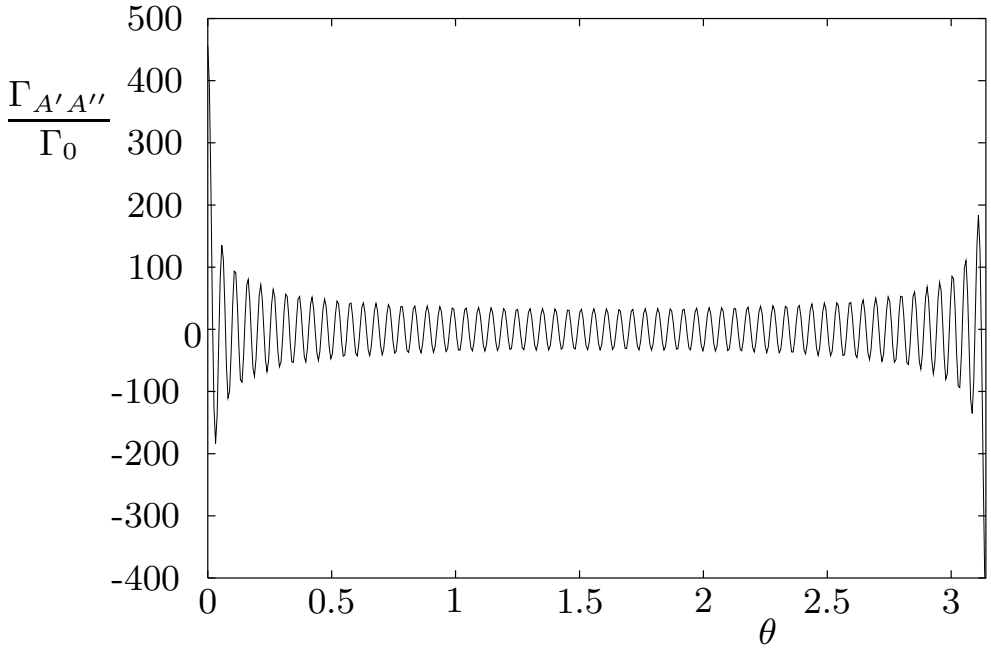


Figure 3.1: The two-atom collective decay rate $\Gamma^{mn}_{AA'}$ [Eq. (3.138), $A' \neq A''$] as a function of the angle θ between the transition dipole moments for $\omega = 1.0501\,\omega_{\rm T}$. The two atoms are at distances $\Delta r \equiv r - R = 0.14\lambda_{\rm T}$ ($\lambda_{\rm T} = 2\pi c/\omega_{\rm T}$) from the surface of a dielectric sphere ($\omega_{\rm P} = 0.5\,\omega_{\rm T}$, $\gamma = 10^{-6}\omega_{\rm T}$, $R = 10\,\lambda_{\rm T}$).

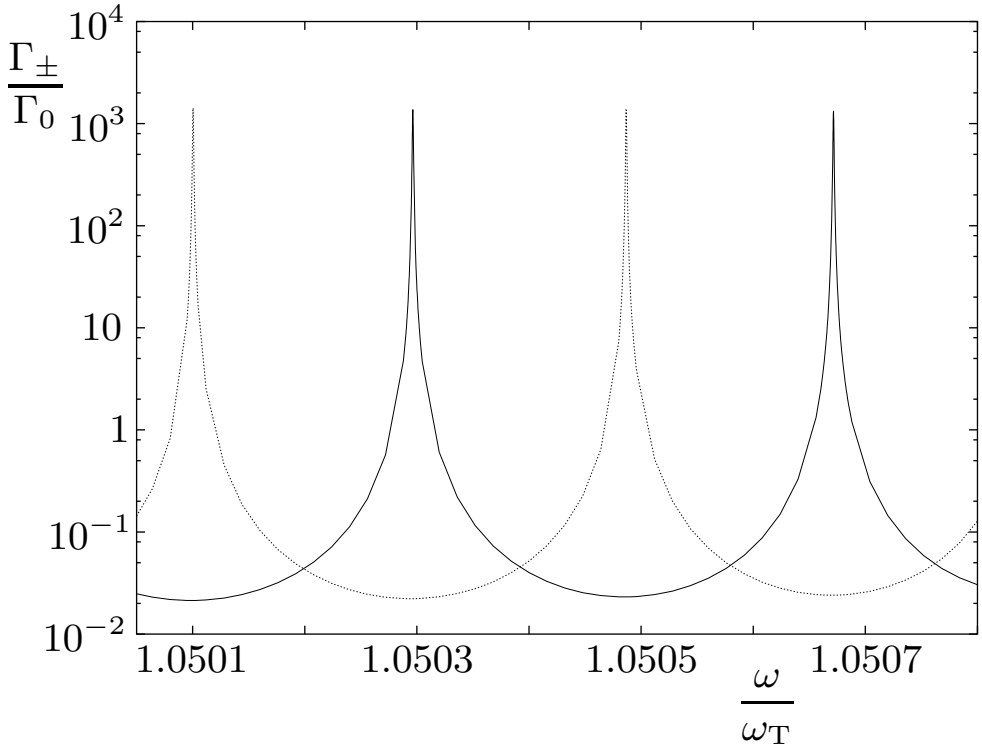


Figure 3.2: The two-atom decay rates $\Gamma_+ = \Gamma_{A'A'} + \Gamma_{A'A''}$ (solid curve) and $\Gamma_- = \Gamma_{A'A'} - \Gamma_{A'A''}$ (dotted curve) for the symmetric and antisymmetric states, respectively, as functions of the transition frequency ω, with $\Gamma_{A'A''}$ from Eq. (3.138) for $\theta = \pi$. The other parameters are the same as in Fig. 3.1].

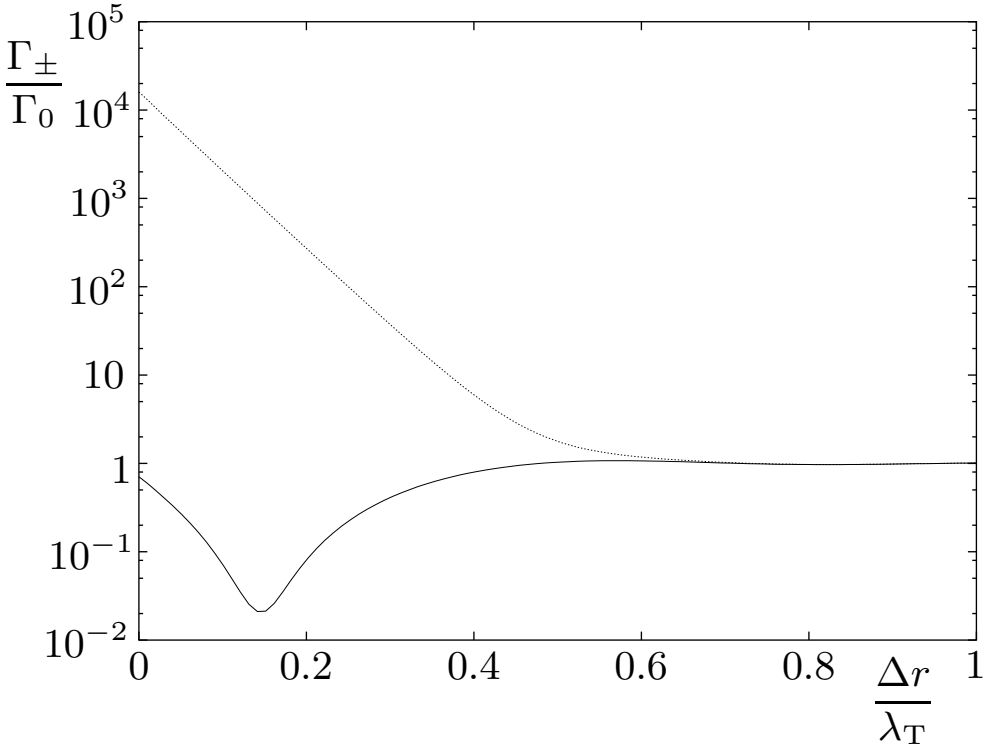


Figure 3.3: The two-atom decay rates $\Gamma_+ = \Gamma_{A'A'} + \Gamma_{A'A''}$ (solid curve) and $\Gamma_- = \Gamma_{A'A'} - \Gamma_{A'A''}$ (dotted curve) for the symmetric and antisymmetric states, respectively, as functions of the atom-sphere surface distance Δr, with $\Gamma_{A'A''}$ from Eq. (3.138) for $\theta = \pi$. The other parameters are the same as in Fig. 3.1].

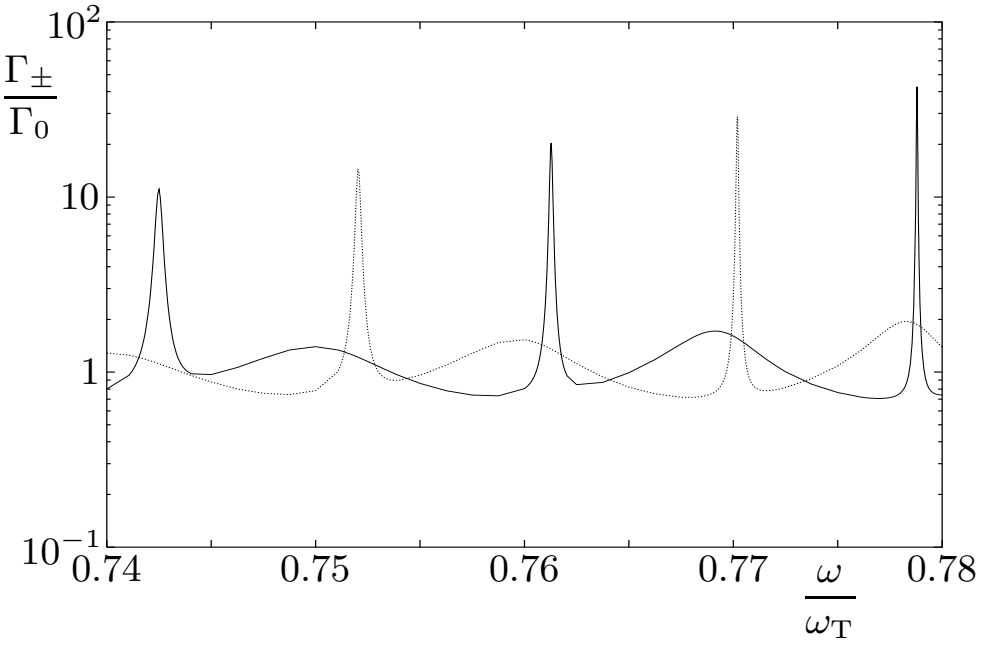


Figure 3.4: The two-atom decay rates $\Gamma_+ = \Gamma_{A'A'} + \Gamma_{A'A''}$ (solid curve) and $\Gamma_- = \Gamma_{A'A'} - \Gamma_{A'A''}$ (dotted curve) for the symmetric and antisymmetric states, respectively, as functions of the transition frequency ω, with $\Gamma_{A'A''}$ from Eq. (3.138) for $\theta = \pi$. The other parameters are the same as in Fig. 3.1].

Figures 3.1–3.3 refer to atomic transition frequencies within bandgap. In this case, the strong two-atom interaction observed when the atoms are at diametrically opposite positions with respect to the sphere is mediated by SG waves. Of course, the effect of enhanced Γ_+ (Γ_-) and simultaneously reduced Γ_- (Γ_+) can also be observed for transition frequencies below the bandgap. In this case, the cavity-assisted field resonances correspond to WG waves. An example is shown in Fig. 3.4. Figures 3.2 and 3.4 also convey a feeling of the sharpness of the field resonances, which ranges from being very sharp to being less so. The sharpness can be improved by increasing the microsphere radius or by reducing the material absorption. Note that WG waves much more suffer from absorption than do SG waves (see, e.g., Ref. [96]).

3.4.2 Entanglement of two Λ-type atoms

The results given in Sec. 3.4.1 show that the optimal positions of two Λ-type atoms A and B, which are desired to entangle with each other near a microsphere, are diametrically opposite with respect to the sphere. Further, the transition frequency $\tilde{\omega}_{A31}{=}\tilde{\omega}_{B31}$ should coincide with the (mid-)frequency ω_C of a sufficiently sharply peaked sphere-assisted field resonance, so that the strong-coupling regime is realized and the first of the conditions (3.117) is satisfied. Finally, the $|3\rangle \leftrightarrow |2\rangle$ transition frequency $\tilde{\omega}_{A32} = \tilde{\omega}_{B32}$ should coincide with the (mid-)frequency of some moderately peaked sphere-assisted field resonance, so that the second of the conditions (3.117) is also satisfied, but the weak-coupling regime applies, thereby giving rise to an irreversible decay channel. As a result, the condition (3.116) can also be expected to be satisfied. By choosing atoms with appropriate transition dipole matrix elements matching right appropriate cavity-assisted field resonances (for more detailed estimations, see Ref. [81]), all the conditions including both the inequalities characterizing the three cases (a)–(c) in Sec. 3.3.3 and the field-preparation conditions (3.136) can be fulfilled. Let us examine the cases (a)–(c) in more detail.

$$\text{(a)} \quad g_{\pm} \gg \Gamma_{AA}^{32} \gg \Delta\omega_{\rm C} \gg g_{\mp}$$

For definiteness, let $\Gamma_{+}^{31} \gg \Gamma_{-}^{31}$ and $\Gamma_{+}^{32} \gg \Gamma_{-}^{32}$. When atom A (or B) changes places with atom D, which provides the initial field excitation, and Eq. (3.136) applies, then Eqs. (3.120) and (3.121) lead to

$$\alpha_{+} \simeq 1, \tag{3.141}$$

$$\alpha_{-} \simeq \frac{\Gamma_{-}^{32}}{2\Gamma_{AA}^{32}} + \frac{2g_{-}^{4}}{g_{+}^{2}\Gamma_{AA}^{32}\Delta\omega_{C}} \ll 1, \tag{3.142}$$

$$\beta \simeq \left(\frac{\Gamma_{AA}^{32} g_{-}}{g_{+}^{2}}\right)^{2} \ll 1. \tag{3.143}$$

Hence, an almost perfectly entangled state is produced, $\hat{\varrho}_{\rm at} \simeq |{+_{12}}\rangle\langle{+_{12}}|$ [see Eq. (3.104)], and, accordingly, $\mathcal{C} \simeq 1$ is achieved. Clearly, $\alpha_{+} = 1$ ($\mathcal{C} = 1$) cannot be exactly realized, because of the losses unavoidably associated with the always

finite width of the field resonance. It is worth mentioning that when the positions of atoms D and A (or B) are different from each other (e.g., when atom D is equidistant from atoms A and B), then the degree of entanglement that can be achieved is smaller than that in case of equal positions in general. Note that when $\Gamma_-^{31} \gg \Gamma_+^{31}$ and $\Gamma_-^{32} \gg \Gamma_+^{32}$, then $\hat{\varrho}_{\rm at} \simeq |+_{12}\rangle\langle+_{12}|$ is also valid. For $\Gamma_\pm^{31} \gg \Gamma_\mp^{31}$ and $\Gamma_\mp^{32} \gg \Gamma_\pm^{32}$, however, the roles of α_+ and α_- are interchanged and $\hat{\varrho}_{\rm at} \simeq |-_{12}\rangle\langle-_{12}|$.

In the scheme, the two-atom system undergoes, e.g., fast $|1,1\rangle \leftrightarrow |+_{13}\rangle$ Rabi oscillations as long as one of the two atoms jumps to state $|2\rangle$, but we do not know which one. Hence, the result is the entangled state between one atom in the state $|2\rangle$ and the other in the state $|1\rangle$. The time after which the stationary limit is established is determined by the lifetime $\sim (\Gamma_{AA}^{32})^{-1}$ of the short-living state $|+_{13}\rangle$, while the long-living state $|-_{13}\rangle$ of lifetime $\sim (\Delta\omega_C)^{-1}$ is practically unpopulated [cf. Eqs. (3.118) and (3.119)].

$$\text{(b)} \quad g_\pm \gg g_\mp \gg \Gamma_{AA}^{32} \gg \Delta\omega_{\rm C}$$

For definiteness, we again assume that $\Gamma_+^{31} \gg \Gamma_-^{31}$ and $\Gamma_+^{32} \gg \Gamma_-^{32}$. From Eqs. (3.123) and (3.124) together with Eq. (3.136) we obtain

$$\alpha_+ \simeq 1, \tag{3.144}$$

$$\alpha_- \simeq \frac{\Gamma_-^{32}}{2\Gamma_{AA}^{32}} + \frac{2g_-^2}{g_+^2} \ll 1, \tag{3.145}$$

$$\beta \simeq \left(\frac{\Gamma_{AA}^{32} g_-}{g_+^2}\right)^2 \ll 1. \tag{3.146}$$

Thus, this coupling regime leaves the two atoms in an entangled state analogous to case (a). However, since the second inequality in (b) tends to conflict with the first, it may be more difficult to realize this regime.

$$\text{(c)} \quad \Gamma_{AA}^{32} \gg g_\pm \gg g_\mp, \Delta\omega_{\rm C}$$

In this case, the irreversible decay from state $|3\rangle$ to state $|2\rangle$ is so dominant that Rabi oscillations are fully suppressed in the time evolution of both C_+^{13} and C_-^{13}

[see Eq. (3.125)]. From Eq. (3.126) we obtain, on again assuming $\Gamma_+^{31} \gg \Gamma_-^{31}$ and $\Gamma_+^{32} \gg \Gamma_-^{32}$ and making use of Eq. (3.136),

$$\alpha_+ \simeq \frac{2g_+^2/\Gamma_{AA}^{32}}{\Delta\omega_C + 2g_+^2/\Gamma_{AA}^{32}}\,. \tag{3.147}$$

To generate the entangled state $|+_{12}\rangle$, i.e., $\alpha_+ \simeq 1$, the additional condition

$$\frac{g_+}{\Gamma_{AA}^{32}} \gg \frac{\Delta\omega_C}{g_+} \tag{3.148}$$

must be required to be satisfied, as can be seen from Eq. (3.147). The parameters α_- and β then read

$$\alpha_- \simeq \frac{\Gamma_-^{32}}{2\Gamma_{AA}^{32}} + \frac{g_-^2}{g_+^2} \ll 1, \tag{3.149}$$

$$\beta \simeq \frac{2g_-^2}{g_+^2} \ll 1. \tag{3.150}$$

In a similar fashion, it can be shown that in case of $\Gamma_\pm^{31} \gg \Gamma_\mp^{31}$ and $\Gamma_\mp^{32} \gg \Gamma_\pm^{32}$ the antisymmetric entangled state $|-_{12}\rangle$ is generated.

The inequality (3.148) can be understood as follows. For $F_+(0) \simeq -g_+$, Eq. (3.125) yields

$$C_+^{13}(t) \simeq -(2g_+/\Gamma_{AA}^{32})\left[e^{-\Delta\omega_C t} - e^{-\Gamma_{AA}^{32}t/2}\right], \tag{3.151}$$

i.e, $C_+^{13}(t) \sim g_+/\Gamma_{AA}^{32}$. Thus, though one can allow for $g_+/\Gamma_{AA}^{32} \ll 1$, this ratio has still to satisfy the inequality (3.148) such that there is a nonvanishing probability that one of the atoms can reach the state $|3\rangle$ from the initial state $|1\rangle$ to jump to the state $|2\rangle$.

3.5 Summary and discussion

We have developed a formalism describing the interaction of multi-level atoms interacting with electromagnetic field in the presence of dispersing-absorbing dielectric bodies and we have proposed a scheme for deterministic preparation of two spatially well separated identical atoms in long-living highly entangled states. The scheme uses Λ-type atoms passing a resonator-like equipment of realistic,

dispersing and absorbing macroscopic bodies which form electromagnetic field resonances, the heights and widths of which are determined by the radiative and nonradiative (absorption) losses. The lowest lying atomic state and the lower lying excited state, which can be the ground state and a metastable state or two metastable states, play the role of the basis states of an atomic qubit. The atoms initially prepared in the lowest lying states, are pumped by a single-excitation "pulse" of the body-assisted electromagnetic field, thereby strongly driving the dipole-allowed transition between the lowest and highest lying atomic states. In this way, one of the two atoms – we do not know which one – can absorb the single-photonic excitation and subsequent irreversible spontaneous decay of the excited atomic state to the lower lying excited state, the transition of which to the lowest lying state is dipole-forbidden, deterministically results in a metastable two-atom entangled state.

To be quite general, we have first developed the theory, without specifying the atoms and the equipment whose body-assisted electromagnetic field is used for the the collective atom-field interaction. For the case of two Λ-type atoms, we have derived the general solution of the coupled field-atom evolution equations and presented special coupling conditions under which high-degree entanglement can be achieved. We have then applied the theory to the problem of entanglement of two Λ-type atoms near a microsphere. In particular, we have shown that the scheme is capable of realizing strong coupling in one arm and weak coupling in the other arm of the Λ configuration. In this context, we have also analyzed the preparation of the initial single-photonic field excitation required for initiating the process of entanglement.

In contrary to the common sense that the existence of dissipation spoils the quantum coherence of a system, dissipation is here essential to transfer the entanglement from the strongly driven transitions to the dipole-forbidden transitions. The fact that only ground or metastable states serve as basis states of the qubits guarantees the long lifetime of the entangled state. It is worth noting that the scheme renders it possible to test nonlocality for a two-atom system. An atomic pair passing by a microsphere and being entangled there, can be separated from each other and one can be sure that in the meantime the entanglement is not

lost.

Chapter 4

Steady-State Entanglement of Two Atoms

The stabilization of entanglement caused by action of a classical driving field in the system of two-level atoms with the dipole interaction accompanied by spontaneous emission is discussed. An exact solution shows that the maximum amount of concurrence that can be achieved in Lamb-Dicke limit is 0.43. Dependence of entanglement on interatomic distance and classical driving field, beyond Lamb-Dicke limit, is examined numerically.

Introduction

The practical applications of entanglement require the *robust* entangled states. This notion includes long enough lifetime of the states and high amount of entanglement (as close to perfect entanglement as possible). However, in many cases entanglement of two-level atoms is not stable enough. In the case of atoms trapped in high-quality cavities, absence of stability is caused mainly by Rabi oscillations. In free space, entanglement related to excited atomic states decays because of the spontaneous emission processes.

To stabilize atomic entanglement, engineered environment can be utilized. For example, it was shown in Refs. [83, 49, 32] that presence of squeezed vacuum field can stabilize entanglement of a pair of two-level atoms with dipole-dipole interaction. The use of bad cavity as a stabilizing environment was considered in [97]. Stabilization in a bad cavity with optical white noise field was discussed in [98]. A scheme of stabilization based on the use of three-level Λ-type atoms in two-mode cavities with leakage and absorption was proposed in [73] and then discussed in [99].

We show that reasonable amount of steady-state entanglement can be achieved in a system of two-level atoms in the weak coupling regime (high losses), in particular for free space, in the presence of a classical driving field. The collective effects, i.e. dipole-dipole interaction and collective spontaneous emission, are the mechanisms responsible for generation of entanglement. However, in the absence of a special environment that compensates the losses of energy caused by spontaneous emission, the entanglement is a transient one. We show that instead of more sophisticated squeezed vacuum field the simple classical driving field can be successfully used for this aim. The classical driving field alone acts only locally on the atoms, so that it cannot create specific quantum correlations between the atoms peculiar for entangled state. However, it continuously provides atomic excitations that are responsible for survival of the collective effects thus enabling a steady-state entanglement[100, 101].

4.1 Steady state entanglement

The system of two identical two-level atoms in free space is governed by the master equation

$$\dot{\rho} = -i[H,\rho] + \frac{1}{2}\sum_{i,j=1}^{2}\Gamma_{ij}(2\sigma_-^i\rho\sigma_+^j - \sigma_+^i\sigma_-^j\rho - \rho\sigma_+^i\sigma_-^j) \qquad (4.1)$$

$$H = \sum_{i=1}^{2}\left[\frac{\omega}{2}\sigma_z^i + E(\sigma_+^i e^{i\vec{k}\cdot\vec{r}_i - i\omega t} + \sigma_-^i e^{-i\vec{k}\cdot\vec{r}_i + i\omega t})\right] + \Omega(\sigma_+^1\sigma_-^2 + \sigma_+^2\sigma_-^1), \qquad (4.2)$$

where the atomic dipoles are alligned in the same direction and driven by a linearly polarized classical field, with dipole coupling constant E. Here $\sigma_+^i = (\sigma_-^i)^\dagger = |e\rangle_i\langle g|_i$ and $\sigma_z^i = |e\rangle_i\langle e|_i - |g\rangle_i\langle g|_i$ with $|e\rangle_i, |g\rangle_i$ denoting the excited and the ground states of the i-th atom, $\Gamma_{ii} = \Gamma$ is the single atom decay rate, $\Gamma = \omega^3|\vec{\mu}|^2/3\pi\hbar\epsilon_0 c^3$, and $\vec{\mu}$ is the atomic dipole moment. The collective decay rates are[102, 103]

$$\Gamma_{12} = \Gamma_{21} = \frac{3}{2}\Gamma\Big\{(1-|\hat{\mu}.\hat{r}|^2)\frac{\sin(kr)}{kr} + (1-3|\hat{\mu}.\hat{r}|^2)\big[\frac{\cos(kr)}{(kr)^2} - \frac{\sin(kr)}{(kr)^3}\big]\Big\} \quad (4.3)$$

and the coupling constant for dipole-dipole interaction has the form

$$\Omega = \frac{3}{4}\Gamma\Big\{-(1-|\hat{\mu}.\hat{r}|^2)\frac{\cos kr}{kr} + (1-3|\hat{\mu}.\hat{r}|^2)\big[\frac{\sin kr}{(kr)^2} + \frac{\cos kr}{(kr)^3}\big]\Big\}, \quad (4.4)$$

where $\hat{\mu}$ is the direction of dipoles, and $\hat{r}$ is the unit vector lying along the interatomic axis.

In our case, when the dipoles are aligned along the interatomic axis, eqs.(4.3,4.4) reduce to,

$$\Gamma_{12} = \Gamma_{21} = -3\Gamma\ \left[\frac{\cos kr}{(kr)^2} - \frac{\sin kr}{(kr)^3}\right], \quad (4.5)$$

and the coupling constant for dipole-dipole interaction has the form

$$\Omega = -\frac{3}{2}\Gamma\left(\frac{\sin kr}{(kr)^2} + \frac{\cos kr}{(kr)^3}\right). \quad (4.6)$$

We are going to consider the case when the classical field is in phase at the atomic locations, namely $\vec{k}\cdot\vec{r}_{12} = 0$. If the density matrix is initially block diagonal

$$\rho = \begin{bmatrix} \rho_T & 0 \\ 0 & \rho_S \end{bmatrix} \rightarrow \rho_T = \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix}, \rho_S = \rho_{44}, \quad (4.7)$$

in the total angular momentum basis, consisting of the four states $\{|ee\rangle$, $(|eg\rangle + |ge\rangle)/\sqrt{2}$,$|gg\rangle$, $(|eg\rangle - |ge\rangle)/\sqrt{2}$ $\}$, then it will always preserve the block diagonal form. Here ρ_T is defined in the triplet part of the Hilbert space spanned by the symmetric vectors in the above basis, while ρ_S corresponds to singlet subspace

with antisymmetric base vector $|a\rangle$. This fact directly follows from the equations of motion for ρ_T and ρ_S,

$$\dot{\rho}_T = -i(H_T\rho_T - \rho_T H_T^\dagger) + \frac{\Gamma+\Gamma_{12}}{2}J_-\rho_T J_+ + (\Gamma-\Gamma_{12})\rho_S|gg\rangle\langle gg|$$
$$\dot{\rho}_S = -(\Gamma-\Gamma_{12})(\rho_S - \langle ee|\rho_T|ee\rangle). \tag{4.8}$$

$J_\pm = \mathbb{P}_T(\sigma_\pm^1 + \sigma_\pm^2)\mathbb{P}_T$ is the raising(lowering) operator projected onto triplet space. Here H_T denotes the non-Hermitian Hamiltonian, corresponding to the interaction picture, which can be represented in the triplet part of the basis as follows

$$H_T = \begin{bmatrix} -i\Gamma & \sqrt{2}E & 0 \\ \sqrt{2}E & \Omega - \frac{i}{2}(\Gamma+\Gamma^{(12)}) & \sqrt{2}E \\ 0 & \sqrt{2}E & 0 \end{bmatrix} \tag{4.9}$$

From (4.8) it is clearly seen that if $\Gamma = \Gamma_{12}$, the population of the antisymmetric state will remain constant, i.e. equations of motion for ρ_T and ρ_S will decouple. In this case, there are two independent steady-state solutions. Otherwise there will be only one solution.

It is evident from (4.9) that in the absence of the classical driving field, all states except $|gg\rangle$ are damped, so that the steady-state entanglement at $E = 0$ is impossible, and the system evolves towards the unentangled ground state $|gg\rangle$.

Because we are interested in the robust entanglement, let us consider the steady-state solutions of the master equation (4.8) for ρ_T. Consider first the Lamb-Dicke limit of short interatomic separation. Then, it follows from the definition of the decay rate (4.5) that

$$\Gamma^{(12)} \approx \Gamma.$$

In this case, assuming that the atoms are initially prepared in their ground states, the steady state density matrix will be determined in the triplet sector as follows

$$\rho_T = \frac{1}{N}\begin{bmatrix} 64E^4 & -16iE^3\sqrt{2} & 8E^2(2i\Omega-1) \\ 16iE^3\sqrt{2} & 8E^2(1+8E^2) & -2E\sqrt{2}(2\Omega+i+8iE^2) \\ -8E^2(2i\Omega+1) & -2E\sqrt{2}(2\Omega-i-8iE^2) & 4(\Omega^2+2E^2+16E^4)+1 \end{bmatrix} \tag{4.10}$$

Here N is the normalization factor s.t. $Tr\rho_T = 1$ and Ω and E are replaced by the dimensionless parameters Ω/Γ and E/Γ, respectively.

To determine the settings, leading to the maximum possible amount of entanglement in the system under consideration, we choose $\Omega = \tau E^2$, where τ is a dimensionless constant to be determined upon the maximization of concurrence. This factor in the Lamb-Dicke limit can be represented as follows

$$\tau = \frac{3}{4\pi\alpha}[(kr)^3 Q\bar{n}V]^{-1}, \tag{4.11}$$

where $\alpha = 1/137$ is the fine structure constant, Q denotes atomic quality factor ($Q = \omega_0 T$, and T is the lifetime of the excited atomic state), $\bar{n}$ is the mean number of photons per unit volume in classical driving field, and V denotes the volume of interaction between atom and field, so that $\bar{n}V$ gives the mean number of photons interacting with atom during the time T.

The concurrence (measure of entanglement in the case of two-qubit system) is defined as follows ([94], See Appendix-A)

$$C = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0), \tag{4.12}$$

where λ denotes the spectrum of matrix $R = (\sqrt{\rho}\bar{\rho}\sqrt{\rho})^{1/2}$ and $\bar{\rho}$ denotes the complex conjugation of (4.10) in the so-called "magic basis" [94]. The maximum entangled state provides $C = 1$, while the unentangled states give $C = 0$.

One can see from Eq. (4.6) that at fixed τ and in the Lamb-Dicke limit $\vec{k}_0 \cdot \vec{r} \ll 1$, both dimensionless parameters $\Omega/\Gamma, E/\Gamma \gg 1$. In this case, the density matrix (4.10) takes the form

$$\rho_T \approx \frac{1}{\tau^2 + 48} \begin{pmatrix} 16 & 0 & 4i\tau \\ 0 & 16 & 0 \\ -4i\tau & 0 & 16 + \tau^2 \end{pmatrix} \tag{4.13}$$

To our surprise, the concurrence (4.12) in this limit turns out to be rational function of τ

$$C(\tau) = \frac{8\tau - 16}{\tau^2 + 48}, \quad \tau \geq 2$$

extended by zero at $\tau \leq 2$. Thus, entanglement is impossible if $\tau \leq 2$. The maximum value of the concurrence

$$C_{max} = \frac{2}{\sqrt{13}+1} \approx 0.43$$

is attained at

$$\tau_{max} = 2 + 2\sqrt{13} \approx 9.21.$$

The corresponding amount of entanglement [94] is

$$\mathcal{E}_{max} = H\left(\frac{1-\sqrt{1-C_{max}^2}}{2}\right) \approx 0.285 \quad \text{ebit.}$$

Taking into account the form of the dimensionless parameter τ given by Eq. (4.11), we can examine the dimensionless interatomic distance $\vec{k}_0 \cdot \vec{r}$, corresponding to the maximum entanglement provided by $\tau_{max} = 9.21$, as a function of the number of photons $\bar{n}V$, which should obey the condition $\bar{n}V \gg 1$ in the case of classical driving field. It is seen that in the case of mean number of photons $\bar{n}V \sim 10$, the interatomic distance should be of the order of $10^{-2}\lambda$ (where λ is the wavelength) to achieve the maximum possible amount of entanglement. Increase of the mean number of photons in the driving field, considered as a coherent state $|\alpha\rangle$ with $|\alpha|^2 \gg 1$, decreases the interatomic distance, which is required to have maximum amount of entanglement.

So far we have discussed the Lamb-Dicke limit. The results of numerical calculations beyond Lamb-Dicke limit for different values of the classical driving field are shown in Fig. 4.1. Both cooperations, the dipole coupling and collective decay are oscillating functions of distance (Eqs. (4.5), (4.6)), and even when one of them becomes zero, the other can still give rise to entanglement(See Figure-4.2). The deviation from Lamb-Dicke limit decreases the cooperation effects, thus decreases steady state entanglement.

4.2 Summary and discussions

We have examined the system of two identical two-level atoms interacting with each other by means of vacuum induced dipole forces and collective decay. The

dissipation of energy in the system is provided by the spontaneous decay of the excited atomic states. The compensation of losses is provided by a classical driving field.

It is shown that in the absence of the classical driving field, system evolves towards an unentangled state (both atomic dipoles are in the ground state). The presence of the classical driving field stabilizes the entanglement.

In the Lamb-Dicke limit of a point-like system, we obtained an exact solution for the steady state density matrix, that manifests high amount of entanglement (the concurrence $C_{max} = 0.43$ and the entanglement $\mathcal{E}_{max} = 0.285$ ebit). This amount is much higher than those obtained in a number of recent proposals. In particular, it is higher than the case when the squeezed vacuum is used for stabilization of entanglement instead of the classical driving field [83].

Outside Lamb-Dicke limit i.e. when $\Gamma_{12} < \Gamma$, both the triplet and the singlet sectors of the density matrix(4.7) are populated, and this leads to a decrease in the amount of entanglement.

In free space small, interatomic distances are required for strong atomic cooperation. However atoms can exhibit collective effects in cavities, or in the vicinity of dielectric bodies[81] even when they are spatially well separated. The prescribed scheme of steady state entanglement generation can as well be applied to these cases.

In the above consideration, we always assumed that the atoms are identical. It seems interesting to extend our consideration to the case of non-identical atoms. In view of the result of Ref. [83], we can expect that this may lead to a significant increase of entanglement.

We also restricted our consideration to the case of polarization of the classical driving field parallel to the interatomic axis. The alternative choice of the polarization perpendicular to the interatomic axis can lead to a strong change of picture as well. First of all, the change of polarization changes the form of the coupling constant (4.6). Then, it causes the consideration of the different values

of the classical driving field in the atomic locations.

The detailed analysis of the above mentioned two extensions of the model deserves a special consideration.

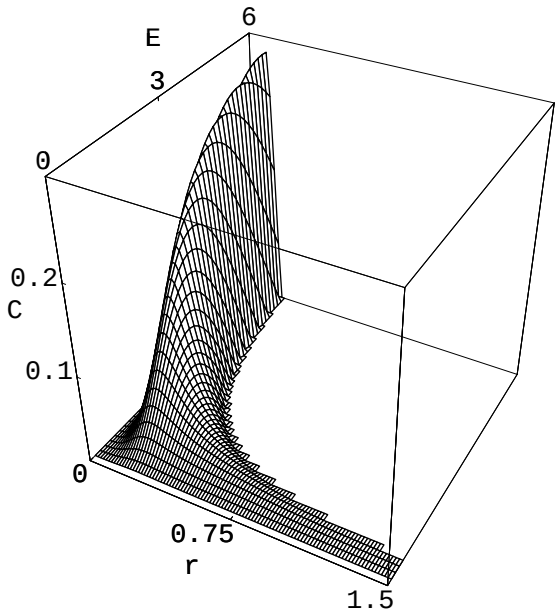


Figure 4.1: Numerical dependence of concurrence on the interatomic distance and classical driving field. The dimensionless quantities r/λ and E/Γ are used here.λ is the wavelength corresponding to atomic transition.

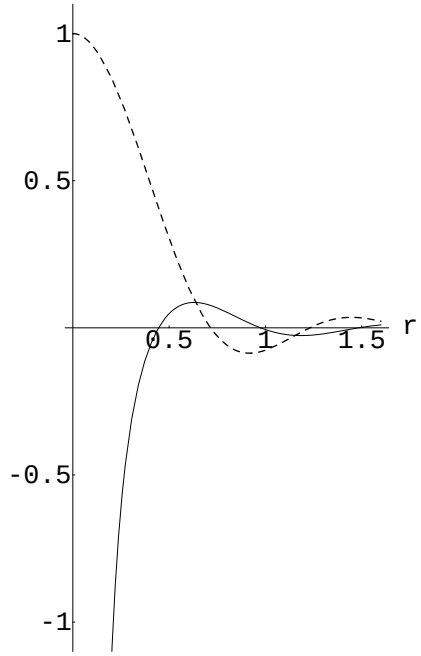


Figure 4.2: The dipole interaction constant Ω (Eq. (4.6))(dashed curve), and collective decay rate Γ_{12} (Eq. (4.5))(solid curve) as a function of interatomic separation r. Here r is given in terms of wavelength corresponding to atomic transition.

Chapter 5

Input-Output Relations for a Cavity with Absorptive Walls

In this chapter we are going to study the modeling of a cavity with absorptive walls. Input-output relations will be formulated, Langevin equation for the cavity field will be obtained in the presence of a source in the cavity and extraction of nonclassical photon states from such cavities will be exemplified.

5.1 Quantization of field in one dimension

In section-3.1 we have discussed the quantization of EM field in dispersing-absorbing medium. In this chapter we are going to consider a one dimensional cavity, and accordingly make the quantization in one dimension.

Consider the EM field propagating in $\pm\hat{x}$ direction, with polarization along $\hat{y}$ direction, the wave equation for the field (3.23)reduces to the following form,

$$(\nabla^2 + \frac{n^2\omega^2}{c^2})A(x,\omega) = \mu_0 j(x,\omega) \tag{5.1}$$

where $j = j^+ + j^-$ is the noise operator, $n = \sqrt{\epsilon(x,\omega)} = \eta + i\kappa$ is the complex refractive index of the medium and $E(x,t) = -\dot{A}(x,t)$. The noise current is

introduced as follows,

$$j^+(x,\omega) = \sqrt{2\epsilon_0\hbar\epsilon_I(x,\omega)\omega^2/S}f(x,\omega)$$
$$[f(x,\omega), f^\dagger(x',\omega')] = \delta(x-x')\delta(\omega-\omega'), \tag{5.2}$$

with $f(x,\omega)$,$f^\dagger(x,\omega)$ being the annihilation and creation operators for the medium-field collective system. S being the transverse area of the system.

For the one dimensional case the equal-time commutation relation reduces to the following form,

$$[A(x,t), -\epsilon_0 E(x',t)] = \frac{i\hbar}{S}\delta(x-x'), \tag{5.3}$$

keeping in mind that $\lim_{\omega\to\infty}\epsilon(x,\omega) = 1$, $\epsilon(x,\omega) = \epsilon^*(x,-\omega)$, and $\epsilon(x,\omega)$ has no poles on the upper half plane[104]. As usual the field at any point may be calculated from the Green function of the differential equation (5.1), and for the bulk case it turns out to be,

$$G(x,x',\omega) = \frac{c}{2in\omega}e^{i\frac{n\omega}{c}|x-x'|}, \tag{5.4}$$

from which the positive frequency part of the field can be expressed simply as,

$$A^+(x,\omega) = \mu_0\int_{-\infty}^{\infty}\mathtt{d}x' G(x,x,\omega)j^+(x',\omega). \tag{5.5}$$

In the bulk medium with refractive index $n = \eta + i\kappa$, the field becomes,

$$\hat{A}^+(x,t) = \int_0^\infty d\omega\sqrt{\frac{\eta\hbar}{4\pi\epsilon_0 n^2\omega c S}}[a_R(x,\omega) + a_L(x,\omega)]e^{-i\omega t}, \tag{5.6}$$

where the right and left propagating modes are identified as follows,

$$a_R(x,\omega) = i\sqrt{\frac{2\kappa\omega}{c}}\int_{-\infty}^{x}\mathtt{d}x' e^{i\frac{n\omega}{c}(x-x')}f(x',\omega)$$
$$a_L(x,\omega) = i\sqrt{\frac{2\kappa\omega}{c}}\int_{x}^{\infty}\mathtt{d}x' e^{-i\frac{n\omega}{c}(x-x')}f(x',\omega). \tag{5.7}$$

The Langevin equation for the field operators follows,

$$\partial_x a_{R,L} = \pm\frac{in\omega}{c}a_{R,L} \pm i\sqrt{\frac{2\kappa\omega}{c}}f(x,\omega), \tag{5.8}$$

with the help of which it is possible to relate an operator to its value at some other position. For a bounded region of space where refractive index is complex, one can introduce free incoming fields which are devoid of noise, propagate freely until they arrive at the absorptive region where they pick up noise as they propagate in the medium,

$$a_{R,L}(x,\omega) = a_{R,L}(x_0)e^{\pm\frac{in\omega}{c}(x-x_0)}a_{R,L}(x,\omega) \pm \sqrt{\frac{2\kappa\omega}{c}}\int_{x_0}^{x} \mathrm{d}x' e^{i\pm\frac{n\omega}{c}(x-x')}. \quad (5.9)$$

The equal time commutation relations for a_R, a_L read,

$$\begin{aligned}[a_{R(L)}(x,\omega), a^\dagger_{R(L)}(x',\omega')] &= e^{\pm\frac{i\eta\omega}{c}(x-x')}e^{-\frac{\kappa\omega}{c}|x-x'|}\delta(\omega-\omega') \\ \left[a_R(x,\omega), a^\dagger_L(x',\omega')\right] &= \frac{2\kappa}{\eta}\sin\frac{\eta\omega}{c}(x-x')e^{-\frac{\kappa\omega}{c}(x-x')}\theta(x-x') \end{aligned} \quad (5.10)$$

where $\theta(x)$ is the Heaviside function. Due to noise right and left going modes become correlated if they had already traversed the same region.

5.1.1 Input-output relations for a dielectric plate

Now we are going to put under scrutiny the quantization of field in the presence of a dispersive-absorptive slab,

$$n(x) = \begin{cases} 1 & x < -d/2 \\ n & -d/2 < x < d/2 \\ 1 & d/2 < x \end{cases} \quad (5.11)$$

One may find the field everywhere from the Green function $G(x,x',\omega)$, then identify incoming and outgoing fields. However one can instead directly write down the solutions of (5.1) and then impose the continuity, and the continuity of derivatives at the boundaries. The solution in three regions is as follows(see Figure-5.1),

$$\hat{A}(x,\omega) = \begin{cases} \sqrt{\frac{\hbar}{4\pi\epsilon_0\omega c}}[a_R(x,\omega) + a_L(x,\omega)] & x < -d/2 \\ \sqrt{\frac{\eta\hbar}{4\pi\epsilon_0 n^2\omega c}}[c_R(x,\omega) + c_L(x,\omega)] & -d/2 < x < d/2 \\ \sqrt{\frac{\hbar}{4\pi\epsilon_0\omega c}}[b_R(x,\omega) + b_L(x,\omega)] & d/2 < x \end{cases} \quad (5.12)$$

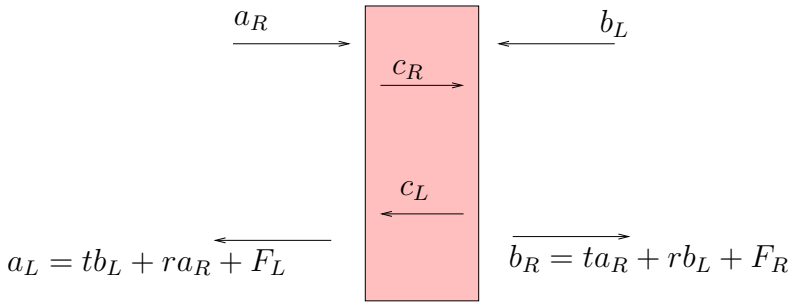


Figure 5.1: Absorbing dielectric slab

With the help of the Langevin equations (5.8) $c_{R,L}(x = d/2)$ may be expressed in terms of $c_{R,L}(x = -d/2)$ and the noise operators,

$$c_{R,L}(d/2,\omega) = c_{R,L}(-d/2,\omega)e^{\pm\frac{in\omega}{c}d} \pm i\sqrt{\frac{2\kappa\omega}{c}}\int_{-d/2}^{d/2} \mathtt{d}x' e^{\pm i\frac{n\omega}{c}(x-x')} f(x',\omega) \quad (5.13)$$

Since the field operator satisfies the wave equation, the field operators and their derivatives should be continuous at the boundaries. The following input-output relations are obtained,

$$\begin{aligned} a_L(\omega) &= tb_L(\omega) + ra_R(\omega) + F_L(\omega) \\ b_R(\omega) &= ta_R(\omega) + rb_L(\omega) + F_R(\omega), \end{aligned} \quad (5.14)$$

$a_{R,L}(\omega) = a_{R,L}(x = -d/2,\omega)$, $b_{R,L}(\omega) = b_{R,L}(x = d/2,\omega)$, and r, t are respectively the reflection and transmission coefficients for the dielectric plate,

$$\begin{aligned} r &= -\frac{(n^2-1)(e^{in\omega d/c} - e^{-in\omega d/c})}{(n-1)^2 e^{in\omega d/c} - (n+1)^2 e^{-in\omega d/c}} \\ t &= -\frac{4n}{(n-1)^2 e^{in\omega d/c} - (n+1)^2 e^{-in\omega d/c}}, \end{aligned} \quad (5.15)$$

The noise operators $F_R,\ F_L$ are defined as,

$$\begin{aligned} F_{R,L}(\omega) &= -\frac{\eta}{2n}t[-(1+n)e^{-i\frac{n\omega}{c}d} f_{R,L} + (1-n)f_{L,R}] \\ f_{R,L}(\omega) &= i\sqrt{\frac{2\kappa\omega}{c}}\int_{-d/2}^{d/2} dx' e^{i\frac{n\omega}{c}(d/2\pm x')} f(x',\omega). \end{aligned} \quad (5.16)$$

It is seen that the outgoing modes pick up noise from the plate, and if there is no absorption in the plate then the noise operators $F_{L,R}$ will vanish.

Incoming fields a_R and b_L satisfy the bosonic commutation relations, and they commute with each other,

$$\begin{aligned}\left[a_R(\omega), a_R^\dagger(\omega')\right] &= [b_L(\omega), b_L^\dagger(\omega')] = \delta(\omega-\omega') \\ \left[a_R(\omega), b_L^\dagger(\omega')\right] &= 0. \end{aligned} \tag{5.17}$$

Outgoing fields a_L and b_R are connected to the incoming fields by a unitary transformation thus have the commutators,

$$\begin{aligned}\left[a_L(\omega), a_L^\dagger(\omega')\right] = \delta(\omega-\omega') = \left[a_R(\omega), a_R^\dagger(\omega')\right] \\ \left[a_L(\omega), b_R^\dagger(\omega')\right] = 0, \end{aligned} \tag{5.18}$$

which impose the following constraints,

$$\begin{aligned}[F_{L,R}(\omega), F_{L,R}^\dagger(\omega')] &= (1-|r|^2-|t|^2)\delta(\omega-\omega') \\ [F_L(\omega), F_R^\dagger(\omega')] &= -(rt^*+r^*t)\delta(\omega-\omega') \end{aligned} \tag{5.19}$$

The dielectric plate might be composed of many layers of different dielectric constants, namely a Distributed Bragg Reflector(DBR) structure, so as to enhance reflection[80] at desired frequencies. The formalism presented here just requires the knowledge of reflection and transmission coefficients of the plate. The input-output relations (5.14) can be retained, however the noise operators $F_{R(L)}$(5.16) will have a much more sophisticated form. Nevertheless one has complete information about the commutation relations (5.19) once r and t are given.

5.2 One sided cavity with absorptive walls

We are going to consider a one dimensional one sided cavity, which will be constructed by introducing a perfect mirror to the left hand side of a dielectric planar structure at position $x=-l$(Figure-5.2). Then the cavity modes a_R and a_L will be related by a π phase difference at the mirror boundary,

$$a_R(\omega)e^{-i\frac{\omega}{c}l} = -a_L(\omega)e^{i\frac{\omega}{c}l}. \tag{5.20}$$

Combining the results of the previously discussed dielectric slab (5.14) and (5.20)

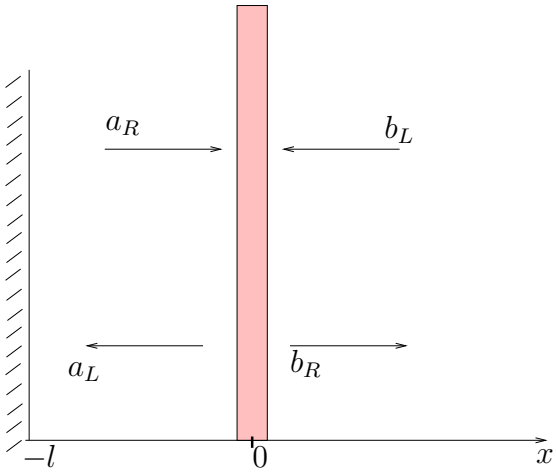


Figure 5.2: One sided cavity

in frequency domain we can express the cavity and outgoing fields as follows,

$$\begin{aligned} a_L(\omega) &= \frac{t}{1+re^{2i\frac{\omega}{c}l}}b_L(\omega) + \frac{F_L(\omega)}{1+re^{2i\frac{\omega}{c}l}} \\ b_R(\omega) &= rb_L(\omega) + ta_R(\omega) + F_R(\omega). \end{aligned} \tag{5.21}$$

So the field everwhere can be described by the three input operators b_L, F_L, F_R. Here the cavity wall is assumed either thin, so that it has no resonances at all, or the frequency of interest is assumed to be far from an internal resonance of the cavity wall. Within these assumptions the reflection coefficient $r(\omega)$ is high and frequency dependence of $r(\omega), t(\omega)$ is very weak.

The cavity resonances are determined by the poles of $a_L(\omega)$(or $a_R(\omega)$),

$$\begin{aligned} 1 + re^{i\frac{2\omega}{c}l} = 0 \rightarrow \omega &= \omega_n - i\frac{\gamma}{2} \\ \omega_n &= \frac{c}{2l}[(2n+1)\pi - \phi_r] \\ \gamma &= \frac{c}{2l}(1-|r|^2) \end{aligned} \tag{5.22}$$

where $r = |r|e^{i\phi_r}, t = |t|e^{i\phi_t}$, further it is assumed that the cavity wall is highly reflective and absorption is small i.e. $1 \gg 1 - |r|^2$. A photon entering the cavity leaves it after a time of flight

$$\tau_{fl} = \frac{2l}{c}, \tag{5.23}$$

thus $\gamma = |t|^2/\tau_{fl}$ is just the probability of a photon to pass out of the cavity or to be absorbed by the cavity walls per unit time, provided that time scales shorter than τ_{fl} are not resolved.

The cavity mode can be expressed as follows,

$$a_n(\omega) = \sqrt{\frac{2l}{c}} a_L(\omega) = \frac{\sqrt{\gamma_1} e^{i\phi_t} b_L(\omega) + \sqrt{\gamma_2}\tilde{F}_L(\omega)}{-i(\omega - \omega_n + i\gamma/2)}$$
$$\gamma_1 = \frac{c}{2l}|t|^2$$
$$\gamma_2 = \frac{c}{2l}(1 - |r|^2 - |t|^2)$$
$$\gamma = \gamma_1 + \gamma_2 \tag{5.24}$$

where $\gamma = \gamma_1 + \gamma_2$ and γ_1, γ_2 are, respectively, absorptive and radiative loss rates of a cavity photon, and $\tilde{F}_L(\omega)$ is the annihilation operator for the absorption channel (5.30).

The field inside the cavity can be expressed as follows,

$$A^+(x,t) = -i \sum_n \int_{[\omega_n]} \mathrm{d}\omega \sqrt{\frac{\hbar}{2\pi\epsilon_0 \omega l S}} \sin\left[\frac{\omega}{c}(x+l)\right] e^{i\frac{\omega}{c}l} a_n(\omega) e^{-i\omega t} \tag{5.25}$$

where $[\omega_n]$ denotes an integration from $\omega_n - \Delta\omega/2$ to $\omega_n + \Delta\omega/2$, with $\Delta\omega = c\pi/l$ being the distance between the resonances.

At the time scales $t \gg \tau_{fl} \sim 1/\Delta\omega$, any interaction that takes place inside the cavity will distinguish different frequencies ω_n's and will select the resonant frequency. On the other hand if the time scale of interest is much smaller than the decay time of the cavity $1 \gg \gamma t$ then one can discretize the integral (5.25), since the frequencies at the interval $[\omega_n - \gamma/2, \omega_n + \gamma/2]$ will not be resolved.

So at the time scales $1/\gamma \gg t \gg 1/\Delta\omega$ the cavity field (5.25) can be discretized as follows,

$$A^+(x,t) = \sum_n \sqrt{\frac{\hbar}{2\epsilon_0 \omega_n S}} f_n(x) \hat{a}_n(t) \tag{5.26}$$

where

$$f_n(x) = (-1)^n \sqrt{\frac{2}{l}} e^{-i\frac{\phi_r}{2}} \sin\left[\frac{\omega_n}{c}(x+l)\right] \tag{5.27}$$

are the normalized mode functions and if the system is freely evolving(i.e. if no source is present),

$$a_n(t) = \int \frac{\mathtt{d}\omega}{\sqrt{2\pi}} a_n(\omega) e^{-i\omega t}. \tag{5.28}$$

5.2.1 Langevin Equation for the cavity mode

Let there be a source in the cavity switched on at $t = 0$. The free field and interaction Hamiltonians constitute the total Hamiltonian,

$$\begin{aligned} H_0 &= \int \mathtt{d}\omega\ \hbar\omega \left[b_L^\dagger(\omega) b_L(\omega) + \tilde{F}_L^\dagger(\omega)\tilde{F}_L(\omega) + \tilde{F}_R^\dagger(\omega)\tilde{F}_R(\omega) \right] \\ V_I &= \int_0^l \mathtt{d}x' \hat{j}_S(x') \hat{A}(x', t=0) \\ &= \hbar\hat{\Omega}(a_n + a_n^\dagger) \end{aligned} \tag{5.29}$$

where $a_n = a_n(t=0)(5.28)$ and $\Omega = \int_{-l}^0 \mathtt{d}x' \sqrt{\hbar/(2\epsilon_0\omega_C S)} f(x') \hat{j}_S(x')$ is the source term. ω_C is the resonant frequency of the cavity that we are interested in. Here $\tilde{F}_L(\omega)$,$\tilde{F}_R(\omega)$ are commuting bosonic operators describing the absorption channels,

$$\begin{aligned} \tilde{F}_L(\omega) &= \frac{1}{\sqrt{1-|r|^2-|t|^2}} F_L(\omega) \\ \tilde{F}_R(\omega) &= N\left[F_R(\omega) + \frac{rt^* + r^*t}{1-|r|^2-|t|^2} F_L(\omega) \right]. \end{aligned} \tag{5.30}$$

For the input fields, here follows the Heisenberg equations of motion from the Hamiltonian (5.29),

$$\begin{aligned} i\dot{b}_L(\omega) &= \omega b_L(\omega) + \frac{\sqrt{\gamma_1} e^{i\phi_t}}{i(\omega - \omega_n - i\frac{\gamma}{2})} \hat{\Omega} \\ i\dot{F}_1(\omega) &= \omega \tilde{F}_L(\omega) + \frac{\sqrt{\gamma_1} e^{i\phi_t}}{i(\omega - \omega_n - i\frac{\gamma}{2})} \hat{\Omega} \\ i\dot{F}_2(\omega) &= \omega \tilde{F}_R(\omega) \end{aligned} \tag{5.31}$$

which can be integrated to yield,

$$b_L(\omega,t) = b_L(\omega)e^{-i\omega t} + \frac{\sqrt{\gamma_1}e^{i\phi_t}}{i(\omega-\omega_n-i\frac{\gamma}{2})}\int_0^t e^{-i\omega\tau}\Omega(t-\tau)$$
$$\tilde{F}_L(\omega,t) = \tilde{F}_L(\omega)e^{-i\omega t} + \frac{\sqrt{\gamma_2}}{i(\omega-\omega_n-i\frac{\gamma}{2})}\int_0^t e^{-i\omega\tau}\Omega(t-\tau)$$
$$\tilde{F}_R(\omega,t) = \tilde{F}_R(\omega)e^{-i\omega t}. \tag{5.32}$$

The next step is to write the equation of motion for the cavity field. At the first step the solution (5.32) are substituted into $a_n(t)$ in (5.26),

$$a_n(t) = \int \frac{\mathtt{d}\omega}{\sqrt{2\pi}}\frac{\sqrt{\gamma_1}e^{i\phi_t}b_L(\omega,t)+\sqrt{\gamma_2}\tilde{F}_L(\omega,t)}{-i(\omega-\omega_n+i\gamma/2)} \tag{5.33}$$

then time derivative of $a_n(t)$ yields the Langevin equation for the cavity field,

$$\dot{a}(t) = -i(\omega_C - i\gamma/2)a(t) + [a(t),V_I(t)] + \sqrt{\gamma_1}e^{i\phi_t}b_{L,in}(t) + \sqrt{\gamma_2}F_{in}(t). \tag{5.34}$$

Noise operators which account for dissipation in (5.34) are as follows,

$$b_L(t) = \int \frac{\mathtt{d}\omega}{\sqrt{2\pi}}b_L(\omega)e^{-i\omega t}$$

$$F_1(t) = \int \frac{\mathtt{d}\omega}{\sqrt{2\pi}}F_1(\omega)e^{-i\omega t}, \tag{5.35}$$

which satisfy the following commutation relations,

$$\big[b_L(t),b_L^\dagger(t')\big] = \delta(t-t')$$
$$\big[F_1(t),F_1^\dagger(t')\big] = \delta(t-t'). \tag{5.36}$$

The solution of the Langevin equation (5.34) is,

$$a(t) = a(t_0)e^{-i(\omega_C-i\frac{\gamma}{2})(t-t_0)} + \int_0^{t-t_0}\mathtt{d}\tau e^{-i(\omega_C-i\frac{\gamma}{2})\tau}\hat{\Omega}(t-\tau) +$$
$$+\int_0^{t-t_0}\mathtt{d}\tau e^{-i(\omega_C-i\frac{\gamma}{2})\tau}\big[\sqrt{\gamma_1}e^{i\phi_t}b_L(t-\tau)+\sqrt{\gamma_2}F_1(t-\tau)\big]. \tag{5.37}$$

The causality is guaranteed by the commutation relations,

$$\big[a(t),b_L(t')\big] = \big[a(t),b_L^\dagger(t')\big] = 0,$$
$$\big[a(t),F_1(t')\big] = \big[a(t),F_1^\dagger(t')\big] = 0 \ \ \mathtt{for} \ \ t<t'. \tag{5.38}$$

5.2.2 Extraction of cavity states

We have studied the dynamics of cavity fields, and now the outgoing fields will be under consideration. From (5.6,5.21) the incoming and outgoing fields outside the cavity follows,

$$\begin{aligned} A^{+}_{in}(x,t) &= \int \frac{\mathrm{d}\omega}{\sqrt{2\pi}}\sqrt{\frac{\hbar}{2\epsilon_0\omega cS}}b_L(\omega)e^{-i\frac{\omega}{c}(x+ct)} \\ A^{+}_{out}(x,t) &= \int \frac{\mathrm{d}\omega}{\sqrt{2\pi}}\sqrt{\frac{\hbar}{2\epsilon_0\omega cS}}\Big\{b_L(\omega)e^{-i\frac{\omega}{c}(x+ct)} + \big[rb_L(\omega)+F_R(\omega)\big]e^{-i\frac{\omega}{c}(-x+ct)}\Big\} \\ &\quad + \sqrt{\frac{\hbar}{2\epsilon_0\omega_n cS}}e^{-i(\phi_r-\phi_t)}\sqrt{\gamma_1}a(t^- - \frac{x}{c}). \end{aligned} \tag{5.39}$$

Here it is seen that any dynamics that can take place inside the cavity is manifest only in the cavity field that is extracted out, viz. $a(t^- - x/c)$. $t^- = t - \lim_{\varepsilon\to 0}\varepsilon$ takes into account the fact that there is some delay in the cavity field that is transmitted outside due to passage time through the cavity wall and flight of photon in the cavity.

When the incoming fields are in their vacuum state or pumped at a very narrow spectral width $[\omega_c - \delta\omega/2, \omega_c + \delta\omega/2]$ where $\Delta\omega \gg \delta\omega \gg \gamma$, only in this interval there will be contribution to the correlation functions of the type (5.41), thus we can formally retain the whole spectral range. From an operational point of view one can introduce the outgoing field at operator $x = 0$ as follows,

$$\tilde{b}_R(t) = \frac{1}{\sqrt{\Delta\omega}}\big[rb_L(t) + F_R(t)\big] + \frac{t}{\sqrt{2}}e^{-i\phi_r}a(t^-). \tag{5.40}$$

where $b_R(t)$, $F_R(t)$, $b_L(t)$ are the Fourier transforms of $b_R(\omega)$, $F_R(\omega)$, $b_L(\omega)$ into the time domain(5.35).

5.2.3 Characterization of the cavity field

For the outgoing fields, the correlation functions of the type,

$$G^{(mn)}(x_1,t_1;\ldots;x_{m+n},t_{m+n}) = \Big\langle T_- \prod_{i=1}^{m} E^-(x_i,t_i)T_+ \prod_{j=1}^{n} E^+(x_j,t_j)\Big\rangle \tag{5.41}$$

can be calculated in terms of correlation functions of the intra-cavity field and the incoming field. Here $T_{+(-)}$ stand for (anti)time ordering. Assuming that the $b_R(t)$, $F_R(t)$, $b_L(t)$incoming fields are in their vacuum state, the correlation function can be cast into a correlation function of intracavity field,

$$G^{(mn)} = \Big(\frac{\hbar\omega_0\gamma_1}{2\epsilon_0\omega_0 cS}\Big)^{\frac{n+m}{2}} \Big\langle T_- \prod_{i=1}^{m} a^-(t_i - \frac{x_i}{c}) T_+ \prod_{j=1}^{n} a^+(t_j - \frac{x_j}{c}) \Big\rangle. \tag{5.42}$$

A useful tool in characterizing field states, especially their quantum nature is the quasiprobability distributions[77]. The characteristic function is defined to be,

$$\chi(\xi) = Tr\big(\rho e^{\xi c^\dagger - \xi^* c}\big) \tag{5.43}$$

whose Fourier transform yields the *Wigner function*,

$$W(\alpha) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathtt{d}^2\xi\ \chi(\xi) e^{\alpha\xi^* - \alpha^*\xi}. \tag{5.44}$$

Wigner function is a normalized, real valued distribution function analogous to a classical phase space probability distribution. Here ρ is the density matrix, and c,$c^\dagger$ are the annihilation and creation operators for the single mode field. Any symmetric ordered moment can be computed by an integral weighted by the Wigner function,

$$< S(c^{\dagger m} c^n) >= \int_{-\infty}^{\infty} \mathtt{d}^2\alpha\ W(\alpha)\alpha^{*m}\alpha^n. \tag{5.45}$$

It is also possible to evaluate (anti)normal ordered moments, by putting the characteristic function (5.43) into (anti)normal ordered form.

Assuming that the incoming fields are in their vacuum state, and at some time $t < 0$ an interaction is switched on in the cavity such that, at time $t = 0$ a cavity state ρ_{cav} is prepared and the interaction is switched off. The characteristic function of the outgoing field at time $t > 0$ reads,

$$\begin{aligned}
\chi(\xi) &= e^{-|\xi|^2/2}\big\langle 0|e^{\xi b_R^\dagger(t)} e^{\xi b_R(t)}|0\big\rangle \\
&= e^{-|\xi|^2/2}\big\langle 0|e^{\xi a^\dagger(t)} e^{\xi a(t)}|0\big\rangle \\
&= e^{-|\xi|^2/2}\Big\{\rho_{cav}\exp\big[\frac{\xi T}{\sqrt{2}} a^\dagger e^{i(\omega-\omega_0+i\frac{\gamma}{2})t+i\phi_r}\big]\exp\big[-\frac{\xi T}{\sqrt{2}} a e^{-i(\omega-\omega_0-i\frac{\gamma}{2})t-i\phi_r}\big]\Big\},
\end{aligned} \tag{5.46}$$

where we have used the symbol T for the transmission coefficient of the cavity wall.

Example

Assume that the cavity is initially prepared in n-photon Fock state, then the characteristic function (5.46) reads,

$$\begin{aligned}\chi(\xi) &= \exp(-|\xi|^2/2) L_n(\frac{|T\xi|^2}{2} e^{-\gamma t}) \\ &= \exp(-|\xi|^2/2)\Big\{\big[1 - \frac{n|T|^2 e^{-\gamma t}}{2}\big] L_0(|\xi|^2) + \frac{n|T|^2 e^{-\gamma t}}{2} L_1(|\xi|^2)\Big\}. \end{aligned} \tag{5.47}$$

$L_m(x)$ stands for *Laguerre polynomial* of mth order[105], where we have used $L_0(x) = 1$, $L_1(x) = 1 - x$. One can calculate the corresponding Wigner function for the outgoing fields, for a cavity initially prepared in n-photon Fock state,

$$W(\alpha) \simeq \big[1 - \frac{n|T|^2 e^{-\gamma t}}{2}\big] W_0(\alpha) + \frac{n|T|^2 e^{-\gamma t}}{2} W_1(\alpha). \tag{5.48}$$

W_n is the Wigner function for the n-photon Fock state,

$$W_n(\alpha) = \frac{2}{\pi}(-1)^n e^{-2|\alpha|^2} L_n(4|\alpha|^2). \tag{5.49}$$

For outgoing field, the Wigner function is a mixture of vacuum and single photon states whereas the density matrix of intra cavity field is a mixture of $n, \ldots, 1, 0$ Fock states if the cavity is initially prepared in n photon Fock state.

5.3 Summary and discussion

We have studied the dynamics of a cavity with absorptive wall and derived an input-output relation relating the intra-cavity and outgoing fields. In particular extraction of Fock states is studied. The absorption channel behaves as an auxiliary port of the cavity which is not detected, and in this sense is similar to a two sided cavity[106, 107]. This simplistic cavity model actually might handle cases of experimental interest. In an actual cavity the sources will make spontaneous

emission into the free transverse fields. However once the desired field state is prepared in the cavity, our model might be capable of handling realistic cases.

Chapter 6

Conclusions

We have made a number of proposals for the generation of robust, long-living atomic entangled states and studied their realization within the contemporary experimental techniques. A scheme for deterministic preparation of long-living maximally entangled states of two identical three-level Λ type atoms is proposed and studied. An irreversible evolution is shown to take place from the initially unentangled state to the maximally entangled state with respect to the ground and the metastable states between which electric-dipole transition is not allowed, resulting in a considerably long lifetime. The atoms prepared in their ground states are pumped by a single excitation driving one of the atoms to the excited state -we do not know which one- subsequently followed by an irreversible decay to the metastable state. The strong collective effects are essential in this scheme, i.e. "which atom" information should be absent in the atom-field interaction. This scheme can be realized in free space if the atoms are pumped by a single excitation pulse, while they are positioned much closer than the wavelength. However in a resonator the scheme can be realized even when the atoms are spatially well separated. This scheme is also generalized to multipartite case, when all the atoms are interacting collectively with the field.

The interaction of multi-level atoms with quantized electro-magnetic field in the presence of dispersing-absorbing dielectric bodies is studied in the most general case. The master equation governing the system is obtained, and collective

spontaneous decay and environment induced dipole-dipole interactions and level shifts are identified for weakly coupled atomic transitions. In the case of strong coupling to the electromagnetic field Jaynes-Cumming type atom-field Hamiltonians arise in the Master equation giving rise to damped Rabi oscillations. The exact coupling constants can be plugged into the equations of motion once the classical Green function for the electric field in dielectric medium is known, which depends on the geometry and the material properties of the system. If the system is properly engineered, then one can obtain either strong or weak coupling for different atomic transition frequencies as desired. Then this formalism is applied to generation of robust entanglement in Λ type atoms passing by a dielectric microsphere. Whispering gallery resonances below a band gap and surface guided waves inside a bandgap give rise to resonances. At a resonant frequency, when the atoms are at the opposite ends of a diameter then atoms will cooperate, either of the symmetric or the antisymmetric state will be in superradiant regime while the other in irradiant regime. The widths of resonances - determined by the radiative and absorption losses - signify either irreversible spontaneous emission(weak coupling regime) or damped field-atom Rabi oscillations(strong coupling regime). Either of these two regimes can be attained by the correct choice of size and material properties of the microsphere, thus it is possible to attain strong coupling in one arm and weak coupling in the other arm of the Λ atomic configuration. Also in chapter-2 the initial single photonic excitation was tacitly assumed to be present. In the case a of microsphere, it is shown that initial photonic excitation can be deposited by a two-level excited atom passing by a microsphere, which later on, couples collectively with the two Λ type atoms.

In the current technology, usually superconducting cavities are used to obtain strong atom-field interaction which is essentially the standing modes of light confined between two mirrors (see [18, 54] and references therein). This kind of cavities can only support field in the microwave region, and usually experiments are carried out with Rydberg atoms. Strong coupling in the optical region might be possible using high Q ($> 10^9$) dielectric microspheres.

The environment can be engineered in order to stabilize entanglement. The

stabilization of entanglement of two dipoles in free space with the help of classical driving field is discussed. The environment induced collective effects, i.e. dipole coupling and collective spontaneous decay, which are present even in free space, give rise to entanglement whereas the classical driving field stabilizes entanglement. In free space considerable amount of entanglement can be realized in Lamb-Dicke limit namely when the dipoles are close to each other. Spatially well separated atoms in a single-mode cavity which is driven and heavily damped[108] is equivalent to the model that we have discussed in chapter-4, and in this case it is possible to obtain a high amount of steady state entanglement even when the atoms are spatially well separated.

Absorption effects associated with the extraction of photonic quantum states from cavities is explored. It is shown that when the incoming fields are in their vacuum state, absorption effects can be modeled by some simple input-output relations. In this model the absorptive effects associated with a cavity wall can be regarded as one of the output channels of the cavity in which no detection is made. Absorptive effects are exemplified for the extraction of photon Fock states states.

Appendix A

Entanglement

Entanglement is one of the key notions which distinguish quantum information from classical information[109, 110]. The idea of entanglement has been proposed by Einstein, Podolsky and Rosen(hereafter EPR) in which by locality arguments they were led to the conclusion that "quantum mechanical description of physical reality is not complete"[3]. The principle of *Einstein locality* asserts that the events occurring in a given space-time region are independent of events occurring in space-like separated regions. In 1964 J.S. Bell proved that *if one adopts Einstein locality then there is an upper limit to the correlation of distant events*[1]. For polarization entangled photons these correlations were tested in 1982 by Aspect et. al.[2], which ruled out *hidden variable theories*[111] and verified the predictions of quantum mechanics.

A quantum state in a given Hilbert space $\mathcal{H}$ of dimension d may be described by the $d \times d$ density matrix ρ, which can be written as a convex combination of pure states $\{\psi_i \in \mathcal{H}\}$,

$$\rho = \sum_i \; p_i \; |\psi_i\rangle\langle\psi_i| \tag{A.1}$$

where convexity implies $p_i > 0$ that sum up to unity $\sum_i p_i = 1$. The decomposition given by (A.1) is not unique and all convex decompositions are physically equivalent. From (A.1) it is evident that the density matrix should be normalized to unity $Tr(\rho) = 1$, be hermitian $\rho = \rho^\dagger$, and should have a non-negative

spectrum (positivity) $\texttt{spec}(\rho) \geq 0$. A density matrix corresponding to a pure state may be written in the form $\rho = |\Psi\rangle\langle\Psi|$ which implies that in this case ρ is a projection operator $\rho^2 = \rho$, otherwise the state is a *mixed state*. Hereafter the density matrix ρ will be called the quantum state of the system of interest.

A bipartite system is associated with the Hilbert space $\mathcal{H}$ given by the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of predefined Hilbert spaces. Separable states defined in the direct product Hilbert space $\mathcal{H}$ can be written as convex combination of product states,

$$\rho = \sum_i p_i \ \rho_i^{(1)} \otimes \rho_i^{(2)}. \tag{A.2}$$

Those states which cannot be written in the convex decomposition form (A.2) are defined to be *entangled* states.

The definition of entangled states can easily be extended to multipartite systems the Hilbert space of which consist of more than two subsystems. An N-partite system is characterized by the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \mathcal{H}_N$. A *pure state* is separable if it can be written as a direct product of N states each belonging to a different subsystem. A mixed state of N-partite system is separable if it can be expressed as a convex sum of product of N states. A mixed state is called *ν-separable* if it can expressed in the form

$$\rho = \sum_i p_i \ \rho_i^{(1)} \otimes \ldots \rho_i^{(\nu)}, \tag{A.3}$$

and if $\nu = N$ the state is completely separable thus unentangled.

The simplest example of entanglement is a bipartite system consisting of two level systems, for instance

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle) \tag{A.4}$$

is the quantum state in a direct product Hilbert space of two-dimensional Hilbert spaces. In fact this is algebraically equivalent to a two spin-1/2 particle system for each of which there exists three observables $\sigma_x, \sigma_y, \sigma_z$ in a two dimensional Hilbert space,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.5}$$

in the basis $\{|+\rangle, |-\rangle\}$. From (A.5) it is seen that if $\sigma_z^{(1)}$ is measured to be $+1$ then $\sigma_z^{(2)}$ measurement will yield -1 and vice versa. Here local observables correspond to direct tensor products $\sigma_z^{(1)} = \sigma_z \otimes \mathbb{1}$, $\sigma_z^{(2)} = \mathbb{1} \otimes \sigma_z$. According to *Einstein locality principle* if the particles are not interacting anymore, no real change can take place at the 2nd party upon measurement. One may instead choose to measure $\sigma_x^{(1)}$ thus determine the value of $\sigma_x^{(2)}$ with certainty and likewise for σ_z. As a result all three spin components $\sigma_x^{(2)}, \sigma_y^{(2)}, \sigma_z^{(2)}$ can be measured without ever disturbing the second system, which is in contradiction with quantum mechanics. According to quantum mechanics only one spin component can be determined with certainty, since two different spin components(A.5) are incompatible(noncommuting).

A.0.1 Detection of entanglement

Given a quantum state the very first natural step is detecting entanglement, i.e. whether the state is separable or not. Any pure bipartite state may be cast into the Schmidt form([110]),

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i} |\psi_i\rangle_1 \otimes |\psi_i\rangle_2, \tag{A.6}$$

where the basis $\{|\psi_i\rangle_1 \otimes |\psi_i\rangle_2\}$ constitutes an orthonormal basis. Without loss of generality $d = \dim(\mathcal{H}_1) \leq dim(\mathcal{H}_2)$ is assumed. Schmidt coefficients may be computed from the reduced density matrices $\rho_1 = Tr_2|\Psi\rangle\langle\Psi|, \rho_2 = Tr_1|\Psi\rangle\langle\Psi|$ where trace operation is performed, respectively, in basis of $\mathcal{H}_1$ and $\mathcal{H}_\in$. The nonzero Schmidt coefficients consists of nonzero elements of spectrum of ρ_1. A bipartite state is separable if the Schmidt vector has only one nonzero component $\lambda = [1, 0, ..., 0]$ and for an entangled state Schmidt vector has at least two nonzero components. The state with Schmidt vector $\lambda = [1/d, ..., 1/d]$ is said to be *maximally entangled*.

The separability of pure states can easily be checked, however this turns out to be quite difficult for mixed states. For mixed states entanglement may be detected by *entanglement witnesses* [109, 112, 113].

A.0.2 Quantification of entanglement

The basic idea for a quantitative description of entanglement is to classify all kinds of operations that can only create or increase classical correlations but none of quantum nature. Naturally this type of operations are local operations and they are usually called *local operations assisted by classical communication(LOCC).* Any scalar quantity assigned to a state which does not increase under such operations is called an *entanglement monotone* and can serve for a quantification of entanglement[114](for a review see [115] and references therein). For two dimensional bipartite systems a widely used entanglement monotone is *entanglement of formation.* Given a density matrix ρ for a bipartite system, consider all possible convex decomposition of the density matrix for all ensembles of ψ_i with probabilities p_i,

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|. \tag{A.7}$$

For each pure bipartite state ψ the *entanglement of formation* is given by the reduced entropy

$$E(\psi) = -Tr\rho_1 \log \rho_1 = -Tr\rho_2 \log \rho_2, \tag{A.8}$$

which gives the asymptotic conversion rate from ψ to a standart Bell state (A.4) via LOCC[116]. Then the entanglement of formation for the mixed state ρ is given as the averaged entanglement over the pure states minimized over all possible convex decompositions of ρ,

$$E(\rho) = \min \sum p_i E(\psi_i). \tag{A.9}$$

For a pair of qubits the entanglement of formation(A.9), Wootters[94] has been able to obtain an analytic expression,

$$E(\rho) = \mathcal{E}(C(\rho)), \tag{A.10}$$

where C is the concurrence defined as,

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}. \tag{A.11}$$

λ_i's are the eigenvalues, in decreasing order, of the Hermitian matrix

$$\sqrt{\sqrt{\rho}\sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y \sqrt{\rho}} \tag{A.12}$$

and $\mathcal{E}$ is given as

$$\begin{aligned} \mathcal{E}(C) &= h\left(\frac{1+\sqrt{1-C^2}}{2}\right) \\ h(x) &= -x\log x - (1-x)\log(1-x). \end{aligned} \tag{A.13}$$

Appendix B

Dissipative Processes

The boundary between the classical and quantum behavior has been of great interest regarding both fundamental and technological issues. Interaction of a quantum system with the environment is the key issue in dissipation and quantum decoherence. In this section we are going to put some simple models under scrutiny to study decoherence and dissipation. The interaction of a system comprised of discrete states with an external continua of states will be reviewed. A thorough treatment can be found in Ref. [77].

B.0.3 Master equations

Consider a system characterized by the Hamiltonian $H_0(\hat{s})$ interacting with an environment characterized by a continuum of modes. The system is described by the following Hamiltonian in the rotating wave approximation,

$$H = \hbar\omega_0 \hat{s}^\dagger \hat{s} + \hbar \int \mathtt{d}\omega \ \omega b^\dagger b + i\hbar \int \mathtt{d}\omega \ \ W(\omega)\{\hat{s}^\dagger b - b^\dagger \hat{s}\}, \tag{B.1}$$

where $\hat{s}$ is the system operator and $b(\omega)$ are a continuum of bosonic operators, s.t. $[b(\omega), b^\dagger(\omega')] = \delta(\omega - \omega')$. W is the frequency dependent coupling constant. In the interaction picture,

$$\hat{H}_I(t) = i\hbar\big(\hat{s}^\dagger F(t) - F^\dagger(t)\hat{s}\big) \tag{B.2}$$

where

$$F(t) = \int \mathtt{d}\omega \; W(\omega) b \exp^{-i(\omega-\omega_0)t} \tag{B.3}$$

is the Langevin noise operator. The equation of motion for the total density matrix comprised of the system and the environment is,

$$\dot{\rho}_T(t) = -\frac{i}{\hbar}[H_I(t), \rho_T(t)], \tag{B.4}$$

which can be integrated to yield,

$$\dot{\rho}_T(t) = -\frac{i}{\hbar}[H_I(t), \rho_T(0)] - \frac{1}{\hbar^2}\int_0^t \mathtt{d}t'[H_I(t), [H_I(t'), \rho(t')]]. \tag{B.5}$$

At this step the environment and the system are decorrelated following the assumption,

$$\rho_T(t) = \rho_e(0) \otimes \rho(t) \tag{B.6}$$

where $\rho_e(0)$ is the initial density matrix of the environment and $\rho(t)$ is the density matrix for the system. The assumption (B.6) implies that the environment has a large number of degrees of freedom and therefore the change in environment can be ignored at this order. Tracing out the environment one obtains the density matrix for the system, viz. $\rho(t) = \mathtt{Tr}_e \rho_T(t)$,

$$\dot{\rho}(t) = \int_0^t \mathtt{d}t' \; \mathtt{Tr}_e[\hat{s}^\dagger F(t) + F^\dagger(t)\hat{s}, [\hat{s}^\dagger F(t') + F^\dagger(t')\hat{s}, \rho_T(t')]], \tag{B.7}$$

which can be cast into the form,

$$\begin{aligned}
\dot{\rho}(t) = \int \mathtt{d}t' \; & \big\{ \big[\hat{s}\rho\hat{s}^\dagger - \hat{s}^\dagger\hat{s}\rho\big]\langle F(t)F^\dagger(t')\rangle + \big[\hat{s}\rho\hat{s}^\dagger - \rho\hat{s}^\dagger\hat{s}\big]\langle F(t')F^\dagger(t)\rangle \\
& + \big[\hat{s}^\dagger\rho\hat{s} - \hat{s}\hat{s}^\dagger\rho\big]\langle F^\dagger(t)F(t')\rangle + \big[\hat{s}^\dagger\rho\hat{s} - \rho\hat{s}\hat{s}^\dagger\big]\langle F^\dagger(t')F(t)\rangle \\
& - \big[\hat{s}^\dagger\rho\hat{s}^\dagger - \hat{s}^{\dagger 2}\rho\big]\langle F(t)F(t')\rangle - \big[\hat{s}^\dagger\rho\hat{s}^\dagger - \rho\hat{s}^{\dagger 2}\big]\langle F(t')F(t)\rangle \\
& - \big[\hat{s}\rho\hat{s} - \hat{s}^2\rho\big]\langle F^\dagger(t)F^\dagger(t')\rangle - \big[\hat{s}\rho\hat{s} - \rho\hat{s}^2\big]\langle F^\dagger(t')F^\dagger(t)\rangle \big\}.
\end{aligned} \tag{B.8}$$

Here the next step is to estimate the expectation values of correlation functions in (B.8). In most cases the correlation terms involving the pairs $F(t), F(t')$ or $F^\dagger(t), F^\dagger(t')$ will vanish, in particular for a thermal environment. However in a rigged (squeezed) reservoir these correlations may survive.

For a thermal reservoir, for the Bose statistics,

$$\begin{aligned}\bar{n}(\omega) &= \langle b^{\dagger}(\omega)b(\omega)\rangle \\ &= \frac{1}{exp(\hbar\omega/kT)-1},\end{aligned} \tag{B.9}$$

from which the nonzero correlation functions in (B.8)can be computed as follows,

$$\begin{aligned}\langle F^{\dagger}(t)F(t')\rangle &= \int \mathtt{d}\omega\ W^2(\omega)\bar{n}(\omega)e^{-i(\omega-\omega_0)(t-t')} \\ \langle F(t)F^{\dagger}(t')\rangle &= \int \mathtt{d}\omega\ W^2(\omega)(\bar{n}(\omega)+1)e^{-i(\omega-\omega_0)(t-t')}.\end{aligned} \tag{B.10}$$

The next step is to convert the integro-differential equations in (B.8) to a differential equation by performing the time integrals within the Markov approximation. If the coupling constant is a slowly varying function of frequency then the frequency integrals in (B.10) will yield a strongly localized function in time at $t = t'$ approximating a Dirac-delta function. In this case $\rho(t')$ in (B.8) may be replaced by $\rho(t)$ and taken outside the integral. In the time integration the contribution will be prominently at $t' \simeq t$, so the limit of the time integral may be extended to $t \to \infty$. The Markov approximation and beyond in (B.8) with the substitutions (B.10), may be illustrated as follows,

$$\begin{aligned}&\int_0^{\infty} \mathtt{d}\omega\ G(\omega) \int_0^{t} \mathtt{d}t'\ \rho(t')e^{-i(\omega-\omega_0)(t-t')} \\ &\qquad \simeq \rho(t)\int_0^{\infty} \mathtt{d}\omega\ G(\omega)\int_0^{t}\mathtt{d}t'\ e^{-i(\omega-\omega_0)(t-t')} \\ &\qquad \simeq \rho(t)\lim_{\delta\to 0}\lim_{t\to\infty}\int_0^{\infty}\mathtt{d}\omega G(\omega)\int_0^{t}\mathtt{d}t'\ e^{-i(\omega-\omega_0-i\delta)(t-t')} \\ &\qquad = \rho(t)\Big[\pi G(\omega_0)+\mathcal{P}\int_0^{\infty}\mathtt{d}\omega\frac{G(\omega)}{\omega-\omega_0}\Big]\end{aligned} \tag{B.11}$$

where $\mathcal{P}$ denotes the principal part,

$$\mathcal{P}\int_0^{\infty}\mathtt{d}\omega\frac{G(\omega)}{\omega-\omega_0} = \lim_{\delta\to 0}\int_0^{\infty}\mathtt{d}\omega\frac{G(\omega)(\omega-\omega_0)}{(\omega-\omega_0)^2+\delta^2}. \tag{B.12}$$

Markov approximation in (B.11) holds if $1 \gg G'(\omega)$, for $\omega \in [\omega_0 - G(\omega), \omega_0 + G(\omega)]$.

Now the differential equation for the master equation can be obtained from

(B.8),

$$\begin{aligned}\dot{\rho}(t) = &- i\delta\omega[\hat{s}^\dagger\hat{s}, \rho(t)] - i\delta\omega_{th}[[\hat{s}^\dagger, \hat{s}], \rho(t)] \\ &+ \Gamma(\bar{n}(\omega_0) + 1)(2\hat{s}\rho\hat{s}^\dagger - \hat{s}^\dagger\hat{s}\rho - \rho\hat{s}^\dagger\hat{s}) + \Gamma\bar{n}(\omega_0)(2\hat{s}^\dagger\rho\hat{s} - \hat{s}\hat{s}^\dagger\rho - \rho\hat{s}\hat{s}^\dagger),\end{aligned} \tag{B.13}$$

where $\Gamma = \pi W^2(0)$, and

$$\begin{aligned}\delta\omega &= -\mathcal{P}\int_0^\infty \mathtt{d}\omega \frac{W^2(\omega)}{\omega - \omega_0} \\ \delta\omega_{th} &= -\mathcal{P}\int_0^\infty \mathtt{d}\omega \frac{\bar{n}(\omega)W^2(\omega)}{\omega - \omega_0}\end{aligned} \tag{B.14}$$

are the frequency shifts. The first expression in (B.14) is the Lamb shift induced by vacuum, and the second expression is the thermally induced Lamb shift. Practically these frequency shifts can be incorporated into the free Hamiltonian, so that the natural frequency of the system is redefined. For a harmonic oscillator, $[s, s^\dagger] = 1$, and the frequency shift is $-\delta\omega$ i.e. there will be no effect of temperature on the natural frequency. For a two level system the frequency shift will be $\delta\omega + 2\delta\omega_{th}$.

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