QUALITY AND PRODUCTION CONTROL WITH OPPORTUNITIES AND EXOGENOUS RANDOM SHOCKS

A Ph.D. Dissertation

by
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Ankara
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QUALITY AND PRODUCTION CONTROL
WITH OPPORTUNITIES AND EXOGENOUS RANDOM SHOCKS

The Institute of Economics and Social Sciences
of
Bilkent University

by

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ABSTRACT

QUALITY AND PRODUCTION CONTROL WITH OPPORTUNITIES AND EXOGENOUS RANDOM SHOCKS

Ayhan Özgür Toy

Ph.D. Dissertation in Business Administration

Supervisor: Asst. Prof. Dr. Emre Berk

September 2005

In a production process, opportunities arise due to exogenous or indigenous factors, for cost reduction. In this dissertation, we consider such opportunities in quality control chart design and production planning for the lot sizing problem. In the first part of the dissertation, we study the economic design of $\overline{X}$ control charts for a single machine facing exogenous random shocks, which create opportunities for inspection and repair at reduced cost. We develop the expected cycle cost and expected operating time functions, and invoking the renewal reward theorem, we derive the objective function to obtain the optimum values for the control chart design parameters. In the second part, we consider the quality control chart design for the multiple machine environment operating under jidoka (autonomation) policy, in which the opportunities are due to individual machine stoppages. We provide the exact model derivation and an approximate model employing the single machine model developed in the first part. For both models, we conduct extensive numerical studies and observe that modeling the inspection and repair opportunities provide considerable cost savings. We also show that partitioning of the machines as opportunity takers and opportunity non-takers yields further cost savings. In the third part, we consider the dynamic lot sizing problem with finite capacity and where there are opportunities to keep the process warm at a unit variable cost for the next period if more than a threshold value has been produced. For this warm/cold process, we develop a dynamic programming formulation of the problem and establish theoretical results on the optimal policy structure. For a special case, we show that forward solution algorithms are available, and provide rules for identifying planning horizons. Our numerical study indicates that utilizing the undertime option results in significant cost savings, and it has managerial implications for capacity planning and selection.

ÖZET

FIRSATLAR VE DIŞSAL RASSAL ŞOKLAR ALTINDA KALİTE VE ÜRETİM KONTROLÜ

Ayhan Özgür Toy

İşletme Doktora Tezi

Tez Yöneticisi: Yrd. Doç. Dr. Emre Berk

Eylül 2005

İçsel ve dışsal etkenler, bir üretim sürecinde, maliyet azaltan fırsatlar yaratır. Bu tezde, bu tür fırsatları, kalite kontrol diyagramlarının tasarımı ve kafile büyüklüğü belirleme problemi kapsamında ele alıyoruz. Tezin ilk kısmında, $\overline{X}$ kontrol diyagramlarının ekonomik tasarımını, daha düşük maliyetli inceleme ve onarım fırsatları yaratan dışsal ve rassal şokların bulunduğu bir ortamda inceliyoruz. Beklenen çevrim maliyeti ve beklenen çalışma zamanı fonksiyonlarını elde ederek, yenilenen ödül (renewal reward) teoremi sayesinde, kontrol diyagramı parametrelerini bulmak için eniyileme probleminin amaç fonksiyonunu oluşturuyoruz. İkinci kısımda, jidoka politikası uygulanan, fırsatların makinaların tek başına sistemi durdurmalarından kaynaklandığı, çok makinalı bir üretim ortamı için kontrol diyagramı tasarımını ele alıyoruz. Gerçek modelin nasıl çıkarılacağını gösteriyoruz; daha sonra, ilk kısımda geliştirilen tek makina modelini kullanan yaklaşık bir model öneriyoruz. Her iki model için, kapsamlı bir sayısal çalışma yapıyoruz. Modellemeye inceleme ve onarım fırsatlarının dahil edilmesinin, dikkate değer maliyet tasarrufları sağladığını gözlemliyoruz. Ayrıca, makinaların fırsatları kullananlar ve kullanmayanlar olarak iki kümeye ayrılmalarının daha da fazla maliyet tasarrufu sağlayacağını gösteriyoruz. Tezin üçüncü kısmında, kapasitenin kısıtlı olduğu ve sürecin, üretim miktarının bir eşik değerinden daha fazla olduğu periyottan bir sonraki periyoda birim değişken maliyetle sıcak tutulma olanağının bulunduğu bir üretim ortamını ele alıyoruz. Bu sıcak/soğuk süreç için, dinamik kafile büyüklüğünün belirlenmesi problemini, dinamik programlama yöntemiyle kuruyoruz. En iyi politikanın yapısal özelliklerini belirliyoruz. Özel maliyet yapısı altında, baştan sona doğru ilerleyen çözüm algoritmalarının (forward algorithms) varlığını ispat ediyoruz. Bu yapı altında, ayrıca planlama ufkunu belirleyen kuralları elde ediyoruz. Sayısal çalışmamızda, kapasitenin altında üretmenin önemli maliyet tasarrufları sağladığı ve kapasite planlaması ve seçimi konularına ilişkin yönetim kararlarında kullanılabileceği gözlenmiştir.

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Chapter 1

General Introduction

In a production process, there may be exogenous and indigenous factors creating opportunities for cost reduction. One such opportunity comes from the jidoka operating policy. In jidoka policy, when there is an indication of defective production in any of the machines, the whole system is forced to cease production. Any kind of operations that will be performed on the machines such as calibrating, cleaning, inspecting, component replacement, etc., can be rescheduled to utilize the idle time and repair assets, which become available following the system stoppage instances. Another such opportunity is the setup carryover. There may be opportunities for maintaining the readiness level of the process for production so that it does not require many operations and the production can start immediately in the next period. Keeping the process ready for the production results in smaller number of setups, which may yield cost reduction. We study the quality control chart design problem and dynamic lot-sizing problem under such opportunities to exploit the possible cost savings.

This dissertation is composed of three parts. In the first part, we develop a model for the economic design of quality control charts for a single machine facing opportunities for inspection and repair that incur reduced cost. We assume that inspection opportunities come exogenously as random shocks. Using a renewal theoretic approach, we develop the expressions for the operating characteristics of the system and then construct the quality control (QC) chart under the objective of minimizing the expected cost rate. Through an extensive numerical study, we conduct *(i)* a sensitivity analysis of the control parameters, *(ii)* an investigation of the cost breakdown structure of the optimum cost rate, and *(ii)* an analysis of the cost improvements provided by the opportunistic inspections and repairs over the control chart designed by the classical model of the economic design.

In the second part, we consider the economic design of quality control charts for the multiple machine environment which exploits the inspection/repair opportunities that arise due to individual machine stoppages. In case of a production line with multiple machines, the stoppage frequency that any machine faces can no longer be modeled as an exogenous parameter. When the machines in the production line have different characteristics, i.e. different cost parameters and different reliability, opportunistic inspections may be decreasing the cost for some machines whereas increasing the cost for the others. We presume that, in addition to the control parameters of the individual machines, in the optimal control plan, partitioning of the machines as *opportunity takers* and *opportunity non-takers* is another decision variable. This partitioning depends on the reliability and the cost characteristics of the machines. First we show that exact model of the multiple machine environment can be derived by formulating the problem as an embedded Markov chain. Then we provide an approximate model which employs the single machine model developed in the first part. We also provide an algorithm for the solution of the control chart design problem for the joint system in an iterative fashion. In a numerical study, *(i)* we analyze the optimum control parameters and partitioning under various settings of machines with respect to their cost parameters and reliability, *(ii)* we provide a comparison of the jointly optimized system cost rate with the cost rate obtained from individual machine optimizations by the classical control chart model, and *(iii)* we conduct a simulation study to observe the performance of the approximations we make.

In our numerical studies for the single machine and multiple machine environments, we have observed that significant cost savings can be achieved when opportunities are incorporated into the model and that the control parameters of the classical model.

In the third part of this dissertation, we study a dynamic lot sizing problem, where demands are deterministic and known and there is a capacity over the production quantity in a period. We introduce a model which incorporates the *opportunities* of keeping the process at a unit variable cost for the next period only if more than a threshold value has been produced, the process would be *cold*, otherwise. We *(i)* develop a dynamic programming formulation of the dynamic lot sizing problem for a warm/cold process, *(ii)* establish the structure of the optimal policy, *(iii)* show that polynomial and linear time solution algorithms exist, *(iv)* provide several planning horizon rules in the presence of warm/cold process, and *(v)* examine, via a numerical study, the sensitivity of the optimal production schedule and total cost to various system parameters, illustrate that restricting or ignoring the use of undertime (warming) option results in substantial savings, and study the horizon length that allows problem partitioning for each planning horizon rule. Our numerical study indicates that utilizing the undertime option (*i.e.,* keeping the process warm via reduced production rates) results in significant cost savings, and it has managerial implications for capacity planning and selection.

The rest of this dissertation is organized as follows. In Chapter 2 we provide an introduction to the quality control, lean manufacturing, statistical process control and control charts, we also provide a review of the relevant literature. In Chapter 3, we develop the single machine model with exogenous opportunistic inspection/repairs, and provide the results of a numerical study of this model in Chapter 4. We next move on the multiple machine environment and develop the model in Chapter 5. Numerical study for the multiple machine model is provided in Chapter 6. Chapter 7 comprises an introduction the dynamic lot sizing problem with warm/cold setups, a review of the literature follows in Chapter 8. In Chapter 9 we provide assumptions, formulation and structural results of the lot sizing problem we consider. A special case of the problem is when the warm setup costs are negligible. In Chapter 10 we consider this special case, and we show that the special case allows us to develop forward solution algorithms. Moreover, we also prove that the planning horizons developed for the classical dynamic lot sizing problem may be implemented, and show that we can construct additional planning horizon rules in the existence of warm process. We exhibit the results of a numerical study for the dynamic lot sizing problem with warm/cold processes in Chapter 11. Finally in Chapter 12, we provide a summary, conclusion and suggestions for the future work.

Chapter 2

Introduction to QC Chart Design with Opportunistic Inspections and Literature Review

2.1 Quality Management and Lean Manufacturing

In this chapter we present the basics of the quality control, lean manufacturing, statistical process control (SPC) and QC-charts, and we review some of the literature in the area of the economic design of the QC-charts.

In today's highly competitive global economy, for ensuring customer satisfaction, which is the basis for the Total Quality Management Philosophy, reducing the costs and increasing the quality of the product are becoming the key factors of production.

There are different definitions of quality in the literature. Some of these are: "a measure for excellence", "a desirable characteristic", "the concept of making products fit for a purpose and with the fewest defects", "reduction of variation around the mean", "products and services that meet or exceed customers' expectations", "value to some person", "the totality of features and characteristics of the product, process or service that bear on its ability to satisfy stated or implied needs" (ISO 8402-Quality Management and Quality Assurance Vocabulary) or simply: "fitness for use" (Juran and Godfrey, 1998). Quality (or fitness for use) has two components: quality of design and quality of conformance. Quality of design is mostly related to the physical characteristics of the product that are the result of deliberate engineering and management decisions. Inventors, engineers, architects, and draftsmen are viewed as the responsible people from the quality of design. In the manufacturing process, where the design specifications are transformed into final products, quality of conformance becomes important. Quality of conformance is how closely the final product meets the design specifications. Therefore, the goal of quality of conformance is the production of identical and defect-free items, hence the systematic reduction of variability and elimination of defects. Since fitness for use incorporates reducing the variability in the key parameters, the focus of the quality studies is on the reduction of unnecessary variability in these parameters.

A typical manufacturing facility has production lines consisting of machines working interdependently, in terms their inputs and outputs. Each of these machines receives materials, processes and submits them to another machine as input. Hence, establishing the required coordination among these machines is very important. One of the procedures for establishing this coordination is the *Kanban* control (JIT, Just-in-time management). Kanban control ensures that parts are not processed except in response to a demand. The coordination is provided by circulating cards, between the machine and downstream buffer. In the kanban control a machine must have a card before it can start an operation. Hence, the system works as a pull system. It receives input material out of its upstream buffer, performs the operation, attaches the card to the processed material, and puts it in the downstream buffer. This tight control also implies/results in that the whole line be stopped whenever a single machine is stopped due to a failure.

Moreover, the quality of finished good, coming out of the production process, depends on the quality of each operation performed on every single machine. The sequential nature of production also implies that care must be taken to ensure the detection of the quality problems as early as possible at each machine. Therefore, lack of coordination in a multi-machine environment could intensify the costs due to poor quality.

Lean manufacturing, also known as The Toyota Production System (TPS) (see

Ohno 1988), is frequently modeled as a house with two pillars (see Figure 2-1). The top of the house consists "highest quality, lowest cost, shortest lead time", whereas one of the two pillars represents *just-in-time* (JIT), and the other pillar the concept of *jidoka*. *Jidoka* (translated as autonomation) is a defect detection system which automatically or manually stops the production operation whenever an abnormal or defective condition arises. The manufacturing system introduced will not stand without both of the pillars. Yet many researchers and practitioners focus on the mechanisms of implementation–one piece flow, pull production, tact time, standard work, kanban–without linking those mechanisms back to the pillars that hold up the entire system. Although JIT, one of these pillars, is fairly well understood, *jidoka*, the other pillar is key to making the entire system hold up. We can state that a lot of failed implementations can be traced back to not building this second pillar. In the concept of *jidoka* when a team member encounters a problem in his or her work station, he/she is responsible for correcting the problem by pulling an *andon cord*, which can stop the line. The objective of jidoka can be summed up as: Ensuring quality 100% of the time, preventing equipment breakdowns, and working efficiently.

In his talk at the 2003 Automotive Parts System Solution Fair held in Tokyo, June 18, 2003, Teruyuki Minoura, Toyota's managing director of global purchasing at the time, stated that:

> "It is essential to halt the line when there's a problem. If the line

> doesn't stop, useless, defective items will move on to the next stage. If you don't know where the problem occurred, you can't do anything to fix it. That's where the concept of visual control comes from. The tool for this is the andon electric light board."

2.2 Statistical Process Control

The manufacturing systems benefit from statistical tools for defect detection and quality improvement. Some of the applications of the statistical method in the manufacturing systems for the quality management are: comparison of different materials, components, ingredients; monitoring of a production process which improves the process capability, which is a measure of the proportion of the items produces that conforms the standards/specifications when the process is in statistical control, by minimizing the variability in the process; optimizing processes in order to increase the yield and reduce the manufacturing costs; development of a measurement system which can be readily used for decision making about the process. Among the statistical methods, SPC has been one of the most successful tools in quality management. Most of the production organizations implement programs that incorporates SPC methods in their manufacturing and engineering activities. Statistical Process Control (SPC) is defined as the application of statistical and engineering methods in measuring, monitoring, controlling, and improving quality. An overview of the historical development and current status of the statistical process control is provided by Stoumbos et al. (2000).

The variability in any production process is unavoidable, hence there is a certain probability that the output of a production system will not fit the quality specifications of the final product. The objective of continuos improvement of the process performance and reducing the variability is achieved through statistical process control. Tools of statistical process control are called as "*Magnificent Seven*" in Montgomery (2004). These tools are: histogram, check sheet, Pareto chart, cause and effect diagram, defect concentration diagram, scatter diagram, and control chart. All of these tools, hence the SPC is based on the observation of the process. These tools translate the data collected into a meaningful display for decision making. Our focus in this dissertation is the control charts.

2.3 Quality Control Charts

The concept of statistical control and the use of control charts for the statistical stability are first introduced by Walter A. Shewhart in the 1920s (see Shewhart, 1931 and 1939). His work considered to be the foundation of modern statistical process control and quality control charts. Shewhart (1931) defines the control as follows:

> "A phenomenon will be said to be controlled when, through the use of past experience, we can predict, at least within limits, how

> the phenomenon may be expected to vary in the future. Here it is understood that prediction within limits means that we can state, at least approximately, the probability that the observed phenomenon will fall within the given limits."

He states that a constant and predictable process has only *random causes* (chance causes). He called the unknown causes of variability in the quality of the product which do not belong to a stable (constant) system as *assignable causes* (special causes). He builds a control chart model to distinguish between "chance causes" and "assignable causes" of variability in the process. When only the random causes are present in the process, it is considered that variability is at an acceptable level, hence the outputs conform the specifications. A process that is operating with only random causes of variation present is said to be **in (statistical) control**. Although a production process mostly operates in the in-control status, other kinds of variability may occur in the process. If this kind of variability due to an assignable cause exists than the output of the process does not meet the specifications. The sources of this kind of variability are: improperly adjusted machines, operator errors, or defective raw materials. A process that is operating in the presence of assignable causes is said to be **out of control**. Once the process is in the out-of- control status the source of the variability should promptly be identified and corrective actions should be taken to restore the process to the in control state.

There are two different purposes of the control charts suggested by Shewhart (1931): *(i)* To determine whether a process has achieved a state of statistical control, *(ii)* To maintain current control of a process.

The basis of the QC-charts is sampling from the output of a machine and conducting a hypothesis test to see whether or not the output meets certain specifications. If these specifications are not met, the machine is then inspected for an assignable cause and corrected/adjusted, if necessary. An illustration of a typical control chart is depicted in Figure 2-2. In a control chart, a specific quality characteristic is measured by the samples taken usually at fixed time intervals, and these measurements are plotted on a chart in chronological fashion. The center line represents the average value of the quality characteristic. The lower control limit and the upper control limit determine the bounds of the in-control region, i.e. as long as the measurement of the sample taken falls in between these two lines the process is assumed to be in the in-control status. Only random causes exist if the sample data points fall between the two control limits. If the process shifts to the out-of-control status then we expect that most of the observations are outside the control limits. Moreover, when in-control, the data points plotted on the chart should be evenly distributed between the control limits. Sometimes the data points are located only on one side of the center line and close to each other. This also may be an evidence for a systematic variation, hence, out-of-control state. It is assumed that, at the start of a production run after the last restoration, the production process is in the in control status, producing items of acceptable quality. After a period of time in production, the production process may *shift* to the out-of-control status. A major objective of QC-charts is to quickly and cost effectively detect the occurrence of assignable causes or process shifts so that investigation of the process and corrective action may be undertaken before many nonconforming units are manufactured.

Distance of control limits from the center line are expressed in standard deviation units. Shewhart (1931) suggests the use of 3σ control limits over a control chart, and taking samples of size four or five. He leaves the sampling interval determination to the quality control engineers. This type of control charts are referred as Shewhart Control Charts in the literature.

Control charts are quite popular for the following reasons (Montgomery, 2004):

1. Control charts are a proven technique for improving productivity

2. Control charts are effective in defect prevention

3. Control charts prevent unnecessary process adjustments

4. Control charts provide diagnostic information

5. Control charts provide information about process capability

Determination of the control parameters of the control charts is called design of the control charts. Literature on the design of the control charts can be classified in several different ways; for example, depending on the methods in selecting the control parameters, depending on the data type of the quality characteristic or depending on the class of assumptions made in the problem formulation.

Depending on the method in control parameter selection, control charts can be classified as follows: *(i) Purely statistical approach:* the statistical performance of the chart is the only consideration and purely statistical aspects are considered when selecting the operating values of control parameters. In the statistical design of control charts, the power of the test for detecting an assignable cause, and value for Type I error are set to their predetermined values, and decision variables are calculated such that power and Type I error objectives are achieved. However this approach completely disregards the economical consequences of the design. *(ii) Fully economic approach:* the quality costs and chart maintenance costs are taken into account explicitly in selecting the operating values of the control parameters. Quality costs used in many manufacturing and service organizations are summarized in Table 2.1 (reproduced, Montgomery 2004). *(iii) Economic statistical (semi-economic) approach:* the economical and the statistical performances of the control charts are simultaneously optimized. This approach is due to some criticism raised for the fully economic approaches for their ignorance of the statistical performance of the charts and the difficulty in estimating and collecting the cost data.

The economic models are generally formulated using the total cost per unit time function. The most commonly used objective in the economic design of control charts is minimizing the cost rate and determining the values of the decision variables that satisfy the objective. Overall production time is divided into stochastically identical cycles. Each cycle starts with the production in the in-control status. At some sampling instant, control chart indicates an out-of-control status, as described above. Then, a search for the assignable cause is conducted and if discovered the process is stored to the in-control status. The time between these two time points is called a cycle. Hence the expected cost within this cycle is computed and divided by the expected duration of the cycle. Minimization of this cost rate yields the design parameters of the control chart. There has been an increased interest in the economic design of the control charts in the late 1980's and early 1990's in accordance with developments in lean management of production systems.

Many research have been done during the recent half-century on the economic design of the quality control charts, following the pioneering work of Duncan (1956). Surveys and reviews of this extensive literature can be found in Gibra (1975), Montgomery (1980), Vance (1983), Ho and Case (1994), and Tagaras (1998). They also provide some future directions for the field. We will provide a review of relevant literature, covering only some of the important works that are the closest to our problem setting, and the summaries of the articles reviewing the literature.

Choosing the control chart design parameters by taking the costs into account is first brought up by Duncan (1956). The objective of his study is to maximize the long run average net income per unit time of a process operating under the surveillance of a control chart. The process net income is equal to the difference between the total income and the total cost. The total income is composed of *(i)* income when the process is in-control, and *(ii)* income when the process is out-of-control. The total cost is composed of *(i)* the cost of looking for an assignable cause when none exists, *(ii)* the cost of looking for an assignable cause when it exists, *(iii)* the cost of maintaining the chart. He considers only one assignable cause case (he also provides some introduction to the multiple assignable case). He assumes that assignable cause occurrences follow a Poisson process and causes a shift in the process mean. He assumes that the production continues while investigating and correcting the process. He specifies that he doesn't consider the cost of adjustment and repair and the cost of bringing the process back to the in-control. It is assumed that the rate of production is sufficiently high, so that the possibility of a shift occurrence during the sample taking is negligible. He incorporates the time between taking the sample and plotting it on to the chart (delay in plotting) into the model. He develops expressions for the proportion of time the process is in-control and that when it is out-of-control. The average number of times the process actually goes out of control and the expected number of false alarms are determined. His solution procedure is based on solving numerical approximations to a system of first partial derivatives of the loss-cost with respect to the control parameters.

To explain the objectives of economic design of quality control charts and their classifications we will quote from Saniga (2000).

"In the control chart design problem, the objective is to determine the parameters of a control chart such that cost is minimized or profit is maximized according to an economic model, desired average run length or average time to signal are achieved, or both are achieved simultaneously. These problems are called economic design, statistical design, and economic statistical design, respectively."

When control charts are designed appropriately, taking into account the economical considerations, they contribute to maintaining the desired level of the quality and result in considerable amount of cost savings by reducing the waste and scrap.

Gibra (1975) discusses the developments in the control charts according to the following classification: *(i)* Shewhart control charts, *(ii)* Modifications of Shewhart control charts, *(iii)* Cumulative Sum control charts, *(iv)* Economic design of $\bar{X}$-control charts, *(v)* Acceptance control charts, and *(vi)* Multi-characteristic control charts.

Montgomery (1980), in his review, summarizes the assumptions that are relatively standard for the formulation of the economic design of the control charts. These assumptions that are considered as standard are:

(i) The production process is assumed to be characterized by a single in-control state.

(ii) The process may have more than one out of control states.

(iv) Assignable causes occur according to a Poisson process.

(v) Transitions between states are instantaneous.

(vi) Process is not self-correcting.

He also explains the three categories of the customarily considered cost structures in the formulation of the quality control charts. These categories are: *(i)* the cost of sampling and testing; *(ii)* the cost associated with the investigation of an alarm signal and with the repair or correction of any assignable causes detected; *(iii)* the costs associated with the production of defective items. His conclusions about the optimum economical design are:

1. The optimum sample size is largely determined by the magnitude of the shift.

2. The hourly penalty cost for production in the out-of-control state mainly effects the interval between samples, h.

3. The cost associated with looking for assignable causes mainly affect the width of the control limits.

4. Variation in the costs of sampling affects all three design parameters.

5. Changes in the mean number of occurrences of the assignable cause per hour, λ, primarily affect the interval between samples.

6. The optimum economic design is relatively insensitive to errors in estimating the cost coefficients.

Another classification of the control charts depends on the type of the data collected. When the quality characteristic can be measured and expressed on a continuous scale (length, weight, volume, etc.) then a *variables control chart* is employed. These are the charts for controlling the central tendency and variability of the quality characteristic. The variables control charts are called: $\bar{X}$-control charts if the mean of the subgroup data is measured, R-chart if the range of the subgroup data is measured and is used when the sample size is small (<10), s-chart if the standard deviation of the subgroup data is measured and is used when either the sample size is moderately large (>10 or 12) or the sample size is variable. In order to maintain the process control, both the mean and the variance of the quality characteristics have to be examined. In the variables control charts, the distribution of the quality characteristic is assumed to be normally distributed. However, because of the Central Limit Theorem the results are still approximately correct even if the underlying distribution is not normal. Some of the seminal works for the "*economic design of variables control charts*" are as follows. Duncan (1956), Lorenzen and Vance (1986), Von Collani (1988), Saniga (1989), and many other research we review here, develop models for economic design of $\bar{X}$-control charts. Von Collani and Sheil (1989) develop an economic model for the s-chart. Saniga (1977), Jones and Case (1981) consider the joint economic design of $\bar{X}$ and R control charts, and Rahim, Lashkari, and Banerje

(1988) consider the joint economic design of $\bar{X}$ and s control charts. For monitoring and controlling the processes with small shifts, Cumulative-Sum (CUSUM) and Exponentially Weighted Moving-Average (EWMA) control charts are more effective alternatives. The advantage of CUSUM and EWMA charts is that each plotted point includes several observations, so central limit theorem can be used to say that the average of the points (or the moving average in this case) is normally distributed and the control limits are clearly defined.

If the data collected for the product evaluation is of a count or discrete response type (pass/fail, yes/no, good/bad, number of defectives, etc.) then an *attribute control chart* is employed. In an attribute control chart: if the sample size of the subgroups are not equal and the percentage of the nonconformities are the control parameter then the control chart is called p-chart; if the sample size in the subgroups are equal and the count of the nonconformities are plotted then it is called np-chart (note that this is a special case p-chart); if the number of nonconformities per unit (per day, per square meter, etc.) is the measure then a c-chart is employed; when the inspection unit is not fixed (for example, some inspections are per day, some are per shift, and some are per week) then the number of nonconformities is normalized with respect to the inspection unit and a u-chart is employed. *"Economic design of attribute control charts"* is beyond the scope of our study, excellent reviews of the literature on this topic are provided by Montgomery (1980), Vance (1983), and Ho and Case (1994).

Lorenzen and Vance (1986) present a general method for fully economical approach, that applies to all control charts, for determining the economic design of control charts. The basic feature of their modeling approach is that it is based on the in-control and out-of-control average run lengths (ARL), rather than the Type I and Type II error probabilities. They introduce a model in which the total cost of quality, including cost of producing nonconforming items while in control, is minimized. They make all the standard assumptions in Montgomery (1980). They discuss two assumptions, exponentially distributed assignable cause occurrences, and single assignable cause with known amount of shift. They state that since the occurrences of the assignable causes are rare events and independent of each other, exponential inter-occurrence times assumption is reasonable. They also argue that if a different distribution is assumed and if the process continues after a false alarm as if the false alarm never occurred then the average time in the in-control status is unchanged by the false alarms, hence the effect of relaxing the exponential assumption would be minor. They develop the expressions for estimating the expected time in-control and expected time in out-of-control. They considered that cycle time is the sum of the following components:

(i) The time until the assignable cause occurs

(ii) The time until the next sample is taken

(iii) The time to analyze the sample and chart the result

(iv) The time until the chart gives an out-of-control signal, and

(v) The time to discover the assignable cause and repair the process

The costs considered in the model are those incurred during the in-control and out-of-control periods and are as follows:

(i) Cost per hour of production of defective items while in-control

(ii) Cost per hour of production of defective items while out-of-control

(iii) Cost per false alarm

(iv) Cost for locating and repairing the assignable cause when one exists

(v) Cost of sampling: a fixed cost per sampling and cost per unit sampled.

Expected cost per hour is calculated by dividing expected cost per cycle by expected cycle time in hours. The model has the advantage that it allows other control charts to be incorporated simply by changing the probability distribution function that generates the average run lengths. In order to minimize the expected cost per hour, they use Fibonacci search if the control limits are discrete, and the golden section search if the control limits are continuous. They give an example and present sensitivity analysis results. They observe that the value of the expected cost rate function is sensitive to the (constant) amount of process shift, however sampling plan is not sensitive to the amount of process shift. Hence, they state that control parameters can fairly be approximated.

Goel, Jain, Wu (1968), employ the fully economic approach to the single assignable cause, variables control chart model. They develop an algorithm based on the model described in Duncan (1956) for determining the optimum values of the control parameters. They evaluate two functions to search over policy parameters, one of the functions is an implicit equation in sample size, y and control limits constant, k, i.e. $f(y,k)$, the other one is an explicit function for sampling interval, h, $f(h)$. Using an initial integer value for y, they obtain values for k that satisfy equation $f(y,k)$ as closely as possible, then, for each of the k values they calculate h, from $f(h)$. Then, they substitute the policy parameter value triplet (y,k,h) into the loss-cost function and find the local minimum. They repeat the procedure for different values of y, and finally comparing the loss-cost function values for different values of n, they determine the optimum policy parameters. They provide a sensitivity analysis study. They observe that *(i)* k is linearly increasing in y, *(ii)* h is increasing by following a concave curve in y, *(iii)* loss-cost function surface is relatively insensitive to y, such that when y varies from its optimum value within an interval of ± 2, the change in the loss-cost function value is only 4%. Their results show that there is only one local minimum for each value of y, but for a fixed k there exist two h values, and similarly for a fixed h there exists two k values satisfying the identical loss-cost function value. They also observe that changes in the shift rate primarily affect sampling interval h, changes in the other two parameters are relatively small. They compare their new algorithm with the Duncan's approximate method for 15 examples.

Gibra(1971) employs the fully economic approach to the single assignable cause, variables control chart model. He makes the standard assumptions in

Montgomery(1980). However, he assumes that sum of the times to take samples, inspection, plotting, and discovering and eliminating the assignable cause follows an Erlang distribution. His justification for the choice of Erlang is that it provides a good fit to empirical distributions. For the design of the control chart he focuses on the length of time that elapses between the occurrence of the assignable cause and its detection, and includes the criterion of the permissible mean expected number of defectives produced within a cycle, addition to the economic criterion. He also extends the cost structure of Duncan(1956) by considering: the cost of searching for the assignable cause when a false alarm is raised, the cost of detecting and eliminating the assignable cause, the penalty cost per unit of defective items, the cost of inspection and plotting per unit sample and the overhead cost per inspected sample for maintaining the $\bar{X}$-chart. After developing his model he suggests a trial and error technique to determine the optimum values for y and k, and then computing h through a provided equation by substituting the y and k values obtained from the previous step.

Economic statistical approach and joint optimization for the variables control charts models are developed by researchers such as Saniga (1977, 1989).

Saniga(1977) has developed a model for the joint economic design of $\bar{X}$ and R control charts. He presumes that the process can be in one of three states (note that in previous works researchers considered only two states), and there are two types of assignable causes that generate the shifts. In the first type process mean shifts but the process standard deviation remains the same, in the second type process standard deviation shifts but the process mean remains the same. The control design parameters are: sample size, number of units produced between successive samples, control limits on the $\bar{X}$ chart, and the upper control limit factor on the R chart. He reports solutions to 81 numerical examples. His results indicate that joint optimization of the $\bar{X}$ and R control charts yields less frequent sampling compared to the case where only the $\bar{X}$ control chart optimized.

Saniga (1989) considers the joint economic design of $\bar{X}$ and R charts. He develops a model where the economic-loss cost function is minimized subject to some constraints such that, there are a minimum value for the power and a maximum value for the Type I error probability and for the average time to signal an expected shift. This new formulation is called "Economic Statistical Design (ESD)". He claims that the economic statistical design avoids many of the disadvantages of heuristic, statistical, and economic designs. Advantages of ESD are listed as: improved assurance of long term product quality and maintenance; reduction of the variance of the distribution of the quality characteristic.

The literature we reviewed so far consider the existence of only one assignable cause, hence is called *single assignable cause models*. Standard assumption in the economic design of the control charts is the existence of single assignable cause, however there are also *multiple assignable cause models.* Duncan (1971), Tagaras and Lee (1988), Tagaras and Lee (1989) are among those who consider multiple assignable cause in the economical control chart design.

Duncan (1971) extends the single assignable cause model and considers control of a process when there exist multiple assignable causes. The state of the process is still defined as either in-control or out-of-control. He assumes that there are s assignable causes, and each assignable cause shifts the process mean by a certain amount. Shift times due to each assignable cause is independent and exponentially distributed. He considers two models. In the first model, he assumes that when the process shifts to the out-of-control state due to one of the assignable causes, another type of assignable cause may not occur. In his second model, this assumption is relaxed. He assumes that the process is kept running until the assignable cause is actually discovered. Cost of repair and restoration is not included in the objective function. He develops the models and through a numerical study presents the sensitivity results. The results indicate that for the multiple assignable cause case, the effects of variations in the cost parameters on the optimal design are identical to those of the single cause model. Additionally, he shows that a reasonably good approximation to the multiple assignable cause model can be obtained from a single assignable cause model.

Tagaras and Lee (1988) deal with use of a control chart having multiple control limits defining multiple areas on the chart with different respective correction actions. At fixed intervals of time units, the process is observed by taking fixed amount of samples of the output quality measurement. When an alarm is raised indicating that the process mean has shifted, there are two possible levels of action. The first level corresponds to a minor adjustment of the process, and the second level calls for a major and usually more costly intervention. The classical single assignable cause control chart can be viewed as a special case of the $\bar{X}$-chart with two pairs of control limits. Expected cost per time unit is calculated as the ratio of expected cycle cost to expected cycle length. Then a two-step procedure is used for the optimization of this ratio. In the first step, for a given sample size, optimal values of sampling interval and control limits and the resulting expected cost per time unit is determined, and in the second step the sample size that minimizes expected cost per time unit is calculated. Experimental results of 126 numerical examples are given. They also present the results of a sensitivity analysis.

Results given in sensitivity analyses are: *(i)* As the rate of the assignable cause increases, that is assignable cause is more likely to occur, sample size and sampling interval decreases, *(ii)* Optimal sample size is drastically reduced when shift in process mean is increased, *(iii)* An increase in expected profit result in increase in both sample size and sampling interval. They also provide a comparison of multiple assignable cause model with single assignable cause approximation. Results indicate that control chart with multiple control limits provide a significant improvement on a single state, single response approximation.

When multiple assignable causes require different restoration procedures and the search for the assignable cause in effect is very expensive, control charts with multiple control limits may be preferred. A simplified scheme for approximate economic design of control charts with multiple control limits is proposed in this research. In the semieconomic design probability of true alarm is considered to be given as data, so that number of variables to be optimized reduces, from four(in two control limits) to two. This reduction in the number of variables is due to defining the control limits in terms of the true alarm probability. Expected cost and expected cycle time equations are modified and again two step optimization technique has been used.

Proposed method is tested with 126 numerical examples. Results show that proposed approximate method results in solutions that are very close to the true optima and can be obtained with minimal computational effort.

Tagaras and Lee (1989) propose a simplified procedure for the approximate economic design of control charts with multiple control limits, since finding the optimal control parameter values using the exact cost function derived in Tagaras and Lee (1988) is very complex. The process they consider has three states: one in-control and two out-of-control states (one indicating a minor problem and the other indicating a major problem). There are different control limits associated with each of the out-of-control states. Shift times to the each of the out-of-control states are independent and exponentially distributed. They assume that if the state of the process is correctly identified when an alarm raised, the process can be restored to the in-control state, hence it restarts afresh. However if the control chart indicates that process is in the out-of-control state associated with the minor problem, although there is a major problem, restoration is not possible hence the process maintains its state in restart. They propose a semieconomic approach for the approximation where the true alarm probabilities associated with each of the out-of-control state are given (preset). In this case control limits can be computed from the true alarm probabilities and the number of control parameters reduces to two, sample size and sampling interval. They proceed with making more assumptions since the cost rate function is still very complicated. They present the results of a numerical study in which they compare the cost rate obtained from the approximation with the optimal cost rate. Numerical study is performed over 126 examples. True alarm probabilities are set to 0.8 for the inner out-of-control state and 0.95 for the outer out-of-control state. They conclude that the approximations they provide performs well.

Tagaras (1989) proposes an approximate method for the optimal economic design of process control charts. He provides a log-power approximation for both single assignable cause and multiple assignable cause cases. In his approximation rather then working with preset value for the power of the control charts, he predict the power from the model parameters by using the log-power approximation.

In the literature of the economic design of the quality control charts, assignable cause occurrences are assumed to be follow a Poisson process. Hence, the time between assignable cause occurrences are distributed exponentially. Validity of this assumption is discussed by several authors. We have provided above the argument by Lorenzen and Vance (1986) about the validity of this assumption. The motivation of this assumption is that, due to extensive burn-in tests, beyond some initial age failure function of the machinery is relatively flat. Tagaras and Lee (1988) also discuss that this assumption is also true for the case where equipment failures result from the failure of any of its components and the number of components are fairly large. However, the constant failure rate assumption is not valid when, for example, the assignable causes are due to tool wear and there is the predictability of the assignable causes, mostly for the mechanical system rather then electronic. There are models in the literature assuming non-exponential assignable causes. A comprehensive review of these models can be found in Ho and Case(1994) and Tagaras(1998). One of the pioneering work in relaxing the exponential assumption is by Banerjee and Rahim (1988). They propose a model for the economic design of the $\bar{X}$-control charts where the shift occurrence times follow a Weibull distribution instead of exponential. They allow the sampling intervals vary with time, contrary to the fixed sampling interval assumption of the previous work. They assume that the sampling and plotting times are negligible and production is stopped during the search for the assignable cause and restoration. They perform a numerical study for the implementation of the search algorithm and the sensitivity analysis.

In the last decade, there are studies for the economical design of the control charts for the finite horizon problems. All of the studies and models discussed above, assumes infinite horizon problems. However, one can intuitively state that the optimal control policy and parameters of the control charts would be different if the horizon is finite. Crowder (1992) discusses the economic design of control charts for the short production runs (or finite horizon problem). He derives the model for the finite horizon problem and shows that the control strategy depends on the length of the production run. He also shows that treating a short-run problem incorrectly as an infinite-run problem can significantly increase the expected costs associated with the control strategy. In his model there is only one decision variable, which is the control limits. In his model he allows the control limits change in every time period. He shows that control limits increases as the end of the finite production horizon approaches.

Tagaras (1994) proposes a dynamic programming approach to the economic design of $\bar{X}$-control charts. He concentrates on the economics of process monitoring in finite production runs. He uses the expected value of total process control related costs incurred during a production run of specified, finite length, as the performance criterion. He presents results of 24 numerical examples. The average expected total cost improvement with regard to the optimal static chart is 14.5% in these examples, and in many cases savings are over 20%.

Del Castillo and Montgomery (1996) consider the design of $\bar{X}$ control charts for finite-horizon production runs. They also consider the case of imperfect setups. They assume that there is a probability of having a perfect setup at the beginning of each cycle, and this probability is constant throughout time. They compare their model with the model in Duncan (1956) and the model in Ladany (1973). They also present the results of a numerical study.

Another recent field of study in the economic design of the quality control chart literature is combining the maintenance policies and availability of the maintenance capabilities with the control charts. Preventive and opportunistic maintenance policies may provide a major improvement on the control chart designs. Preventive maintenance is performing proactive maintenance in order to prevent system problems. Opportunistic maintenance is the preventive maintenance performed at opportunities, either by choice or based on the physical condition of the system. Lee and Rosenblatt (1988) consider the costs of different policies for providing detection and restoration capabilities. Four monitoring policies depending on the availability of the detection and restoration capabilities are considered in the paper: *Policy 1:* "continuous detection capability" and " at inspection available restoration capability", *Policy 2:* "periodic detection capability" and "at inspection available restoration capability", *Policy 3:* "continuous detection capability" and "periodically available restoration capability", *Policy 4:* "periodic detection capability" and "periodically available restoration capability". They make the standard assumptions listed in Montgomery (1980). They derive the cost models and the optimal cost for each of the four policies. They provide a comparative analysis of the different monitoring policies, focusing on the impact of *the rate of the shift of the production process* and *the cost of operating in out-of-control state* on the choice of the policies. They illustrate their propositions with a numerical example. The analysis of the monitoring strategies yields following conclusions:

1. When the system is very reliable Policy 4 may be dominant

2. When the process become less reliable, the cost of defective items forces

tighter control of the process, hence continuous inspection is favorable and Policy 3 dominates

3. For very unreliable processes, due to the frequent need, availability of the restoration capabilities becomes more important, hence Policy 1 dominates

4. For the low cost of operating in out-of-control state, continuous monitoring and inspecting the process is not preferable, hence Policy 3 or 4 dominates.

5. For higher cost of out-of-control operation, it is necessary to provide both continuous inspection as well as restoration capabilities, thus Policy 1 dominates.

Ben-Daya and Rahim (2000) provide a study to incorporate the effects of maintenance on quality control charts. They develop a model which allows jointly optimizing the quality control charts and preventive maintenance level. They assume that when a preventive maintenance activity is conducted on the process it reduces the failure rates but not to the level of a fresh process. They model the above described process for increasing failure rate of the shifts. They assume that the preventive maintenance and the sampling are simultaneous. They provide an example which shows that as the preventive maintenance level gets higher, the quality control costs reduce. They propose to increase the preventive maintenance level up to the level in which the savings compensate the added maintenance cost.

Some of the other recent works in the area of study are as follows:

Del Castillo et al. (1996) study multiple-criteria optimal design of $\bar{X}$ control charts. They provide a model without explicitly considering the costs of false alarms and running in the out-of-control state. They formulate the problem as a nonlinear, constrained, multiple-objective programming model. There are three objective functions to be minimized in their model: (1) expected number of false alarms, (2) average time to signal, (3) sampling cost per cycle, and two constraints which limit the probability of Type I and Type II errors. They show that, using their model, control chart designs can be obtained without explicit estimation of the quality related costs, namely cost of operating in the out-of-control state, cost of incurring and investigating a false alarm, and cost of finding an assignable cause, which the single objective economic design models rely on. They also provide a practical illustration through an example.

Costa and Rahim (2001) develop a model for the economic design of $\bar{X}$ charts in which they allow the control parameters, y, k, and h, vary between their minimum and maximum values. They assume Poisson arrivals of the process shifts. They divide the control chart into three regions: the central region, the warning region, and the action region. If a sample points falls into the warning region then the control is tightened in the next sampling by reducing the sampling interval and control limits to their minimum values and increasing the sample size to its maximum value. If, however, the sample point is in the central region then the control is relaxed for the next sampling, by increasing the sampling interval and control limits to their maximum values and decreasing the number of samples taken to its minimum value. Through a numerical analysis they show that variable parameters design is more economical compared to the static parameters design.

Independent from the design of quality control charts, opportunity based age-replacement models have also been studied in the literature. Dekker and Dijkstra (1992) consider the problem where preventive replacements are allowed only at opportunities. Opportunities are due to the failure of the other components in series configuration to the component in consideration. They assume that opportunities arise according to a Poisson process. They use the renewal reward theorem to derive the long-term average cost expression. Since the opportunities occurrences are according to a Poisson process, due to the memoryless property, the renewal cycles end either with a failure or with an opportunity in their model. They derive the optimality equation.

In the first and second part of our research, we focus is on the *variable control charts,* specifically on the economical design of $\bar{X}-$control charts. All of the work, to our knowledge, in this area are tailored for the single machine environment. We consider quality control chart design for the multiple machine environment. Design of QC-charts in the multi-machine environment may have major implications. As alluded to above, in a production line operated with the JIT management philosophy, whenever a machine is stopped for an inspection and/or repair, the whole line is stopped and thereby the production ceases. Each such stoppage results in a profit loss due to the downtime. In previous works on economic design of QC-charts, the negative impact (i.e., downtime cost) has been considered as an explicit cost absorbed into the so-called alarm costs. However, each stoppage of the line also has a positive impact (i.e.,presents an *opportunity*) to inspect / repair the machines that have not triggered the stoppage. This potential positive impact has not been studied analytically before, despite numerous anecdotal evidence pointing to the common practice in industry. When the number of the machines in a line becomes larger, the frequency of the line stoppages and the number of opportunistic inspection/repair instances increases. Hence, one would expect the benefits of the opportunistic inspections/repair to get larger in production processes with considerably large number of workstations. We are not aware of any effort for combining the opportunistic inspection/repair (maintenance) with the quality control chart design that we consider herein.

Chapter 3

Economic Design of $\bar{\text{X}}$ Control Charts With Exogenous Opportunistic Inspection/Repairs

In this chapter, we will consider a single machine subject to exogenous stoppages for opportunistic inspections. We elaborate the economic design of quality control charts that we introduced in the previous chapter. We provide the cycle definition in the presence of opportunistic inspections, and derive the probability functions for each cycle type. Next, we provide the expressions for the expected cycle cost and expected operating time, and introduce the objective function. Finally, we show that the classical economic design of QC chart problem is a special case of the model we develop herein.

3.1 Preliminaries

A production process is intended to generate products with certain specifications. We consider a production process with a single assignable cause and single control limits. Hence, the production process can be characterized by a single in-control status (1) and a single out-of-control status (0). For convenience, we suppose that the process is performed on a single machine so that we can use the terms process and machine interchangeably. It is assumed that, at the start of a production run after the last intervention, the production process is in status 1, producing items of acceptable quality when operational. After some time in production, the production process may *shift* to status 0. From that point on, items of unacceptable quality are produced until an intervention occurs and the process is stopped. We assume that the transition from the in-control status to the out-of-control status is instantaneous, i.e. through arrival of an exogenous shock.

There is a single quality characteristic, X, by which the process is evaluated and controlled. Without loss of generality, we assume that X is continuous and that it is always distributed normally with the mean depending on the status of the process. This assumption relies on the well-known central limit theorem. When the process is in the in-control status, X has the mean μ_0 and the standard deviation σ; in the out-of-control status, the process mean experiences a shift of magnitude $\mp\delta\sigma$, where δ is known and positive. We assume that the standard deviation of the process does not change in the out-of-control status. More explicitly; when in the in-control status: $X \sim N(\mu_0, \sigma)$, and when in the out-of-control status: $X \sim N(\mu_0 \pm \delta\sigma, \sigma)$. Tagaras and Lee (1988) have shown that ignoring the variability of the process variance has almost no effect on the economic design of control charts. Lorenzen and Vance (1986) discuss the validity of the single assignable cause and a shift by a known amount. They rest their case on the studies and conclusions of Duncan (1971), who have studied many assignable cause cases. In summary, they agree that single assignable cause model with a weighted average time to out-of-control state closely approximates the multiple assignable cause model.

Occurrence of the single assignable cause constitutes the shift in the process mean. The assignable cause is assumed to be non-observable so that inference about the status of the process can only be drawn indirectly through observation of a sample statistic of X. The elapsed time for the process to be in the in-control state, before a shift occurs, is assumed to be distributed exponentially with mean $1/\lambda$. Lorenzen and Vance (1986) also provide arguments for the validity of the exponential shift times assumption and discuss that when the assignable causes are not random events, events such as tool wear which exhibit certain degree of predictability, this assumption would not be valid. The exponential assumption for the shift of the production process has been widely used in the quality control literature (see Duncan, 1956; Goel and Wu, 1973; Chiu ,1975, 1975b, 1976; Gibra, 1971; and Lorenzen and Vance, 1986) and is empirically supported (Davis, 1952; and Eppstein, 1958). The primary justification of such an assumption is that production equipment typically have flat failure rates due to extensive initial burn-in tests. For further discussion of the plausibility of exponential shift times, see Drenick (1960), and Rosenblatt and Lee (1986).

The production process we consider is not self-correcting; that is, after a shift occurs, the process cannot improve by itself and remains in the out-of-control status until it is discovered by some detection mechanism and corrected by restorative action. Defective items, if produced, are eventually discarded at a prespecified cost. Although considered in isolation, the production process at hand is assumed to constitute a part of a bigger system operated under the principles of *jidoka* so that it undergoes forced (system-wide) shutdowns originating from the rest of the system. The exogenous shutdowns with a fixed duration of L_O are assumed to arrive according to a Poisson process with mean μ.

Before we proceed with the proposed quality control policy, a few remarks are in order regarding the memoryless nature of the exogenously forced shutdowns. In a complex production system where *jidoka* is employed, there will also be some system-wide forced shutdowns, which originate from the other machines in the system and arise from the alarms signalled on those machines. When there are a large number of machines in the system and/or when the sampling instances are different, due to, for example, different reliability and cost parameters of the machines, it will appear to a particular machine that the shutdowns come randomly. An assembly line typically consists of tens of workstations working in tandem. When autonomation is employed on such a line, population from which the stoppages come is very large. Assuming that the sampling intervals are different, as they would be in general for nonidentical machines as workstations, it is reasonable to assume that there is a positive probability that a shutdown signal may be issued in a small time increment. Given the exponential nature of shift occurrences and the large number of machines involved in the population, it is again reasonable to assume that the stoppage probability over a time increment is stationary.

In this setting, we propose that the process is operated under what we call the *jidoka process control (JPC)*, which is a combination of the standard operation under the surveillance of control charts and randomly occurring opportunity-based inspections.

The basic control mechanism of the proposed *JPC* may be outlined as follows: A sample of a certain size is taken from the process at prespecified intervals. The sample units are analyzed and measured; the sample statistic is computed and plotted on a control chart with prespecified control limits. The value of the sample statistic determines whether or not an adjustment or restoration of the process is called for, depending on the position of the sample statistics in the control chart: If it is outside the control limits, an inspection of the process or a search for the assignable cause is conducted. If the process is indeed in the out-of-control state, the signal is said to result in a *true alarm* followed by a complete restoration of the process to the in-control status; otherwise, the signal results in a *false alarm* which requires no adjustment or restoration. So far, it is supposed that the process stops itself; this is the standard SPC scheme. Under the proposed JPC policy, the process at hand also acts as an *opportunity-taker*. That is, if the process faces an exogenous shutdown, its operator uses this stoppage as an *opportunity* to carry out an inspection of the process although no signals have been received from the control chart to initiate one. Upon inspection, if the process is found to be in the out-of-control status, the opportunity is said to be a *true opportunity* which is followed by a complete restoration of the process to the in-control state; otherwise, the opportunity is a *false opportunity* which requires no adjustment. Thus, under *JPC*, the process stops either by itself via an alarm arising from the inferring procedure or by an exogenous opportunity generated by a system-wide shutdown. Assuming perfect repair/restoration after each stoppage, the process restarts in the in-control status.

The instances at which the process restarts are regeneration points, since, at each restart, the process is in the in-control status, and occurrences of shifts and opportunities are memoryless processes. Therefore, define a regenerative cycle as the time between two consecutive process restarts. Clearly, a cycle is composed of two components: operating time, denoted by τ and down time, denoted by L. Then, we can describe a cycle in terms of how the process is stopped (s), the time of the process shift (x), and the arrival time of an opportunity (z). (Note that, for any cycle realization, x and z can easily be related to the number of samples taken before the shift has occurred (n_1) and the number of samples taken after the shift (n_2) in that cycle, and vice versa.) We identify four cycle classes $s \in$

$S = \{T, F, OT, OF\}$ where, T (as in True alarm) denotes the class of cycles in which an alarm triggers the process stoppage and the process is in the out-of-control status at the time of stoppage; F (as in False alarm) denotes the class of cycles in which an alarm triggers the process stoppage and the process is in the in-control status at the time of stoppage; OT (as in True Opportunity) denotes the class of cycles in which an opportunity triggers the process stoppage and the process is in the out-of-control status at the time of stoppage; and, finally, OF (as in False Opportunity) denotes the class of cycles in which an opportunity triggers the process stoppage and the process is in the in-control status at the time of stoppage. Each class s will have only certain permissible values for x and z (and, thereby, for n_1 and n_2), as we shall discuss shortly.

We assume that the entire process of analysis for a sample (*i.e.*, measurement of the sample units, computing and plotting on the chart of the sample statistic) takes negligible time, whereas, both the search for the assignable cause and the possible restoration of the process necessitate the stoppage of a machine and take non-negligible time. The durations of the search and restoration activities depend on the status of the process. The inferring is done following a three-parameter (y, h, k) policy such that a sample of size y is taken every h time units while the process is operational, and the sample statistic (*i.e.*, sample mean, $\bar{x}$) is checked against the control limits specified as $\pm k\sigma/\sqrt{y}$, all three parameters are non-negative. Then the control limits are:

$$\begin{aligned} \text{Upper Control Limit (UCL)} &= \mu_0 + k\sigma/\sqrt{y} \\ \text{Lower Control Limit (LCL)} &= \mu_0 - k\sigma/\sqrt{y} \end{aligned}$$

There are two types of errors associated with the control process, Type I and Type II. Type I error is denoted by α and is the inference of out-of-control status although the process is in the in-control status in fact. Type I error is illustrated in Figure 3-1.

$$\begin{aligned} \alpha &= P\{\textit{out-of-control signal} \mid \textit{process in-control}\} \\ &= P(X \geq UCL) + P(X \leq LCL) \\ &= P(X \geq \mu_0 + k\sigma/\sqrt{y}) + P(X \leq \mu_0 - k\sigma/\sqrt{y}) \\ &= P(Z \geq k) + P(Z \leq -k) \\ &= 1 - \Phi(k) + \Phi(-k) \\ &= 2\Phi(-k) \qquad (3.1) \end{aligned}$$

We will denote the Type II error by β, and it is the inference that the process is in the in-control status although it is in the out-of-control status. Figure 3-2 depicts Type II error.

$$
\begin{aligned}
\beta &= P\{\textit{in-control signal} \mid \textit{process out–of-control}\} \\
&= P(LCL \leq Y \leq UCL) \\
&= P(Y \leq \mu_0 + k\sigma/\sqrt{y}) - P(Y \leq \mu_0 - k\sigma/\sqrt{y}) \\
&= P\left(Z \leq \frac{(\mu_0 + \frac{k\sigma}{\sqrt{y}}) - (\mu_0 + \delta\sigma)}{\frac{\sigma}{\sqrt{y}}}\right) - P\left(Z \leq \frac{\left(\mu_0 - \frac{k\sigma}{\sqrt{y}}\right) - (\mu_0 + \delta\sigma)}{\frac{\sigma}{\sqrt{y}}}\right) \\
&= \Phi(k - \delta\sqrt{y}) - \Phi(-k - \delta\sqrt{y}) \qquad (3.2)
\end{aligned}
$$

Then the probability that the assignable cause will be detected when it has occurred, or the power of the test is:

$$
\begin{aligned}
(1-\beta) &= 1 - [\Phi(k - \delta\sqrt{y}) - \Phi(-k - \delta\sqrt{y})] \\
&= \Phi(\delta\sqrt{y} - k) + \Phi(-k - \delta\sqrt{y}) \qquad (3.3)
\end{aligned}
$$

We consider the following categories of costs: (i) the costs of sampling and testing, (ii) the costs associated with the production of defective items, and (iii) the costs associated with the investigation and correction of the assignable cause of variation. The cost of sampling and testing is assumed to consist of both fixed and variable components, denoted by u and b, respectively, such that the total cost of sampling and testing for each sample $u + by$. The cost associated with operating in the out-of-control status is taken as a per time unit of the process operating in the out-of-control status, due to, for example, substandard outputs. The costs of investigating an assignable cause and possibly restoring the process to the in-control status consist of out-of-pocket repair or replacement costs due to, for example, scrapped components and destructive inspection, and opportunity costs of foregone profit due to downtime of the machine. For cycle class s ($\in S$), R_s is the out-of-pocket component and the opportunity cost component is computed as πL_s, where π is the profit (lost) per unit of time and L_s denotes the downtime of the process.

In the economic design of the control chart we consider, the objective is to determine the control parameters (y, h, k) which minimize the expected cost per unit produced, $E[TC]$. We assume that any defective item out of the process will be corrected through some rework operations. To this end, we shall use the renewal theoretic approach so that the objective is to minimize the ratio of the expected cycle cost, $E[CC]$, and the expected operating time in a cycle, $E[\tau]$, where a cycle and operating time are as defined before.

Renewal reward process can be summarized as: *long-run average reward is just the expected reward earned during a cycle divided by the expected length of a cycle.* We will state renewal reward theorem once again, proof of which is available in Ross (1993) (pp.318).

Renewal Reward Theorem:

Consider a renewal process $\{N(t), t \geq 0\}$ having inter-arrival times $X_n, n \geq 1$, and each time a renewal occurs we receive a reward R_n. If $R(t)$ represents the total reward earned by time t:

$$R(t) = \sum_{n=1}^{N(t)} R_n$$

Let

$$E[R] = E[R_n], \qquad E[X] = E[X_n]$$

If $E[R] < \infty$ and $E[X] < \infty$, then with probability 1

$$\frac{R(t)}{t} \to \frac{E[R]}{E[X]} \qquad \text{as } t \to \infty$$

3.2 Derivation of the Cost Rate Expression

Let $f(\tau, s, n_1, n_2, x, z)$ denote the joint probability function of operating time, τ, of a cycle of class s, in which n_1 samples have been taken in the in-control state, n_2 samples have been taken in the out–of-control state, the time of the process shift is x and, the time of the opportunity arrival is z. Note that $f(\tau, s, n_1, n_2, x, z)$ is a probability function of a mixed nature, consisting of point mass and density components for certain values of the joint random variable. Define $P_s(\mu)$ as the probability that a cycle ends in cycle class $s \in \{T, F, OT, OF\}$ for a given value of μ. Define also $\Omega(s)$ for $s \in \{T, F, OT, OF\}$ as the set of values that the sextuple

$(\tau, s, n_1, n_2, x, z)$ can assume.

Then, the expected length of the operating time, $E[\tau_s]$, is given by:

$$\begin{aligned} E[\tau_s] &= E[\tau|\text{cycle ends in class } s]P_s(\mu) \\ &\quad \sum_{\{n_1,n_2\} \in \Omega(s)} \int_{\{\tau,s,x,z\} \in \Omega(s)} \tau \cdot f(\tau, s, n_1, n_2, x, z)dxdz \end{aligned} \tag{3.4}$$

Similarly, the expected cycle cost for a cycle of class s, $E[CC_s]$, is given by:

$$\begin{aligned} E[CC_s] &= E[CC|\text{cycle ends in class } s]P_s(\mu) \\ &\quad \sum_{\{n_1,n_2\} \in \Omega(s)} \int_{\{\tau,s,x,z\} \in \Omega(s)} C(\tau, s, n_1, n_2, x, z) \cdot f(\tau, s, n_1, n_2, x, z)dxdz \end{aligned} \tag{3.5}$$

A schematic representation of the evolution of a production process subject to the specified jidoka process control scheme is depicted in Figures 3-3 through 3-6. Next, to construct our objective function, we describe each cycle class and derive the corresponding operating characteristics.

3.2.1 *Case 1:* Cycle ends with a true alarm, $s = T$

In this cycle class, the process is stopped by a signal from the control chart, and is found upon inspection to be in the out-of-control status. A self-stoppage implies that there have been no exogenous opportunity arrivals during this cycle; that is, $z > \tau_T$, where τ_s denotes the operating time of a cycle in class s. Furthermore, since the shift must have occurred before the process is stopped, some of the samples may have been taken before the shift has occurred and some afterwards. (For convenience, we make the customary assumption that the shift does not occur while a sample is being taken and, thereby, that all units in a sample come from the same status of the process.) Considering possible realizations for this class, we have $n_1 \geq 0$, $n_2 \geq 1$, $n_1 h < x < (n_1 + 1)h$, and $z > \tau_T$. The minimum value that n_1 can attain is 0, since process may shift to the out-of-control status before the first sample is taken, for n_2 however, at least one sample must be taken since the cycle ends with an alarm from the chart. Each one of n_1 samples provides accurate inference about the true state of the process whereas, by definition, each one of $n_2 - 1$ samples causes a Type II error (*i.e.*, causes the inference that the process is in the in-control status where, in fact, it is in the out-of-control state) and lets the shift go undetected. The last (${n_2}^{th}$) sample taken after the shift detects the shift and stops the process; that is, $\tau_T = (n_1 + n_2)h$. The downtime required for the inspection/search and restoration activities in this cycle class is L_T. Considering the possible cycle class realizations, we have:

$$\Omega(T) = \{\tau = (n_1+n_2)h, s = T, n_1 \geq 0, n_2 \geq 1, n_1 h < x < (n_1+1)h, z > (n_1+n_2)h\}$$

For the cycle class $s = T$, $f(\tau, s, n_1, n_2, x, z)$ is computed as described below,

$$f(\tau, T, n_1, n_2, x, z) = (1-\alpha)^{n_1}\beta^{(n_2-1)}(1-\beta)\mu\exp[-\mu z]\lambda\exp[-\lambda x] \quad (3.6)$$
$$\text{for } (\tau, s, n_1, n_2, x, z) \in \Omega(T)$$

Then,

$$\begin{aligned} P_T(\mu) &= \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\int_{x=n_1h}^{(n_1+1)h}\int_{z=(n_1+n_2)h}^{\infty}\mu\exp[-\mu z]\lambda\exp[-\lambda x](1-\alpha)^{n_1} \\ &\times \quad \beta^{(n_2-1)}(1-\beta)dzdx \\ &= \frac{e^{-\mu h}(1-\beta)\left(1-e^{-\lambda h}\right)}{\left[1-\beta e^{-\mu h}\right]\left[1-\left\{(1-\alpha)e^{-(\mu+\lambda)h}\right\}\right]} \end{aligned} \quad (3.7)$$

Since $\tau_T = (n_1+n_2)h$, then the expected operating time for this cycle class, $E[\tau_T]$, is given by

$$\begin{aligned} E[\tau_T] &= \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\int_{x=n_1h}^{(n_1+1)h}\int_{z=(n_1+n_2)h}^{\infty}[(n_1+n_2)h](1-\alpha)^{n_1} \\ &\times \quad \mu\exp[-\mu z]\lambda\exp[-\lambda x]\beta^{(n_2-1)}(1-\beta)dzdx \end{aligned} \quad (3.8)$$

Likewise, the expected cost for this cycle class, $E[CC_T]$, is given by

$$\begin{aligned} E[CC_T] &= \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\int_{x=n_1h}^{(n_1+1)h}\int_{z=(n_1+n_2)h}^{\infty}\{[(n_1+n_2)(u+by)] \\ &+ \quad a[(x-n_1h)+n_2h]+\pi L_T+R_T\} \\ &\times \quad \mu\exp[-\mu x]\lambda\exp[-\lambda x](1-\alpha)^{n_1}\beta^{(n_2-1)}(1-\beta)dzdx \end{aligned} \quad (3.9)$$

3.2.2 *Case 2:* Cycle ends with a false alarm, $s = F$

In this class, the cycle ends with a self-stoppage triggered by an out-of-control signal from the control chart, but the process is found upon inspection to be in the in-control status. As in class T, there are no exogenous opportunity arrivals during a cycle of this class. Hence, we have $\tau_F = n_1 h < x < z,\ \ n_1 \geq 1$, and $n_2 = 0$. Note that each one of $n_1 - 1$ samples provides accurate inference about the true state of the process but the last (*i.e.*, n_1^{th}) sample taken causes a Type I error (*i.e.*, causes the inference that the process is in the out-of-control status where, in fact, it is in the in-control status) and stops the process. Thus, $\tau_F = n_1 h$.

$$\Omega(F) = \{\tau = n_1 h, s = F,\ n_1 \geq 1, n_2 = 0, n_1 h < x < \infty, n_1 h < z < \infty\}$$

Then for $s = F$,

$$f(\tau, F, n_1, n_2, x, z) = \alpha(1-\alpha)^{(n_1-1)} \mu \exp[-\mu z] \lambda \exp[-\lambda x] \qquad (3.10)$$

$$\text{for } (\tau, s, n_1, n_2, x, z) \in \Omega(F)$$

For a given μ,

$$
\begin{aligned}
P_F(\mu) &= \sum_{n_1=1}^{\infty} \int_{x=n_1 h}^{\infty} \int_{z=n_1 h}^{\infty} \alpha (1-\alpha)^{n_1-1} \mu \exp[-\mu z] \\
&\times \quad \lambda \exp[-\lambda x] \, dz dx \\
&= \frac{\alpha . e^{-(\mu+\lambda)h}}{1 - \{(1-\alpha) e^{-(\mu+\lambda)h}\}}
\end{aligned} \tag{3.11}
$$

The expected operating time, $E[\tau_F]$, is given by

$$
E[\tau_F] = \sum_{n_1=1}^{\infty} \int_{x=n_1 h}^{\infty} \int_{z=x}^{\infty} [n_1 h] \; \mu \exp[-\mu z] \lambda \exp[-\lambda x] \alpha (1-\alpha)^{(n_1-1)} dz dx \tag{3.12}
$$

Likewise, the expected cost for this cycle class, $E[CC_F]$, is given by

$$
\begin{aligned}
E[CC_F] &= \sum_{n_1=1}^{\infty} \int_{x=n_1 h}^{\infty} \int_{z=x}^{\infty} \{ ([n_1(u+by)] + \pi L_F + R_F) \\
&\times \quad \mu \exp[-\mu x] \lambda \exp[-\lambda x] \alpha (1-\alpha)^{(n_1-1)} dz dx \}
\end{aligned} \tag{3.13}
$$

The above two types of cycles are the cycle types that are also encountered in the classical *SPC* environment. Next, we look at the other two types which are unique to the *JPC* setting.

3.2.3 *Case 3:* Cycle ends with a true opportunity, $s = OT$

In this cycle class, the machine stoppage is triggered by an opportunity arrival and, upon inspection, the process is found to be in the out-of-control status. Unlike a self-stoppage, when the machine is stopped by an opportunity arrival, the stoppage instance *does not* coincide with the sampling instance due to the continuous nature of the inter-arrival times and the machine remains in operation for some additional time after the last sample has been taken. Hence, $\tau_{OT} = z$. Considering possible realizations for this class, we have either (i) $n_1 \geq 0$, $n_2 = 0$, $n_1 h < x < z < (n_1 + 1)h$, or (ii) $n_1 \geq 0$, $n_2 > 0$, $n_1 h < x < (n_1 + 1)h$, $(n_1 + n_2)h < z < (n_1 + n_2 + 1)h$. Each one of n_1 samples accurately infers the in-control status of the process, and each one of n_2 samples (for $n_2 > 0$) causes a Type II error.

$$
\begin{aligned}
\Omega(OT) = \quad & \{\{\tau = z, s = OT, n_1 \geq 0, n_2 = 0, n_1 h < x < z < (n_1 + 1)h\}; \\
& \{\tau = z, s = OT, n_1 \geq 0, n_2 > 0, n_1 h < x < (n_1 + 1)h, \\
& (n_1 + n_2)h < z < (n_1 + n_2 + 1)h\}\}
\end{aligned}
$$

Then for $s = OT$,

$$
f(\tau, OT, n_1, n_2, x, z) = (1-\alpha)^{n_1} \beta^{n_2} \mu \exp[-\mu z] \lambda \exp[-\lambda x] \tag{3.14}
$$

$$
\text{for } (\tau, s, n_1, n_2, x, z) \in \Omega(OT)
$$

Then for a given μ,

$$
\begin{aligned}
P_{OT}(\mu) &= \sum_{n_1=0}^{\infty}\int_{x=n_1h}^{(n_1+1)h}\int_{z=x}^{(n_1+1)h}\mu\exp[-\mu z]\lambda\exp[-\lambda x](1-\alpha)^{n_1}dzdx \\
&+ \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\int_{x=n_1h}^{(n_1+1)h}\int_{z=(n_1+n_2)h}^{(n_1+n_2+1)h}\mu\exp[-\mu z]\lambda\exp[-\lambda x] \\
&\times (1-\alpha)^{n_1}\beta^{n_2}dzdx \\
&= \frac{\left[\left(\beta.e^{-\mu h}\right)\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)\right]}{\left[1-\beta.e^{-\mu h}\right]\left[1-\left\{(1-\alpha)\,e^{-(\mu+\lambda)h}\right\}\right]} \\
&+ \frac{\left[\left(1-e^{-\lambda h}\right)-\frac{\lambda}{(\mu+\lambda)}\left(1-e^{-(\mu+\lambda)h}\right)\right]}{\left[1-\left\{(1-\alpha)\,e^{-(\mu+\lambda)h}\right\}\right]} \qquad (3.15)
\end{aligned}
$$

The expected operating time for this cycle class, $E[\tau_{OT}]$, is given by

$$
\begin{aligned}
E[\tau_{OT}] &= \sum_{n_1=0}^{\infty}\int_{x=n_1h}^{(n_1+1)h}\int_{z=x}^{(n_1+1)h}z\ \mu\exp[-\mu z]\lambda\exp[-\lambda x](1-\alpha)^{n_1}dzdx \\
&+ \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\int_{x=n_1h}^{(n_1+1)h}\int_{z=(n_1+n_2)h}^{(n_1+n_2+1)h}z\ \mu\exp[-\mu z]\lambda\exp[-\lambda x] \\
&\times (1-\alpha)^{n_1}\beta^{n_2}dzdx
\end{aligned}
$$

The expected cost for this cycle class, $E[CC_{OT}]$, is given by

$$\begin{aligned} E[CC_{OT}] &= \sum_{n_1=0}^{\infty} \int_{x=n_1h}^{(n_1+1)h} \int_{z=x}^{(n_1+1)h} \{[\,[n_1(u+by)] + a\,(z-x) \\ &+ \pi L_{OT} + R_{OT}](1-\alpha)^{n_1} \mu \exp[-\mu z] \lambda \exp[-\lambda x] dz dx\} \\ &+ \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{\infty} \int_{x=n_1h}^{(n_1+1)h} \int_{z=(n_1+n_2)h}^{(n_1+n_2+1)h} \{[a\,(z-x) \\ &+ [(n_1+n_2)(u+by)] + \pi L_{OT} + R_{OT}] \\ &\times \mu \exp[-\mu z] \lambda \exp[-\lambda x] (1-\alpha)^{n_1} \beta^{n_2} dz dx\} \end{aligned} \tag{3.16}$$

3.2.4 *Case 4:* Cycle ends with a false opportunity, $s = OF$

Finally, in this cycle class, the machine stoppage is triggered by an opportunity and the process is in the in-control status at the time of stoppage. Since the machine is stopped by an opportunity arrival, $\tau_{OF} = z$. Considering the possible realizations for this class, we have $n_1 (\geq 0)$ samples accurately inferring the in-control state of the process, and because there has been no shift, $n_2 = 0$. Furthermore, $n_1 h < z < (n_1+1)h < x$.

$$\Omega(OF) = \{\tau_{OF} = z, s = OF, n_1 \geq 0, n_2 = 0, n_1 h < z < x < \infty\}$$

Then for $s = OF$,

$$f(\tau, s, n_1, n_2, x, z) = \mu \exp[-\mu z] \lambda \exp[-\lambda x](1-\alpha)^{n_1} \tag{3.17}$$

$$\text{for } (\tau, s, n_1, n_2, x, z) \in \Omega(OF)$$

$$\begin{aligned} P_{OF}(\mu) &= \sum_{n_1=0}^{\infty} \int_{z=n_1 h}^{(n_1+1)h} \int_{x=z}^{\infty} \mu \exp[-\mu z] \lambda \exp[-\lambda x](1-\alpha)^{n_1} dz dx \\ &= \left[\frac{\mu}{\mu+\lambda}\right] \frac{1-e^{-(\mu+\lambda)h}}{[1-\{(1-\alpha)\, e^{-(\mu+\lambda)h}\}]} \end{aligned} \tag{3.18}$$

The expected operating time for this cycle class, $E(\tau_{OF})$, is given by

$$E(\tau_{OF}) = \sum_{n_1=0}^{\infty} \int_{z=n_1 h}^{(n_1+1)h} \int_{x=z}^{\infty} z \mu \exp[-\mu z] \lambda \exp[-\lambda x](1-\alpha)^{n_1} dz dx \tag{3.19}$$

The expected cost for this cycle class, $E[CC_{OF}]$, is given by

$$\begin{aligned} E(CC_{OF}) &= \sum_{n_1=0}^{\infty} \int_{z=n_1 h}^{(n_1+1)h} \int_{x=z}^{\infty} [n_1(u+by) + \pi L_{OF} + R_{OF}] \\ &\times \quad \mu \exp[-\mu z] \lambda \exp[-\lambda x](1-\alpha)^{n_1} dz dx \end{aligned} \tag{3.20}$$

In summary,

$$f(\tau, s, n_1, n_2, x, z) = \begin{cases} (1-\alpha)^{n_1}\beta^{(n_2-1)}(1-\beta)\mu\exp[-\mu z]\lambda\exp[-\lambda x] \\ \qquad\qquad\qquad \text{for } (\tau, s, n_1, n_2, x, z) \in \Omega(T) \\ (1-\alpha)^{(n_1-1)}\alpha\ \mu\exp[-\mu z]\lambda\exp[-\lambda x] \\ \qquad\qquad\qquad \text{for } (\tau, s, n_1, n_2, x, z) \in \Omega(F) \\ (1-\alpha)^{n_1}\beta^{n_2}\mu\exp[-\mu z]\lambda\exp[-\lambda x] \\ \qquad\qquad\qquad \text{for } (\tau, s, n_1, n_2, x, z) \in \Omega(OT) \\ \mu\exp[-\mu z]\lambda\exp[-\lambda x](1-\alpha)^{n_1} \\ \qquad\qquad\qquad \text{for } (\tau, s, n_1, n_2, x, z) \in \Omega(OF) \\ 0 \qquad\qquad\qquad \text{otherwise} \end{cases} \tag{3.21}$$

The cycle cost incurred for a cycle of class s, C_s, is

$$C(\tau, s, n_1, n_2, x, z) = \begin{cases} (n_1+n_2)(u+by) + a[(x-n_1h)+n_2h] + \pi L_T + R_T \\ \qquad\qquad\qquad \text{for } (\tau, s, n_1, n_2, x, z) \in \Omega(T) \\ n_1(u+by) + \pi L_F + R_F \\ \qquad\qquad\qquad \text{for } (\tau, s, n_1, n_2, x, z) \in \Omega(F) \\ (n_1+n_2)(u+by) + a\,(z-x) + \pi L_{OT} + R_{OT} \\ \qquad\qquad\qquad \text{for } (\tau, s, n_1, n_2, x, z) \in \Omega(OT) \\ n_1(u+by) + \pi L_{OF} + R_{OF} \\ \qquad\qquad\qquad \text{for } (\tau, s, n_1, n_2, x, z) \in \Omega(OF) \end{cases} \tag{3.22}$$

The individual activities involved in inspecting the machine to identify the assignable cause of variation and restoring the process to its in-control status are pre-specified, and, hence, their *actual* duration do not depend on whether the machine was stopped by itself or by an opportunity. However, the *effective* durations of those activities will be different for each case. To see this, consider a self-stoppage and an opportunity stoppage. When the machine is stopped by an alarm, the effective durations of the search and possible restoration activities are their actual durations. When the machine is stopped by an opportunity, the process lies idle, by definition, for L_O time units during which the inspection and possible restoration activities are conducted. If L_O is shorter than the time for the required activities, then the effective duration of the performed activities is their actual duration. However, if L_O is longer than the time for the required activities, then the effective duration of the performed activities is the duration of the opportunity stoppage, L_O. Therefore, we have

$$L_s = \begin{cases} L_T & \text{if } s = T \\ L_F & \text{if } s = F \\ L_{OT} = \max\left[L_T, L_O\right] & \text{if } s = OT \\ L_{OF} = \max\left[L_F, L_O\right] & \text{if } s = OF \end{cases} \tag{3.23}$$

The *effective* out-of-pocket repair costs are such that $R_{OT} = R_T$ and $R_{OF} = R_F$ because the activities are prespecified.

3.3 Objective Function

We are now ready to construct the objective cost function. From the renewal reward theorem, the objective is given by:

$$\textbf{Minimize}_{y,h,k>0}\ E[TC] = \frac{E[CC]}{E[\tau]} = \frac{\sum_{s\in S} E[CC_s]}{\sum_{s\in S} E[\tau_s]} \tag{3.24}$$

The expected length of the operating time within a cycle is given by:

$$E[\tau] = \sum_{s\in S} E[\tau_s] \tag{3.25}$$

Carrying out the expectation, we get,

$$E[\tau_T] = \frac{(1-\beta)\left(1-e^{-\lambda h}\right)e^{-\mu h}h\left[1-\{(1-\zeta)(1-\gamma)\}\right]}{\gamma^2\zeta^2} \tag{3.26}$$

$$E[\tau_F] = \frac{\alpha h e^{-(\mu+\lambda)h}}{\zeta^2} \tag{3.27}$$

$$\begin{aligned} E[\tau_{OT}] \;=\; & \left[\frac{(1-\gamma)\left(1-e^{-\lambda h}\right)}{\gamma\zeta}\right]\left(h.\left(1-e^{-\mu h}\right)\left[\frac{(1-\zeta)}{\zeta}+\frac{1}{\gamma}\right]\right. \\ & \left. + \frac{1-e^{-\mu h}}{\mu} - h.e^{-\mu h}\right) \\ & + \frac{1}{\zeta}\left\{\frac{h.\mu.(1-\zeta)}{\zeta}.\left[\frac{1-e^{-\mu h}}{\mu} - \frac{1-e^{-(\mu+\lambda)h}}{\lambda+\mu}\right]\right. \\ & \left. + \left[\frac{\left(1-e^{-\mu h}(1+h\mu)\right)}{\mu} - \frac{\mu.\left(1-e^{-(\mu+\lambda)h}(1+h(\mu+\lambda))\right)}{(\mu+\lambda)^2}\right]\right\} \end{aligned} \tag{3.28}$$

$$
\begin{aligned}
E[\tau_{OF}] &= \frac{\mu}{(\mu+\lambda)\,\zeta}\left[\frac{h.\left(1-e^{-(\mu+\lambda)h}\right)(1-\zeta)}{\zeta}+\frac{1}{\mu+\lambda}\right. \\
&\qquad \left. -\, e^{-(\mu+\lambda)h}\left(h+\frac{1}{\mu+\lambda}\right)\right] \qquad (3.29)
\end{aligned}
$$

Similarly, the expected cycle cost is given by

$$
E[CC]=\sum_{s\in S}E[CC_s] \qquad (3.30)
$$

Carrying out the expectation, we obtain

$$
\begin{aligned}
E[CC_T] &= \frac{(1-\beta)\,e^{-\mu h}\left(1-e^{-\lambda h}\right)}{\zeta\gamma}\left\{\frac{(yb+u)\,(1-\zeta)}{\zeta}+\pi L_T+R_T\right. \\
&\qquad \left. -\,\frac{a}{\lambda}+\frac{a.h.e^{-\lambda h}}{(1-e^{-\lambda h})}+\frac{(yb+u+ah)}{\gamma}\right\} \qquad (3.31)
\end{aligned}
$$

$$
E[CC_F]=\frac{\alpha e^{-(\mu+\lambda)h}}{\zeta}\left[\pi L_F+R_F+\frac{(y.b+u)}{\zeta}\right] \qquad (3.32)
$$

$$
\begin{aligned}
E[CC_{OT}] \;=\; & \left\{ \frac{(1-\gamma)\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)}{\zeta\gamma} \right. \\
& \times \left[\frac{yb+u+ah}{\gamma} + \frac{(yb+u)(1-\zeta)}{\zeta} + \pi L_{OT} \right. \\
& \left. \left. + R_{OT} + \frac{a}{\mu} - \frac{a}{\lambda} \right] + \frac{ah\left(e^{-\lambda h}-e^{-\mu h}\right)(1-\gamma)}{\zeta\gamma} \right\} \\
& + \frac{\lambda(yb+u)(1-\zeta)}{\zeta^2} \left[\frac{1-e^{-(\mu+\lambda).h}}{\lambda+\mu} - \frac{\left(1-e^{-\lambda h}\right)e^{-\mu h}}{\lambda} \right] \\
& + \frac{\lambda}{\zeta} \left[a.e^{-\mu h} \left(\frac{1-e^{-\lambda h}}{\lambda^2} - \frac{h.e^{-\lambda h}}{\lambda} \right) \right. \\
& - \frac{\left(1-e^{-\lambda h}\right)e^{-\mu h}\left(\pi L_{OT}+R_{OT}+a.h+\frac{a}{\mu}\right)}{\lambda} \\
& \left. + \frac{\left(\pi L_{OT}+R_{OT}+\frac{a}{\mu}\right)\left(1-e^{-(\mu+\lambda)h}\right)}{\mu+\lambda} \right]
\end{aligned} \tag{3.33}
$$

$$
E[CC_{OF}] = \frac{\mu.\left(1-e^{-(\mu+\lambda).h}\right)}{(\lambda+\mu).\zeta} . \left[\pi L_{OF} + R_{OF} + \frac{(yb+u)(1-\zeta)}{\zeta} \right] \tag{3.34}
$$

where $\zeta = \left[1 - \left\{(1-\alpha)\, e^{-(\mu+\lambda)h}\right\}\right]$ and $\gamma = \left[1 - \beta.e^{-\mu h}\right]$.

The derivation of these expressions are presented in Appendix B and C.

3.4 A Special Case when $\mu \to 0$

We will show that $lim_{\mu\to 0}E\left[TC\right] = \frac{lim_{\mu\to 0}E[CC_T]+lim_{\mu\to 0}E[CC_F]}{lim_{\mu\to 0}E[\tau_T]+lim_{\mu\to 0}E[\tau_F]}$.

First consider the operating time of the opportunity true cycle type, i.e.

$\lim_{\mu\to 0} E[\tau_{OT}]$:

$$\begin{aligned}
E[\tau_{OT}] &= \left[\frac{(1-\gamma)\left(1-e^{-\lambda h}\right)}{\gamma\zeta}\right]\left(h\left(1-e^{-\mu h}\right)\left[\frac{(1-\zeta)}{\zeta}+\frac{1}{\gamma}\right]+\frac{1-e^{-\mu h}}{\mu}-he^{-\mu h}\right) \\
&\quad +\frac{1}{\zeta}\left\{\frac{h.\mu.(1-\zeta)}{\zeta}.\left[\frac{1-e^{-\mu h}}{\mu}-\frac{1-e^{-(\mu+\lambda)h}}{\lambda+\mu}\right]\right. \\
&\quad \left.+\left[\frac{\left(1-e^{-\mu h}(1+h\mu)\right)}{\mu}-\frac{\mu.\left(1-e^{-(\mu+\lambda)h}(1+h(\mu+\lambda))\right)}{(\mu+\lambda)^2}\right]\right\} \qquad (3.35)
\end{aligned}$$

When take the limit as $\mu \to 0$ in the equation above, we have,

$$\begin{aligned}
\lim_{\mu\to 0} E[\tau_{OT}] &= 0+\lim_{\mu\to 0}\left\{\left[\frac{(1-\gamma)\left(1-e^{-\lambda h}\right)}{\gamma\zeta}\right]\left(\frac{1-e^{-\mu h}}{\mu}-h.e^{-\mu h}\right)\right. \\
&\quad \left.+\frac{1}{\zeta}\left[\frac{\left(1-e^{-\mu h}(1+h\mu)\right)}{\mu}\right]\right\} \\
&= \lim_{\mu\to 0}\left\{\left[\frac{(1-\gamma)\left(1-e^{-\lambda h}\right)}{\gamma\zeta}\right]\left(\frac{1-e^{-\mu h}}{\mu}-h.e^{-\mu h}\right)\right. \\
&\quad \left.+\frac{1}{\zeta}\left[\frac{1-e^{-\mu h}}{\mu}-he^{-\mu h}\right]\right\} \qquad (3.36)
\end{aligned}$$

Applying L'Hopital and taking the derivative of numerator and denominator of expression $\frac{1-e^{-\mu h}}{\mu}$ with respect to μ we get $he^{-\mu h}$, and taking the limit yields $\lim_{\mu\to 0}\frac{1-e^{-\mu h}}{\mu}=h$. Hence $\lim_{\mu\to 0} E[\tau_{OT}]=0$

Next we have

$$\lim_{\mu\to 0} E[\tau_{OF}] = \lim_{\mu\to 0}\left\{\frac{\mu}{(\mu+\lambda)\zeta}\left[\frac{h.\left(1-e^{-(\mu+\lambda)h}\right)(1-\zeta)}{\zeta}+\frac{1}{\mu+\lambda}\right.\right.$$
$$\left.\left. - e^{-(\mu+\lambda)h}\left(h+\frac{1}{\mu+\lambda}\right)\right]\right\} = 0 \tag{3.37}$$

Consider $E[CC_{OT}]$

$$\lim_{\mu\to 0} E[CC_{OT}] = \lim_{\mu\to 0}\left\{\left\{\frac{(1-\gamma)\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)}{\zeta\gamma}\right.\right.$$
$$\times\left[\frac{yb+u+ah}{\gamma}+\frac{(yb+u)(1-\zeta)}{\zeta}+\pi L_{OT}\right.$$
$$\left.\left.+R_{OT}+\frac{a}{\mu}-\frac{a}{\lambda}\right]+\frac{ah\left(e^{-\lambda h}-e^{-\mu h}\right)(1-\gamma)}{\zeta\gamma}\right\}$$
$$+\frac{\lambda(yb+u)(1-\zeta)}{\zeta^2}\left[\frac{1-e^{-(\mu+\lambda).h}}{\lambda+\mu}-\frac{\left(1-e^{-\lambda h}\right)e^{-\mu h}}{\lambda}\right]$$
$$+\frac{\lambda}{\zeta}\left[a.e^{-\mu h}\left(\frac{1-e^{-\lambda h}}{\lambda^2}-\frac{h.e^{-\lambda h}}{\lambda}\right)\right.$$
$$-\frac{\left(1-e^{-\lambda h}\right)e^{-\mu h}\left(\pi L_{OT}+R_{OT}+a.h+\frac{a}{\mu}\right)}{\lambda}$$
$$\left.\left.+\frac{\left(\pi L_{OT}+R_{OT}+\frac{a}{\mu}\right)\left(1-e^{-(\mu+\lambda)h}\right)}{\mu+\lambda}\right]\right\} \tag{3.38}$$

$$
\begin{aligned}
= & \lim_{\mu \to 0} \left\{ \frac{(1-\gamma)\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)}{\zeta\gamma^2} (yb+u+ah) \right. \\
& + \frac{(1-\gamma)(1-\zeta)\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)}{\zeta^2\gamma} (yb+u) \\
& + \frac{(1-\gamma)\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)}{\zeta\gamma} (\pi L_{OT}+R_{OT}) \\
& + \frac{(1-\gamma)\left(1-e^{-\lambda h}\right)}{\zeta\gamma} a \left(\frac{1-e^{-\mu h}}{\mu} - he^{-\mu h} \right) \\
& - \frac{(1-\gamma)\left(1-e^{-\mu h}\right)}{\zeta\gamma} a \left(\frac{1-e^{-\lambda h}}{\lambda} - he^{-\lambda h} \right) \\
& + \frac{\lambda (yb+u)(1-\zeta)}{\zeta^2} \left[\frac{1-e^{-(\mu+\lambda).h}}{\lambda+\mu} - \frac{\left(1-e^{-\lambda h}\right)e^{-\mu h}}{\lambda} \right] \\
& + \frac{\lambda}{\zeta} \left[a.e^{-\mu h} \left(\frac{1-e^{-\lambda h}}{\lambda^2} - \frac{h.e^{-\lambda h}}{\lambda} \right) \right. \\
& - \frac{\left(1-e^{-\lambda h}\right)e^{-\mu h}\left(\pi L_{OT}+R_{OT}+a.h+\frac{a}{\mu}\right)}{\lambda} \\
& \left. \left. + \frac{\left(\pi L_{OT}+R_{OT}+\frac{a}{\mu}\right)\left(1-e^{-(\mu+\lambda)h}\right)}{\mu+\lambda} \right] \right\}
\end{aligned} \tag{3.39}
$$

then $\lim_{\mu \to 0}\left(1-e^{-\mu h}\right) = 0$ and $\lim_{\mu \to 0} \frac{1-e^{-\mu h}}{\mu} = h$. Hence,

$$
\begin{aligned}
= & \; 0 + \lim_{\mu \to 0} \left\{ \frac{\lambda (yb+u)(1-\zeta)}{\zeta^2} \left[\frac{1-e^{-(\mu+\lambda).h}}{\lambda+\mu} - \frac{\left(1-e^{-\lambda h}\right)e^{-\mu h}}{\lambda} \right] \right. \\
& + \frac{\lambda}{\zeta} \left[a.e^{-\mu h} \left(\frac{1-e^{-\lambda h}}{\lambda^2} - \frac{h.e^{-\lambda h}}{\lambda} \right) \right. \\
& - \frac{\left(1-e^{-\lambda h}\right)e^{-\mu h}\left(\pi L_{OT}+R_{OT}+a.h+\frac{a}{\mu}\right)}{\lambda} \\
& \left. \left. + \frac{\left(\pi L_{OT}+R_{OT}+\frac{a}{\mu}\right)\left(1-e^{-(\mu+\lambda)h}\right)}{\mu+\lambda} \right] \right\}
\end{aligned} \tag{3.40}
$$

$$
\begin{aligned}
E[CC_{OT}] \quad = \quad & \left\{ \frac{(1-\gamma)\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)}{\zeta\gamma}\frac{a}{\mu} + \frac{ah\left(e^{-\lambda h}-e^{-\mu h}\right)(1-\gamma)}{\zeta\gamma} \right\} \\
& + \frac{\lambda}{\zeta}\left[a.e^{-\mu h}\left(\frac{1-e^{-\lambda h}}{\lambda^2} - \frac{h.e^{-\lambda h}}{\lambda}\right) \right. \\
& \left. - \frac{\left(1-e^{-\lambda h}\right)e^{-\mu h}\left(a.h+\frac{a}{\mu}\right)}{\lambda} + \frac{\left(\frac{a}{\mu}\right)\left(1-e^{-(\mu+\lambda)h}\right)}{\mu+\lambda}\right] = 0 \qquad (3.41)
\end{aligned}
$$

Consider $E[CC_{OF}]$

$$
\lim_{\mu\to 0} E[CC_{OF}] = \lim_{\mu\to 0}\left\{ \frac{\mu\left(1-e^{-(\mu+\lambda).h}\right)}{(\lambda+\mu)\,\zeta}\left[\pi L_{OF} + R_{OF} + \frac{(yb+u)(1-\zeta)}{\zeta}\right]\right\} = 0 \qquad (3.42)
$$

Similarly,

$$
\lim_{\mu\to 0} E[\tau_T] = \frac{(1-\beta)\left(1-e^{-\lambda h}\right)h\left[1-\{(1-\zeta)(1-\gamma)\}\right]}{\gamma^2\zeta^2} \qquad (3.43)
$$

$$
\lim_{\mu\to 0} E[\tau_F] = \lim_{\mu\to 0}\left\{\frac{\alpha e^{-(\mu+\lambda)h}h}{\zeta^2}\right\} = \frac{\alpha.e^{-\lambda h}h}{\zeta^2} \qquad (3.44)
$$

$$
\begin{aligned}
\lim_{\mu\to 0} E[CC_T] \quad = \quad & \lim_{\mu\to 0}\left\{\frac{(1-\beta)\,e^{-\mu h}\left(1-e^{-\lambda h}\right)}{\zeta\gamma}\left\{\frac{(yb+u)(1-\zeta)}{\zeta} + \pi L_T + R_T \right.\right. \\
& \left.\left. - \frac{a}{\lambda} + \frac{ahe^{-\lambda h}}{\left(1-e^{-\lambda h}\right)} + \frac{(yb+u+ah)}{\gamma}\right\}\right\} \\
= \quad & \frac{(1-\beta)\left(1-e^{-\lambda h}\right)}{\zeta\gamma}\left\{\frac{(yb+u)(1-\zeta)}{\zeta} + \pi L_T + R_T \right. \\
& \left. - \frac{a}{\lambda} + \frac{ahe^{-\lambda h}}{\left(1-e^{-\lambda h}\right)} + \frac{(yb+u+ah)}{\gamma}\right\} \qquad (3.45)
\end{aligned}
$$

$$\begin{aligned}\lim_{\mu \to 0} E[CC_F] &= \lim_{\mu \to 0} \left\{ \frac{\alpha e^{-(\mu+\lambda)h}}{\zeta} \left[\pi L_F + R_F + \frac{(yb+u)}{\zeta} \right] \right\} \\ &= \frac{\alpha e^{-\lambda h}}{\zeta} \left[\pi L_F + R_F + \frac{(y.b+u)}{\zeta} \right] \qquad (3.46)\end{aligned}$$

Thus, it can be verified that the classical setting of the QC chart design is a special case of our problem.

Chapter 4

Numerical Study: Single Machine Model

4.1 Introduction and Search Algorithm

We have implemented the single machine model in *Microsoft Visual C++ Version 6.0.* We have run the codes on a PC (*Pentium III* computer). Primarily, we have investigated the change in the cost rate with respect to the decision variables, i.e. y, k, h. For a preliminary experimental set, we have conducted an exhaustive search with respect to the decision variables, changing their values one at a time. The search results indicate that, the cost rate is unimodal for all the decision variables. Hence, there exists a global minimum of the cost rate function. Figures 4-1 and 4-2 are the contour plots of the cost rate function for two selected experiments and illustrate the unimodality of the function. Parameter set for these graphics is: $\pi = 500$, $L_T = L_F = 0.1$, $L_O = 0.1$, $a = 100$, $u = 5$, $b = 0.1$, $\lambda = 0.05$, $\mu = 0.25$ in both of them, but sample size, y is 1 in Figure 4-1 and 6 in Figure 4-2. The minimum cost rate for the first illustrative experiment is 18.22, and the minimizing values for k and h are 1.65 and 2.35, respectively. For the second illustrative experiment the minimum cost rate is 14.83, and the minimizing values for k and h are 2.9 and 1.8, respectively.

Goel, Jain, Wu (1968) show that cost rate function of the classical model, which is a special case of the model developed herein for $\mu \longrightarrow 0$, is unimodal. Similarly, we have observed numerically that the cost rate function of the model we develop is unimodal. Hence, we have implemented a search algorithm in order to find the optimal values for the policy parameters y, k, and h. Among the policy parameters y is integer where as k and h are real numbers. In our algorithm, for a wide range of parameter values, we have used the *golden section search* (GSS) method for determining the optimum values of k and h, and exhaustive search over y. The search domain for k, h and y are $[0.001, 50]$, $[0.001, 50]$ and $[1, 50]$ respectively.

GSS is a method of locating a minimum (see Bazaraa et al. 1993, pp. 270). It involves evaluating the function at some point x in the larger of the two intervals (a, b) or (b, c). If $f(x) < f(b)$ then x replaces the midpoint b, and b becomes an end point. If $f(x) > f(b)$ then b remains the midpoint with x replacing one of the end points. Either way the width of the bracketing interval will reduce and the position of the minima will be better defined (See Figure 4-3). The procedure is then repeated until the width achieves a desired tolerance. The new test point, x, is chosen to be a fraction 0.38197 from one end, and 0.61803 from the other end (these fractions are called *golden section*), then the width of the full interval will reduce at an optimal rate. The Golden Section Search requires no information about the derivative of the function.

A brief pseudo code for the implemented algorithm is presented below. (An expended version is in Appendix D) We will call this algorithm $OPTIMIZE\,()$.

Algorithm- $OPTIMIZE\,(y_{\min}, y_{\max}, h_{\min}, h_{\max}, k_{\min}, k_{\max})$

for $y = y_{\min} \rightarrow y_{\max}\{$

set $k = k_{\min}$

while $(TCR_{old} - TCR > \epsilon)\{$

SEARCH over h in the interval $[h_{\min}, h_{\max}]$ to find $\hat{h}_{\min}$ and $\hat{h}_{\max}$

set $h = (\hat{h}_{\min} + \hat{h}_{\max})/2$

SEARCH over k in the interval $[k_{\min}, k_{\max}]$ to find $\hat{k}_{\min}$ and $\hat{k}_{\max}$

set $k = (\hat{k}_{\min} + \hat{k}_{\max})/2$

compute α, β, and CR

$TCR_{old} = TCR$

$TCR = E\,[TC]\}$

if $(TCR \leq TCR_{opt})\{$

$TCR_{opt} = TCR$

$h_{opt} = h$

$k_{opt} = k$

$y_{opt} = y\}\}$

4.2 Design of the Numerical Study and the Data Set

In our numerical study for the single machine setting, we have carried out a number investigations. *(i)* In our first set of experiments, we have conducted a sensitivity analysis in order to provide some insight on how the control parameters and the total cost rate changes when we vary the cost parameters and the opportunity rate. Our observations in the sensitivity analysis are in line with the results in the literature. *(ii)* In the second set of experiments we have analyzed the cost breakdown structure of the optimum cost rate. We have investigated how the cost components change with respect to the opportunity arrival rate, tried to identify if any particular cost component is more reactive to the opportunities. *(iii)* In the final set of experiments we have provided a comparison of the cost rate of a machine which utilizes the exogenous shut downs with that of a machine which doesn't utilize the exogenous shut downs for investigation and repair of the process.

The parameter set we used in the experiments is presented in Table 4.1. For the opportunity rate we used multiples of the shift rate, such that $\mu \in \{0,\ 0.5\lambda,\ \lambda,\ 1.5\lambda,\ 2\lambda,\ 5\lambda,\ 10\lambda\}$. Our data set comprises three different values for

a, b, u, L_F, L_T, L_o. Overall, we have generated 5103 different experiment instances for our numerical study.

In the reports below optimum values of the control parameters are denoted by y^*, h^* and k^*.

4.3 Sensitivity Analyses

In order to provide some useful managerial insights and to observe the behavior of the control parameters to the changes in the model parameters we have conducted a series of experiments. In these experiments we have changed the value of one of the parameters at a time while keeping all the others fixed. Some sample results and our observations are given below.

4.3.1 Sensitivity with respect to cost parameters

We observe that, as the fixed component of the sampling cost, u increases, optimal cost rate increases, the percentage savings in the cost rate fits in a concave curve, where the savings have an increasing trend for smaller values of L_o and decreasing trend for the L_o values closer to L_t (or L_f); control limit coefficient, k^*, does not have consistent behavior with respect to u; the sample size, y^* tends to increase while u increases, however, the rate of increase reduces when u gets larger; sampling interval, h^*, increases. Since the sampling becomes more costly, taking more samples but less less frequently is more desirable. A set of examples is depicted in Table 4.2.

As the per unit sampling cost, b increases, optimal cost rate and sampling interval increases;.both the sample size and the control limits coefficient decrease. Since, an increase in the per unit sampling cost motivates to take smaller samples and less frequently, in order to achieve a certain degree of power the control limits must become more tight. This can be followed from the example set presented in Table 4.3.

When the cost of operating in the out-of-control status, a increases, optimum cost rate increases; control limits coefficient does not change significantly, however we observe slight decrease in some experiment instances; sample size does not changes significantly; sampling interval, h^*, decreases. When operating in out-of-control status become more costly, system will be better off detecting the assignable cause as soon as possible. (See Table 4.4)

L_O is the forced shutdown duration in case of a stoppage by an opportunity. Total cost rate decreases as L_o increases. Sample size and control limits are insensitive to the changes in L_O. For small values of L_T, when L_o increases h^* decreases. However, for large L_T, h^* has an increasing trend with respect to L_o. For example, when $L_T = 0.1$ and $L_F = 0.5$, $h^*|_{L_o=0.1} > h^*|_{L_o=0.25} > h^*|_{L_o=0.5}$. However, when $L_T = 0.25$ and $L_F = 0.5$, $h^*|_{L_o=0.1} \leq h^*|_{L_o=0.25} > h^*|_{L_o=0.5}$, and when $L_T = 0.5$ and $L_F = 0.5$, $h^*|_{L_o=0.1} \leq h^*|_{L_o=0.25} < h^*|_{L_o=0.5}$.

Since, exogenous opportunities do not incur any cost, the policy parameters of the process remain the same, however as the free time for inspection and repair increases, the total cost rate decreases as expected.

We observe that when the inspection and repair time in case of a true alarm, L_T increases, k^* and y^* don't change significantly, but both h^* and the total cost rate increase. Inspection and repair time only contributes to the cost in our cost rate function. Hence, if all the policy parameters are fixed, the total cost rate would increase, which explains our observation in this case. However, to compensate for this increase and to reduce the cost without sacrificing much from the cost of operating in the out-of-control status, sampling interval increases slightly.

When the inspection and repair time in case of a true alarm, L_T increases, the total cost rate increases; all the policy parameters y^*, k^* and h^* increase. When L_F increases the total cost rate increases. In order to decrease the false alarm, i.e. Type I error probability (see Equation 3.1), control limit coefficient, k^* increases, yielding more relaxed control limits. However, as k^* increases, Type II error probability increases as well (see Equation 3.3). To compensate the increasing effect of k^* on the Type II error probability the sample size, y^* increases as well, which follows from Equation 3.3. Moreover, due to the effect of increasing k^* and y^* over the cost rate, we observe that sampling frequent decreases, i.e. h^* increases.

A summary of the sensitivity study results are presented on Table 4.8.

4.3.2 Impact of opportunistic inspection rate μ

We observe that changes in the optimum cost rate with respect to μ depend on the relationship between L_O and L_F. Specifically, when L_O is strictly less then L_F $(L_O < L_F)$, independent of the relationship either between L_O and L_T or between L_F and L_T, the total cost rate increases in μ. In all of the other cases the total cost rate decreases in μ.

To explain the rationale behind this observation, first consider the case $(L_O \geq L_F)$ and $(L_O \geq L_T)$. Recall that, when the system stoppage is triggered by an opportunity, the machine incurs lost profit cost only for the additional time it delays the system restart, i.e. $L_{OT}\left(= [L_T - L_O]^+\right)$ or $L_{OF}\left(= [L_F - L_O]^+\right)$. Hence, the conditions $L_O \geq L_F$ and $L_O \geq L_T$ imply that the machine will be ready for the operation before the opportunistic inspection/repair time, L_O elapses. Then, in case of a stoppage by an opportunity, the machine will be restored to the in-control status free of charge. Therefore, under the conditions above, more frequent opportunities are beneficial in order to keep the machine in the in-control status and to provide savings in cost of operating in out-of-control status and cost of inspection and repair. Thereby, the overall cost rate decreases.

Next consider the case $(L_O \geq L_F)$ and $(L_O < L_T)$. A similar argumentation applies. Since the duration of opportunity is longer than the false alarm restoration duration, opportunities arriving when the machine is in the in-control status can be taken at no cost. Although $L_O < L_T$, more frequent opportunities contribute to the early detection of the out-of-control status with less cost, i.e. $\pi.(L_O - L_T)$. Hence, more frequent opportunities decreases the cost rate.

Finally consider the case $(L_O < L_F)$ with either $(L_O < L_T)$ or $(L_O > L_T)$. In this case stoppages due to opportunities are more costly, since the restoration time takes longer than the opportunistic inspection/restoration duration if the system is in the in-control status at the stoppage instant. When the opportunity rate increases, it is more likely that the process will be in the in-control status at an opportunistic stoppage instant. Hence, the overall expected cost rate increases as the opportunity rate increases whenever the above conditions apply.

Our numerical results indicate that k^* and y^* are very insensitive to the opportunity rate μ. However, the sampling interval h^* increases as the opportunity rate increases. This is due to the fact that, when there are more frequent exogenous stoppages for inspection and repair, the process status can be assessed without inference from sampling. Therefore, sampling less frequently yields a lower sampling cost, resulting in a lower total cost rate.

4.4 Cost Breakdown

In this section of the numerical studies we present the cost breakdown of the optimal cost rate. As discussed in Chapter 3, overall cost of the system consists of costs of each cycle type $s \in \{T, F, OT, OF\}$. We would expect that, as the opportunity rate increases, it is more likely that a cycle will be of type opportunity true or opportunity false. Hence, cost components corresponding to the true and false cycle type would decrease in μ.

The cost breakdown of the selected experiments are presented in Tables 4.13 through 4.18. The tabulated values show cost component of each cycle type as the percentage of the overall cost rate, i.e. $\%E[CR_s] = 100 \times \frac{E[CC_s]/E[\tau_s]}{E[TC]}$ for $s \in \{T, F, OT, OF\}$. We have observed that when μ is small, almost 98% of the total cost rate can be attributed to true cycle cost rate. When μ increases, the proportion of the true cycle cost rate decreases. The slope of this decrease is steeper when L_O is small. Increase in opportunity arrival rate indicates more frequent machine stoppages, therefore it is more likely that the machine will be in the in-control status when an opportunity arrives. For all values of L_O, when the proportion of true cycle cost rate decreases, the proportion of opportunity false cost rate tends to increase. The proportion of the false cycle cost rate is always the smallest. The effect of both the variable and fixed parts of the sampling cost over the cost proportions is very small.

4.5 Advantages of JPC

In this section we explore the advantages of jidoka process control versus the classical SPC. Before presenting our methodology and study results, a discussion of the no opportunity ($\mu = 0$) special case is in line. As shown in Section 3.4, the model introduced herein reduces to the classical single machine setting considered in the literature. In the absence of the model we provide here, a decision maker would use the classical model to obtain the control parameters of the QC-chart, which does not take the opportunities into account. Hence, when opportunistic inspection and repairs are available indeed, control chart would operate with the suboptimal control parameters. Therefore, we compare the optimal cost rate with the cost rate obtained with the classical model, in order to evaluate the improvement provided by the opportunistic model.

Let $\left(\hat{y}, \hat{k}, \hat{h}\right)$ triplet denote the global minimizer values of the arguments of the expected cost rate function at $\mu = 0$, i.e. $\left(\hat{y}, \hat{k}, \hat{h}\right) = \arg\min_{(y,k,h)} \left(\lim_{\mu \to 0} E\left[TC\right]\right)$. Similarly let (y^*, k^*, h^*) triplet denote the global minimizer values of the arguments of the expected cost rate function at $\mu = \mu'$, i.e. $(y^*, k^*, h^*) = \arg\min_{(y,k,h)} E\left[TC\right]|_{\mu=\mu'}$. Then, the percentage improvement provided by optimization with the opportunistic model is:

$$\%\Delta = 100 \times \frac{E\left[TC\right]|_{\mu=\mu',\left(\hat{y},\hat{k},\hat{h}\right)} - E\left[TC\right]|_{\mu=\mu',(y^*,k^*,h^*)}}{E\left[TC\right]|_{\mu=\mu',\left(\hat{y},\hat{k},\hat{h}\right)}} \tag{4.1}$$

In the Tables 4.2 through 4.7 we report percentage improvements for the selected experiment sets. For these selected cases the mean of the percentage savings is 2.37%, the median is 0.415%, and the maximum and the minimum savings are 43.48% and 0% respectively.

We observed that as b and u increase $\%\Delta$ increases in general, however there are instances when $a = 50$, the percentage savings decrease for large μ as b and u

increase. The percentage improvement decreases as a increases. We have observed that in the following settings $\%\Delta$ decreases, *(i)* $(L_T < L_O; L_F < L_O; L_T < L_F)$, *(ii)* $(L_T < L_O; L_F > L_O; L_T < L_F)$, *(iii)* $(L_T > L_O; L_F > L_O; L_T = L_F)$, *(iv)* $(L_T > L_O; L_F = L_O; L_T > L_F)$, *(v)* $(L_T = L_O; L_F > L_O; L_T < L_F)$. In all of the other settings $\%\Delta$ increases.

Table 4.19 depicts the summary statistics of the $\%\Delta$ for each opportunity rate and for the overall experimental set. In this table we observe that the improvements increases as the opportunity rate increases. For example, when the opportunity rate is five times more than the shift arrival rate, the mean of the percentage savings is 3.527%, and there are improvements up to 34.5%. For the overall experimental set, the mean of the savings is 1.92%, the standard deviation is 5.98%, and the maximum and minimum are 59.77% and 0%. The maximum percentage saving we observed occurs when $L_T = 0.5$, $L_F = 0.1$, $L_O = 0.5$, $a = 50$, $u = 0$, $b = 0.1$, $\mu = 0.5$. Figure 5.5 shows the changes in the mean and median of the percentage improvements with respect to the opportunity rate, μ.

Experimental results indicate that when opportunities exist, employing the model developed herein is always beneficial. Moreover, with the increase in true restoration time, L_T, savings increase drastically. This is because of the relatively lower restoration time, L_{OT}. Notice that, cost of operating in the out-of-control status, a, is constant. Therefore depending on L_T, lost profit cost due to the idle time $(\pi.L_T)$ increases, and as a result, relative cost of operating in the out-of-control status decreases. Per unit sampling cost has the similar effect on the savings. Thus, if opportunities are included in the model, one is always better off.

Chapter 5

Economic Design of $\bar{\mathbf{X}}$ Control Chart Design: Multiple Machine Setting

In the single machine model, inspection opportunities were exogenous. In this chapter, we develop the multiple machine model, where the inspection/repair opportunities are no longer exogenous to the system at hand. The opportunities are now due to individual machine stoppages. That is, every time an alarm is raised and the system stoppage is triggered by a machine, it creates an inspection opportunity for the rest of the machines in the system.

A typical production facility consists of machines working in coordination, such as output of one of these machines is input for another machine. Although variety of layouts are available for a production facility, a line is the most common layout. For all practical purposes, the system we consider in this chapter can be represented by a production line.

In the sequel, we will: *(i)* provide an exact derivation of the multiple machine model, and *(ii)* through a series of approximations develop a model that uses the results obtained in the single machine with opportunities model.

Let M denote the set of machines working in coordination in the production line, and let $|M|$, i.e. the number of machines, be m. These machines can be identical or non-identical. We use the same notation for the multiple machine model as for the single machine case, but since there are multiple machines, we introduce the superscript $^{(i)}$ notation to denote the parameters of machine i ($\in M$). Every machine i in the system is subject to control with an $\bar{X}$-control chart, i.e. sample of size $y^{(i)}$ is taken at every $h^{(i)}$ time intervals, and the mean of the quality specification of the sample, $\bar{x}$ is plotted on a control chart, if the sample mean falls outside the control limits defined by $\mu_0^{(i)} \pm k^{(i)}\sigma^{(i)}/\sqrt{y^{(i)}}$, then the line is stopped and searched for the assignable cause and restoration activities, if necessary, are conducted.

We assume that the production process is operated in accordance with the *Jidoka* concept. *Jidoka* (translated as autonomation) is a defect detection system which automatically or manually stops the production operation whenever an abnormal or defective condition arises. In the concept of *jidoka* when a team member encounters a problem in his or her work station, he/she is responsible for correcting the problem by pulling an *andon cord*, which can stop the line. Hence, when an alarm is issued by any machine belonging to the set M, a system-wide shut down is triggered and the production is ceased until the inspection/restoration of the triggering machine. Here, we assume that holding WIP (work-in-process) inventory between the machines, that keeps upstream and downstream of the line working during restoration of an intermediate machine, is not feasible or undesirable.

Since, raising of an alarm by a machine causes a system-wide shutdown, during the inspection and restoration of the stoppage triggering machine the rest of the line is idle. It is likely that some of the machines have shifted to the out-of-control status but not issued an alarm yet. Hence, the idle period provides an *opportunity* to inspect the other machines and restore those shifted to the out-of-control status. We assume that these opportunistic inspections are conducted by the operating personnel, hence no additional repair assets are required for the inspection. However, if there is a malfunction detected, i.e. the machine is in the out-of-control status, then the repair assets are requested at some positive cost for the restoration of that machine.

In our setting, we allow the machines in the system to be non-identical. Hence, when the machines have different reliability, restoration times, sampling interval etc., inspection opportunities may be beneficial, i.e., resulting in cost reduction, for some machines, it may be not beneficial, i.e., increasing the cost, for the others. As an example consider a very reliable machine for which the shift rate is very low compared to other machines. Then inspecting this machine at every system stoppage instant will increase the cost it incurs, hence will increase the overall cost of the system. On the other hand, in the case of a machine with very low reliability, inspecting the machine at every possible stoppage would decrease the sampling and operating in the out-of-control state cost. We will designate a machine that utilizes the opportunities as an *opportunity taker,* and a machine that does not utilizes the opportunities as an *opportunity non-taker.* The set of opportunity taker machines will be denoted by M_{TK}, and the set of opportunity non-taker machines will be denoted by M_{NTK}. Then, $M = M_{TK} \cup M_{NTK}$ and $M_{TK} \cap M_{NTK} = 0$.

In the multiple machine model we want to determine: *(i)* optimum control parameters, $y^{(i)}$, $h^{(i)}$ and $k^{(i)}$ for each machine $i\,(\in M)$, *(ii)* optimum partitioning of the machines into the opportunity taker and opportunity non-taker sets, both of which minimize the long run expected cost per unit produced. Let $E\,[TC]$ denote the expected per unit cost, $E\,[CC]$ denote the expected cycle cost, and $E\,[\tau]$ denote the expected operating time in a cycle. (We will discuss and provide the definition of a cycle below.) Then assuming constant production rate and invoking the Renewal Reward Theorem discussed in Chapter 3 objective function can be written as:

$$Minimize_{y^{(i)},k^{(i)},h^{(i)}\ \forall i\in M}\ E\left[TC\right] = \frac{E\left[CC\right]}{E\left[\tau\right]} \tag{5.1}$$

The multiple machine case is different from the single machine case in a number of respects:

1. In the single machine model, we have assumed that opportunity arrival times follow an exponential distribution with rate μ , which was given. We have argued that, given the exponential nature of shift occurrences and the large number of machines involved in the population, it is reasonable to assume that the stoppage probability over a time increment is stationary, hence inter-arrival times are exponential as well. However, in the multiple machine setting we need to compute the opportunity rate observed by each machine, $\mu^{(i)}$ for $i \in M$. The opportunity rate for machine $i \in M_{TK}$ is the sum of the stoppage rates generated by all of the machines except itself. Let $\gamma^{(i)}$ denote the stoppage rate that is *generated* by machine i. Then $\mu^{(i)} = \sum_{j\in M\backslash\{i\}} \gamma^{(j)}$ for the opportunity taker machines, where $M\backslash\{i\}$ denotes all of the machines except i. We will discuss the details of computing the stoppage rate later in this chapter. Clearly, the opportunity rate for any opportunity non-taker machine is effectively zero, since they do not utilize system stoppages. Without loss of generality we will indicate the stoppage rate observed by an opportunity non-taker machine as 0. Hence;

$$\mu^{(i)} = \begin{cases} \sum_{j \in M \setminus \{i\}} \gamma^{(j)} & \text{for } i \in M_{TK} \\ 0 & \text{for } i \in M_{NTK} \end{cases} \tag{5.2}$$

2. The system regeneration points in the single machine case are identified as the machine restart instances, and in that setting, following every system stoppage the machine is restored to the in-control status, hence the machine starts every cycle in the in-control status. In the multiple machine setting, consider the system restart instants as possible regeneration points. When there are opportunity taker and opportunity non-taker machines together in the system, those opportunity non-taker machines which are in the out-of-control status at the stoppage instant will not be restored to the in control state. For an opportunity non-taker machine to be in the in-control status at a system restart either it must be in the in-control status at the previous stoppages instance or the stoppage must be triggered by itself. Thus, system restarts, by themselves, may not always correspond to the regeneration points. This issue is elaborated with an example in Section 5.1.

3. The cost computation of the multiple machine case also requires some care. In the multiple machine environment, the sampling cost and the cost of operating in the out-of-control status are still incurred by individual machines, however the idleness (lost profit) cost, unlike in the single machine case, is incurred by the overall system. Hence, following a system-wide stoppage, the stoppage triggering machine incurs an idleness cost for the duration of its inspection and repair time, and if the longest inspection and repair time among the opportunity taker machines is longer than that of the stoppage triggering machine, the machine with the longest inspection and repair time incurs an idleness cost for the extra duration until it will become ready. We discuss this issue in detail in Section 5.4.

5.1 Exact Derivation

Alluded to above, in the multiple machine setting, the state of the system is not identical at each system restart. Opportunity taker machines are restored to the in-control status, whereas opportunity non-taker machines retain their status at the system stoppage instant, unless the stoppage is triggered by an opportunity non-taker machine, in which case the status of the triggering machine is restored to the in-control status. Moreover, although the regenerative cycle is completed for the opportunity taker machines at every system restart, as in the single machines case, the regenerative cycle is not completed for the opportunity non-taker machines, since they maintain their status as at the previous system stoppage. Additionally, as a consequence, at a system restart, the time to first sampling may be less than the fixed sampling interval, $h^{(i)}$, for the opportunity non-taker machines. Therefore, the information about the time left to the next sampling at the previous system stoppage is needed in the system restart, and the analysis must take this into account.

We provide an illustration of the multiple machine process in Figure 5-1, for a three-machine-sytem. In this illustration, first and second machines are opportunity taker machines and the third machine is an opportunity non-taker machine. Numbers in the circles are the status of each machine at that time instant, where 1 denotes the in-control status and 0 denotes the out-of-control status. Cross marks (X) indicate a system stoppage and the triggering machine. Observe in the figure that opportunity taker machines are in the in-control status at every system restart whereas the opportunity non-taker machine is in the in-control status either when it is in the in-control status at the previous stoppage instant or when the stoppage is triggered by itself.

We model the described process as a semi-markov process where time between two consecutive system restarts are embedded cycles. A semi-markov process is a stochastic process where the number of the states that the process can visit is finite, transition between states are with fixed probability and takes random amount of time (see Ross, 1983 and Tijms 1994).

Next, we will describe this semi-Markov process. Define the system state, $(\boldsymbol{\phi};\boldsymbol{\eta}) = \left\{\left(\phi^{(1)};\eta^{(1)}\right),\left(\phi^{(2)};\eta^{(2)}\right),\cdots,\left(\phi^{(m)};\eta^{(m)}\right)\right\}$ at each system <u>restart</u> by *(i)* status of the machine i, $\phi^{(i)} \in \{0,1\}$, $\phi^{(i)} = 1$ when machine i is in the in-control status and $\phi^{(i)} = 0$ when machine i is in the out-of-control status at the system restart, *(ii)* time from the system restart to the first sampling instant for machine i, $\eta^{(i)}$ where $0 \leq \eta^{(i)} \leq h^{(i)}$. Clearly, for an opportunity taker machine $i \in M_{TK}$, $\left(\phi^{(i)} = 1;\eta^{(i)} = h^{(i)}\right)$ and for an opportunity non-taker machine $j \in M_{NTK}$ $\phi^{(j)} \in \{0,1\}$ and $0 \leq \eta^{(j)} \leq h^{(j)}$. Additionally, define $\varpi^{(i)}$ as the status of machine i at a <u>stoppage</u> instant, $\varpi^{(i)}$, $\varpi^{(i)} = 1$ when machine i is in the in-control status and $\varpi^{(i)} = 0$ when machine i is in the out-of-control status at the system stoppage instant, and $\boldsymbol{\varpi} = \left\{\varpi^{(1)}, \varpi^{(2)}, \cdots, \varpi^{(m)}\right\}$ is the associated system status vector.

A particularity of the machine that triggers the system stoppage is that, independent of whether it is an opportunity taker or opportunity non-taker machine it will be restored to the in-control state in the next system restart. All of the other machines in the system operate independently except for the triggering of system stoppage and system restart. Therefore, the state transition probability for each machine can be written separately.

Define $e\left[\left(\phi^{(i)}; \eta^{(i)}\right), \left(\acute{\phi}^{(i)}; \acute{\eta}^{(i)}\right); j\right]$ to be the state transition probability of machine i from state $\left(\phi^{(i)}; \eta^{(i)}\right)$ to $\left(\acute{\phi}^{(i)}; \acute{\eta}^{(i)}\right)$ when the system stoppage is triggered by machine j. We can group the machines into three, depending on their state transition behavior, such that: *Group 1: the* stoppage triggering machine, *Group 2:* opportunity taker machines except the stoppage triggering machine, and *Group 3:* opportunity non-taker machines except the stoppage triggering machine. State transition probabilities of individual machines will be derived through this grouping. Before deriving the transition probabilities, for ease of notation, we define a binary indicator variable $I(x)$ such that $I(x) = 1$ if $x \geq \tau^{(j)}$, and $I(x) = 0$ otherwise. For convenience, we also introduce the notation $\xi_j^{(i)}$ to denote the maximum number of samples taken by machine i, during the operating time of stoppage triggering machine j, therefore $\xi_j^{(i)} = \left\lfloor \frac{\tau^{(j)}}{h^{(i)}} \right\rfloor$ if $\eta^{(i)} = 0$ and $\xi_j^{(i)} = \left(\left\lfloor \frac{\tau^{(j)} - \eta^{(i)}}{h^{(i)}} \right\rfloor + 1\right)$ if $\eta^{(i)} > 0$.

Group 1: Since j itself triggers the system-wide stoppage, its status in the next system restart will be in-control, and the time to the next sampling is its fixed sampling interval, hence, $\acute{\phi}^{(j)} = 1$, and $\acute{\eta}^{(j)} = h^{(j)}$.

Therefore, for $j \in M$, $e\left[\left(\phi^{(j)}; \eta^{(j)}\right), \left(\acute{\phi}^{(j)}; \acute{\eta}^{(j)}\right); j\right]$ can be derived as follows.

First consider the transition probability for machine i from $\left(\phi^{(j)} = 1; \eta^{(j)}\right)$ to $\left(\acute{\phi}^{(j)} = 1; \acute{\eta}^{(j)}\right)$ where $0 \leq \eta^{(j)} \leq h^{(j)}$, $\acute{\eta}^{(j)} = h^{(j)}$. This transition may occur in two different ways: *(i)* the status of the machine j may have shifted to the out-of-control state before the system stoppage, i.e. $\varpi^{(j)} = 0$ and been restored to the in-control status, or *(ii)* machine j may be in the in-control status at the system stoppage, i.e. $\varpi^{(j)} = 1$ and retains its status at the system restart. The transition probability is the sum of the probabilities of these two alternatives. For either $\varpi^{(j)} = 0$ or $\varpi^{(j)} = 1$ the total number of samplings is $n_1^{(j)} + n_2^{(j)} = \xi_j^{(j)}$. If $\varpi^{(j)} = 0$, then there are $n_1^{(j)}$ correct inference about the in-control status, i.e. $(1 - \alpha^{(j)})^{n_1^{(j)}}$, process shifts to the out-of-control status sometime between the samplings $n_1^{(j)}$ and $n_1^{(j)} + 1$, i.e. $\int_{x=\left[\eta^{(j)}+\left(n_1^{(j)}-1\right)h^{(j)}\right]^+}^{\eta^{(j)}+n_1^{(j)}h^{(j)}} \lambda^{(j)} e^{-\lambda^{(j)}x} dx$, after the shift there are $(n_2^{(j)} - 1)$ type II error, i.e. $\left[\beta^{(j)}\right]^{(n_2^{(j)}-1)}$, and one correct inference about the out-of-control status, i.e. $(1 - \beta^{(j)})$. If $\varpi^{(j)} = 1$, the process does not shift during the operating time of machine j, i.e. $e^{-\lambda^{(j)}\tau^{(j)}}$, there are $\xi_j^{(j)} - 1$ correct inference about the in-control status i.e. $(1 - \alpha^{(j)})^{\xi_j^{(j)}-1}$ and one type I error, i.e.

$\alpha^{(j)}$. Hence,

$$e\left[\left(\phi^{(j)};\eta^{(j)}\right),\left(\acute{\phi}^{(j)};\acute{\eta}^{(j)}\right);j\right] = \left[1-I\left(\eta^{(j)}\right)\right] \tag{5.3}$$
$$\times\left\{\sum_{n_1^{(j)}=0}^{\xi_j^{(j)}-1}\int_{x=\left[\eta^{(j)}+\left(n_1^{(j)}-1\right)h^{(j)}\right]^+}^{\eta^{(j)}+n_1^{(j)}h^{(j)}}\left[(1-\alpha^{(j)})^{n_1^{(j)}}\left[\beta^{(j)}\right]^{(\xi_j^{(j)}-n_1^{(j)}-1)}\right.\right.$$
$$\left.\left.\times(1-\beta^{(j)})\lambda^{(j)}e^{-\lambda^{(j)}x}\right]dx\ +\alpha^{(j)}(1-\alpha^{(j)})^{\xi_j^{(j)}-1}e^{-\lambda^{(j)}\tau^{(j)}}\right\}$$
$$\text{for } \phi^{(j)}=\acute{\phi}^{(j)}=1 \text{ and}$$
$$0\le\eta^{(j)}\le h^{(j)},\ \acute{\eta}^{(j)}=h^{(j)}$$

Next consider the transition from $\left(\phi^{(j)}=0;\eta^{(j)}\right)$ to $\left(\acute{\phi}^{(j)}=1;\acute{\eta}^{(j)}\right)$ where $0\le\eta^{(j)}\le h^{(j)}$, $\acute{\eta}^{(j)}=h^{(j)}$. This transition explicitly implies that machine j is an opportunity non-taker machine, since its status at the system restart is out-of-control. The machine is in the out-of-control status at the system restart, so that it must be in the out-control status at the system stoppage, i.e. $\varpi^{(j)}=0$, because the process is not self correcting. Hence, there are $\xi_j^{(j)}-1$ type II error, i.e. $\left(\beta^{(j)}\right)^{\xi_j^{(j)}-1}$ and one correct inference about the out-of-control state, i.e. $(1-\beta^{(j)})$. Hence:

$$e\left[\left(\phi^{(j)};\eta^{(j)}\right),\left(\acute{\phi}^{(j)};\acute{\eta}^{(j)}\right);j\right] = \left[1-I\left(\eta^{(j)}\right)\right](1-\beta^{(j)})\left(\beta^{(j)}\right)^{\xi_j^{(j)}-1} \tag{5.4}$$
$$\text{for } \phi^{(j)}=0\ ,\ \acute{\phi}^{(j)}=1 \text{ and}$$
$$0\le\eta^{(j)}\le h^{(j)},\ \acute{\eta}^{(j)}=h^{(j)}$$

The machine that triggers a stoppage can never be in the out-of-control status in the system restart, and only possible value $\acute{\eta}^{(j)}$ can take is $h^{(j)}$, hence:

$$e\left[\left(\phi^{(j)};\eta^{(j)}\right),\left(\acute{\phi}^{(j)};\acute{\eta}^{(j)}\right);j\right]=0 \qquad \text{for } \acute{\phi}^{(j)}=0 \text{ or } \acute{\eta}^{(j)}<h^{(j)} \tag{5.5}$$

Group 2: Since i is an opportunity taker machine it is restored to the in-control status whenever there is a system-wide stoppage, and the time to the next sampling is always equal to its fixed sampling interval, hence, $\phi^{(i)}=\acute{\phi}^{(i)}=1$, and $\eta^{(i)}=\acute{\eta}^{(i)}=h^{(i)}$. Therefore, for $i\in M_{TK}$, $i\neq j$ and $j\in M$, $e\left[\left(\phi^{(i)};\eta^{(i)}\right),\left(\acute{\phi}^{(i)};\acute{\eta}^{(i)}\right);j\right]$ can be derived as follows.

First consider the transition probability for machine i from $\left(\phi^{(i)}=1;\eta^{(i)}=h^{(i)}\right)$ to $\left(\acute{\phi}^{(i)}=1;\acute{\eta}^{(i)}=h^{(i)}\right)$. This transition may occur in three different ways: *(i)* the status of the machine i may have shifted to the out-of-control state before the system stoppage, i.e. $\varpi^{(i)}=0$ where shift and system stoppages occur in different sampling intervals, *(ii)* the status of the machine i may have shifted to the out-of-control state before the system stoppage, i.e. $\varpi^{(i)}=0$ where shift and system stoppages occur within the same sampling interval, *(iii)* machine i may be in the in-control status at the system stoppage, i.e. $\varpi^{(i)}=1$ and retains its status at the system restart. The transition probability is the sum of the probabilities of these three alternatives. For $\varpi^{(i)}=0$ and the system stoppage and the shift occur in different sampling intervals, $n_1^{(i)}$ can take on values between 0 and $\xi_j^{(i)}$, then $n_2^{(i)}=\xi_j^{(i)}-n_1^{(i)}$. In this case, there are $n_1^{(i)}$ correct inferences about the in-control status, i.e. $(1-\alpha^{(i)})^{n_1^{(i)}}$, process shifts to the out-of-control status between the samplings $n_1^{(i)}$ and $n_1^{(i)}+1$, i.e. $\int_{x=\left[\eta^{(i)}+\left(n_1^{(i)}-1\right)h^{(i)}\right]^+}^{\eta^{(i)}+n_1^{(i)}h^{(i)}} \lambda^{(i)} e^{-\lambda^{(i)}x} dx$, there are $n_2^{(i)}$ Type II errors, i.e. $\left[\beta^{(i)}\right]^{(\xi_j^{(i)}-n_1^{(i)})}$. For $\varpi^{(i)}=0$ and the system stoppage and the shift occur within the same sampling interval, then $n_1^{(i)}=\xi_j^{(i)}$ and $n_2^{(i)}=0$, there are $n_1^{(i)}$ correct inferences about the in-control status, i.e. $(1-\alpha^{(i)})^{n_1^{(i)}}$, and the process shifts to the out-of-control status between the last sampling and system stoppage, i.e. $\int_{x=\left[\eta^{(i)}+\left(n_1^{(i)}-1\right)h^{(i)}\right]^+}^{\tau^{(j)}} \lambda^{(i)} e^{-\lambda^{(i)}x} dx$. For $\varpi^{(j)}=1$, the process does not shift before the system stoppage, i.e. $e^{-\lambda^{(i)}\tau^{(j)}}$, and there are $\xi_j^{(i)}$ correct inferences about the in-control status i.e. $(1-\alpha^{(i)})^{\xi_j^{(i)}}$. Hence,

$$
\begin{aligned}
e\left[\left(\phi^{(i)};\eta^{(i)}\right),\left(\acute{\phi}^{(i)};\acute{\eta}^{(i)}\right);j\right] &= \left[1-I\left(\eta^{(i)}\right)\right](1-\alpha^{(i)})^{\xi_j^{(i)}} e^{-\lambda^{(i)}\tau^{(j)}} \\
&+ I\left(\eta^{(i)}\right) e^{-\lambda^{(i)}\tau^{(j)}} + \left[1-I\left(\eta^{(i)}\right)\right] \left\{ \sum_{n_1^{(i)}=0}^{\xi_j^{(i)}} \left(1-I\left(\eta^{(i)}+n_1^{(i)}h^{(i)}\right)\right) \right. \\
&\times \left[\int_{x=\left[\eta^{(i)}+\left(n_1^{(i)}-1\right)h^{(i)}\right]^+}^{\eta^{(i)}+n_1^{(i)}h^{(i)}} (1-\alpha^{(i)})^{n_1^{(i)}} \right. \\
&\times \left. \left[\beta^{(i)}\right]^{(\xi_j^{(i)}-n_1^{(i)})} \lambda^{(i)} e^{-\lambda^{(i)}x} dx \right] + I\left(\eta^{(i)}+n_1^{(i)}h^{(i)}\right) \qquad (5.6) \\
&\times \left. \left[\int_{x=\left[\eta^{(i)}+\left(n_1^{(i)}-1\right)h^{(i)}\right]^+}^{\tau^{(j)}} (1-\alpha^{(i)})^{n_1^{(i)}} \lambda^{(i)} e^{-\lambda^{(i)}x} dx \right] \right\} \\
&+ I\left(\eta^{(i)}\right)\left(1-e^{-\lambda^{(i)}\tau^{(j)}}\right) \qquad \text{for } \phi^{(i)}=\acute{\phi}^{(i)}=1 \text{ and} \\
&\qquad \eta^{(i)}=\acute{\eta}^{(i)}=h^{(i)}
\end{aligned}
$$

Opportunity taker machines are restored to the in-control status following every system stoppage, therefore any initial out-of-control status for the opportunity taker machines is not possible. Hence,

$$e\left[\left(\phi^{(i)};\eta^{(i)}\right),\left(\acute{\phi}^{(i)};\acute{\eta}^{(i)}\right);j\right] = 0 \qquad \text{for } \phi^{(i)}=0 \text{ or } \acute{\phi}^{(i)}=0 \qquad (5.7)$$
$$\text{or } \eta^{(i)}<h^{(i)} \text{ or } \acute{\eta}^{(i)}<h^{(i)}$$

Group 3: the state transition probability for the opportunity non-taker machine i, where the stoppage is triggered by machine j, i.e. $e\left[\left(\phi^{(i)};\eta^{(i)}\right),\left(\acute{\phi}^{(i)};\acute{\eta}^{(i)}\right);j\right]$ where $i\neq j$. Since i is an opportunity non-taker machine, at the system restart it retains its status in the previous system stoppage. For $i\in M_{NTK}$, $i\neq j$ and $j\in M$ state transition probabilities can be derived as follows.

First consider the transition probability for machine i from $\left(\phi^{(i)}=1;\eta^{(i)}\right)$ to $\left(\acute{\phi}^{(i)}=1;\acute{\eta}^{(i)}\right)$ where $0\leq\eta^{(i)}\leq h^{(i)}$, $0\leq\acute{\eta}^{(i)}\leq h^{(i)}$. This transition implies that the status of the machine i at the previous stoppage instance is in-control. Then, there are $\xi_j^{(i)}$ correct inferences about the in-control status, i.e.$(1-\alpha^{(i)})^{\xi_j^{(i)}}$, and the process does not shift before the system stoppage, i.e. $e^{-\lambda^{(i)}\tau^{(j)}}$. Hence:

$$
\begin{aligned}
e\left[\left(\phi^{(i)};\eta^{(i)}\right),\left(\acute{\phi}^{(i)};\acute{\eta}^{(i)}\right);j\right] &= \left(1-I\left(\eta^{(i)}\right)\right)\left((1-\alpha^{(i)})^{\xi_j^{(i)}}e^{-\lambda^{(i)}\tau^{(j)}}\right) \\
&\quad + I\left(\eta^{(i)}\right)e^{-\lambda^{(i)}\tau^{(j)}} \qquad \text{for } \phi^{(i)}=1, \acute{\phi}^{(i)}=1 \text{ and} \qquad (5.8) \\
&\qquad 0 \le \eta^{(i)} \le h^{(i)},\ \acute{\eta}^{(i)} = \eta^{(i)} + \xi_j^{(i)}h^{(i)} - \tau^{(j)}
\end{aligned}
$$

Next, consider the transition probability for machine i from $\left(\phi^{(i)}=0;\eta^{(i)}\right)$ to $\left(\acute{\phi}^{(i)}=0;\acute{\eta}^{(i)}\right)$ where $0 \le \eta^{(i)} \le h^{(i)}$, $0 \le \acute{\eta}^{(i)} \le h^{(i)}$. This transition implies that there are $\xi_j^{(i)}$ Type II error, i.e.$\left[\beta^{(i)}\right]^{\xi_j^{(i)}}$. Hence:

$$
\begin{aligned}
e\left[\left(\phi^{(i)};\eta^{(i)}\right),\left(\acute{\phi}^{(i)};\acute{\eta}^{(i)}\right);j\right] &= I\left(\eta^{(i)}\right)\left[\beta^{(i)}\right]^{\xi_j^{(i)}} + \left(1-I\left(\eta^{(i)}\right)\right) \\
&\qquad \text{for } \phi^{(i)} = \acute{\phi}^{(i)} = 0 \text{ and} \qquad (5.9) \\
&\qquad 0 \le \eta^{(i)} \le h^{(i)},\ \acute{\eta}^{(i)} = \eta^{(i)} + \xi_j^{(i)}h^{(i)} - \tau^{(j)}
\end{aligned}
$$

Now, consider the transition probability for machine i from $\left(\phi^{(i)}=1;\eta^{(i)}\right)$ to $\left(\acute{\phi}^{(i)}=0;\acute{\eta}^{(i)}\right)$ where $0 \le \eta^{(i)} \le h^{(i)}$, $0 \le \acute{\eta}^{(i)} \le h^{(i)}$. This transition may occur in two different ways: *(i)* the process shift and system stoppages occur in different sampling intervals, *(ii)* the process shift and system stoppages occur within the same sampling interval. The transition probability is the sum of the probabilities of these two alternatives. For the case where system stoppage and the shift occur in different sampling intervals, $n_1^{(i)}$ can take on values between 0 and $\xi_j^{(i)}$, then

$n_2^{(i)} = \xi_j^{(i)} - n_1^{(i)}$. In this case, there are $n_1^{(i)}$ correct inferences about the in-control status, i.e. $(1-\alpha^{(i)})^{n_1^{(i)}}$, process shifts to the out-of-control status between the samplings $n_1^{(i)}$ and $n_1^{(i)}+1$, i.e. $\int_{x=\left[\eta^{(i)}+\left(n_1^{(i)}-1\right)h^{(i)}\right]^+}^{\eta^{(i)}+n_1^{(i)}h^{(i)}} \lambda^{(i)} e^{-\lambda^{(i)}x}dx$, there are $n_2^{(i)}$ Type II errors, i.e. $\left[\beta^{(i)}\right]^{(\xi_j^{(i)}-n_1^{(i)})}$. For the case where the system stoppage and the shift occur within the same sampling interval, there are $n_1^{(i)} = \xi_j^{(i)}$ correct inferences about the in-control status, i.e. $(1-\alpha^{(i)})^{n_1^{(i)}}$, and the process shifts to the out-of-control status between the last sampling and system stoppage, i.e. $\int_{x=\left[\eta^{(i)}+\left(n_1^{(i)}-1\right)h^{(i)}\right]^+}^{\tau^{(j)}} \lambda^{(i)} e^{-\lambda^{(i)}x}dx$. Hence:

$$
\begin{aligned}
e\left[\left(\phi^{(i)};\eta^{(i)}\right),\left(\acute{\phi}^{(i)};\acute{\eta}^{(i)}\right);j\right] &= \left[1-I\left(\eta^{(k)}\right)\right] \\
&\times \Bigg\{ \sum_{n_1^{(i)}=0}^{\xi_j^{(i)}} \left(1-I\left(\eta^{(i)}+n_1^{(i)}h^{(i)}\right)\right) \qquad (5.10)\\
&\times \Bigg[\int_{x=\left[\eta^{(i)}+\left(n_1^{(i)}-1\right)h^{(i)}\right]^+}^{\eta^{(i)}+n_1^{(i)}h^{(i)}} (1-\alpha^{(i)})^{n_1^{(i)}} \\
&\quad \times \left[\beta^{(i)}\right]^{(\xi_j^{(i)}-n_1^{(i)})} \lambda^{(i)} e^{-\lambda^{(i)}x} dx \Bigg] + I\left(\eta^{(i)}+n_1^{(i)}h^{(i)}\right) \\
&\quad \times \left[\int_{x=\left[\eta^{(i)}+\left(n_1^{(i)}-1\right)h^{(i)}\right]^+}^{\tau^{(j)}} (1-\alpha^{(i)})^{n_1^{(i)}} \lambda^{(i)} e^{-\lambda^{(i)}x} dx \right] \Bigg\} \\
&+ I\left(\eta^{(i)}\right)\left(1-e^{-\lambda^{(i)}\tau^{(j)}}\right)
\end{aligned}
$$

$$\text{for } \phi^{(i)}=1, \acute{\phi}^{(i)}=0 \text{ and}$$

$$0 \le \eta^{(i)} \le h^{(i)},\ \acute{\eta}^{(i)} = \eta^{(i)} + \xi_j^{(i)}h^{(i)} - \tau^{(j)}$$

The process is not self correcting, therefore transition from out-of-control status to the in-control status is impossible. Hence:

$$e\left[\left(\phi^{(i)};\eta^{(i)}\right),\left(\acute{\phi}^{(i)};\acute{\eta}^{(i)}\right);j\right] = 0 \qquad \text{for } \phi^{(i)}=0, \acute{\phi}^{(i)}=1 \text{ or} \tag{5.11}$$
$$\acute{\eta}^{(i)} \neq \eta^{(i)} + \xi_j^{(i)} h^{(i)} - \tau^{(j)}$$

Define $\mathbf{Q}$ as the one step transition matrix, where each entry is the transition probability from state $(\boldsymbol{\phi};\boldsymbol{\eta})$ at the stoppage instant l, to state $\left(\acute{\boldsymbol{\phi}};\acute{\boldsymbol{\eta}}\right)$ at the stoppage instant $l+1$, when the system stoppage is triggered by machine j. We denote element of $\mathbf{Q}$ by $Q\left[(\boldsymbol{\phi};\boldsymbol{\eta}),\left(\acute{\boldsymbol{\phi}};\acute{\boldsymbol{\eta}}\right);j\right]$. Then,

$$Q\left[(\boldsymbol{\phi};\boldsymbol{\eta}),\left(\acute{\boldsymbol{\phi}};\acute{\boldsymbol{\eta}}\right);j\right] = \int\limits_{\tau^{(j)}=0}^{\infty} \prod_{i=1}^{m} e\left[\left(\phi^{(i)};\eta^{(i)}\right),\left(\acute{\phi}^{(i)};\acute{\eta}^{(i)}\right);j\right] d\tau^{(j)} \tag{5.12}$$

Define the probability matrix $\tilde{\pi}(\boldsymbol{\phi};\boldsymbol{\eta})$ of stationary probability of the system being in state $(\boldsymbol{\phi};\boldsymbol{\eta})$, which can be obtained from the one step transition probability matrix $Q\left[(\boldsymbol{\phi};\boldsymbol{\eta}),\left(\acute{\boldsymbol{\phi}};\acute{\boldsymbol{\eta}}\right);j\right]$. Each element of stationary probability matrix $\tilde{\pi}(\boldsymbol{\phi};\boldsymbol{\eta})$ can be obtained by solving the set of equations below:

$$\tilde{\pi}\left(\acute{\boldsymbol{\phi}};\acute{\boldsymbol{\eta}}\right) = \tilde{\pi}\left(\boldsymbol{\phi};\boldsymbol{\eta}\right)\sum_{j=1}^{m} Q\left[\left(\boldsymbol{\phi};\boldsymbol{\eta}\right),\left(\acute{\boldsymbol{\phi}};\acute{\boldsymbol{\eta}}\right);j\right] \tag{5.13}$$

$$\sum_{\left(\acute{\boldsymbol{\phi}};\acute{\boldsymbol{\eta}}\right)} \tilde{\pi}\left(\acute{\boldsymbol{\phi}};\acute{\boldsymbol{\eta}}\right) = 1 \tag{5.14}$$

Although the embedded cycles we define are between two consecutive system restarts, system state at the end of a cycle is different than the system state at the immediate next system start, because of the repair and restoration activities over the opportunity taker and the stoppage triggering machines. In our model construction below, we will take this into account and define events that are particular realizations of the process, and compute the event probabilities and expected event costs.

First, consider the following event, $\{E_T^{(j)}|\left(\boldsymbol{\phi};\boldsymbol{\eta}\right)\}$: A cycle, where system state is $(\boldsymbol{\phi};\boldsymbol{\eta})$ at the previous system stoppage and with operating time length $\tau^{(j)}$ ends when the machine j triggers a system-wide stoppage by signaling a true alarm, i.e. $\varpi^{(j)} = 0$ and, at the time of stoppage, the opportunity taker machines grouped in set $\Psi_{OF}^{(j)}$ are in the in-control state, i.e., $\varpi^{(i)} = 1$, $\forall i \in \Psi_{OF}^{(j)}$, and the machines grouped in set $\Psi_{OT}^{(j)}$ are in the out–of-control state, i.e., $\varpi^{(i)} = 0$, $\forall i \in \Psi_{OT}^{(j)}$. Hence, $M_{TK}\backslash\{j\} = \Psi_{OT}^{(j)} \cup \Psi_{OF}^{(j)}$. The opportunity non-taker machines are grouped in set $\Psi_F^{(j)}$ are in the in-control state, i.e., $\varpi^{(i)} = 1$, $\forall i \in \Psi_F^{(j)}$and machines grouped in set $\Psi_T^{(j)}$ are in the out-of-control state, i.e., $\varpi^{(i)} = 0$, $\forall i \in \Psi_T^{(j)}$. Hence, $M_{NTK}\backslash\{j\} = \Psi_T^{(j)} \cup \Psi_F^{(j)}$. The machines in the set $\Psi_T^{(j)}$ are also grouped in to two depending on their state at the previous system start instant, such that machines that were in the in-control state when the system start but in the out-of-control state at the stoppage, $\left(\Psi_T^{(j)}; \phi^{(r)} = 1\right)$ and machines that were in the out-of-control state when the system start and in the out-of-control state at the stoppage instant $\left(\Psi_T^{(j)}; \phi^{(r)} = 0\right)$.Then,

$$
\begin{aligned}
\Pr\{E_T^{(j)} | (\boldsymbol{\phi}; \boldsymbol{\eta})\} \quad = \quad & \Pr \left\{ \begin{array}{c} \text{machine } j \text{ signals a true} \\ \text{alarm at time } \tau^{(j)} | \left(\phi^{(j)}; \eta^{(j)}\right) \end{array} \right\} \\
& \times \Pi_{i \in \Psi_{OF}^{(j)}} \Pr \left\{ \begin{array}{c} \text{machine } i \text{ is stopped at time } \tau^{(j)} \\ \text{and } \varpi^{(i)} = 1 \mid \left(\phi^{(i)}; \eta^{(i)}\right) \end{array} \right\} \\
& \times \Pi_{k \in \Psi_{OT}^{(j)}} \Pr \left\{ \begin{array}{c} \text{machine } k \text{ is stopped at time } \tau^{(j)} \\ \text{and } \varpi^{(k)} = 0 \mid \left(\phi^{(k)}; \eta^{(k)}\right) \end{array} \right\} \\
& \times \Pi_{n \in \Psi_{F}^{(j)}} \Pr \left\{ \begin{array}{c} \text{machine } n \text{ is stopped at time } \tau^{(j)} \\ \text{and } \varpi^{(n)} = 1 | \left(\phi^{(n)}; \eta^{(n)}\right) \end{array} \right\} \\
& \times \Pi_{r \in \Psi_{T}^{(j)}, \phi^{(r)}=1} \Pr \left\{ \begin{array}{c} \text{machine } r \text{ is stopped at time } \tau^{(j)} \\ \text{and } \varpi^{(r)} = 0 \mid \left(\phi^{(r)} = 1; \eta^{(r)}\right) \end{array} \right\} \\
& \times \Pi_{w \in \Psi_{T}^{(j)}, \phi^{(w)}=0} \Pr \left\{ \begin{array}{c} \text{machine } w \text{ is stopped at time } \tau^{(j)} \\ \text{and } \varpi^{(w)} = 0 \mid \left(\phi^{(w)} = 0; \eta^{(w)}\right) \end{array} \right\}
\end{aligned}
\tag{5.15}
$$

Substituting the previously derived probabilities we get:

$$
\begin{aligned}
\Pr\{E_T^{(j)} | (\boldsymbol{\phi}; \boldsymbol{\eta})\} = & \Bigg\{ \left[1 - I\left(\eta^{(j)}\right)\right] \left[\sum_{n_1^{(j)}=0}^{\xi_j^{(j)}-1} \int_{x=\left[\eta^{(j)}+\left(n_1^{(j)}-1\right)h^{(j)}\right]^+}^{\eta^{(j)}+n_1^{(j)}h^{(j)}} \right. \\
& \left. \times (1-\alpha^{(j)})^{n_1^{(j)}} \left[\beta^{(j)}\right]^{(\xi_j^{(j)}-n_1^{(j)}-1)} (1-\beta^{(j)})\lambda^{(j)} e^{-\lambda^{(j)}x} dx \right] \\
& + \left[1 - I\left(\eta^{(j)}\right)\right] (1-\beta^{(j)}) \left(\beta^{(j)}\right)^{\xi_j^{(j)}-1} \Bigg\} \\
& \times \prod_{i \in \Psi_{OF}^{(j)}} \Bigg\{ \left[1 - I\left(\eta^{(i)}\right)\right] (1-\alpha^{(i)})^{\xi_j^{(i)}} e^{-\lambda^{(i)}\tau^{(j)}} \\
& + I\left(\eta^{(i)}\right) e^{-\lambda^{(i)}\tau^{(j)}} \Bigg\} \\
& \times \prod_{k \in \Psi_{OT}^{(j)}} \Bigg\{ \left[1 - I\left(\eta^{(k)}\right)\right] \left[\sum_{n_1^{(k)}=0}^{\xi_j^{(k)}} \left(1 - I\left(\eta^{(k)} + n_1^{(k)}h^{(k)}\right)\right) \right. \\
& \times \int_{x=\left[\eta^{(k)}+\left(n_1^{(k)}-1\right)h^{(k)}\right]^+}^{\eta^{(k)}+n_1^{(k)}h^{(k)}} (1-\alpha^{(k)})^{n_1^{(k)}} \\
& \times \left[\beta^{(k)}\right]^{(\xi_j^{(k)}-n_1^{(k)})} \lambda^{(k)} e^{-\lambda^{(k)}x} dx + I\left(\eta^{(k)} + n_1^{(k)}h^{(k)}\right) \\
& \left. \times \int_{x=\left[\eta^{(k)}+\left(n_1^{(k)}-1\right)h^{(k)}\right]^+}^{\tau^{(j)}} (1-\alpha^{(k)})^{n_1^{(k)}} \lambda^{(k)} e^{-\lambda^{(k)}x} dx \right] \\
& + I\left(\eta^{(k)}\right) \left(1 - e^{-\lambda^{(k)}\tau^{(j)}}\right) \Bigg\} \\
& \times \prod_{n \in \Psi_F^{(j)}} \Bigg\{ \left(1 - I\left(\eta^{(n)}\right)\right) (1-\alpha^{(n)})^{\xi_j^{(n)}} e^{-\lambda^{(n)}\tau^{(j)}} \\
& + I\left(\eta^{(n)}\right) e^{-\lambda^{(n)}\tau^{(j)}} \Bigg\}
\end{aligned} \tag{5.16}
$$

$$
\begin{aligned}
&\times \prod_{r\in\Psi_T^{(j)},\phi^{(r)}=1} \left\{ \left(1-I\left(\eta^{(r)}\right)\right) \left[\sum_{n_1^{(r)}=0}^{\xi_j^{(r)}} \left(1-I\left(\eta^{(r)}+n_1^{(r)}h^{(r)}\right)\right) \right.\right.\\
&\times \left[\int_{x=\left[\eta^{(r)}+\left(n_1^{(r)}-1\right)h^{(r)}\right]^+}^{\eta^{(r)}+n_1^{(r)}h^{(r)}} (1-\alpha^{(r)})^{n_1^{(r)}} \left[\beta^{(r)}\right]^{\left(\xi^{(r)}-n_1^{(r)}\right)} \right.\\
&\left. \times \lambda^{(r)}e^{-\lambda^{(r)}x}dx \right] + I\left(\eta^{(r)}+n_1^{(r)}h^{(r)}\right)\\
&\left.\times \left[\int_{x=\left[\eta^{(r)}+\left(n_1^{(r)}-1\right)h^{(r)}\right]^+}^{\tau^{(j)}} (1-\alpha^{(r)})^{n_1^{(r)}} \lambda^{(r)}e^{-\lambda^{(r)}x}dx \right]\right]\\
&\left. + I\left(\eta^{(r)}\right)\left(1-e^{-\lambda^{(r)}\tau^{(j)}}\right) \right\}\\
&\times \prod_{w\in\Psi_T^{(j)},\phi^{(r)}=0} \left\{ I\left(\eta^{(w)}\right)\left[\beta^{(w)}\right]^{\xi_j^{(w)}} + \left(1-I\left(\eta^{(w)}\right)\right) \right\}
\end{aligned}
$$

and $\Pr\{E_F^{(j)}|\,(\boldsymbol{\phi};\boldsymbol{\eta})\}$ can be written in a similar way as follows:

$$
\begin{aligned}
\Pr\{E_F^{(j)}|\,(\boldsymbol{\phi};\boldsymbol{\eta})\} \;=\; & \left\{ \left[1-I\left(\eta^{(j)}\right)\right]\alpha^{(j)}(1-\alpha^{(j)})^{\xi_j^{(j)}-1}e^{-\lambda^{(j)}\tau^{(j)}} \right\}\\
& \times \prod_{i\in\Psi_{OF}^{(j)}} \left\{ \left[1-I\left(\eta^{(i)}\right)\right](1-\alpha^{(i)})^{\xi_j^{(i)}}e^{-\lambda^{(i)}\tau^{(j)}} \right.\\
& \left. + I\left(\eta^{(i)}\right)e^{-\lambda^{(i)}\tau^{(j)}} \right\} \qquad (5.17)
\end{aligned}
$$

$$
\times \prod_{k\in\Psi_{OT}^{(j)}} \left\{ \left[1 - I\left(\eta^{(k)}\right)\right] \left[\sum_{n_1^{(k)}=0}^{\xi_j^{(k)}} \left(1 - I\left(\eta^{(k)} + n_1^{(k)} h^{(k)}\right)\right) \right. \right.
$$

$$
\times \int_{x=\left[\eta^{(k)}+\left(n_1^{(k)}-1\right)h^{(k)}\right]^+}^{\eta^{(k)}+n_1^{(k)}h^{(k)}} (1-\alpha^{(k)})^{n_1^{(k)}}
$$

$$
\times \left[\beta^{(k)}\right]^{(\xi_j^{(k)}-n_1^{(k)})} \lambda^{(k)} e^{-\lambda^{(k)}x} dx + I\left(\eta^{(k)} + n_1^{(k)} h^{(k)}\right)
$$

$$
\left. \times \int_{x=\left[\eta^{(k)}+\left(n_1^{(k)}-1\right)h^{(k)}\right]^+}^{\tau^{(j)}} (1-\alpha^{(k)})^{n_1^{(k)}} \lambda^{(k)} e^{-\lambda^{(k)}x} dx \right]
$$

$$
\left. + I\left(\eta^{(k)}\right) \left(1 - e^{-\lambda^{(k)}\tau^{(j)}}\right) \right\}
$$

$$
\times \prod_{n\in\Psi_{F}^{(j)}} \left\{ \left(1 - I\left(\eta^{(n)}\right)\right) (1-\alpha^{(n)})^{\xi_j^{(n)}} e^{-\lambda^{(n)}\tau^{(j)}} \right.
$$

$$
\left. + I\left(\eta^{(n)}\right) e^{-\lambda^{(n)}\tau^{(j)}} \right\}
$$

$$
\times \prod_{r\in\Psi_{T}^{(j)},\phi^{(r)}=1} \left\{ \left(1 - I\left(\eta^{(r)}\right)\right) \left[\sum_{n_1^{(r)}=0}^{\xi_j^{(r)}} \left(1 - I\left(\eta^{(r)} + n_1^{(r)} h^{(r)}\right)\right) \right. \right.
$$

$$
\times \left[\int_{x=\left[\eta^{(r)}+\left(n_1^{(r)}-1\right)h^{(r)}\right]^+}^{\eta^{(r)}+n_1^{(r)}h^{(r)}} (1-\alpha^{(r)})^{n_1^{(r)}} \left[\beta^{(r)}\right]^{\left(\xi^{(r)}-n_1^{(r)}\right)} \right.
$$

$$
\left. \times \lambda^{(r)} e^{-\lambda^{(r)}x} dx \right] + I\left(\eta^{(r)} + n_1^{(r)} h^{(r)}\right)
$$

$$\times \left[\int_{x=\left[\eta^{(r)}+\left(n_1^{(r)}-1\right)h^{(r)}\right]^+}^{\tau^{(j)}} (1-\alpha^{(r)})^{n_1^{(r)}} \lambda^{(r)} e^{-\lambda^{(r)}x} dx \right] \Bigg]$$

$$\left. + I\left(\eta^{(r)}\right)\left(1 - e^{-\lambda^{(r)}\tau^{(j)}}\right) \right\}$$

$$\times \prod_{w \in \Psi_T^{(j)}, \phi^{(r)}=0} \left\{ I\left(\eta^{(w)}\right) \left[\beta^{(w)}\right]^{\xi_j^{(w)}} + \left(1 - I\left(\eta^{(w)}\right)\right) \right\}$$

Now we can derive the expression for the embedded cycle length and embedded cycle cost.

Define $C(\tau, s^{(i)}, n_1^{(i)}, n_2^{(i)}, x^{(i)}, z^{(i)}, \boldsymbol{\varpi})$ as the total cost incurred by machine i, for a given operating time τ of a cycle class $s^{(i)} \in \{T, F, OT, OF\}$, where the number of samplings taken before the shift is $n_1^{(i)}$, the number of samplings after the shift is $n_2^{(i)}$, the process shifts to the out-of-control status at time $x^{(i)}$, opportunity arrival time is $z^{(i)}$, and status of the machines at the system stoppage are given by the vector $\boldsymbol{\varpi}$. Let $\left(C_T^{(j)} | (\boldsymbol{\phi}; \boldsymbol{\eta})\right)$denote the cost associated with the event $\left(E_T^{(j)} | (\boldsymbol{\phi}; \boldsymbol{\eta})\right)$ given that the system state at the previous system stoppage is $(\boldsymbol{\phi}; \boldsymbol{\eta})$ and $\left(C_F^{(j)} | (\boldsymbol{\phi}; \boldsymbol{\eta})\right)$ denote the cost associated with the event $\left(E_F^{(j)} | (\boldsymbol{\phi}; \boldsymbol{\eta})\right)$ given that the system state at the previous system stoppage is $(\boldsymbol{\phi}; \boldsymbol{\eta})$, then the expected embedded cycle cost for the given state is :

$$
\begin{aligned}
E\left(C_T^{(j)}|\left(\boldsymbol{\phi};\boldsymbol{\eta}\right)\right) = & \left\{ \left[1-I\left(\eta^{(j)}\right)\right]\left[\sum_{n_1^{(j)}=0}^{\xi_j^{(j)}-1}\int_{x=\left[\eta^{(j)}+\left(n_1^{(j)}-1\right)h^{(j)}\right]^+}^{\eta^{(j)}+n_1^{(j)}h^{(j)}}\right.\right. \\
& C(\tau^{(j)},T,n_1^{(j)},\xi_j^{(j)}-n_1^{(j)},x,\left(z^{(j)}>\tau^{(j)}\right),\boldsymbol{\varpi}) \\
& \left.\times(1-\alpha^{(j)})^{n_1^{(j)}}\left[\beta^{(j)}\right]^{(\xi_j^{(j)}-n_1^{(j)}-1)}(1-\beta^{(j)})\lambda^{(j)}e^{-\lambda^{(j)}x}dx\right] \\
& +\left[1-I\left(\eta^{(j)}\right)\right]C(\tau^{(j)},T,0,\xi_j^{(j)},x,\left(z^{(j)}>\tau^{(j)}\right),\boldsymbol{\varpi}) \\
& \left.(1-\beta^{(j)})\left(\beta^{(j)}\right)^{\xi_j^{(j)}-1}\right\} \\
& \times\prod_{i\in\Psi_{OF}^{(j)}}\left\{\left[1-I\left(\eta^{(i)}\right)\right]C(\tau^{(j)},OF,\xi_j^{(i)},0,\left(x>\tau^{(j)}\right),\tau^{(j)},\boldsymbol{\varpi})\right. \\
& \times(1-\alpha^{(i)})^{\xi_j^{(i)}}e^{-\lambda^{(i)}\tau^{(j)}} \qquad (5.18) \\
& \left.+I\left(\eta^{(i)}\right)C(\tau^{(j)},OF,0,0,\left(x>\tau^{(j)}\right),\tau^{(j)},\boldsymbol{\varpi})e^{-\lambda^{(i)}\tau^{(j)}}\right\} \\
& \times\prod_{k\in\Psi_{OT}^{(j)}}\left\{\left[1-I\left(\eta^{(k)}\right)\right]\left[\sum_{n_1^{(k)}=0}^{\xi_j^{(k)}}\left(1-I\left(\eta^{(k)}+n_1^{(k)}h^{(k)}\right)\right)\right.\right. \\
& \times\int_{x=\left[\eta^{(k)}+\left(n_1^{(k)}-1\right)h^{(k)}\right]^+}^{\eta^{(k)}+n_1^{(k)}h^{(k)}}C(\tau^{(j)},OT,n_1^{(k)},\xi_j^{(k)}-n_1^{(k)},x,\tau^{(j)},\boldsymbol{\varpi}) \\
& \times(1-\alpha^{(k)})^{n_1^{(k)}}\left[\beta^{(k)}\right]^{(\xi_j^{(k)}-n_1^{(k)})}\lambda^{(k)}e^{-\lambda^{(k)}x}dx+I\left(\eta^{(k)}+n_1^{(k)}h^{(k)}\right) \\
& \times\int_{x=\left[\eta^{(k)}+\left(n_1^{(k)}-1\right)h^{(k)}\right]^+}^{\tau^{(j)}}C(\tau^{(j)},OT,n_1^{(k)},0,x,\tau^{(j)},\boldsymbol{\varpi})
\end{aligned}
$$

$$\times \left(1-\alpha^{(k)}\right)^{n_1^{(k)}} \lambda^{(k)} e^{-\lambda^{(k)} x} dx \Bigg]$$

$$+ I\left(\eta^{(k)}\right) C(\tau^{(j)}, OT, 0, 0, x, \tau^{(j)}, \boldsymbol{\varpi}) \left(1 - e^{-\lambda^{(k)}\tau^{(j)}}\right) \Bigg\}$$

$$\times \prod_{n \in \Psi_F^{(j)}} \Bigg\{ \left(1 - I\left(\eta^{(n)}\right)\right) C(\tau^{(j)}, OF, \xi_j^{(n)}, 0, \left(x > \tau^{(j)}\right), \tau^{(j)}, \boldsymbol{\varpi})$$

$$\times \left(1-\alpha^{(n)}\right)^{\xi_j^{(n)}} e^{-\lambda^{(n)}\tau^{(j)}}$$

$$+ I\left(\eta^{(n)}\right) C(\tau^{(j)}, OF, 0, 0, \left(x > \tau^{(j)}\right), \tau^{(j)}, \boldsymbol{\varpi}) e^{-\lambda^{(n)}\tau^{(j)}} \Bigg\}$$

$$\times \prod_{r \in \Psi_T^{(j)}, \phi^{(r)}=1} \Bigg\{ \left(1 - I\left(\eta^{(r)}\right)\right) \Bigg[\sum_{n_1^{(r)}=0}^{\xi_j^{(r)}} \left(1 - I\left(\eta^{(r)} + n_1^{(r)} h^{(r)}\right)\right)$$

$$\times \Bigg[\int_{x=\left[\eta^{(r)} + \left(n_1^{(r)}-1\right) h^{(r)}\right]^+}^{\eta^{(r)} + n_1^{(r)} h^{(r)}} C(\tau^{(j)}, OT, n_1^{(r)}, \xi_j^{(r)} - n_1^{(r)}, x, \left(z^{(r)} > \tau^{(j)}\right), \boldsymbol{\varpi})$$

$$\times \left(1-\alpha^{(r)}\right)^{n_1^{(r)}} \left[\beta^{(r)}\right]^{\left(\xi^{(r)} - n_1^{(r)}\right)} \lambda^{(r)} e^{-\lambda^{(r)} x} dx \Bigg] + I\left(\eta^{(r)} + n_1^{(r)} h^{(r)}\right)$$

$$\Bigg[\int_{x=\left[\eta^{(r)} + \left(n_1^{(r)}-1\right) h^{(r)}\right]^+}^{\tau^{(j)}} C(\tau^{(j)}, OT, \xi_j^{(r)}, 0, x, \left(z^{(r)} > \tau^{(j)}\right), \boldsymbol{\varpi}) \Bigg]$$

$$\times \left(1-\alpha^{(r)}\right)^{n_1^{(r)}} \lambda^{(r)} e^{-\lambda^{(r)} x} dx \Bigg]$$

$$+ I\left(\eta^{(r)}\right) C(\tau^{(j)}, OT, 0, 0, x, \left(z^{(r)} > \tau^{(j)}\right), \boldsymbol{\varpi}) \left(1 - e^{-\lambda^{(r)}\tau^{(j)}}\right) \Bigg\}$$

$$\times \prod_{w \in \Psi_T^{(j)}, \phi^{(r)}=0} \left\{ I\left(\eta^{(w)}\right) C(\tau^{(j)}, OT, 0, \xi_j^{(w)}, 0, \left(z^{(w)} > \tau^{(j)}\right), \boldsymbol{\varpi}) \right\}$$

$$\times \left[\beta^{(w)}\right]^{\xi_j^{(w)}} + \left(1 - I\left(\eta^{(w)}\right)\right) C(\tau^{(j)}, OT, 0, 0, 0, \left(z^{(w)} > \tau^{(j)}\right), \boldsymbol{\varpi})$$

$E\left(C_F^{(j)} | (\boldsymbol{\phi}; \boldsymbol{\eta})\right)$ can be written in a similar way.

$$\Pr\{E_F^{(j)} | (\boldsymbol{\phi}; \boldsymbol{\eta})\} = \left\{ \left[1 - I\left(\eta^{(j)}\right)\right] C\left(\tau^{(j)}, F, \xi_j^{(j)}, 0, x, \left(z > \tau^{(j)}\right), \boldsymbol{\varpi}\right) \right. \tag{5.19}$$

$$\left. \times \alpha^{(j)} (1 - \alpha^{(j)})^{\xi_j^{(j)} - 1} e^{-\lambda^{(j)} \tau^{(j)}} \right\}$$

$$E\left(C_F^{(j)} | (\boldsymbol{\phi}; \boldsymbol{\eta})\right) = \left\{ \left[1 - I\left(\eta^{(j)}\right)\right] C\left(\tau^{(j)}, F, \xi_j^{(j)}, 0, x, \left(z > \tau^{(j)}\right), \boldsymbol{\varpi}\right) \right.$$

$$\left. \times \alpha^{(j)} (1 - \alpha^{(j)})^{\xi_j^{(j)} - 1} e^{-\lambda^{(j)} \tau^{(j)}} \right\} \tag{5.20}$$

$$\times \prod_{i \in \Psi_{OF}^{(j)}} \left\{ \left[1 - I\left(\eta^{(i)}\right)\right] C(\tau^{(j)}, OF, \xi_j^{(i)}, 0, \left(x > \tau^{(j)}\right), \tau^{(j)}, \boldsymbol{\varpi}) \right.$$

$$\times (1 - \alpha^{(i)})^{\xi_j^{(i)}} e^{-\lambda^{(i)} \tau^{(j)}}$$

$$
\left. + I\left(\eta^{(i)}\right) C(\tau^{(j)}, OF, 0, 0, \left(x > \tau^{(j)}\right), \tau^{(j)}, \boldsymbol{\varpi}) e^{-\lambda^{(i)}\tau^{(j)}} \right\}
$$

$$
\times \prod_{k \in \Psi_{OT}^{(j)}} \left\{ \left[1 - I\left(\eta^{(k)}\right)\right] \left[\sum_{n_1^{(k)}=0}^{\xi_j^{(k)}} \left(1 - I\left(\eta^{(k)} + n_1^{(k)} h^{(k)}\right)\right) \right. \right.
$$

$$
\times \int_{x=\left[\eta^{(k)}+\left(n_1^{(k)}-1\right)h^{(k)}\right]^+}^{\eta^{(k)}+n_1^{(k)}h^{(k)}} C(\tau^{(j)}, OT, n_1^{(k)}, \xi_j^{(k)} - n_1^{(k)}, x, \tau^{(j)}, \boldsymbol{\varpi})
$$

$$
\times (1 - \alpha^{(k)})^{n_1^{(k)}} \left[\beta^{(k)}\right]^{(\xi_j^{(k)} - n_1^{(k)})} \lambda^{(k)} e^{-\lambda^{(k)}x} dx
$$

$$
+ I\left(\eta^{(k)} + n_1^{(k)} h^{(k)}\right)
$$

$$
\times \int_{x=\left[\eta^{(k)}+\left(n_1^{(k)}-1\right)h^{(k)}\right]^+}^{\tau^{(j)}} C(\tau^{(j)}, OT, n_1^{(k)}, 0, x, \tau^{(j)}, \boldsymbol{\varpi})
$$

$$
\left. \times (1 - \alpha^{(k)})^{n_1^{(k)}} \lambda^{(k)} e^{-\lambda^{(k)}x} dx \right]
$$

$$
\left. + I\left(\eta^{(k)}\right) C(\tau^{(j)}, OT, 0, 0, x, \tau^{(j)}, \boldsymbol{\varpi}) \left(1 - e^{-\lambda^{(k)}\tau^{(j)}}\right) \right\}
$$

$$
\times \prod_{n \in \Psi_F^{(j)}} \left\{ \left(1 - I\left(\eta^{(n)}\right)\right) C(\tau^{(j)}, OF, \xi_j^{(n)}, 0, \left(x > \tau^{(j)}\right), \tau^{(j)}, \boldsymbol{\varpi}) \right.
$$

$$
\times (1 - \alpha^{(n)})^{\xi_j^{(n)}} e^{-\lambda^{(n)}\tau^{(j)}}
$$

$$
\left. + I\left(\eta^{(n)}\right) C(\tau^{(j)}, OF, 0, 0, \left(x > \tau^{(j)}\right), \tau^{(j)}, \boldsymbol{\varpi}) e^{-\lambda^{(n)}\tau^{(j)}} \right\}
$$

$$\times \prod_{r\in\Psi_T^{(j)},\phi^{(r)}=1} \left\{ \left(1-I\left(\eta^{(r)}\right)\right) \left[\sum_{n_1^{(r)}=0}^{\xi_j^{(r)}} \left(1-I\left(\eta^{(r)}+n_1^{(r)}h^{(r)}\right)\right) \right.\right.$$

$$\times \left[\int_{x=\left[\eta^{(r)}+\left(n_1^{(r)}-1\right)h^{(r)}\right]^+}^{\eta^{(r)}+n_1^{(r)}h^{(r)}} C(\tau^{(j)},OT,n_1^{(r)},\xi_j^{(r)}-n_1^{(r)},x,\left(z^{(r)}>\tau^{(j)}\right),\boldsymbol{\varpi}) \right.$$

$$\left. \times \left(1-\alpha^{(r)}\right)^{n_1^{(r)}} \left[\beta^{(r)}\right]^{\left(\xi^{(r)}-n_1^{(r)}\right)} \lambda^{(r)} e^{-\lambda^{(r)}x} dx \right] + I\left(\eta^{(r)}+n_1^{(r)}h^{(r)}\right)$$

$$\left[\int_{x=\left[\eta^{(r)}+\left(n_1^{(r)}-1\right)h^{(r)}\right]^+}^{\tau^{(j)}} C(\tau^{(j)},OT,\xi_j^{(r)},0,x,\left(z^{(r)}>\tau^{(j)}\right),\boldsymbol{\varpi}) \right]$$

$$\left. \times \left(1-\alpha^{(r)}\right)^{n_1^{(r)}} \lambda^{(r)} e^{-\lambda^{(r)}x} dx \right]$$

$$\left. + I\left(\eta^{(r)}\right) C(\tau^{(j)},OT,0,0,x,\left(z^{(r)}>\tau^{(j)}\right),\boldsymbol{\varpi}) \left(1-e^{-\lambda^{(r)}\tau^{(j)}}\right) \right\}$$

$$\times \prod_{w\in\Psi_T^{(j)},\phi^{(r)}=0} \left\{ I\left(\eta^{(w)}\right) C(\tau^{(j)},OT,0,\xi_j^{(w)},0,\left(z^{(w)}>\tau^{(j)}\right),\boldsymbol{\varpi}) \right\}$$

$$\times \left[\beta^{(w)}\right]^{\xi_j^{(w)}} + \left(1-I\left(\eta^{(w)}\right)\right) C(\tau^{(j)},OT,0,0,0,\left(z^{(w)}>\tau^{(j)}\right),\boldsymbol{\varpi})$$

We will introduce the renewal reward theorem for the Semi-Markov process below (Theorem 3.5.1, after Tijms (1992) p.219) Suppose that the embedded Markov chain $\{X_n\}$ associated with policy R has no two disjoint closed sets. Then

$$\lim_{t\to\infty} \frac{Z(t)}{t} = g(R) \text{ with probability 1}$$

for each initial state $X_0 = i$, where the constant $g(R)$ is given by

$$g(R) = \frac{\sum_j c_j(R_j)\,\pi_j(R)}{\sum_j \tau_j(R_j)\,\pi_j(R)}$$

where $Z(t)$ is the total cost incurred up to time t, $c_i(a)$ is the expected cost incurred until the next decision epoch if action a is chosen in the present state i, $\tau_i(a)$ is the expected time until the next decision epoch if action a is chosen in the present state i, $\pi_i(R)$ is the stationary distribution of Markov chain $\{X_n\}$ associated with policy R.

Hence, the long run expected cost per unit produced, $E[TC]$ can be written as follows by invoking the above theorem:

$$E[TC] = \frac{E[CC]}{E[\tau]} \tag{5.21}$$

where

$$E[CC] = \int_{\tilde{\pi}} \tilde{\pi}(\boldsymbol{\phi};\boldsymbol{\eta}) \sum_{j=1}^{m} \int_{\tau^{(j)}=0}^{\infty} \left[E\left(C_F^{(j)} | (\boldsymbol{\phi};\boldsymbol{\eta}) \right) + E\left(C_T^{(j)} | (\boldsymbol{\phi};\boldsymbol{\eta}) \right) \right] \tag{5.22}$$

and

$$E\left[\tau\right]=\int_{\tilde{\pi}}\tilde{\pi}\left(\boldsymbol{\phi};\boldsymbol{\eta}\right)\sum_{j=1}^{m}\int\limits_{\tau^{(j)}=0}^{\infty}\tau^{(j)}\left[\Pr\left\{E_F^{(j)}|\left(\boldsymbol{\phi};\boldsymbol{\eta}\right)\right\}+\Pr\left\{\left(E_T^{(j)}|\left(\boldsymbol{\phi};\boldsymbol{\eta}\right)\right)\right\}\right] \tag{5.23}$$

Since $\eta^{(i)}$ is a continuous parameter, we can discretize the Markov process by dividing a sampling interval $h^{(i)}$ into smaller time segments $\Delta h^{(i)}$. Let d be the number of time intervals that a sampling interval divided into, i.e. $d=\frac{h^{(i)}}{\Delta h^{(i)}}$. Then the dimension of the transition probability matrix Q would be $(2^m d^m)\times(2^m d^m)\times m$, a size that cannot be used for the practical purposes. Next, we will consider some approximations for the multiple machine environment which will allow us to use the results obtained from the single machine model which is constructed in Chapter 3.

5.2 Approximate Model

Depending on the partitioning of the machines in the system, there are three possible cases: (i) *All non-taker case*: all of the machines are opportunity non-takers, (ii) *All taker case*: all of the machines are opportunity takers or (iii) *Mixed case:* some of the machines are opportunity takers and some of them are opportunity non-takers We will construct the approximate model by considering these three cases separately.

5.2.1 All non-taker case:

We first consider the case where all of the machines are opportunity non-takers, i.e.$M_{TK} = \emptyset$. In the all non-taker case, machines are inspected and repaired only through self-stoppages. Assuming that there exists a Least Common Multiple (LCM) of the individual sampling intervals, at the intervals which constitute these LCMs, all of the machines will take samples simultaneously. At each of these instances, there is a positive probability that a system stoppage will be triggered at least by all of the machines in the out-of-control state; hence, all the machines will restart in the in-control status and the time to the next sampling instance for each machine will be exactly $h^{(i)}$. These LCM instances, although very infrequent, constitute the system regeneration points. Between two consecutive system regeneration points, each restart of a machine will also constitute a regeneration point only for that particular machine. The machines may then be viewed as going through regenerative cycles independently. By invoking the Renewal Reward Theorem, the long run cost rate of the system can be computed by the expected cost rate of a system cycle. Since within each system cycle, there are individual cycles for particular machines, the system can be analyzed by considering each machine separately. As shutdowns are not utilized for inspecting the processes, the cost of downtime as foregone profit will be charged only to the machine that stops the system. The expected cost rate for the m machine system can, then, be written as the sum of expected cost rates incurred for individual machines facing no opportunities:

$$E[TC] = \sum_{\forall i \in M} \frac{\lim_{\mu^{(i)} \to 0} E[CC^{(i)}]}{\lim_{\mu^{(i)} \to 0} E[\tau^{(i)}]} \tag{5.24}$$

Expressions $E[CC^{(i)}]$ and $E[\tau^{(i)}]$ are as given in the single machine model in equations 3.30 and 3.25. Since in the all non-taker case, machines do not utilize the opportunities, they operate and stop independently and individually, therefore, cycle cost and operating time functions should be evaluated at $\mu^{(i)} = 0$.

5.2.2 All taker case

Now, we consider the case where all of the machines are opportunity takers, i.e. $M_{NTK} = \emptyset$. When all the machines are opportunity takers, at each system restart all of the machines are in the in-control status. The time to the first sampling instance is exactly $h^{(i)}$, $\forall i \in M$. Therefore each system restart is a regeneration point for all of the machines and hence, for the overall system. Therefore we can use the exact model derived in Section 5.1.

Consider the following event, $\{E_T^{(j)}|\boldsymbol{\phi}, \boldsymbol{\varpi}\}$: A cycle with operating time length $\tau^{(j)}$ ends when machine j triggers a system-wide stoppage by signaling a true alarm, i.e. $\varpi^{(j)} = 0$, and, the status of the machines at the start of the cycle is given by the vector $\boldsymbol{\phi}$ and their status at the stoppage instant is given by the vector $\boldsymbol{\varpi}$, the machines grouped in set $\Psi_{OF}^{(j)}$ are in the in-control state, i.e. $\varpi^{(i)} = 1$

$\forall i \in \Psi_{OF}^{(j)}$, and the machines grouped in set $\Psi_{OT}^{(j)}$ are in the out–of-control state, i.e. $\varpi^{(i)} = 0\ \forall i \in \Psi_{OT}^{(j)}$. Then, define:

$$\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} &= \Pr\left\{\begin{array}{c}\text{machine } j \text{ signals a true} \\ \text{alarm at time } \tau^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\end{array}\right\} \\
&\quad \times \prod_{i\in\Psi_{OF}^{(j)}} \Pr\left\{\begin{array}{c}\text{machine } i \text{ is stopped at time } \tau^{(j)} \\ \text{and } \varpi^{(i)} = 1|\boldsymbol{\phi},\boldsymbol{\varpi}\end{array}\right\} \qquad (5.25) \\
&\quad \times \prod_{k\in\Psi_{OT}^{(j)}} \Pr\left\{\begin{array}{c}\text{machine } k \text{ is stopped at time } \tau^{(j)} \\ \text{and } \varpi^{(k)} = 0|\boldsymbol{\phi},\boldsymbol{\varpi}\end{array}\right\}
\end{aligned}$$

$$\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} &= \sum_{n_1^{(j)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(j)}}\right\rfloor} \int_{x=n_1^{(j)}h^{(j)}}^{(n_1^{(j)}+1)h^{(j)}} \left[(1-\alpha^{(j)})^{n_1^{(j)}} \left[\beta^{(j)}\right]^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(j)}}\right\rfloor - n_1^{(j)} - 1\right)}\right. \\
&\quad \left. \times (1-\beta^{(j)})\lambda^{(j)} e^{-\lambda^{(j)}x}\right] dx \\
&\quad \times \prod_{i\in\Psi_{OF}^{(j)}} \left\{(1-\alpha^{(i)})^{\left\lfloor \frac{\tau^{(j)}}{h^{(i)}}\right\rfloor} e^{-\lambda^{(i)}\tau^{(j)}}\right\} \qquad (5.26)
\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{k\in\Psi_{OT}^{(j)}} \left\{ \sum_{n_1^{(k)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(k)}} \right\rfloor} \left[\left(1 - I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right)\right) \right.\right. \\
&\times \int_{x=n_1^{(k)}h^{(k)}}^{(n_1^{(k)}+1)h^{(k)}} (1-\alpha^{(k)})^{n_1^{(k)}} \left[\beta^{(k)}\right]^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(k)}} \right\rfloor - n_1^{(k)} - 1\right)} \lambda^{(k)} e^{-\lambda^{(k)}x} dx \\
&\left.\left. + I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right) \int_{x=n_1^{(k)}h^{(k)}}^{\tau^{(j)}} (1-\alpha^{(k)})^{n_1^{(k)}} \lambda^{(k)} e^{-\lambda^{(k)}x} dx \right] \right\}
\end{aligned}
$$

Similarly for the event $\{E_F^{(j)}|\boldsymbol{\phi}, \boldsymbol{\varpi}\}$ in which the machine j triggers the system stoppage with a false alarm the probability is as follows:

$$
\begin{aligned}
\Pr\{E_F^{(j)}|\boldsymbol{\phi}, \boldsymbol{\varpi}\} &= \alpha^{(j)}(1-\alpha^{(j)})^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor - 1\right)} e^{-\lambda^{(j)}\tau^{(j)}} \\
&\times \prod_{i\in\Psi_{OF}^{(j)}} \left\{ (1-\alpha^{(i)})^{\left\lfloor \frac{\tau^{(j)}}{h^{(i)}} \right\rfloor} e^{-\lambda^{(i)}\tau^{(j)}} \right\} \\
&\times \prod_{k\in\Psi_{OT}^{(j)}} \left\{ \sum_{n_1^{(k)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(k)}} \right\rfloor} \left[\left(1 - I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right)\right) \right.\right. \qquad (5.27) \\
&\times \int_{x=n_1^{(k)}h^{(k)}}^{(n_1^{(k)}+1)h^{(k)}} (1-\alpha^{(k)})^{n_1^{(k)}} \left[\beta^{(k)}\right]^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(k)}} \right\rfloor - n_1^{(k)} - 1\right)} \lambda^{(k)} e^{-\lambda^{(k)}x} dx \\
&\left.\left. + I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right) \int_{x=n_1^{(k)}h^{(k)}}^{\tau^{(j)}} (1-\alpha^{(k)})^{n_1^{(k)}} \lambda^{(k)} e^{-\lambda^{(k)}x} dx \right] \right\}
\end{aligned}
$$

The expressions above give the exact probability of the events defined. However, it is quite tedious to compute them, especially if we need to compute the expectation over the given number of samplings taken before and after the shift. Even though the state space for the all taker case is smaller compared to that of the general exact model, $(2^m vs. 2^m d^m)$, the size increases exponentially in the number of machines in the system, which may easily become beyond the manageable sizes. However, through some reasonable approximations we can develop an approximate model for practical purposes, that uses the model for single machine with opportunities. Therefore, we would like to approximate this expression with the help of the single machine results obtained in Chapter 3.

Now consider the probability of event $\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\}$, i.e. $\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\}$, and multiply and divide the probability function of each machine i that were stopped, by $\left(\Gamma-\gamma^{(i)}\right)e^{-\left(\Gamma-\gamma^{(i)}\right)\tau^{(j)}}$, where Γ denote the system stoppage rate.

Hence we have,

$$\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \quad = \quad & \sum_{n_1^{(j)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor} \int_{x=n_1^{(j)}h^{(j)}}^{(n_1^{(j)}+1)h^{(j)}} \left[(1-\alpha^{(j)})^{n_1^{(j)}} \left[\beta^{(j)}\right]^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor - n_1^{(j)} - 1\right)} \right. \\
& \left. \times (1-\beta^{(j)})\lambda^{(j)} e^{-\lambda^{(j)}x} \right] dx \\
& \times \prod_{i\in\Psi_{OF}^{(j)}} \left\{ (1-\alpha^{(i)})^{\left\lfloor \frac{\tau^{(j)}}{h^{(i)}} \right\rfloor} e^{-\lambda^{(i)}\tau^{(j)}} \frac{\left(\Gamma-\gamma^{(i)}\right)e^{-\left(\Gamma-\gamma^{(i)}\right)\tau^{(j)}}}{\left(\Gamma-\gamma^{(i)}\right)e^{-\left(\Gamma-\gamma^{(i)}\right)\tau^{(j)}}} \right\} \\
& \times \prod_{k\in\Psi_{OT}^{(j)}} \left\{ \sum_{n_1^{(k)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(k)}} \right\rfloor} \left[\left(1 - I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right)\right) \right. \right. \qquad (5.28)
\end{aligned}$$

$$\times \int_{x=n_1^{(k)}h^{(k)}}^{(n_1^{(k)}+1)h^{(k)}} (1-\alpha^{(k)})^{n_1^{(k)}} \left[\beta^{(k)}\right]^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(k)}} \right\rfloor - n_1^{(k)} - 1\right)}$$

$$\times \lambda^{(k)} e^{-\lambda^{(k)}x} dx \frac{\left(\Gamma - \gamma^{(k)}\right) e^{-\left(\Gamma - \gamma^{(k)}\right)\tau^{(j)}}}{\left(\Gamma - \gamma^{(k)}\right) e^{-\left(\Gamma - \gamma^{(k)}\right)\tau^{(j)}}}$$

$$+ I\left(\left(n_1^{(k)}+1\right) h^{(k)}\right) \int_{x=n_1^{(k)}h^{(k)}}^{\tau^{(j)}} (1-\alpha^{(k)})^{n_1^{(k)}}$$

$$\left. \left. \times \lambda^{(k)} e^{-\lambda^{(k)}x} dx \frac{\left(\Gamma - \gamma^{(k)}\right) e^{-\left(\Gamma - \gamma^{(k)}\right)\tau^{(j)}}}{\left(\Gamma - \gamma^{(k)}\right) e^{-\left(\Gamma - \gamma^{(k)}\right)\tau^{(j)}}} \right] \right\}$$

$$\begin{aligned} \Pr\{E_T^{(j)}|\boldsymbol{\phi}, \boldsymbol{\varpi}\} \quad = \quad & \sum_{n_1^{(j)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor} \int_{x=n_1^{(j)}h^{(j)}}^{(n_1^{(j)}+1)h^{(j)}} \left[(1-\alpha^{(j)})^{n_1^{(j)}} \left[\beta^{(j)}\right]^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor - n_1^{(j)} - 1\right)} \right. \\ & \left. \times (1-\beta^{(j)}) \lambda^{(j)} e^{-\lambda^{(j)}x} \right] dx \\ & \times \prod_{i \in \Psi_{OF}^{(j)}} \left\{ \frac{\int_{x=\tau^{(j)}}^{\infty} f^{(i)}\left(\tau^{(j)}, OF, \left\lfloor \frac{\tau^{(j)}}{h^{(i)}} \right\rfloor, 0, x, \tau^{(j)} \mid \left(\Gamma - \gamma^{(i)}\right)\right) dx}{\left(\Gamma - \gamma^{(i)}\right) e^{-\left(\Gamma - \gamma^{(i)}\right)\tau^{(j)}}} \right\} \end{aligned}$$

$$\times \prod_{k\in\Psi_{OT}^{(j)}} \left\{ \sum_{n_1^{(k)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(k)}} \right\rfloor} \left[\left(1 - I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right)\right) \right. \right. \tag{5.29}$$

$$\times \frac{\int_{x=n_1^{(k)}h^{(k)}}^{(n_1^{(k)}+1)h^{(k)}} f^{(k)}\left(\tau^{(j)}, OT, n_1^{(k)}, \left\lfloor \frac{\tau^{(j)}}{h^{(k)}} \right\rfloor - n_1^{(k)}, x, \tau^{(j)} \mid \left(\Gamma - \gamma^{(k)}\right)\right) dx}{\left(\Gamma - \gamma^{(k)}\right) e^{-\left(\Gamma - \gamma^{(k)}\right)\tau^{(j)}}}$$

$$+I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right)$$

$$\left. \left. \times \frac{\int_{x=n_1^{(k)}h^{(k)}}^{\tau^{(j)}} f^{(k)}\left(\tau^{(j)}, OT, n_1^{(k)}, 0, x, \tau^{(j)} \mid \left(\Gamma - \gamma^{(k)}\right)\right) dx}{\left(\Gamma - \gamma^{(k)}\right) e^{-\left(\Gamma - \gamma^{(k)}\right)\tau^{(j)}}} \right] \right\}$$

Now define the point probability $p_s^{(i)}\left(\tau, \left(\Gamma - \gamma^{(i)}\right)\right)$ to be the probability that machine $i \in M_{TK}$, faced with an opportunity rate $\left(\Gamma - \gamma^{(i)}\right)$, is stopped at time τ. Hence,

$$p_{OF}^{(i)}\left(\tau, \left(\Gamma - \gamma^{(i)}\right)\right) = \int_{x=\tau}^{\infty} f^{(i)}\left(\tau, OF, \left\lfloor \frac{\tau}{h^{(i)}} \right\rfloor, 0, x, \tau \mid \left(\Gamma - \gamma^{(i)}\right)\right) dx \tag{5.30}$$

and

$$p_{OT}^{(k)}\left(\tau,\left(\Gamma-\gamma^{(k)}\right)\right) = \sum_{n_1^{(k)}=0}^{\left\lfloor\frac{\tau}{h^{(k)}}\right\rfloor}\Bigg[\left(1-I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right)\right) \tag{5.31}$$

$$\times\int_{x=n_1^{(k)}h^{(k)}}^{(n_1^{(k)}+1)h^{(k)}} f^{(k)}\left(\begin{array}{c}\tau^{(j)},OT,n_1^{(k)},\left\lfloor\frac{\tau^{(j)}}{h^{(k)}}\right\rfloor-n_1^{(k)},\\ x,\tau^{(j)}\mid\left(\Gamma-\gamma^{(k)}\right)\end{array}\right)dx$$

$$+I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right)$$

$$\times\int_{x=n_1^{(k)}h^{(k)}}^{\tau^{(j)}} f^{(k)}\left(\tau,OT,n_1^{(k)},0,x,\tau|\left(\Gamma-\gamma^{(k)}\right)\right)dx\Bigg]$$

Then by substituting we can rewrite:

$$\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} = \sum_{n_1^{(j)}=0}^{\left\lfloor\frac{\tau^{(j)}}{h^{(j)}}\right\rfloor}\int_{x=n_1^{(j)}h^{(j)}}^{(n_1^{(j)}+1)h^{(j)}}(1-\alpha^{(j)})^{n_1^{(j)}}\left[\beta^{(j)}\right]^{\left(\left\lfloor\frac{\tau^{(j)}}{h^{(j)}}\right\rfloor-n_2^{(j)}-1\right)}$$

$$\times(1-\beta^{(j)})\lambda^{(j)}e^{-\lambda^{(j)}x}dx$$

$$\times\prod_{i\in\Psi_{OF}^{(j)}}\left\{\frac{p_{OF}^{(i)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(i)}\right)\right)}{\left(\Gamma-\gamma^{(i)}\right)e^{-\left(\Gamma-\gamma^{(i)}\right)\tau^{(j)}}}\right\} \tag{5.32}$$

$$\times\prod_{k\in\Psi_{OT}^{(j)}}\left\{\frac{p_{OT}^{(k)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(k)}\right)\right)}{\left(\Gamma-\gamma^{(k)}\right)e^{-\left(\Gamma-\gamma^{(k)}\right)\tau^{(j)}}}\right\}$$

Then, we can approximate

$$p_s^{(i)}(\tau,\left(\Gamma-\gamma^{(i)}\right))\approx\left(\Gamma-\gamma^{(i)}\right)e^{-\left(\Gamma-\gamma^{(i)}\right)\tau}e^{-\gamma^{(i)}\tau}\frac{P_{s\in\{OT,OF\}}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right)+P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \tag{5.33}$$

and

$$p_s^{(i)}(\tau,0) \approx \gamma^{(i)} e^{-\gamma^{(i)}\tau} \frac{P_{s\in\{T,F\}}^{(i)}(0)}{P_T^{(i)}(0)+P_F^{(i)}(0)} \tag{5.34}$$

where $P_s^{(i)}(\cdot)$ is as defined before in Chapter 3 for $s \in \{T, F, OT, OF\}$. The term $\frac{P_{s\in\{OT,OF\}}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right)+P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}$ approximates the probability that machine i, facing an opportunity rate of $\left(\Gamma - \gamma^{(i)}\right)$, has the cycle class $s \in \{OT, OF\}$, having the status at the system stoppage $\varpi^{(i)} \in \{0, 1\}$, and the term $\left(\Gamma - \gamma^{(i)}\right) e^{-\left(\Gamma-\gamma^{(i)}\right)\tau} e^{-\gamma^{(i)}\tau}$ approximates the probability that the machine i does not trigger a stoppage up to time τ and opportunity arrives at time τ. Similarly for $p_s^{(i)}(\tau, 0)$, the term $\frac{P_{s\in\{T,F\}}^{(i)}(0)}{P_T^{(i)}(0)+P_F^{(i)}(0)}$ approximates the probability that the triggering machine i has the cycle class $s \in \{T, F\}$, having the status at the system stoppage $\varpi^{(i)} \in \{0, 1\}$, and the term $\gamma^{(i)} e^{-\gamma^{(i)}\tau}$ approximates the probability that the machine i triggers a stoppage at time τ. Hence we have introduced our first key assumption of approximation at this point. Specifically, we assume that, for each machine, time from the system start to the stoppage time of each machine is exponentially distributed. Since the time until the shift to the out-of-control state is exponentially distributed and the power of the control chart is high enough (implying that the probability of correct inference about the process status at a sampling instant is close to 1), it is reasonable to approximate the time to stoppage by an exponential distribution. Then;

$$
\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \approx\; & \sum_{n_1^{(j)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor} \int_{x=n_1^{(j)}h^{(j)}}^{(n_1^{(j)}+1)h^{(j)}} (1-\alpha^{(j)})^{n_1^{(j)}} \left[\beta^{(j)}\right]^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor - n_2^{(j)} - 1\right)} \\
& \times (1-\beta^{(j)})\lambda^{(j)} e^{-\lambda^{(j)}x} dx \\
& \times \prod_{i\in\Psi_{OF}^{(j)}} \left\{ \frac{\left(\Gamma-\gamma^{(i)}\right) e^{-\left(\Gamma-\gamma^{(i)}\right)\tau^{(j)}} e^{-\gamma^{(i)}\tau^{(j)}}}{\left(\Gamma-\gamma^{(i)}\right) e^{-\left(\Gamma-\gamma^{(i)}\right)\tau^{(j)}}} \right. \\
& \qquad \left. \times \frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right) + P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \right\} \\
& \times \prod_{k\in\Psi_{OT}^{(j)}} \left\{ \frac{\left(\Gamma-\gamma^{(k)}\right) e^{-\left(\Gamma-\gamma^{(k)}\right)\tau^{(j)}} e^{-\gamma^{(k)}\tau^{(j)}}}{\left(\Gamma-\gamma^{(k)}\right) e^{-\left(\Gamma-\gamma^{(k)}\right)\tau^{(j)}}} \right. \\
& \qquad \left. \times \frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right) + P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)} \right\}
\end{aligned}
\tag{5.35}
$$

$$
\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \approx\; & \sum_{n_1^{(j)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor} \int_{x=n_1^{(j)}h^{(j)}}^{(n_1^{(j)}+1)h^{(j)}} (1-\alpha^{(j)})^{n_1^{(j)}} \left[\beta^{(j)}\right]^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor - n_2^{(j)} - 1\right)} \\
& \times (1-\beta^{(j)})\lambda^{(j)} e^{-\lambda^{(j)}x} dx \\
& \times e^{\left(\Gamma-\gamma^{(j)}\right)\tau^{(j)}} \prod_{i\in\Psi_{OF}^{(j)}} \frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right) + P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \\
& \times \prod_{k\in\Psi_{OT}^{(j)}} \frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right) + P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}
\end{aligned}
$$

$$
\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \;\approx\;& p_T^{(j)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(i)}\right)\right) \\
&\times \prod_{i\in\Psi_{OF}^{(j)}} \frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right)+P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \\
&\times \prod_{k\in\Psi_{OT}^{(j)}} \frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)+P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}
\end{aligned} \tag{5.36}
$$

We define event $\{E_F^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\}$ to be a cycle with operating time length $\tau^{(j)}$, ends when machine j triggers a system-wide stoppage by signaling a false alarm. Following the similar derivation procedure and approximations we obtain:

$$
\begin{aligned}
\Pr\{E_F^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \;\approx\;& p_F^{(j)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(i)}\right)\right) \\
&\times \prod_{i\in\Psi_{OF}^{(j)}} \frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right)+P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \\
&\times \prod_{k\in\Psi_{OT}^{(j)}} \frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)+P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}
\end{aligned} \tag{5.37}
$$

Thus, we can approximate the probability of event $\{E_s^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\}$ by the probability that: machine j, which is a single machine facing exogenous opportunity rate $\left(\Gamma-\gamma^{(j)}\right)$, has the cycle class s (T or F) at time $\tau^{(j)}$; and, each machine i of those in the set $\Psi_{OF}^{(j)}$, which is a single machine facing exogenous opportunity rate $\left(\Gamma-\gamma^{(i)}\right)$, is stopped exogenously and is in the in-control status at the time of stoppage; and, each machine k of those in the set Ψ_{OT}^{j} , which is a single machine facing exogenous opportunity rate $\left(\Gamma-\gamma^{(k)}\right)$, is stopped exogenously and is in the out–of-control status at the time of stoppage.

Next, consider in isolation one of the machines that have been stopped exogenously, say of index m. Then, we can approximate the probability of event $\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\}$ as follows:

$$
\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \quad = \quad & \sum_{n_1^{(j)}=0}^{\left\lfloor\frac{\tau^{(j)}}{h^{(j)}}\right\rfloor} \int_{x=n_1^{(j)}h^{(j)}}^{(n_1^{(j)}+1)h^{(j)}} (1-\alpha^{(j)})^{n_1^{(j)}} \left[\beta^{(j)}\right]^{\left(\left\lfloor\frac{\tau^{(j)}}{h^{(j)}}\right\rfloor - n_2^{(j)}-1\right)} \\
& \times (1-\beta^{(j)})\lambda^{(j)}e^{-\lambda^{(j)}x}dx \\
& \times \left(1-\alpha^{(m)}\right)^{\left\lfloor\frac{\tau^{(j)}}{h^{(m)}}\right\rfloor} e^{-\lambda^{(m)}\tau^{(j)}} \\
& \times \prod_{i\in\Psi_{OF}^{(j)}\setminus m} \left\{ (1-\alpha^{(i)})^{\left\lfloor\frac{\tau^{(j)}}{h^{(i)}}\right\rfloor} e^{-\lambda^{(i)}\tau^{(j)}} \right\} \qquad\qquad (5.38) \\
& \times \prod_{k\in\Psi_{OT}^{(j)}} \left\{ \sum_{n_1^{(k)}=0}^{\left\lfloor\frac{\tau^{(j)}}{h^{(k)}}\right\rfloor} \left[\left(1 - I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right)\right) \right.\right. \\
& \times \int_{x=n_1^{(k)}h^{(k)}}^{(n_1^{(k)}+1)h^{(k)}} (1-\alpha^{(k)})^{n_1^{(k)}} \left[\beta^{(k)}\right]^{\left(\left\lfloor\frac{\tau^{(j)}}{h^{(k)}}\right\rfloor - n_1^{(k)}-1\right)} \lambda^{(k)}e^{-\lambda^{(k)}x}dx \\
& \left.\left. + I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right) \int_{x=n_1^{(k)}h^{(k)}}^{\tau^{(j)}} (1-\alpha^{(k)})^{n_1^{(k)}} \lambda^{(k)}e^{-\lambda^{(k)}x}dx \right] \right\}
\end{aligned}
$$

Multiply and divide the probability function of each machine $i \in \Psi_{OF}^{(j)}\setminus\{m\}$ and $i \in \Psi_{OT}^{(j)}$ by $\left(\Gamma-\gamma^{(i)}\right)e^{-\left(\Gamma-\gamma^{(i)}\right)\tau^{(j)}}$; and machine m by $\left(\Gamma-\gamma^{(m)}\right)$ we get:

$$
\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \;=\; & \sum_{n_1^{(j)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor} \int_{x=n_1^{(j)}h^{(j)}}^{(n_1^{(j)}+1)h^{(j)}} (1-\alpha^{(j)})^{n_1^{(j)}} \left[\beta^{(j)}\right]^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor - n_2^{(j)} - 1\right)} \\
& \times (1-\beta^{(j)})\lambda^{(j)} e^{-\lambda^{(j)}x} dx \\
& \times \left(1-\alpha^{(m)}\right)^{\left\lfloor \frac{\tau^{(j)}}{h^{(m)}} \right\rfloor} e^{-\lambda^{(m)}\tau^{(j)}} \frac{\left(\Gamma-\gamma^{(m)}\right)}{\left(\Gamma-\gamma^{(m)}\right)} \\
& \times \prod_{i\in\Psi_{OF}^{(j)}\setminus m} \left\{ (1-\alpha^{(i)})^{\left\lfloor \frac{\tau^{(j)}}{h^{(i)}} \right\rfloor} e^{-\lambda^{(i)}\tau^{(j)}} \frac{\left(\Gamma-\gamma^{(i)}\right) e^{-\left(\Gamma-\gamma^{(i)}\right)\tau^{(j)}}}{\left(\Gamma-\gamma^{(i)}\right) e^{-\left(\Gamma-\gamma^{(i)}\right)\tau^{(j)}}} \right\} \\
& \times \prod_{k\in\Psi_{OT}^{(j)}} \left\{ \sum_{n_1^{(k)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(k)}} \right\rfloor} \left[\left(1 - I\left(\left(n_1^{(k)}+1\right) h^{(k)}\right)\right) \right.\right. \qquad (5.39) \\
& \times \int_{x=n_1^{(k)}h^{(k)}}^{(n_1^{(k)}+1)h^{(k)}} (1-\alpha^{(k)})^{n_1^{(k)}} \left[\beta^{(k)}\right]^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(k)}} \right\rfloor - n_1^{(k)} - 1\right)} \lambda^{(k)} e^{-\lambda^{(k)}x} dx \\
& \times \frac{\left(\Gamma-\gamma^{(k)}\right) e^{-\left(\Gamma-\gamma^{(k)}\right)\tau^{(j)}}}{\left(\Gamma-\gamma^{(k)}\right) e^{-\left(\Gamma-\gamma^{(k)}\right)\tau^{(j)}}} + I\left(\left(n_1^{(k)}+1\right) h^{(k)}\right) \\
& \left.\left. \times \int_{x=n_1^{(k)}h^{(k)}}^{\tau^{(j)}} (1-\alpha^{(k)})^{n_1^{(k)}} \lambda^{(k)} e^{-\lambda^{(k)}x} dx \frac{\left(\Gamma-\gamma^{(k)}\right) e^{-\left(\Gamma-\gamma^{(k)}\right)\tau^{(j)}}}{\left(\Gamma-\gamma^{(k)}\right) e^{-\left(\Gamma-\gamma^{(k)}\right)\tau^{(j)}}} \right] \right\}
\end{aligned}
$$

By using the approximations we have introduced above, we get:

$$
\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \;=\; & \sum_{n_1^{(j)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor} \int_{x=n_1^{(j)}h^{(j)}}^{(n_1^{(j)}+1)h^{(j)}} (1-\alpha^{(j)})^{n_1^{(j)}} \left[\beta^{(j)}\right]^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor - n_2^{(j)} - 1\right)} \\
& \times (1-\beta^{(j)})\lambda^{(j)} e^{-\lambda^{(j)}x} dx \\
& \times \left(1-\alpha^{(m)}\right)^{\left\lfloor \frac{\tau^{(j)}}{h^{(m)}} \right\rfloor} e^{-\lambda^{(m)}\tau^{(j)}} \frac{\left(\Gamma-\gamma^{(m)}\right)}{\left(\Gamma-\gamma^{(m)}\right)} \\
& \times \prod_{i\in\Psi_{OF}^{(j)}\setminus\{m\}} \left\{ \frac{\left(\Gamma-\gamma^{(i)}\right) e^{-\left(\Gamma-\gamma^{(i)}\right)\tau^{(j)}} e^{-\gamma^{(i)}\tau^{(j)}}}{\left(\Gamma-\gamma^{(i)}\right) e^{-\left(\Gamma-\gamma^{(i)}\right)\tau^{(j)}}} \right. \\
& \qquad \left. \times \frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right) + P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \right\} \\
& \times \prod_{k\in\Psi_{OT}^{(j)}} \left\{ \frac{\left(\Gamma-\gamma^{(k)}\right) e^{-\left(\Gamma-\gamma^{(k)}\right)\tau^{(j)}} e^{-\gamma^{(k)}\tau^{(j)}}}{\left(\Gamma-\gamma^{(k)}\right) e^{-\left(\Gamma-\gamma^{(k)}\right)\tau^{(j)}}} \right. \\
& \qquad \left. \times \frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right) + P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)} \right\}
\end{aligned} \tag{5.40}
$$

$$
\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \;=\; & \sum_{n_1^{(j)}=0}^{\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor} \int_{x=n_1^{(j)}h^{(j)}}^{(n_1^{(j)}+1)h^{(j)}} (1-\alpha^{(j)})^{n_1^{(j)}} \left[\beta^{(j)}\right]^{\left(\left\lfloor \frac{\tau^{(j)}}{h^{(j)}} \right\rfloor - n_2^{(j)} - 1\right)} \\
& \left. \times (1-\beta^{(j)})\lambda^{(j)} e^{-\lambda^{(j)}x} \right] dx \\
& \times \left(1-\alpha^{(m)}\right)^{\left\lfloor \frac{\tau^{(j)}}{h^{(m)}} \right\rfloor} e^{-\lambda^{(m)}\tau^{(j)}} \frac{\left(\Gamma-\gamma^{(m)}\right)}{\left(\Gamma-\gamma^{(m)}\right)} e^{-\left(\Gamma-\gamma^{(j)}-\gamma^{(m)}\right)\tau^{(j)}} \\
& \times \prod_{i\in\Psi_{OF}^{(j)}\setminus\{m\}} \frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right) + P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \\
& \times \prod_{k\in\Psi_{OT}^{(j)}} \frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right) + P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}
\end{aligned} \tag{5.41}
$$

Approximating the probability associated with machine j by

$$\gamma^{(j)} e^{-\gamma^{(j)}\tau^{(j)}} \frac{P_T^{(j)}(0)}{P_T^{(j)}(0) + P_F^{(j)}(0)}$$

we get:

$$\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi}, \boldsymbol{\varpi}\} \approx\ & \gamma^{(j)} e^{-\gamma^{(j)}\tau^{(j)}} \frac{P_T^{(j)}(0)}{P_T^{(j)}(0) + P_F^{(j)}(0)} \frac{e^{\gamma^{(j)}\tau^{(j)}}}{(\Gamma - \gamma^{(m)})} \\
& \times \left(1 - \alpha^{(m)}\right)^{\left\lfloor \frac{\tau^{(j)}}{h^{(m)}} \right\rfloor} e^{-\lambda^{(m)}\tau^{(j)}} \left(\Gamma - \gamma^{(m)}\right) e^{-\left(\Gamma - \gamma^{(m)}\right)\tau^{(j)}} \\
& \times \prod_{i \in \Psi_{OF}^{(j)} \setminus \{m\}} \frac{P_{OF}^{(i)}\left(\Gamma - \gamma^{(i)}\right)}{P_{OT}^{(i)}(\Gamma - \gamma^{(i)}) + P_{OF}^{(i)}(\Gamma - \gamma^{(i)})} \\
& \times \prod_{k \in \Psi_{OT}^{(j)}} \frac{P_{OT}^{(k)}\left(\Gamma - \gamma^{(k)}\right)}{P_{OT}^{(k)}(\Gamma - \gamma^{(k)}) + P_{OF}^{(k)}(\Gamma - \gamma^{(k)})}
\end{aligned} \tag{5.42}$$

$$\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi}, \boldsymbol{\varpi}\} \approx\ & \left(1 - \alpha^{(m)}\right)^{\left\lfloor \frac{\tau^{(j)}}{h^{(m)}} \right\rfloor} e^{-\lambda^{(m)}\tau^{(j)}} \left(\Gamma - \gamma^{(m)}\right) e^{-\left(\Gamma - \gamma^{(m)}\right)\tau^{(j)}} \\
& \times \frac{\gamma^{(j)}}{(\Gamma - \gamma^{(m)})} \frac{P_T^{(j)}(0)}{P_T^{(j)}(0) + P_F^{(j)}(0)} \\
& \times \prod_{i \in \Psi_{OF}^{(j)} \setminus \{m\}} \frac{P_{OF}^{(i)}\left(\Gamma - \gamma^{(i)}\right)}{P_{OT}^{(i)}(\Gamma - \gamma^{(i)}) + P_{OF}^{(i)}(\Gamma - \gamma^{(i)})} \\
& \times \prod_{k \in \Psi_{OT}^{(j)}} \frac{P_{OT}^{(k)}\left(\Gamma - \gamma^{(k)}\right)}{P_{OT}^{(k)}(\Gamma - \gamma^{(k)}) + P_{OF}^{(k)}(\Gamma - \gamma^{(k)})}
\end{aligned} \tag{5.43}$$

$$
\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \approx\ & p_{OF}^{(j)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(i)}\right)\right) \\
& \times\frac{\gamma^{(j)}}{\left(\Gamma-\gamma^{(m)}\right)}\frac{P_T^{(j)}(0)}{P_T^{(j)}(0)+P_F^{(j)}(0)} \\
& \times\prod_{i\in\Psi_{OF}^{(j)}\setminus\{m\}}\frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right)+P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \\
& \times\prod_{k\in\Psi_{OT}^{(j)}}\frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)+P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}
\end{aligned}
\tag{5.44}
$$

Similarly for the machine $m\in\Psi_{OT}^{j}$ in isolation the event probability is:

$$
\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \approx\ & \sum_{n_1^{(m)}=0}^{\left\lfloor\frac{\tau^{(j)}}{h^{(m)}}\right\rfloor}\Bigg[\left(1-I\left(\left(n_1^{(m)}+1\right)h^{(m)}\right)\right) \\
& \times\int_{x=n_1^{(m)}h^{(m)}}^{\left(n_1^{(m)}+1\right)h^{(m)}}\left(1-\alpha^{(m)}\right)^{n_1^{(m)}}\left[\beta^{(m)}\right]^{\left(\left\lfloor\frac{\tau^{(j)}}{h^{(m)}}\right\rfloor-n_1^{(m)}-1\right)} \\
& \times\lambda^{(m)}e^{-\lambda^{(m)}x}\left(\Gamma-\gamma^{(m)}\right)e^{-\left(\Gamma-\gamma^{(m)}\right)\tau^{(j)}}dx \\
& +I\left(\left(n_1^{(m)}+1\right)h^{(m)}\right)\int_{x=n_1^{(m)}h^{(m)}}^{\tau^{(j)}}\left(1-\alpha^{(m)}\right)^{n_1^{(m)}} \\
& \times\lambda^{(m)}e^{-\lambda^{(m)}x}\left(\Gamma-\gamma^{(m)}\right)e^{-\left(\Gamma-\gamma^{(m)}\right)\tau^{(j)}}dx\Bigg] \\
& \times\frac{\gamma^{(j)}}{\left(\Gamma-\gamma^{(m)}\right)}\frac{P_T^{(j)}(0)}{P_T^{(j)}(0)+P_F^{(j)}(0)} \\
& \times\prod_{i\in\Psi_{OF}^{(j)}}\frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right)+P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \\
& \times\prod_{k\in\Psi_{OT}^{(j)}\setminus\{m\}}\frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)+P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}
\end{aligned}
\tag{5.45}
$$

$$
\begin{aligned}
\Pr\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \approx\ & p_{OT}^{(j)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(i)}\right)\right) \\
& \times\frac{\gamma^{(j)}}{(\Gamma-\gamma^{(m)})}\frac{P_T^{(j)}(0)}{P_T^{(j)}(0)+P_F^{(j)}(0)} \\
& \times\prod_{i\in\Psi_{OF}^{(j)}}\frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}(\Gamma-\gamma^{(i)})+P_{OF}^{(i)}(\Gamma-\gamma^{(i)})} \\
& \times\prod_{k\in\Psi_{OT}^{(j)}\setminus\{m\}}\frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}(\Gamma-\gamma^{(k)})+P_{OF}^{(k)}(\Gamma-\gamma^{(k)})}
\end{aligned}
\tag{5.46}
$$

Above probabilities for machine m in the event $\{E_F^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\}$ are as follows:

$$
\begin{aligned}
\Pr\{E_F^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \approx\ & p_{OF}^{(j)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(i)}\right)\right) \\
& \times\frac{\gamma^{(j)}}{(\Gamma-\gamma^{(m)})}\frac{P_F^{(j)}(0)}{P_T^{(j)}(0)+P_F^{(j)}(0)} \\
& \times\prod_{i\in\Psi_{OF}^{(j)}}\frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}(\Gamma-\gamma^{(i)})+P_{OF}^{(i)}(\Gamma-\gamma^{(i)})} \\
& \times\prod_{k\in\Psi_{OT}^{(j)}\setminus\{m\}}\frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}(\Gamma-\gamma^{(k)})+P_{OF}^{(k)}(\Gamma-\gamma^{(k)})}
\end{aligned}
\tag{5.47}
$$

$$
\begin{aligned}
\Pr\{E_F^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \approx\ & p_{OF}^{(j)}\left(\tau^{(j)}, \left(\Gamma-\gamma^{(i)}\right)\right) \\
& \times \frac{\gamma^{(j)}}{\left(\Gamma-\gamma^{(m)}\right)} \frac{P_F^{(j)}(0)}{P_T^{(j)}(0)+P_F^{(j)}(0)} \\
& \times \prod_{i\in\Psi_{OF}^{(j)}} \frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right)+P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \\
& \times \prod_{k\in\Psi_{OT}^{(j)}\setminus\{m\}} \frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)+P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}
\end{aligned}
\tag{5.48}
$$

$$
\begin{aligned}
\Pr\{E_F^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\} \approx\ & p_{OT}^{(j)}\left(\tau^{(j)}, \left(\Gamma-\gamma^{(i)}\right)\right) \\
& \times \frac{\gamma^{(j)}}{\left(\Gamma-\gamma^{(m)}\right)} \frac{P_F^{(j)}(0)}{P_T^{(j)}(0)+P_F^{(j)}(0)} \\
& \times \prod_{i\in\Psi_{OF}^{(j)}\setminus\{m\}} \frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right)+P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \\
& \times \prod_{k\in\Psi_{OT}^{(j)}} \frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)+P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}
\end{aligned}
\tag{5.49}
$$

Thus, we can approximate the probability of event $\{E_s^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\}$ by the probability that: machine m, which is a single machine facing exogenous opportunity rate $\left(\Gamma-\gamma^{(m)}\right)$, is stopped at time $\tau^{(j)}$, and its cycle class is of OT or OF at the time of stoppage; and, the stoppage was signaled by machine j; and, machine j, which is a single machine facing no exogenous opportunities with a cycle class s

(T or F); and, each machine i of those in the set Ψ^{j}_{OF} , which is a single machine facing exogenous opportunity rate $\left(\Gamma - \gamma^{(i)}\right)$, is stopped exogenously and is in the in-control status at the time of stoppage; and, each machine k of those in the set Ψ^{j}_{OT} , which is a single machine facing exogenous opportunity rate $\left(\Gamma - \gamma^{(k)}\right)$, is stopped exogenously and is in the out-of-control status at the time of stoppage.

Similarly, we can write the expected cost incurred during the cycle which ends with the occurrence of an event. Let $\{C^{(j)}_{T}|\boldsymbol{\phi}, \boldsymbol{\varpi}, \tau^{(j)}\}$ and $\{C^{(j)}_{F}|\boldsymbol{\phi}, \boldsymbol{\varpi}, \tau^{(j)}\}$ be the costs associated with the events $\{E^{(j)}_{T}|\boldsymbol{\phi}, \boldsymbol{\varpi}\}$ and $\{E^{(j)}_{F}|\vec{\phi}^{(i)}\}$ respectively. Then, we can write the expected cost of these events as follows:

$$
\begin{aligned}
E\left[C_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi},\tau^{(j)}\right] &= \sum_{n_1^{(j)}=0}^{\left\lfloor\frac{\tau^{(j)}}{h^{(j)}}\right\rfloor}\int_{x=n_1^{(j)}h^{(j)}}^{(n_1^{(j)}+1)h^{(j)}} C\left(\begin{array}{c}\tau^{(j)},T,n_1^{(j)},\left(\left\lfloor\frac{\tau^{(j)}}{h^{(j)}}\right\rfloor-n_1^{(j)}\right)\\ ,x,\left(z^{(j)}>\tau^{(j)}\right),\boldsymbol{\varpi}\end{array}\right)\\
&\times(1-\alpha^{(j)})^{n_1^{(j)}}\left[\beta^{(j)}\right]^{(n_2^{(j)}-1)}(1-\beta^{(j)})\lambda^{(j)}e^{-\lambda^{(j)}x}dx\\
&\times\prod_{i\in\Psi_{OF}^{(j)}}\left\{C\left(\tau^{(j)},OF,\left\lfloor\frac{\tau^{(j)}}{h^{(i)}}\right\rfloor,0,x>\tau^{(j)},\tau^{(j)},\boldsymbol{\varpi}\right)\right.\\
&\left.\times(1-\alpha^{(i)})^{\left\lfloor\frac{\tau^{(j)}}{h^{(i)}}\right\rfloor}e^{-\lambda^{(i)}\tau^{(j)}}\right\}\\
&\times\prod_{k\in\Psi_{OT}^{(j)}}\left\{\sum_{n_1^{(k)}=0}^{\left\lfloor\frac{\tau^{(j)}}{h^{(k)}}\right\rfloor}\left[\left(1-I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right)\right)\right.\right. \qquad (5.50)\\
&\times\int_{x=n_1^{(k)}h^{(k)}}^{(n_1^{(k)}+1)h^{(k)}} C\left(\tau^{(j)},OT,n_1^{(k)},\left\lfloor\frac{\tau^{(j)}}{h^{(k)}}\right\rfloor-n_1^{(k)},x,\tau^{(j)},\boldsymbol{\varpi}\right)\\
&\times(1-\alpha^{(k)})^{n_1^{(k)}}\left[\beta^{(k)}\right]^{\left(\left\lfloor\frac{\tau^{(j)}}{h^{(k)}}\right\rfloor-n_1^{(k)}-1\right)}\lambda^{(k)}e^{-\lambda^{(k)}x}dx\\
&+I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right)\int_{x=n_1^{(k)}h^{(k)}}^{\tau^{(j)}} C\left(\tau^{(j)},OT,n_1^{(k)},0,x,\tau^{(j)},\boldsymbol{\varpi}\right)\\
&\left.\left.\times(1-\alpha^{(k)})^{n_1^{(k)}}\lambda^{(k)}e^{-\lambda^{(k)}x}dx\right]\right\}
\end{aligned}
$$

and,

$$
\begin{aligned}
E\left[C_F^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi},\tau^{(j)}\right] \;=\;& C\left(\tau^{(j)},F,\left\lfloor\frac{\tau^{(j)}}{h^{(j)}}\right\rfloor,0,x>\tau^{(j)},\left(z^{(j)}>\tau^{(j)}\right),\boldsymbol{\varpi}\right)\\
&\times\alpha^{(j)}(1-\alpha^{(j)})^{\left(\left\lfloor\frac{\tau^{(j)}}{h^{(j)}}\right\rfloor-1\right)}e^{-\lambda^{(j)}\tau^{(j)}}\\
&\times\prod_{i\in\Psi_{OF}^{(j)}}\left\{C\left(\tau^{(j)},OF,\left\lfloor\frac{\tau^{(j)}}{h^{(j)}}\right\rfloor,0,x>\tau^{(j)},\tau^{(j)},\boldsymbol{\varpi}\right)\right.\\
&\left.\times\left(1-\alpha^{(i)}\right)^{\left\lfloor\frac{\tau^{(j)}}{h^{(i)}}\right\rfloor}e^{-\lambda^{(i)}\tau^{(j)}}\right\} \qquad (5.51)\\
&\times\prod_{k\in\Psi_{OT}^{(j)}}\left\{\sum_{n_1^{(k)}=0}^{\left\lfloor\frac{\tau^{(j)}}{h^{(k)}}\right\rfloor}\left[\left(1-I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right)\right)\right.\right.\\
&\times\int_{x=n_1^{(k)}h^{(k)}}^{(n_1^{(k)}+1)h^{(k)}}C\left(\tau^{(j)},OT,n_1^{(k)},\left\lfloor\frac{\tau^{(j)}}{h^{(k)}}\right\rfloor-n_1^{(k)},x,\tau^{(j)},\boldsymbol{\varpi}\right)\\
&\times(1-\alpha^{(k)})^{n_1^{(k)}}\left[\beta^{(k)}\right]^{\left(\left\lfloor\frac{\tau^{(j)}}{h^{(k)}}\right\rfloor-n_1^{(k)}-1\right)}\lambda^{(k)}e^{-\lambda^{(k)}x}dx\\
&+I\left(\left(n_1^{(k)}+1\right)h^{(k)}\right)\int_{x=n_1^{(k)}h^{(k)}}^{\tau^{(j)}}C\left(\tau^{(j)},OT,n_1^{(k)},0,x,\tau^{(j)},\boldsymbol{\varpi}\right)\\
&\left.\left.\times\left(1-\alpha^{(k)}\right)^{n_1^{(k)}}\lambda^{(k)}e^{-\lambda^{(k)}x}dx\right]\right\}
\end{aligned}
$$

Following the steps in the event probability generation and approximation we get:

$$E\left[C_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi},\tau^{(j)}\right] \approx C_T^{(j)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(i)}\right)\right) \times \prod_{i\in\Psi_{OF}^{(j)}} \frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right)+P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \times \prod_{k\in\Psi_{OT}^{(j)}} \frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)+P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)} \tag{5.52}$$

$$E\left[C_F^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi},\tau^{(j)}\right] \approx C_F^{(j)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(i)}\right)\right) \times \prod_{i\in\Psi_{OF}^{(j)}} \frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right)+P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \times \prod_{k\in\Psi_{OT}^{(j)}} \frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)+P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)} \tag{5.53}$$

For the machine $m \in \Psi_{s\in\{OT,OF\}}^{(j)}$ in isolation the expected cost of event $\{E_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\}$ is:

$$
\begin{aligned}
E\left[C_T^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi},\tau^{(j)}\right] &\approx C_s^{(m)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(m)}\right)\right) \\
&\times\frac{\gamma^{(j)}}{\left(\Gamma-\gamma^{(m)}\right)}\frac{P_T^{(j)}(0)}{P_T^{(j)}(0)+P_F^{(j)}(0)} \\
&\times\prod_{i\in\Psi_{OF}^{(j)}\setminus\{m\}}\frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right)+P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \\
&\times\prod_{k\in\Psi_{OT}^{(j)}\setminus\{m\}}\frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)+P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}
\end{aligned} \tag{5.54}
$$

For the machine $m\in\Psi_{s\in\{OT,OF\}}^{(j)}$ in isolation the expected cost of event $\{E_F^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi}\}$ is:

$$
\begin{aligned}
E\left[C_F^{(j)}|\boldsymbol{\phi},\boldsymbol{\varpi},\tau^{(j)}\right] &\approx C_s^{(m)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(m)}\right)\right) \\
&\times\frac{\gamma^{(j)}}{\left(\Gamma-\gamma^{(m)}\right)}\frac{P_F^{(j)}(0)}{P_T^{(j)}(0)+P_F^{(j)}(0)} \\
&\times\prod_{i\in\Psi_{OF}^{(j)}\setminus\{m\}}\frac{P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)}{P_{OT}^{(i)}\left(\Gamma-\gamma^{(i)}\right)+P_{OF}^{(i)}\left(\Gamma-\gamma^{(i)}\right)} \\
&\times\prod_{k\in\Psi_{OT}^{(j)}\setminus\{m\}}\frac{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}{P_{OT}^{(k)}\left(\Gamma-\gamma^{(k)}\right)+P_{OF}^{(k)}\left(\Gamma-\gamma^{(k)}\right)}
\end{aligned} \tag{5.55}
$$

Define

$$\begin{aligned}
\hat{\Xi}(s;m;\psi_{OF}^{(j)}\ ;\psi_{OT}^{(j)}) \quad : &= \frac{\gamma^{(j)}}{(\Gamma-\gamma^{(m)})}\frac{P_s^{[j]}(0)}{P_T^{[j]}(0)+P_F^{[j]}(0)} \\
&\times \prod_{\substack{i\in\psi_{OF}^j \\ i\neq m}} \frac{P_{OF}^{[i]}(\Gamma-\gamma^{[i]})}{P_{OT}^{[i]}(\Gamma-\gamma^{[i]})+P_{OF}^{[i]}(\Gamma-\gamma^{[i]})} \qquad (5.56) \\
&\times \prod_{\substack{k\in\psi_{OT}^j \\ k\neq m}} \frac{P_{OT}^{[k]}(\Gamma-\gamma^{[k]})}{P_{OT}^{[k]}(\Gamma-\gamma^{[k]})+P_{OF}^{[k]}(\Gamma-\gamma^{[k]})} \qquad (5.57)
\end{aligned}$$

$\left(\text{for } \hat{\Xi} : s \in \{T,F\}\,, m \neq j, m \notin \{\psi_{OF}^j \cup \psi_{OT}^j\}\right)$

and

$$\begin{aligned}
\Xi(s;j;\psi_{OF}^{(j)};\psi_{OT}^{(j)}) \quad : &= \prod_{i\in\psi_{OF}^j} \frac{P_{OF}^{[i]}(\Gamma-\gamma^{[i]})}{P_{OT}^{[i]}(\Gamma-\gamma^{[i]})+P_{OF}^{[i]}(\Gamma-\gamma^{[i]})} \qquad (5.58) \\
&\times \prod_{k\in\psi_{OT}^j} \frac{P_{OT}^{[k]}(\Gamma-\gamma^{[k]})}{P_{OT}^{[k]}(\Gamma-\gamma^{[k]})+P_{OF}^{[k]}(\Gamma-\gamma^{[k]})}
\end{aligned}$$

$\left(\text{for } \hat{\Xi} : s \in \{T,F\}\right)$

The expected cycle cost is, then, the sum over all machines of the expected cycle costs that each machine, say j incurs (i) when it stops the system and (ii) when it is stopped exogenously by another machine. Then, integrating and summing over the appropriate variable domains, the expected cycle cost $E\left[CC\right]$ can be written as:

$$
\begin{aligned}
E\left[CC\right] \approx & \sum_{j\in M}\left[\sum_{s\in\{T,F\}}\int_{\tau^{(j)}} C_s^{(j)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(i)}\right)\right)\Xi(s;j;\psi_{OF}^{(j)};\psi_{OT}^{(j)})\,d\tau^{(j)}\right.\\
& +\sum_{m=(i\in M\setminus\{j\})}\sum_{s'\in\{OT,OF\}}\sum_{s\in\{T,F\}}\int_{\tau^{(j)}} C_{s'}^{(m)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(m)}\right)\right) \qquad (5.59)\\
& \left.\times\,\hat{\Xi}(s;m;\psi_{OF}^{(j)};\psi_{OT}^{(j)})d\tau^{(j)}\right]
\end{aligned}
$$

Similarly expected operating time in a cycle $E\left[\tau\right]$ can be written as follows:

$$
\begin{aligned}
E\left[\tau\right] \approx & \sum_{j=(i\in M)}\left[\sum_{s\in\{T,F\}}\int_{\tau^{(j)}} \tau^{(j)}p_s^{(j)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(i)}\right)\right)\Xi(s;j;\psi_{OF}^{(j)};\psi_{OT}^{(j)})\,d\tau^{(j)}\right.\\
& +\sum_{m=(i\in M\setminus\{j\})}\sum_{s'\in\{OT,OF\}}\sum_{s\in\{T,F\}}\int_{\tau^{(j)}} \tau^{(j)}p_{s'}^{(j)}\left(\tau^{(j)},\left(\Gamma-\gamma^{(i)}\right)\right) \qquad (5.60)\\
& \left.\times\,\hat{\Xi}(s;m;\psi_{OF}^{(j)};\psi_{OT}^{(j)})d\tau^{(j)}\right]
\end{aligned}
$$

Note that since all of the machines are opportunity-takers, they start and end their cycles simultaneously; therefore, the expected cycle length for all machines are identical.

The fact that all expected cycle lengths for individual machines are identical allows us to express the expected cost rate for the whole system as the sum of the expected cost rates for individual machines. Hence,

$$E[TC] \approx \frac{E[CC]}{E[\tau]} = \sum_{i=1}^{m} \frac{E[CC^{(i)}]}{E[\tau^{(i)}]} = \sum_{i=1}^{m} \frac{E[CC^{(i)}]}{E[\tau]} \tag{5.61}$$

5.2.3 Mixed case

Finally, consider the cases where some of the machines are opportunity takers and some of them are opportunity non-takers, i.e. $M_{NTK} \neq \emptyset$ and $M_{TK} \neq \emptyset$. In the presence of opportunity taker and opportunity non-taker machines together in the system, system regeneration points are similar to those in the all non-taker case, i.e. LCM instances constitute the regeneration points. Each system restart instant is still a regeneration point for the opportunity taker machines, in which opportunity non-taker machines maintain their existing states. In the mixed case system stoppage can be triggered either by an opportunity taker machine or by an opportunity non-taker machine. Due to the complex structure of the exact model described in the previous section we will develop an approximate model.

Before proceeding any further, we will demonstrate the mixed case by an example. Suppose in a system with three machines, machines #1 and #2 are opportunity takers and #3 is an opportunity non-taker machine. Suppose, at time t_1 machine #1 raises a false alarm, and suppose also that, machines #2 and #3 have already shifted to their out-of-control status without having been detected before the system is stopped. Since machine #2 is an opportunity taker, it is inspected and restored to the in-control state. However, machine #3, being a non-taker is not inspected; therefore, it remains in the out-of-control state. Hence, when the system is started, machines #1 and #2 will be in the in-control state, but machine #3 will be in the out-of-control state. Therefore, this stoppage instant does not constitute a regeneration point.

Let's demonstrate what happens when machine #3 triggers the system stoppage. The stoppage triggered by machine #3 presents an inspection opportunity for machines #1 and #2 since they are opportunity takers, they will be inspected and repaired along with the restoration of machine #3. All three of them will be in-control when the production is resumed. The stoppage triggered by machine #3 is a regeneration point for the system, however in the existence of more than one opportunity non-taker machines, any stoppage wouldn't be regeneration point, unless coincidentally.

For the mixed case we will use the Renewal Reward Theoretic approach to model the system. We will use an approximation for the computation of the system cost rate. For the opportunity taker machines we will use the cost rate computation described in the "all taker" section, and for the opportunity non-takers we will compute their individual cost rates as described in all non-taker case. To obtain the overall cost rate of the system we will add up all these cost rates.

$$E[TC] = \sum_{i \in M_{TK}} \frac{E[CC^{(i)}]}{E[\tau]} + \sum_{i \in M_{NTK}} \frac{\lim_{\mu^{(i)} \to 0} E[CC^{(i)}]}{\lim_{\mu^{(i)} \to 0} E[\tau^{(i)}]} \tag{5.62}$$

5.3 Computation of Stoppage Rates

Previously, we have introduced the notion of stoppage rate generated by machine $i\,(\in M)$, and the notation $\gamma^{(i)}$. The stoppage rate generated for a machine is a function of its operating time, such that;

$$\gamma^{(i)} = \frac{1}{E[\tau^{(i)}]} \tag{5.63}$$

As discussed above, when all the machines in the system are opportunity takers then every machine stoppage is a regeneration point for all the machines. Therefore the expected operating time for machine i is the sum of the expected operating time of its true cycle and the expected operating time of its false cycle, i.e. $E[\tau^{(i)}] = E[\tau_T^{(i)}] + E[\tau_F^{(i)}]$.

However, when there are opportunity non-taker machines in the system along with the opportunity takers, system stoppages are no longer regeneration points for all the machines. An opportunity non-taker machine $i \in M_{NTK}$ will maintain its state at a stoppage instant unless it triggers the stoppage itself. Moreover, due to the fact that there is no restriction on $h^{(i)}$, if system stops at t time units after the last sampling of machine i, when the system restarts, first sampling interval will be in $\eta^{(i)} = h^{(i)} - t$ time units for machine i. Therefore, following a system-wide shutdown the operating time for the opportunity non-taker machines will depend on their status and the time remaining to their next sampling. Hence, although for the computation of operating times of the opportunity taker machines it is sufficient to know that they are opportunity takers , for the opportunity non-taker machines we need information on whether they were in the in-control status or in the out-of-control status at the previous system restart and the time to the first sampling $\eta^{(i)}$. Hence, we will introduce an approximation to overcome the difficulty in operating time computation for the opportunity non-taker machines.

We will approximate the operating times for the opportunity non-taker machines by implementing the "alternating renewal process theorem". Alternating renewal process considers that a system can be in one of two states: *ON* or *OFF*. Initially it is *ON* and it remains on for a time T_1, it then goes *OFF* and remains *OFF* for a time Z_1. It then goes *ON* for a time T_2, then *OFF* for a time Z_1. The process continues in this fashion. Let $Time_{ON}$ denote the time in the *ON* state and $Time_{OFF}$ denote the time in the *OFF* state. Random vectors (T_n, Z_n), $n \geq 1$ are independent and identically distributed. Then the station probability that system is on, P_{ON}, and the system is off, P_{OFF} is given as follows:

$$P_{ON} = \frac{E[Time_{ON}]}{E[Time_{ON}]+E[Time_{OFF}]}; \quad P_{OFF} = \frac{E[Time_{OFF}]}{E[Time_{ON}]+E[Time_{OFF}]}$$

More detailed information on the alternating renewal process can be found in Ross (1993).

In our problem setting, the in-control status corresponds to the *ON* state, and the out-of-control status corresponds to *OFF* state in the alternating renewal process described above. Expected time in the in-control status is given by the expected time to shift, then for machine $i \in M_{NTK}$: $E[Time_{in-control}] = \frac{1}{\lambda^{(i)}}$. The expected time in the out-of-control state is the time interval from occurrence of the shift to the system stoppage. Following the previously introduced notation, after the shift there will be $n_2\,(\geq 1)$ sampling, in $(n_2 - 1)$ of which there will be Type II error and in the last one a correct inference about the process' state. Let t be the time from the last sampling to the shift, then for machine $i \in M_{NTK}$:

$$\begin{aligned}
E[Time_{out-of-control}] &= \int_{t=0}^{h^{(i)}} \sum_{n_2=1}^{\infty} \left[h^{(i)} - t + (n_2 - 1)\, h^{(i)}\right] \left(\beta^{(i)}\right)^{(n_2-1)} \\
&\quad \times \left(1 - \beta^{(i)}\right) \lambda^{(i)} \exp\left[-\lambda^{(i)} t\right] dt \qquad (5.64) \\
&= \frac{h^{(i)}\left(1 - \exp\left[-\lambda^{(i)} h^{(i)}\right]\right)}{1 - \beta^{(i)}} + h^{(i)} \exp\left[-\lambda^{(i)} h^{(i)}\right] \\
&\quad - \frac{\left(1 - \exp\left[-\lambda^{(i)} h^{(i)}\right]\right)}{\lambda}
\end{aligned}$$

For the machines that are in the in-control status at the system restart, expected operating time is given by $E[\tau_T^{(i)}] + E[\tau_F^{(i)}]$ for $i \in M$. An opportunity non-taker machine, however, can be in the out-of-control status at a system restart, then the operating time will be different and computed such:

$$
\begin{aligned}
\lim_{\mu^{(i)}\to 0} E[\tau^{(i)}] &= E\left[\eta^{(i)}\right] + \sum_{n_2=1}^{\infty} (n_2-1)\, h^{(i)} \left(\beta^{(i)}\right)^{(n_2-1)} \left(1-\beta^{(i)}\right) \\
&= \left(h^{(i)}/2\right) + h^{(i)} \left(\frac{\beta^{(i)}}{1-\beta^{(i)}}\right) \qquad (5.65)
\end{aligned}
$$

where $\eta^{(i)}$ denotes the time to the first sampling from a system restart for an opportunity non-taker machine, as before. Note that the occurrence time of the shift, given that it occurs between 0 and $h^{(i)}$ has the probability density function $\frac{1}{h^{(i)}}$, since the event occurrences are distributed uniformly within the given interval. Therefore, $E\left[\eta^{(i)}\right] = h^{(i)}/2$.

Now, we are ready to compute the stoppage rate generated by an opportunity non-taker machine:

$$
\begin{aligned}
\gamma^{(i)} &= P_{ON} \cdot \lim_{\mu^{(i)}\to 0} E[\tau^{(i)}] + P_{OFF} \cdot \lim_{\mu^{(i)}\to 0} E[\tau^{(i)}] \\
&= \frac{P_{ON}}{\lim_{\mu^{(i)}\to 0} E[\tau_T^{(i)}] + \lim_{\mu^{(i)}\to 0} E[\tau_F^{(i)}]} + \frac{P_{OFF}}{(h^{(i)}/2) + \left(\frac{h^{(i)}\beta^{(i)}}{1-\beta^{(i)}}\right)} \qquad (5.66)
\end{aligned}
$$

Hence;

$$\gamma^{(i)} = \begin{cases} \frac{1}{E[\tau_T^{(i)}]+E[\tau_F^{(i)}]} & \text{for } i \in M_{TK} \\ \frac{P_{ON}}{\lim_{\mu^{(i)}\to 0} E[\tau_T^{(i)}]+\lim_{\mu^{(i)}\to 0} E[\tau_F^{(i)}]} + \frac{P_{OFF}}{\left(h^{(i)}/2\right)+\left(\frac{h^{(i)}\beta^{(i)}}{1-\beta^{(i)}}\right)} & \text{for } i \in M_{NTK} \end{cases} \tag{5.67}$$

Note that the operating time of an opportunity non-taker machine is underestimated for the case when it starts in the in-control status. To be precise, for the opportunity non-taker machines we considered that the time to the first sampling instance, $\eta^{(i)}$ is always equal to the fixed sampling interval $h^{(i)}$ when they start the cycle in the in-control status, which is not always the case.

5.4 Repair Times and Costs

In the multi-machine model, the cost of sampling, the cost of operating in the out-of-control state cost and the cost of inspection/repair are incurred by machines individually, identical to the single machine case. But the idleness (lost profit) cost is incurred by the overall system. Therefore, the idleness cost should be incurred by only one machine.

Example: Let's illustrate this situation by the 3-machine-example we have introduced previously, where $M_{TK} = \{1, 2\}$ and $M_{NTK} = \{3\}$.

First suppose machine #1 triggers a stoppage. Let's presume that restoration time of machine #2 is longer than the restoration time of machine #1, then although machine #1 provides an opportunity for the inspection of machine #2, the total time the system remains out of operation is as much as the restoration time of machine #2. Since machine #1 triggers the stoppage it incurs the lost profit cost for the duration until it becomes ready, however machine #2 causes a profit loss for the additional time until it becomes ready. Machine #3 will be waiting until machine #2 is ready for the production.

Next suppose machine #3 triggers the stoppage and let's presume that restoration of machine #2 takes longer than the restoration time of machines #1 and #3, then again lost profit cost will be incurred by machine #3 for the time duration until it becomes ready, but the lost profit cost for the additional time will be incurred by machine #2. No lost profit cost will be incurred by machine #1.

In the single machine model we have introduced the notation L_o for the exogenously given, forced shutdown duration by the opportunities. This parameter was given for the single machine model. However, in the multiple machine model, L_o is a random variable taking values depending on the repair time of the machine j that triggers a system wide stoppage. $L_{OT}^{(i)}$ and $L_{OF}^{(i)}$ are the additional times required to complete the inspection and repair for machine i under consideration if its inspection and repair time is longer than those of all the other opportunity taker machines (except the one triggers the stoppage) in the system.

Define $L_{\max}(\boldsymbol{\varpi}, j)$ as the maximum inspection and repair time of the overall system, when the system stoppage is triggered by machine j and the status of the machines are given by the vector $\boldsymbol{\varpi}$; and define $i_{\max}(\boldsymbol{\varpi}, j)$ as the index of the machine with the maximum inspection and repair time $L_{\max}(\boldsymbol{\varpi}, j)$. Then,

$$L_{\max}(\boldsymbol{\varpi}, j) = \max \left\{ \begin{array}{c} L^{(j)}_{s\in\{T,F\}}, \max\left\{L^{(i)}_T, \forall i \in M_{TK\backslash\{j\}} \text{ and } \varpi^{(i)} = 0\right\}, \\ \max\left\{L^{(i)}_F, \forall i \in M_{TK\backslash\{j\}} \text{ and } \varpi^{(i)} = 1\right\} \end{array} \right\} \tag{5.68}$$

and

$$i_{\max}(\boldsymbol{\varpi}, j) = \arg\max_i \; \{L_{\max}(\boldsymbol{\varpi}, j)\} \tag{5.69}$$

Now we are ready to provide the inspection and repair time parameters for the system. When machine j triggers the system-wide stoppage:

$$L_O = \begin{cases} L^{(j)}_T & \text{if } \varpi^{(j)} = 0 \\ L^{(j)}_F & \text{if } \varpi^{(j)} = 1 \end{cases} \tag{5.70}$$

$$L^{(i)}_{OT} = \begin{cases} \left[L^{(i)}_T - L_o\right]^+ & \text{if } i = i_{\max} \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases} \tag{5.71}$$

and,

$$L^{(i)}_{OF} = \begin{cases} \left[L^{(i)}_F - L_o\right]^+ & \text{if } i = i_{\max} \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases} \tag{5.72}$$

Then the idleness or lost profit cost of the overall system in the multiple machine setting is given by $\pi \cdot L_{\max}(\boldsymbol{\varpi}, j) = \pi \left(L_O + L_s^{(i_{\max})}\right)$ where $s = OT$ if $\varpi^{(i_{\max})} = 0$ and $s = OF$ if $\varpi^{(i_{\max})} = 1$.

Then the cost expression for the stoppage triggering machine $j \in M$ is given by:

$$
\begin{aligned}
&C(\tau^{(j)}, s, n_1^{(j)}, n_2^{(j)}, x^{(j)}, \left(z > \tau^{(j)}\right), \boldsymbol{\varpi}) = \\
&\qquad \left(n_1^{(j)} + n_2^{(j)}\right)\left(y^{(j)} b^{(j)} + u^{(j)}\right) + a\left(\tau^{(j)} - x^{(j)}\right)^+ + \pi L_O + R_s \\
&\qquad\qquad \text{for } j \in M \text{ and } s = T \text{ if } \varpi^{(j)} = 0 \\
&\qquad\qquad\qquad \text{and } s = F \text{ if } \varpi^{(j)} = 1
\end{aligned}
\tag{5.73}
$$

for an opportunity taker machine $i \in M_{TK} \backslash \{j\}$:

$$
\begin{aligned}
&C(\tau^{(j)}, s, n_1^{(i)}, n_2^{(i)}, x^{(i)}, \tau^{(j)}, \boldsymbol{\varpi}) = \\
&\qquad \left(n_1^{(i)} + n_2^{(i)}\right)\left(y^{(i)} b^{(i)} + u^{(i)}\right) + a\left(\tau^{(j)} - x^{(i)}\right)^+ + \pi L_s + R_s \\
&\qquad\qquad \text{for } i \in M_{TK}, i \neq j \text{ and } s = OT \text{ if } \varpi^{(i)} = 0 \\
&\qquad\qquad\qquad \text{and } s = OF \text{ if } \varpi^{(i)} = 1
\end{aligned}
\tag{5.74}
$$

Note that for the opportunity non-taker machines, $i \in M_{NTK}$ cost expression is as provided above for the stoppage triggering machine.

Since we have defined system stoppage rate, opportunity rate and the cost parameters for each machine, we are ready to provide the objective function:

$$\min_{y^{(i)},h^{(i)},k^{(i)}\ \forall i\in M} E[TC] \tag{5.75}$$

where

$$E[TC] = \sum_{i\in M_{TK}} \frac{E[CC^{(i)}]}{E[\tau]} + \sum_{i\in M_{NTK}} \frac{\lim_{\mu^{(i)}\to 0} E[CC^{(i)}]}{\lim_{\mu^{(i)}\to 0} E[CL^{(i)}]} \tag{5.76}$$

as described before.

Chapter 6

Numerical Study: Multiple Machine Model

6.1 Introduction

We have conducted numerical studies for the multiple machine model we have developed in the previous Chapter 5. Through the studies, we intend:

1. To analyze the optimum control parameters and partitioning under various setting of machines with respect to their cost parameters and reliability;

2. To provide some insight on the comparison of the jointly optimized system cost rate and the system cost rate where opportunities are not utilized and machines are individually optimized; and,

3. To analyze the performance of our approximations through a simulation.

First, to demonstrate the size of the problem we will revisit the objective function and its construction. We have introduced the algorithm $OPTIMIZE$ for the single machine model in Chapter 3 which implements the golden section search method. In that algorithm opportunity rates and duration of the idle period in case of a stoppage by an opportunity were given and the search was over three control parameters, namely y, k, and h. In the multiple machine model with m machines, there are 2^m different partitioning of the machines into the opportunity taker and non-taker set, and for each of these 2^m partitioning to find the minimum system cost rate the function must be searched over $3m$ control parameters, $y^{(i)}$, $k^{(i)}$, and $h^{(i)}$ $i \in M$. Moreover, the opportunity rate $\mu^{(i)}$ that each machine has to be computed from the individual stoppage rates of the machines which need to be revised whenever the values for $y^{(i)}$and $k^{(i)}$ change, as described in Chapter 5.

We have implemented two search algorithms, one for obtaining the best partitioning of the machines and the other one is for the search on the surface of the total cost rate function. In the next section we will describe these algorithms.

6.2 Algorithms

6.2.1 Cost rate convergence algorithm

In the multiple machine setting, finding the optimal values of $y^{(i)}, k^{(i)}, h^{(i)}\ \forall i \in M$ requires joint optimization of the m machines, since the opportunity rates $\mu^{(i)}\ \forall i$ depend on the other machines policy parameters. We have developed an algorithm for joint optimization of the multiple machine model which uses the single machine model. In this algorithm we independently optimize each machine to get $y^{(i)}, k^{(i)}, h^{(i)}\ \forall i$. Based on this optimization, we compute the opportunity rates $\mu^{(i)}$'s they observe from the optimal policy parameters. We compute the system cost rate by substituting $\mu^{(i)}, y^{(i)}, k^{(i)}, h^{(i)}$ into the cost rate expressions introduced in Chapter 5. We repeat this procedure until we obtain a sufficient convergence in the cost rate between each iteration. Although cost convergence is achieved for all the experiments in our numerical study, there is no guarantee of convergence. We call this algorithm $CONVERGE\,()$. This algorithm takes the sets of opportunity taker machines and opportunity non-taker machines as input. A step by step description of this algorithm is below. The more detailed and unified pseudo code for the multiple machine is provided in Appendix D.

Algorithm $CONVERGE(M_{TK}, M_{NTK})$:

Step 1: Let $\mu^{(i)} = 0$ for $i = 1, ...m$. OPTIMIZE: get $\hat{y}^{(i)}, \hat{k}^{(i)}, \hat{h}^{(i)}$ - optimal (y, k, h) triplet

Step 2: Compute $\mu^{(i)}$ using $\hat{y}^{(i)}, \hat{k}^{(i)}, \hat{h}^{(i)}$ for all $i = 1, ..., m$.

Step 3: Using $\mu^{(i)}$ and $\hat{y}^{(i)}, \hat{k}^{(i)}, \hat{h}^{(i)}$ compute $E[TC]_{old}$

Step 4: Using $\mu^{(i)}$ OPTIMIZE: get $\hat{y}^{(i)}, \hat{k}^{(i)}, \hat{h}^{(i)}$

Step 5: Compute $\mu^{(i)}$ using $\hat{y}^{(i)}, \hat{k}^{(i)}, \hat{h}^{(i)}$ for all $i = 1, ..., m$.

Step 6: Using $\mu^{(i)}$ and $\hat{y}^{(i)}, \hat{k}^{(i)}, \hat{h}^{(i)}$ compute $E[TC]_{new}$

Step 7: If $(|E[TC]_{new} - E[TC]_{old}| / E[TC]_{old} > \epsilon)$

set $E[TC]_{old} = E[TC]_{new}$ and go to Step 4.

else STOP.

6.2.2 Partitioning heuristic

In the multiple machine setting, in addition to the operating parameters $y^{(i)}$, $k^{(i)}$ and $h^{(i)}$, whether a machine will be opportunity taker or opportunity non-taker is a decision variable as well. For finding the optimal partition of the m machines to the sets M_{TK} and M_{NTK} total cost rate for 2^m possible combination should be computed by implementing the previously derived cost rate functions and compared to find the best partitioning. However, the computational time increases exponentially in the number of machines, hence infeasible. This problem is identified as "set partitioning problem" in the literature, and is known to be NP-complete.

Instead of complete enumeration we implement a simple heuristic search algorithm for separation of the machines. The algorithm works as follows:

As the initial solution we assign all the machines to the opportunity taker set and compute the corresponding cost rate. Then, we assign machines one by one to the opportunity non-taker set and compute the cost rate. If the minimum of these cost rates is less than the all taker cost rate then we include the machine, assignment of which yields the minimum cost, to the non-taker set. We repeat the process for the machines remaining in the opportunity taker set, and compare the best cost rate with the optimum cost rate of the previous partitioning. We continue in the similar fashion until either we get an empty taker set or no improvement can be achieved by including another machine into the opportunity non-taker set. We call this algorithm $PARTITION$, and provide below its pseudo code.

Algorithm-$PARTITION$

Step 1: set $i \in M_{TK}$, $\forall i$ and compute $E[TC]_{old}$

Step 2: for $i = 1 \rightarrow m$

if $i \in M_{TK}$ then

set $M'_{NTK} = M_{NTK} \cup \{i\}\,; M'_{TK} = M_{TK} \backslash \{i\}$

compute $E^{(i)}[TC]$ by $CONVERGE(M_{TK}, M_{NTK})$

if $E^{(i)}[TC] < E[TC]_{\min}$ then

$E[TC]_{\min} = E^{(i)}[TC]$

$I_{\min} = i$

Step 3: if $(E[TC]_{\min} < E[TC]_{old})$ then

$E[TC]_{old} = E[TC]_{\min}$

$M_{NTK} = M_{NTK} \cup \{I_{\min}\}$

$$M_{TK} = M_{TK} \backslash \{I_{\min}\}$$

else STOP.

Step 4: if $(M_{TK} = \emptyset)$ STOP

else go to STEP2.

6.3 Implementation and Data Set

We have implemented the multiple machine models in *Microsoft Visual C++ Version 6.0.* We have run the codes on a PC (*Pentium III* computer). For the experiments in the multiple machine environment, we assume that the production facility consists of 8 machines. We use the algorithms $PARTITION$, $CONVERGE$ and $OPTIMIZE$ introduced above and in Chapter 3. We presume that separation of the machines would be a consequence of the differences in their reliability (λ), restoration times (L_T and L_F), and cost of operating in out-of-control state (a). In the experiments presented below we looked at $L_T = L_F$ case only. Therefore, we will denote values corresponding to L_T and L_F by L, in the rest of this chapter. In our experimental set we fixed the per unit cost of sampling $b = 0.1$, fixed cost per sampling $u = 5$, and the repair costs $R_T = R_F = 0$. The rest of the parameters of the experimental set are as follows:

$$
\begin{aligned}
\pi &\in \{500, 1500\} \\
\lambda &\in \{0.01, 0.03, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1\} \\
L &\in \{0.025, 0.05, 0.075, 0.1, 0.125, 0.15, 0.2, 0.25\} \\
a &\in \{50, 150, 250, 300, 350, 400, 450, 500\}
\end{aligned}
$$

We consider 14 different experiments depending on the patterns of the values of λ, a, and L. Table 5.1 summarizes these 14 experiments. In this table ($\leftrightarrow$) indicates that corresponding parameter is identical for all of the machines, ($\nearrow$) indicates that values of the corresponding parameter assigned to the machines in an increasing trend, such that machine #1 has the smallest value and machine #8 has the largest value of the corresponding parameter, likewise ($\searrow$) indicates that the machine #1 has the largest and machine #8 has the smallest parameter value. are assigned to as the index of the machine increases When the parameters are identical throughout the machines they have the following values: $\lambda = 0.05, L = 0.15, a = 250$. To illustrate, consider a few examples.

For the experiment #1: all of the machines have identical values for all three of the parameters, such that $\lambda^{(i)} = 0.05, L^{(i)} = 0.15$, and $a^{(i)} = 250\ \forall i$.

For the experiment #5: $\lambda^{(1)} = 0.01$, $\lambda^{(2)} = 0.03$, $\cdots$, $\lambda^{(8)} = 0.1$; $L^{(1)} = 0.025$, $L^{(2)} = 0.05$, $\cdots$, $L^{(8)} = 0.25$; $a^{(i)} = 250\ \forall i$.

For the experiment #13: $\lambda^{(1)} = 0.1$, $\lambda^{(2)} = 0.09$, $\cdots$, $\lambda^{(8)} = 0.01$; $L^{(1)} = 0.025$,

$L^{(2)} = 0.05, \cdots, L^{(8)} = 0.25$; $a^{(1)} = 500$, $a^{(2)} = 450, \cdots, a^{(8)} = 50$.

6.4 Test for the Poisson Opportunity Arrival Process

Recall that we approximated the system stoppages by an exponential distribution. In fact, the sampling process for each machine is not continuous, and this process can be better approximated by a geometric distribution. Then, for the multiple machine case, the system stoppage rate would be computed as the minimum of geometric random variables. However, we assume that, when the number of machines in the system is large enough, system stoppages can be approximated by a continuos distribution, namely a Poisson process. For some multiple machine configurations it will be difficult to capture this continuous behavior. When, for example, all of the machines are identical, sampling time of every machine will be identical and every system stoppage will be at some sampling instant for every machine. Therefore, probability of a system stoppage at times other then the sampling instances will be zero, and our exponential approximation would fail. In order to test how good is the exponential assumption, we have conducted a Goodness-of-Fit Test by using the stoppage times obtained from simulation for the eight machine system. As the maximum likelihood estimator for the exponential distribution mean we took the average of the operating times. We present, in Figures 5-2 through 5-7, observed system stoppage time distribution vs. the cumulative density function of the fitted exponential distribution for the selected experiment sets. According to test results we don't have enough evidence to accept the null hypothesis that the system stoppages are distributed exponentially. However, by looking at the cost savings provided by the approximate model we still believe that the exponential assumption can be used by practical purposes. Moreover, we also believe that as the number of machines in the system increases, exponential assumption will become a good approximation for the operating time distribution.

Since, tests for Poisson opportunity arrival process failed, we have implemented a simulation in order to identify the impact of the model developed herein. We implemented the simulation code in *Microsoft Visual C++ Version 6.0.* This simulation code allows us to find the exact cost rate of the system for a given machine partitioning and control parameter values, and analyze and compare the analytic solution with the simulated solution. In our numeric study, we solved each experiment with the analytic model, then we exported the optimum control parameter values and optimal partitioning into the simulation code. We picked simulation run lengths long enough to reach saturation. The simulation run length we use was 50000 time units. We didn't do any replications for the simulations.

6.5 Results

6.5.1 Machine partitioning

First we report the partitioning of the machines into the opportunity taker and opportunity non-taker sets. The resulting partitioning of the machines is summarized for $\pi = 500$ in Table 5.2 In this table T denotes that a machine is an opportunity taker machine and **N** denotes that a machine is an opportunity non-taker machine. For both $\pi = 500$ and $\pi = 1500$ in 5 out of the 14 experiments all the machines are opportunity takers. Only in one experiment, experiment number 6 for $\pi = 1500$, we have an all opportunity non-taker solution, i.e. $M_{TK} = \emptyset$.

Partitioning is primarily determined by parameter L, and the machines with larger L tend to be opportunity takers. For example, compare experiments 7, 11, and 13. In all of these experiments λ and a have the same pattern with respect to the machine indices, however L is fixed in experiment number 7, increases in experiment number 11 and decreases in experiment number 13. We observe that in experiment number 7 all of the machines are opportunity takers, whereas in experiments 11 and 13 machines with larger L tend to be opportunity non-takers. The machines with larger L, would cause the system idle time, hence the lost profit cost, to increase, if they were opportunity takers. Therefore, independent of their reliability the system is better off if they are opportunity non-takers.

The parameters, λ and a have no or little effect on the partitioning. For example, compare experiments 3, 9, and 10. In all of these there cases L increases as the index of the machine increases however a remains fixed in experiment number 3, increases in experiment number 9, and decreases in experiment number 10. The optimal partitioning is the same for all of these experiments. Although λ does not affect the partitioning by itself, it has a magnifying effect on the partitioning, so that the machines with the smaller λ tend to be opportunity non-taker. The machines with smaller shift rate, λ are more reliable. Therefore, in a system stoppage instant it is more likely that those machines will be in the in-control status. If these machines utilize the opportunities, and if they have higher repair time L, they would cause the system to stop for longer time periods yielding an increase in the system cost rate.

The resulting partitioning of the machines in the experiments we consider indicates that (except the experiment number 6 for $\pi = 1500$) there are opportunity taker machines in the system. Hence, joint consideration of the machines pays off.

6.5.2 Cost rates

Next, we discuss the impact on cost rates. We present, on the tables 5.3 and 5.4, “all non-taker”, “all taker”, and “optimum partition” cost rates and percentage deviation from the cost rate obtained from the simulation. The percentage deviations are computed as

$$\%\Delta = \frac{100 * (E[TC_{simulation}] - E[TC_{analytic}])}{E[TC_{simulation}]}$$

Since for the all non-taker case, analytical cost rates are obtained from the individual machine optimizations for $\mu^{(i)} = 0$ one would expect to get the same cost rate from the simulation; hence, the deviations would be due to the simulation errors. However, this argument would only be true if the simulation run time could capture enough of the system regeneration points, which are LCM of the sampling intervals of the machines, as we discussed in Chapter 5. Optimum control parameters of the machines for the selected experiments are depicted in Table 5.7 and LCMs of the sampling intervals for these selected experiments are presented in Table 5.8. In these tables we present both from the analytical model, denoted by A, and simulation, denoted by S. Due to the difficulty in guaranteeing the existence of enough system regeneration points, for some experiments, we observe a significant amount of deviation between analytical cost rate and simulation cost rate for the all non-taker case. The summary statistics of the percentage deviation in the all non-taker case are as follows: mean=-0.07%, median=-0.03% for $\pi = 500$; mean=0.27%, median=0.35% for $\pi = 1500$.

Percentage deviations between analytical and simulation solutions for the all-taker and optimum cases are indicators of the performance of the approximations we make (in addition to the simulation errors). Comparison of the analytical and simulation results show that there is no consistent over or under estimation in our model. However, percentage deviations hence the approximations worsen when $\pi = 1500$. We observe that as the number of opportunity non-taker machines increases the difference between the analytical solution and the simulation solution decreases.

We observe that when all the machines are identical the analytical model has the worst performance. This is due to the fact that assumption of the Poisson arrival of the opportunities fails in this case since all the machines have identical operating parameters and opportunity arrivals are at the sampling instances only.

It is interesting to compare the optimum cost rate with the all opportunity non-taker case in order to evaluate the improvement provided by the introduction of the Jidoka Process Control. If the production line is operated in the classical fashion, all machines would be naturally opportunity non-takers. Hence comparison of the all opportunity non-taker case with the optimum partition case gives us the percentage improvement provided by JPC. Table 5.5 depicts these improvements. Entries of this table are $100 \times \frac{E[TC|Case1]-E[TC|Case3]}{E[TC|Case1]}$, where in Case 1 all of the machines are opportunity takers and in Case 3 machines are assigned to opportunity taker and opportunity non-taker sets according to the optimal partitioning. We provide the percentage improvements for both the analytically computed costs and simulation costs. We observe that in the presence of opportunistic inspections, there is a significant improvement in the cost rate. Summary statistics of the percentage improvements for the 14 experiments are depicted in Table 5.6. According to the simulation results, there improvements provided by JPC are between 1.03% and 14.98% with mean 5.65% and median 5.01% for $\pi = 500$ and up to 18.32% with mean 6.15% and median 3.05% for $\pi = 1500$.

Chapter 7

Introduction: Dynamic Lot Sizing Problem for a Warm/Cold Process

Inventory replenishment processes, whether they are direct production or purchasing in a supply chain, typically involve setups. In a manufacturing setting, a setup is a certain set of activities to prepare the process for production which may include cleaning, warming up and calibrating the equipment, readying the shop-floor and the work force along with other operations. In a purchasing setting, the fixed set of activities performed with each order may include identification of suppliers, legal and clerical documentation, customs clearance of imports, shipment of goods, inspection of incoming goods, unloading etc. Associated with each of these activities, an out-of-pocket setup cost may be incurred. The dynamic lot sizing problem is the management of such a replenishment process by determining the production (purchasing) plan which minimizes the total setup, production (purchasing) and holding costs for an inventorable item, facing known demands over a finite number of time periods. More specifically, dynamic lot sizing problem's objective is to determine a production plan which minimizes the total cost of the system.

In some cases, it may be possible to avoid some of the activities typically included in a setup by carrying over the setup through keeping the process "warm" for the next time period. Then, a smaller portion of the set of setup activities (such as only cleaning), if any, are performed at the beginning of the next period. Thus, one can speak of a major setup, which involves the original set of readying activities, for a cold process and a minor setup, which involves a smaller subset of readying activities, for a warm process. Agra and Constantino (1999) consider a single item setting in which a minor setup cost is incurred if the process is ready (i.e., if it was set up for production in the previous period) and a major setup cost, otherwise. In their formulation, it is assumed that if a setup was performed for the item in a period, the process will be ready for use in the following period regardless of the quantity produced. However, as we discuss below, this may not be feasible and/or desirable in certain production/replenishment environments. In this part of the dissertation, we consider the dynamic lot sizing problem with finite capacity and where the process could be kept warm for the next period only if a minimum amount has been produced and would be cold, otherwise.

The lot sizing problem setting that we investigate herein is encountered in a number of environments. The process industries such as glass, steel and ceramic production provide the foremost examples in which the physical nature of the production processes dictates that the processes be literally kept warm in certain periods to avoid expensive shutdown/startups. A particularly striking example with which we are familiar comes from the glass industry: In some periods, production of glass is continued in order to avoid substantial shutdown/startup costs but the produced glass is deliberately broken on the production line and fed back into the furnace! In this case, the process is being kept warm at an additional cost of breakage (plus some costs for non-reusable materials consumed). Similar practices are employed in foundries; ceramic and brick ovens are also kept warm sometimes even though no further production is done in the current period to avoid costly cooling-and-reheating procedures. Aside from such literal manifestations, a process can be kept warm in an abstract sense, as well. Robinson and Sahin(2001) cite specific examples in food and petrochemical industries where certain cleanup and inspection operations can be avoided in the next period if the quantity produced in the current period exceeds a certain threshold (that is, the current production continues onto the next period). This may be done through either overtime or undertime. The treatment of the overtime option is outside the scope of our analysis; however, deliberate undertime practices can be studied within our context of warm/cold processes. With undertime, processes can be kept warm by reducing the "nominal" or "calibrated" production rate within a prespecified range. As an illustration, suppose that the process is capable of producing at most R units at a nominal production rate in a certain time period. Further suppose that its production rate can be reduced so that, within the same time period, the process can produce Q $(< R)$ units at the slowest rate. Thus, it is possible to keep this process warm by having it operate at rates lower than nominal so long as the quantity to be produced is between Q and R. Such variable production rates are quite common in both process and discrete item manufacturing industries - feeder mechanisms can be adjusted so as to set almost any pace to a line; some chemical operations such as electroplating and fermentation can be decelerated deliberately (within certain bounds), and, manual operations can be slowed down by inserting idle times between units. Depending on the nature of the operations involved, the reduction in production rate can be obtained at either zero or positive additional cost. This additional variable cost is then the variable cost of keeping the process warm onto the next period. Furthermore, if one can use the process for multiple purposes (e.g., different products) there may be additional variable costs due to keeping the process idle for the remainder of the current period (e.g., foregone profit on other product(s) nor produced). Then, the cost of keeping the process warm would include such idleness costs, as well. Aside from direct economic calculations, a managerial decision on a warm process threshold may also be made with non-economic considerations such as safety of mounted tools and fixtures left idle on the machinery, impact on worker morale of engaging them in non-productive activities for a longer duration, impact of learning/forgetting phenomena on subsequent runs, etc. Hence, there may be managerially imposed policies in place that dictate the process be kept warm onto the next period only if the production quantity in the current period exceeds a certain level, say, Q.

Another example of the setting we consider herein can be found in a replenishment environment where the supplier and/or shipper offers rebates that can be exercised in the next period if the amount ordered in the current period exceeds a certain quantity. In this procurement setting, the replenishment process is kept warm by ordering in quantities larger than a pre-specified amount, say, Q, in a certain period. The additional cost of keeping the replenishment process warm onto the next period is then zero for periods with ordering quantities larger than Q. Although such rebate structures would have significant impact on the operational performance of supply chains via coordination and smoothing of orders between echelons, they have not received any attention in the literature. We believe that the model herein provides a building block in the analysis and design of such two-party contracts, as well. Note that the production processes cited above need to be modeled as capacitated, whereas, the replenishment processes in the last may be uncapacitated.

As the above examples illustrate, the dynamic lot-sizing problem in the presence of production quantity-dependent warm/cold processes is a rather common problem. However, to the best of our knowledge, this problem has not been studied before.

The works that are most closely related to ours are those that consider reserving a period for production with the option of not producing anything in that period. This setting also occurs as a subproblem of the multi-item capacitated lot-sizing problem with the Lagrangean multipliers as the reservation costs for each of the periods. Although the models on lot-sizing with reservation options employ the notion of a warm process, they do not consider a lower bound on the quantity produced for keeping the process warm onto the next period. Thus, previous results are not readily applicable to a setting with positive warm process thresholds which we consider herein. Similarly, we cannot rely on the results in the vast literature on the multi-item capacitated lot-sizing problems with sequence-dependent setups which consider warm processes but assumes only warm process thresholds of zero production.

Another body of work that uses the notion of warm processes is the discrete lot-sizing and scheduling problem (DLSP) literature (e.g. Fleischmann 1990, Bruggemann and Jahnke 2000, Loparic *et al.* 2003). This group of work differs from ours in the use of small bucket approach (i.e. $R = 1$ in every period) and, more importantly, in that the process can be kept warm only if there has been capacitated production in the current period (i.e. $Q = R$ in every period). Thus, the results in the DLSP literature are not readily applicable to our general setting, as well.

Our model differs from the works on lot sizing with undertime option in that our setting employs periodic review over a finite horizon and has a single product with deterministic but variable demands. Due to the variable nature of demand, we do not obtain a single, stationary solution as in other works but rather establish the structure of the optimal production plan and conditions on the existence of forward solutions.

Planning horizons and forecast horizons have been of interest in the dynamic lot sizing literature, due to their contribution to computational efficiency. A period t is said to form a planning horizon if the optimal production plan for periods 1 through t remains unchanged in an optimal solution to the $t+j$-period problem for all $j > 0$. A forecast horizon is that, beyond which the information on the demand and cost structure does not effect the production decision in the initial period. We show that planning horizon rules developed by Wagner and Within(1958), Zabel (1964), and Lundin and Morton (1975) still hold.

To the best of our knowledge, this is the first work that considers warm/cold processes in the presence of warm process thresholds which depend on the production quantities in the previous period. We believe that our main contribution lies in establishing the structure of the optimal solution and proving a number of other properties of the dynamic lot sizing problem with warm process thresholds. Our numerical results also provide managerial insights into capacity selection decisions for warm/cold processes.

Chapter 8

Literature Review

Below, we briefly review related works in the vast dynamic lot-sizing problem, its variants and planning horizons literature. We also devote a section for the heuristics literature in the area.

The first formulation of the dynamic lot-sizing problem is by Wagner and Whitin (1958) and assumes uncapacitated production and no shortages; hence, it is also called the Wagner-Whitin problem. We shall henceforth refer to this problem and its setting as the "classical problem". Wagner and Whitin (1958) provided a dynamic programming solution algorithm and structural results on the optimal solution of the classical problem. Their fundamental contribution lies in the identification of planning horizons, which makes forward solution algorithms possible. The Planning Horizon Theorem identifies the points where the problem will stabilize such that the production plan up to that point remains unchanged in an optimal solution to the complete problem. Planning horizons provide considerable computational savings for the forward solution algorithm.

Manne (1958) introduces the dynamic lot sizing problem simultaneously with Wagner and Whitin (1958) in a similar setting. He provides a linear programming formulation. He starts with the multiple item problem and by aggregating those reaches the same formulation of the problem.

Zabel (1964) extends the planning horizon theorem of Wagner and Within (1958) by providing additional rules. His rules allow planning horizon to be discovered in shorter horizon subproblem solutions. He also introduces a general procedure for the cases in which the zero initial inventory level assumption of Wagner and Within (1958) is relaxed. If t denotes the number of periods, demands of which can be met by the initial inventory, then he suggests setting the demands of first t periods to zero, and reducing the $(t+1)$st period's demand as much as the remaining initial inventory and solving the problem with the same algorithm. The solution of the new problem yields the same plan, which would be obtained from the original problem. Additionally, he studies the dynamic lot sizing problem when variable unit ordering costs are incorporated. He provides a backward dynamic programming solution algorithm for the new cost structure, due to the fact that the planning horizon theorem vanishes in the existence of variable unit ordering costs.

Zangwill (1966) provides a model for the dynamic lot sizing problem with backordering. He considers a backlog limit α; that is, all units must be delivered within α periods after the scheduled delivery. When $\alpha = 0$, no backordering is allowed, which is the assumption in the classical problem. The study assumes concave holding and shortage costs. He develops a dynamic programming algorithm for a commonly found cost structure which selects an optimal vector from the set of all possible production plans.

Eppen et al. (1969) extend the classical problem model by incorporating marginal production cost and develop a forward algorithm to determine the optimal production plan. They develop “the general planning horizon” based on the “violator set”; such that, if $a_{i,j}$ denotes the cost of producing one item in period i and carrying it to j, then a period t is in the violator set $m(t_1, t_2)$, if $a_{t,t_2} < a_{t_1,t_2}$. In their first theorem they state that, in an optimal solution to the $t + 1$-period problem, last setup occurs either in $l(t)$ or in the violator set $m(l(t), t + 1)$. They also define the “strong minimum”, such that the period with minimum $a_{i,t}$ for $i \leq t$ is called a strong minimum. Then, their planning horizon theorem states that, if the last period with setup for a t-period problem, $l(t)$, is also a strong minimum for t, then periods 1 through $l(t) - 1$ constitute a planning horizon.

Florian and Klein (1971) provide methods for production planning when there exists a capacity constraint - Capacitated Lot Sizing Problem (CLSP). They study the cases both with and without backordering when the production and the inventory holding costs are concave. The structure of optimal plans are described; optimal plans consist of independent subplans in which (a) inventory level is zero in the last period and non-zero in all the others; (b) the production level is at capacity when there is production except at most one period. A shortest path algorithm is also developed based on the above characterizations, for the problems in which the production capacities are identical.

Jagannathan and Rao (1973) extend the capacitated lot sizing model of Florian and Klein (1971) to a more general cost structure, where the production cost function is neither convex nor concave.

Love (1973) considers the capacitated lot sizing problem with piecewise concave production and inventory holding costs, when the production and inventory levels in each period are required to lie within certain upper and lower bounds. He defines an "inventory point" as a period in which the inventory carried to that period is at the level of y_i^L, 0, or y_i^U, where y_i^L is the maximum amount of backorders allowed, and y_i^U is the maximum amount of on-hand inventory allowed at the end of a period (note that in the classical problem $y_i^U = \infty$, and $y_i^L = 0$). He refers to a production plan "extreme" if there is an inventory point between any two periods of nonzero production. He develops a solution algorithm for this bounded inventory model, based on the characteristics of the set of extreme schedules.

Blackburn and Kunreuther (1974) incorporate the backordering into the classical problem. They assume concave backordering and inventory holding costs. They define the "regeneration point", "production point" and "planning period": A period is a regeneration point, if the inventory carried to that period is zero; a production point, if there is a positive production in that period; and, a planning period if there is no speculative motive for producing in any other period to satisfy the demand in that period. If, in a t-period problem, $k(t)$ and $l(t)$ denote the next-to-last regeneration point, and the last production point, respectively, then they show that when $l(t)$ is a planning period $k(t^*) \geq k(t)$ and $l(t^*) \geq l(t)$ for all t^*-period problems, such that $t^* > t$. Furthermore, when $k(t) = t - 1$ and t is a planning period, then $t - 1$ is a planning horizon. They also provide a recursive forward solution algorithm in their study.

Lundin and Morton (1975) develops "protective planning horizon" procedures. They observed that, if at period t, the first regeneration point, denoted by $f(.)$, of the subproblems with length from $l(t)$ ($l(t)$ is the last regeneration period for t period problem) through $t - 1$, is the same , then, periods from 1 to $f(t - 1)$ constitute a planning horizon. They proposes a simpler rule (heuristic) in order to estimate the horizon length such that the time that EOQ, calculated with the average forecasted demand, spans is multiplied by 5, which gives the estimated planning horizon length.

Baker et al. (1978) consider the dynamic lot sizing problem with time varying capacity constraint and they provide a tree search solution algorithm. They present the properties of an optimal solution, which are; (1) if the ending inventory in the previous period is positive then production in the present period is either at capacity or zero; (2) if there is production in present period and the production level is less than the capacity then ending inventory in the previous period should be zero; (3) in seeking an optimal solution it is sufficient to consider only plans in which the last production quantity is equal to capacity or to demand for the periods remaining in the problem. They also present results of a numerical study which show the performance of the tree search algorithm. They empirically show that, although the problem is in the class of NP-complete problems, the algorithm performs quite well.

Bitran and Yanasee(1982) provide a study on the computational complexity of the capacitated lot sizing problems. They group the dynamic lot sizing problems into families according to the cost parameters' (setup cost, holding cost, production cost) and the capacity's assumed patterns, such that: constant, nonincreasing, nondecreasing, zero or no pre-specified pattern. They show the computational complexity of the each family.

Bitran and Matsuo (1986) provide approximate formulations for the NP-hard single item capacitated lot sizing problem. They provide two different reformulation with two different assumptions. The first approximation is about a restriction on the production quantity in any period. They assume that production in any period can only be multiples of a known constant K. Their second assumption is, what they call, "softness of demand". They assume that the standard deviation of the forecast errors is large enough to allow the demands rounded-up to the nearest multiple of K. They provide algorithms for both of the approximations. They observe that the relative error bound of the algorithms is proportional to $1/m$ while the order of calculations is proportional to m or m^2, where m is a nonnegative integer such at mK is the capacity in a period.

Bean, Smith and Yano (1987) study the conditions for the existence of planning and forecast horizons in a discounted dynamic lot sizing problem setting. They assume the existence of a speculative motive for carrying inventory (i.e. unit production cost in a period is greater than the sum of unit production and unit holding cost of previous period). They provide a rolling variable-horizon algorithm.

Karmarkar et al. (1987) consider the dynamic lot sizing problem with start-up and reservation costs. They assume that there production in a period is capacitated, and when the process is switched from "off" to "on", start-up cost is incurred, however the setup of the process can be kept by producing at the capacity or by incurring a positive reservation cost. The capacitated lot sizing problem is a special case of the problem introduced in this work, hence, since the CLSP is NP-complete, this general version is also NP-hard. They employ a Lagrangian relaxation strategy to find bounds on the problem, and use one of the bounds in a branch-and-bound algorithm.

Eppen and Martin (1987) describe to use and develop mixed integer programming models to solve multi-item capacitated lot sizing problems. They start with providing an overview and background of mixed integer linear programming, then consider the single item capacitated lot sizing problem similar to the one considered by Karmarkar et al. (1987). They reformulate the problem as a shortest path problem. Then they move on to the multi item setting, and provide formulations of this problem as small bucket and big bucket problems. They conduct a numerical study.

Chand, Sethi, and Proth (1990) study the existence of the forecast horizons for the undiscounted dynamic lot sizing problem. They assume constant demand throughout the horizon. They propose that $T = m(m+1)$ is a forecasts horizon where m is the planning horizon such that the number of periods the EOQ order spans.

Chand, Sethi, and Sorger (1992) study the forecast horizons under discounted dynamic lot sizing problem setting, where the demand is constant. They show that $T = m(m+1)$, the forecast horizon for the undiscounted case still hold. However, due to the increasing nature of EOQ with respect to the discount factor, they conclude that the forecast horizon decreases in discounting factor.

Aggarwal and Park (1993) provide algorithms for uncapacitated dynamic lot sizing problem. They assume that the demands are deterministic and known, and backordering is possible. They provide a review, description and results of the previous models and algorithms with the above assumptions. The algorithms they develop are implementations of dynamic programming and array searching (namely Monge arrays). They compare the complexity of their algorithm with those of the previous algorithms, and they conclude that provided algorithms improve the running time of the previous algorithms by factors of n or $n/\log(n)$, where n is the horizon length in terms of number of periods.

Wolsey (1995), in this invited review, provides historical overview of the uncapacitated economic lot sizing problem. He mentions some of the important extensions and reformulations of the classical problem. He also provides information on the development of dynamic programming algorithms and their complexity for the economic lot sizing problem.

Hindi (1995) considers the capacitated lot sizing problem with start-up and reservation costs, introduced by Karmarkar et.al. (1987). He formulates the problem by redefinition of the variables as a shortest path problem. Then he uses a tailored version of the formulations which has smaller number of variables and constraints to obtain lower bounds. Then he suggests a procedure to obtain an upper bound by repeatedly solving a small trans-shipment problem. He also provides some computational results.

Pochet and Wolsey (1995) in their comprehensive work present algorithms and reformulations for lot sizing problems. They first consider the single item lot sizing problems by first presenting dynamic programming algorithms for those problems solvable in polynomial time, then they present reformulations such as multicomodity and facility location reformulations. They proceed developing a branch and cut algorithm by solving separation problem. Later on they move on to constant capacity and variable capacity lot sizing problem reformulations.

Sox (1997) incorporates the random demand and non-stationary costs into the dynamic lot sizing problem. He assumes that the cumulative demand distribution for each period is known, backorders are allowed, and all the costs are time-varying. He provides an optimal solution procedure.

Agra and Constantino (1999) consider dynamic lot sizing problem with back-ordering. There are two types of costs associated with the setups in their model: setup cost and start-up cost. Setup cost is incurred when the process is ready and able to produce, however the start-up cost is incurred if the process is setup in the present period and is not setup in the previous period. They develop the model for the Wagner-Whitin cost structure. Their solution procedure relies on the extreme points.

Allahverdi et al. (1999) provide a comprehensive review of the literature on scheduling problems involving setup times and costs. They group the relevant problems into groups and reviews, such as: single machine models, parallel machine models, flowshop problems, and job shop problems.

Eiamkanchanalai and Banerjee (1999) examine the lot sizing problem under deterministic conditions with variable production rate. They assume constant demand, and convex manufacturing cost in production rate. They consider production rate as one of the decision variables. They allow for bidirectional changes in the production rate. In their setting, once a production rate is chosen it cannot be changed after production starts. They also incorporate the idle capacity cost into their model. They provide an example and sensitivity analysis, which shows the trade offs involved in simultaneously determining production run lengths and output rates.

Karimi et al. (2003) provide a review of the models and algorithms on the capacitated lot sizing problem focusing on the research since the late 1980's. The later works and their review can be found in a recent work by Brahimi et al. (2006).

Chapter 9

Model: Assumptions and Formulation

In this chapter we present the basic assumptions of our model, formulate the optimization problem, and we provide theoretical results on the structure of the optimal solution.

We assume that the length of the problem horizon, N is finite and known. The amount of demand in period t is denoted by D_t $(t = 1, 2, \cdots, N)$. All of the demands are non-negative and known in advance, but may be different over the problem horizon. No shortages or backordering are allowed; that is, the amount demanded in a period has to be produced in or before its period. The amount of production in period t is denoted by x_t. For every item produced in period t, a positive unit production cost c_t is incurred. The inventory on hand at the end of period t is denoted by y_t, since we don't allow backordering and the demand in period t must be satisfied by production during that period or during earlier periods, then y_t may attain only nonnegative values. A positive inventory holding cost h_t is incurred for every unit of ending inventory in period t. Without loss of generality, we assume that the initial inventory level is zero. In any problem violating this assumption, by invoking the Proposition 2 below, it can be converted to an equivalent problem which satisfies the zero initial inventory assumption.

The production in a period is non-negative with a maximum capacity, R_t. We assume that physical capacities are non-decreasing, i.e. $R_{t-1} \leq R_t$ for all t and make no assumption on the demand structure other than that $\sum_{i=t}^{j} D_i \leq \sum_{i=t}^{j} R_i$ for $1 \leq t \leq j \leq N$. Suppose that the demand structure does not satisfy the above inequality, i.e. $\sum_{i=t}^{j} D_i > \sum_{i=t}^{j} R_i$, which implies that the total demand up to period j is larger than the total capacity up to j. Since we don't allow backordering or lost sales then any solution to the problem would be infeasible due to some unsatisfied demand. Hence, the inequality $\sum_{i=t}^{j} D_i \leq \sum_{i=t}^{j} R_i$ ensures the feasibility.

We consider warm and cold production processes: The production process may be kept warm onto the beginning of period t if $x_{t-1} \geq Q_{t-1}$; otherwise, the process cannot be kept warm and is cold. In order to keep the process warm onto period t, ω_{t-1} is charged for every unit of unused capacity in period $t-1$. That is, the warming cost incurred in period $t-1$ would be $\omega_{t-1}(R_{t-1} - x_{t-1})$ monetary units. Note that even if the quantity produced in period $t-1$ is at least Q_{t-1}, it may not be optimal to keep the process warm onto the next period if during the next period, there would not be any production; in such instances, there will be no warming costs incurred although $x_{t-1} \geq Q_{t-1}$(since $x_t = 0$). In other words, there are two necessary conditions to keep the process warm onto period t; the production quantity in period $t-1$ must be at least Q_{t-1}, and production quantity in period t must be positive.

A warm process requires a warm setup with an incurred cost k_t, and a cold process requires a cold setup with an incurred cost K_t, if production is to be done in period t; $K_t \geq k_t$ for all t. We assume that all setup costs are non-negative, with $K_{t+1} \leq K_t$ and $k_{t+1} \leq k_t$ for all t. It may occur that considering constant setup costs over time is more realistic, however our assumption of non-increasing setup cost does not preclude this scenario. By treating the setup costs as nonstationary we are able to provide a more general setting. Introduction of nonstationary costs also allows us to handle the objective of discounted costs. Thus, one can think of a cost in period t, K_t as $K\alpha^{(t-1)}$ where K is the constant setup cost and $\alpha\,(< 1)$ is the discount factor per period. Similar argument applies for the warm setup cost, k_t.

Furthermore, in the sequel, we assume $\max(0, \hat{Q}_t) < Q_t \leq R_t$ where $\hat{Q}_t$ denotes the point of indifference for a cold setup and is defined as $R_t - \left(\frac{K_{t+1}-k_{t+1}}{\omega_t}\right)$ for all t. In words: the cost of producing as much as $\hat{Q}_t$ in period t, keeping to process warm onto $t+1$, and making a warm setup in $t+1$ $\left(c_t\hat{Q}_t + \omega_t\left(R_t - \hat{Q}_t\right) + k_{t+1}\right)$is equal to the cost of producing as much as $\hat{Q}_t$ in period t, and making a cold setup in $t+1$ $\left(c_t\hat{Q}_t + K_{t+1}\right)$. We discuss the consequences of relaxing this assumption in Section 9.2.

Clearly, for $Q_t \geq R_t$ and $k_t = K_t$, we have the classical CLSP setting; for $Q_t = R_t$ we have the CLSP with start-up costs setting; and, as $Q_t (= R_t) \rightarrow \infty$, we get the classical (uncapacitated) problem setting.

The single-item capacitated lot-sizing problem with complex setup structures is known to be NP-hard (Bitran and Yanasse 1982). Therefore, it is very difficult to optimally solve large instances of the problem. In fact, the solution time grows exponentially as the number of planning periods increase. However, for certain cost structures, it is possible to obtain analytical results on the structural and computational properties of the optimal production plan. Hence, we consider only the so-called Wagner-Whitin-type cost structures over the horizon of the problem. Specifically, we assume $c_t + h_t > c_{t+1}$, $c_t + h_t - \omega_t > c_{t+1}$, $c_t + h_t - \omega_t > c_{t+1} - \omega_{t+1}$ for all t. This cost structure ensures that Wagner-Whitin-type costs are incurred for productions not exceeding the warm thresholds in both of the consecutive periods, for productions exceeding the threshold in one but not in the other, and for productions exceeding in both of the consecutive periods, respectively. Eppen et al. (1987) have shown that assuming $c_t + h_t > c_{t+1}$ (i) provides simplifications in forward algorithm for computing optimal schedules, and (ii) guarantees that partitioning will occur in the classical problem. Our additional assumptions ensure that this condition is satisfied at all levels of production (that is, below and above the threshold in successive periods). This, in turn, enables us to derive the analytical properties of the optimal production plan. The assumed conditions hold in practice, as well. When $c_t = c\ \forall t$, the condition $c_t + h_t > c_{t+1}$ is valid for any realistic setting with $h_t > 0$. Similarly, if we allow for discounting, we have $c_t > c_{t+1}$; and, hence, the condition holds. Therefore, the first cost structure that we assume holds in general. For $c_t = c\ \forall t$, $c_t + h_t > c_{t+1} + \omega_t$ holds when $h_t > \omega_t$. As we noted in Chapter 7, ω_t can also be viewed as the cost of idleness for the production system with undertime incurred during the production time equivalent of a unit. Hence, in settings where warehousing costs are significant, this condition is also satisfied. A similar argument also holds when there is discounting. In the presence of stationary costs, the condition $c_t + h_t - \omega_t > c_{t+1} - \omega_{t+1}$ holds for $h_t = h > 0$. When there is discounting, this condition also follows immediately from the first two conditions. Although the assumed cost structure is realistic, clearly, there may be particular instances where they do not hold. In such cases, some of our results do not hold. Therefore, the above cost structure is essential for the results obtained in this research.

The objective is to find a production schedule $x_t \geq 0$ $(t = 1, 2, \cdots, N)$ (timing and amount of production), such that all demands are met at minimum total cost. We develop a dynamic programming formulation of the problem (P). Let $f_t^N(x_{t-1}, y_{t-1})$ denote the minimum total cost under an optimal production schedule for periods t through N, where x_t is the production quantity and y_{t-1} is the starting inventory for period t. Then,

$$f_t^N(x_{t-1}, y_{t-1}) = \min_{\substack{0 \le x_t \le R_t \\ x_t + y_{t-1} \ge D_t}} \left[\begin{array}{c} K_t \cdot \delta_t \cdot z_t \\ +[k_t + \omega_{t-1}(R_{t-1} - x_{t-1})] \cdot \delta_t \cdot (1 - z_t) \\ +c_t \cdot x_t + \ h_t \cdot y_t + f_{t+1}^N(x_t, y_t) \end{array} \right] \tag{9.1}$$

where

$$y_t \quad = y_{t-1} + x_t - D_t \qquad \text{for} \quad t = 1, 2, \cdots, N \tag{9.2}$$

$$x_t \quad \ge 0 \qquad \text{for} \quad t = 1, 2, \cdots, N \tag{9.3}$$

$$y_t \quad \ge 0 \qquad \text{for} \quad t = 1, 2, \cdots, N \tag{9.4}$$

$$\delta_t \quad = \begin{cases} 0 & \text{if } x_t = 0 \\ 1 & \text{if } \ x_t > 0 \end{cases} \qquad \text{for} \quad t = 1, 2, \cdots, N \tag{9.5}$$

$$z_{t+1} \quad = \begin{cases} 0 & \text{if } \ x_t \ge Q_t \\ 1 & \text{if } \ x_t < Q_t \end{cases} \qquad \text{for} \quad t = 1, 2, \cdots, N-1 \tag{9.6}$$

with the boundary condition in period N:

$$f_N^N(x_{N-1}, y_{N-1}) = \min_{\substack{0 \le x_N \le R_N \\ x_N + y_{N-1} \ge D_N}} \left[\begin{array}{c} K_N \cdot \delta_N \cdot z_N \\ +[k_N + \omega_{N-1}(R_{N-1} - x_{N-1})] \cdot \delta_N \cdot (1 - z_N) \\ +c_N \cdot x_N + h_N \cdot y_N \end{array} \right] \tag{9.7}$$

The constraint (9.2) is the conservation of flow constraint and it requires that the sum of the inventory at the start of a period and the production during that period equals the sum of the demand during that period and the inventory at the start of the next period. The constraints (9.3) and (9.4) limit production and inventory to nonnegative values. The constraint (9.5) ensures that the binary variable δ_t attains values 0 or 1 depending on the production in period t; if there is a production $\delta_t = 1$, otherwise 0. The binary variable z_t indicates whether the process can kept warm onto period t or not, hence, the constraint (9.6) ensures that it is equal to 1 if the process cannot be kept warm, 0 otherwise.

The optimal solution is found using the above recurrence and $f_1^N(0{,}0)$ denotes the minimum cost of supplying the demand for periods 1 through N (where we arbitrarily set $x_{t-1} = 0$). We are now ready to examine some of the structural properties of the optimal solution to the above formulation. (Without loss of generality we assume throughout that $y_0 = y_N = 0$ and, for convenience, $R_0 = \omega_0 = 0$).

9.1 Structural Results

In this section, we present structural results on the optimal production plan for the lot sizing problem with a warm/cold process. In particular, we establish the conditions under which production is to be done and the amount of production in a period. Furthermore, we show that certain production plans enable one to partition the original problem into independently solvable subproblems.

First, we provide an equivalence property which will simplify our development of further structural results.

Proposition 1 *If problem (P) is feasible, it can be written as an equivalent capacitated lot sizing problem where in each period the demand is not greater than the capacity.*

Proof. Proof is similar to that in Bitran and Yanasse (1982) and consists of defining a new inventory variable, I_{t-1} and rewriting the objective function in terms of the new inventory variable. To this end, for every feasible production plan of problem (P) with production quantities x_t and inventory levels y_t, define a new production plan for $t = 1, 2, \cdots, N$ such that production quantities are equal to the those of the original plan and inventory levels given as:

$$I_{t-1} = y_{t-1} - \max_{\tau=0,\cdots,N-t}\{0, \sum_{j=t}^{t+\tau}(D_j - R_j)\}$$

Since the original plan is feasible then $y_{t-1} \geq \max_{\tau=0,\cdots,N-t}\{0, \sum_{j=t}^{t+\tau}(D_j - R_j)\}$ and consequently $I_{t-1} \geq 0$. Let $D'_t = I_{t-1} + x_t - I_t$ then,

$$\begin{aligned} I_{t-1} + x_t - I_t &= x_t + y_{t-1} - \max_{\tau=0,\cdots,N-t}\left\{0, \sum_{j=t}^{t+\tau}(D_j - R_j)\right\} \\ &\quad -y_t + \max_{\tau=1,\cdots,N-t}\left\{0, \sum_{j=t+1}^{t+\tau}(D_j - R_j)\right\} \end{aligned}$$

since $y_{t-1} + x_t - y_t = D_t$, we have

$$\begin{aligned} I_{t-1} + x_t - I_t &= D_t - \max_{\tau=0,\cdots,N-t}\left\{0, \sum_{j=t}^{t+\tau}(D_j - R_j)\right\} \\ &\quad + \max_{\tau=1,\cdots,N-t}\left\{0, \sum_{j=t+1}^{t+\tau}(D_j - R_j)\right\} \\ &= R_t - \max_{\tau=1,\cdots,N-t}\left\{R_t - D_t, \sum_{j=t+1}^{t+\tau}(D_j - R_j)\right\} \\ &\quad + \max_{\tau=1,\cdots,N-t}\left\{0, \sum_{j=t+1}^{t+\tau}(D_j - R_j)\right\} \end{aligned}$$

Hence, $D'_t \geq 0$ and $D'_t \leq R_t$. Substituting y_t as a function of I_t in the objective function we get the new objective function. The new objective function differs from the original one only by a constant, hence the result. ■

Therefore, without loss of generality, we shall assume in the sequel that $D_t \leq R_t$ for all t; this, naturally, ensures the feasibility condition. An important property that plays a key role in developing algorithms to solve lot-sizing problems is the one that states when to do a setup and to produce. In the absence of warm/cold processes, Bitran and Yanasse(1982) introduce a notation $\alpha/\beta/\gamma/\nu$ in order to classify the special families of the capacitated lot sizing problem, where α, β, γ and ν specify respectively a special structure for setup costs, holding costs, production costs, and capacities. Each of these parameters may be equal to G (no prespecified) pattern, C (constant), ND (nondecreasing), NI (nonincreasing) and Z (zero). They provide a property of the optimal solution which states that, for capacitated settings where, over the horizon, no prespecified pattern exists for setup costs, unit holding costs and capacities, and unit production costs are non-increasing ($G/G/NI/G$ setting in their notation) production is done in a period only if there is not enough inventory to satisfy the demand of the period (Proposition 2.4 in Bitran and Yanasse 1982). In the presence of warm/cold processes, this property no longer holds. Below, we present an extension of their result to the instance when there are quantity-dependent warm processes.

Theorem 1 *An optimal production plan has the property* $z_t \cdot x_t \cdot [y_{t-1} - D_t]^+ = 0$ *for* $t = 1, 2, \cdots, N$ *where* z_t, x_t *and* y_{t-1} *are as given in (9.2)-(9.6).*

Proof. Suppose the contrary (i.e., $z_t \cdot x_t \cdot [y_{t-1} - D_t]^+ > 0$). That is, suppose a proposed production plan suggests $x_{t-1} < Q_{t-1}$, $[y_{t-1} - D_t]^+ > 0$, and $x_t > 0$. Since the production in period $t-1$ is less than the warm process threshold, the process will be cold in period t. As in the classical lot sizing problem, the solution can be improved; hence, it cannot be optimal. Therefore, whenever $z_t = 1$ production is done if and only if the starting inventory is strictly less than the demand for that period. With $z_t = 0$, however, there is no such restriction on the production plan; hence optimal production plan satisfies the stated condition. ■

As expected, Theorem 1 reduces to Proposition 2.4 in Bitran and Yanasse(1982) when $k_t = K_t$ (i.e., $z_t = 1$) for all t. In the presence of warm/cold processes, however, we see that it may be optimal to produce even in a period of zero demand, which is not the case for the classical CLSP setting (see Corollary 2.1 in Bitran and Yanasse 1982).

In the classical problem setting, it is established that, in an optimal production plan, the values that the production quantities can take on in any period are either zero or exactly equal to a sum of demands for a finite number of periods into the future. In CLSP, however, the optimal production plan is composed of subplans in which the production quantities in any period are either zero or at capacity, except for at most one period in which it is less than capacity. In the presence of quantity-dependent warm/cold processes, these fundamental results no longer hold in general for all periods. In the sequel, we establish certain structural properties of the optimal plans and gradually develop the structure of the optimal solution for the capacitated lot-sizing problem with quantity-dependent warm/cold processes.

We introduce the following definitions. Let $X = \{x_1, \cdots, x_N\}$ denote a feasible production plan constructed over periods 1 through N then;

Definition 1 *Period t is "a regeneration point" if $y_{t-1} = 0$, $z_t = 1$ and $x_t > 0$.*

Definition 2 *Define "a production series", $\Psi_{uv|I_{u-1},I_{v-1}}$, which is a subset of X between two consecutive cold setups u and v with given starting and ending inventories given such that $y_{u-1} = I_{u-1}$ and $y_{v-1} = I_{v-1}$. We have*

$$\begin{aligned}\Psi_{uv|I_{u-1},I_{v-1}} &= \{x_i|x_i > 0, i = u, \cdots, m;\ x_i = 0 \text{ for } i = m+1, \cdots, v-1; \\ &\quad z_u = 1 = z_v; z_i = 0 \text{ for } u+1 \leq i \leq m; \\ &\quad z_{m+1} \geq 0 \text{ and } z_i = 1 \text{ for } m+2 \leq i \leq v-1\}\end{aligned}$$

where m denotes the latest period in which production is done between u and $v-1$ for $0 \leq u \leq m \leq v-1 \leq N$.

Definition 3 *A period t for $u+1 \leq t \leq m-1$ is called "an intermediate production period".*

Note that, a production series may begin and end with positive inventory, i.e. $I_{u-1} \geq 0$, $I_{v-1} \geq 0$. Therefore, the first period of a production series is not necessarily a regeneration point as defined above (i.e. for it be regeneration point we must have $y_{u-1} = 0$, $z_u = 1$ and $x_u > 0$). However, from Theorem 1, for cold setups to exist in periods u and v, we must have $I_{t-1} < D_t$ for $t = u$, v. (It is possible to form feasible series which violate this condition, but they may safely be ignored due to their suboptimality.) By using the following result, we will further simplify our development and, henceforth, consider only production series that have zero starting and ending inventories.

Proposition 2 *If $\Psi_{uv|I_{u-1},I_{v-1}}$ (with $I_{u-1} < D_u$ and $I_{v-1} < D_v$) is a subset of an optimal plan for problem (P) with demands D_t over periods u through $v-1$, then $\Psi'_{uv|0,0}$, which has the same production schedule, is a subset of an optimal plan for problem (P') with demands $D'_u = D_u - I_{u-1}$ and $D'_{v-1} = D_{v-1} + I_{v-1}$ ceteris paribus.*

Proof. With respect to problem (P), we only change the demands in periods u and $v-1$ in (P'). From Proposition 1, feasibility is ensured for problem (P'). In both problems, under the same production schedule, the net demands in periods u through $v-1$ are the same. Therefore, the costs are also the same. Hence, the result. ■

It follows from above that a series denoted by $\Psi_{uv|I_{u-1},I_{v-1}}$can be substituted by $\Psi_{uv|0,0}$, which we will refer to in shorthand as Ψ_{uv}. Feasibility of a production series implies that the physical capacity constraint and no backordering assumption are not violated. Hence, in the optimal plan $[D_i - y_{i-1}]^+ \leq x_i \leq R_i$ for $u \leq i \leq m$. Furthermore, from the definition of a warm setup one intuitively obtains $x_i \geq Q_i$ for $u \leq i \leq m-1$. Thus, for any optimal series;

$$\max(Q_i, [D_i - y_{i-1}]^+) \leq x_i \leq R_i \text{ for } u \leq i \leq m-1$$

For exposition purposes, we make a distinction between two instances of production at capacity. We shall refer to the production instance $x_i = R_i$ as *capaci-*

tated production only if $[D_i - y_{i-1}]^+ < R_i$. Hence $x_i = [D_i - y_{i-1}]^+ = R_i$ will not be referred to as capacitated production but will simply be called *production at capacity.* (As it will be clear below, we make this distinction to identify the successive capacitated periods which emerge from/are found in the end of a production series.)

As we establish in the following lemma, in addition to the physical capacity in period t, there is also an economic bound, E_t on the production quantity in the presence of warm/cold processes. That is, E_t is such a quantity that, producing more than this quantity in period t for a future period, is more costly than producing the excess quantity in period $t+1$.

Lemma 1 *In an optimal production plan,*

(i) $x_t \leq E_t$ *where* $E_t = \frac{\max(Q_t,[D_t - y_{t-1}]^+)\cdot(c_t + h_t - c_{t+1} - \omega_t) + k_{t+1} + R_t \cdot \omega_t}{c_t + h_t - c_{t+1}}$, $\forall t$,

(ii) $x_t = R_t$ *only if* $E_t \geq R_t$.

Proof. *(i)* Consider the production series Ψ_{uv} and let m be the last period with production within this series. First, suppose $u \leq t = m = v-1$. In this case, $x_t = [D_t - y_{t-1}]^+$ and, by definition, $E_t > [D_t - y_{t-1}]^+$; therefore, $x_t \not> E_t$. Hence, the result. Next, suppose $u \leq t \leq m < v-1$ or $u \leq t < m = v-1$. We will prove by contradiction. In particular, we will show that any production series that violates the above result can be improved and, hence cannot be optimal. Suppose that series Ψ_{uv} is feasible but violates the above lemma, such that $R_t \geq$

$x_t > E_t$ for some t $(u \leq t \leq m)$ where $t = \max\{i : x_i > E_i\}$. Consider another production series Ψ'_{uv} such that $\Psi'_{uv} = \{x'_u = x_u, \cdots, x'_{t-1} = x_{t-1}, x'_t = E_t + e_0, x'_{t+1} = x_{t+1} + e_1,\ x'_{t+2} = x_{t+2} + e_2, \cdots, x'_{v-1} = x_{v-1} + e_{v-1-t}\}$ where $x'_t < x_t$, $0 \leq e_i := x'_{t+i} - x_{t+i} \leq s_i := [\min\left(R_{t+i}, E'_{t+i}\right) - x_{t+i}]^+$ for $1 \leq i \leq v-1-t$ and $\sum_{i=1}^{v-1-t} e_i = (x_t - x'_t)$. (Throughout in our proofs, all entities with notation $(')$ retain their original definition and indicate recomputation for a new sequence.)

Note that in this new production series, for $t+1 \leq i \leq v-1$, $y'_i = y_i - (x_t - x'_t) + \sum_{j=t+1}^{i}\left(x'_j - x_j\right)$, hence, $y'_i \leq y_i$;therefore, $\left[D_i - y'_{i-1}\right]^+ \geq [D_i - y_{i-1}]^+$ implying $E'_i \geq E_i$. Furthermore, due to the construction of the new series, $x'_i \leq E'_i$. Let Δ denote the cost difference between production series Ψ'_{uv} and Ψ_{uv}. Then,

$$\begin{aligned}
\Delta \;=\; & -\left(x_t - x'_t\right)\left(c_t + h_t - \omega_t\right) + \sum_{i=t+1}^{v-1} c_i\left(x'_i - x_i\right) \\
& + \sum_{i=t+1}^{v-1} h_i\left(y'_i - y_i\right) - \sum_{i=t+1}^{m} \omega_i\left(x'_i - x_i\right) \\
& + \sum_{i=m+1}^{v-1} (k_i + \omega_i(R_i - x'_i)\delta'_{i+1})\delta'_i
\end{aligned}$$

where δ'_i is a binary variable indicating whether or not production is done in period i. Noting that $(x_t - x'_t) = \sum_{i=t+1}^{v-1}\left(x'_i - x_i\right)$ and $y'_i = y_i - (x_t - x'_t) +$

$\sum_{j=t+1}^{i}\left(x'_j - x_j\right)$, and arranging the terms, we get

$$\begin{aligned}
\Delta &= -\sum_{i=t+1}^{v-1}\left[\left(x'_i - x_i\right)\left(c_t + h_t - \omega_t + \sum_{j=t+1}^{i-1} h_j\right)\right] \\
&\quad -\sum_{i=t+1}^{v-1}\left(x'_i - x_i\right)\sum_{j=i}^{v-1} h_j + \sum_{i=t+1}^{v-1} c_i\left(x'_i - x_i\right) \\
&\quad -\sum_{i=t+1}^{m-1}\omega_i\left(x'_i - x_i\right) - \omega_m\left(x'_m - x_m\right)\delta'_{m+1} \\
&\quad +\sum_{i=t+1}^{v-1} h_i\sum_{j=t+1}^{i}\left(x'_j - x_j\right) \\
&\quad +\sum_{i=m+1}^{v-1}(k_i + \omega_i(R_i - x'_i)\delta'_{i+1})\delta'_i
\end{aligned}$$

Using the equivalence between $\sum_{i=t+1}^{v-1}\left(x'_i - x_i\right)\sum_{j=i}^{v-1} h_j$ and $\sum_{i=t+1}^{v-1} h_i \sum_{j=t+1}^{i}\left(x'_j - x_j\right)$, we finally obtain,

$$\begin{aligned}
\Delta &= \sum_{i=t+1}^{v-1}\left[\left(x'_i - x_i\right)\left(c_i - \left(c_t + h_t - \omega_t + \sum_{j=t+1}^{i-1} h_j\right)\right)\right] \\
&\quad -\sum_{i=t+1}^{m}\omega_i\left(x'_i - x_i\right)\delta'_{i+1} + \sum_{i=m+1}^{v-1}(k_i + \omega_i(R_i - x'_i)\delta'_{i+1})\delta'_i
\end{aligned}$$

If $\sum_{i=1}^{m-t} s_i \geq x_t - E_t$, then one can construct a feasible series Ψ'_{uv} such that $m' = m$ (where m' is defined like m for the series Ψ'_{uv}), $e_i = 0$ for $m - t + 1 \leq i \leq v - 1 - t$ and $\sum_{i=1}^{m-t} e_i \geq x_t - E_t$ with $e_0 \leq 0$. Then,

$$\Delta = \sum_{i=t+1}^{m}\left[\left(x'_i - x_i\right)\left(c_i - \omega_i\delta'_{i+1} - \left(c_t + h_t - \omega_t + \sum_{j=t+1}^{i-1} h_j\right)\right)\right]$$

Since $c_i - \omega_i \delta'_{i+1} < \left(c_t + h_t + \sum_{j=t+1}^{i-1} h_j\right)$ due to the assumed marginal production cost structure and $(x'_i - x_i) \geq 0$ for all $i \geq t+1$, we have $\Delta < 0$. Thus, Ψ'_{uv} with $x'_t \leq E_t$ yields a lower cost than Ψ_{uv}; therefore, Ψ_{uv} cannot be optimal. If, however, $\sum_{i=1}^{m-t} s_i < x_t - E_t$, one can construct another Ψ'_{uv} such that $m' = m+1$, $e_0 = \max(Q_t, [D_t - y_{t-1}]^+) - E_t$, $e_i = s_i$ for $1 \leq i \leq m-t$, and $e_{m+1-t} = \left[\frac{k_{t+1} + R_t\omega_t - \max(Q_t, [D_t - y_{t-1}]^+)\omega_t}{c_t + h_t - c_{t+1}} + q\right]$ where $q = x_t - E_t - \sum_{i=1}^{m-t} e_i$, and $e_i = 0$ for $m - t + 2 \leq i \leq v - 1 - t$. Note that this series is feasible since $e_{m+1-t} \leq R_{m+1}$ due to non-decreasing capacities. (Also note that we construct a series such that $e_0 \leq 0$ in this case as well.) Then,

$$
\begin{aligned}
\Delta \; = \; & \sum_{i=t+1}^{m-1} \left[(x'_i - x_i) \left(c_i - \omega_i - \left(c_t + h_t - \omega_t + \sum_{j=t+1}^{i-1} h_j \right) \right) \right] \\
& + (x'_m - x_m) \left(c_m - \left(c_t + h_t - \omega_t + \sum_{j=t+1}^{m-1} h_j \right) \right) + k_{m+1} + (R_m - x'_m)\omega_m \\
& + \left(\left[\frac{k_{t+1} + R_t\omega_t - \max\left(Q_t, [D_t - y_{t-1}]^+\right)\omega_t}{c_t + h_t - c_{t+1}} + q \right] \left[c_{m+1} - c_t + \omega_t - \sum_{j=t}^{m} h_j \right] \right)
\end{aligned}
$$

We need to consider the two possible values that s'_{m-t} may take on. First, suppose that $s'_{m-t} = R_m - x'_m$, which implies that $x'_m = R_m$. Then, due to the assumed cost structure, $\Delta < 0$. Next, suppose that $s'_{m-t} = E'_m - x'_m$. Then, $x'_m = E'_m$; using the defining expression for E'_m, it is again easy to show that $\Delta < 0$. Hence, Ψ'_{uv} yields a lower cost than Ψ_{uv}; therefore, Ψ_{uv} cannot be optimal. If the newly constructed Ψ'_{uv} has $x'_{m+1} \leq E'_{m+1}$, we conclude our proof. Otherwise, we carry the same argumentation over periods $m+1$ and onward successively. Hence, the result.

(ii) Follows immediately from *(i)*. ■

Lemma 2 *A production series Ψ_{uv}, in which there is at least one period t such that $x_t = R_t(> Q_t)$, $0 < x_{t+1} < R_{t+1}$, and $y_t > 0$, cannot be optimal.*

Proof. We will prove by contradiction. Suppose a feasible production series Ψ_{uv} where for some t, $x_t = R_t(> Q_t)$, $0 < x_{t+1} < R_t$ and $y_t > 0$. Since $y_t > 0$, production in period t covers some of the demands in periods later than t; therefore, one can construct another feasible production series,

$$\begin{aligned}\Psi'_{um'v,r'} &= \{x'_u = x_u, \cdots, x'_{t-1} = x_{t-1}, x'_t = R_t - e, \\ &\quad x'_{t+1} = x_{t+1} + e, x'_{t+2} = x_{t+2}, \cdots, x'_{v-1} = x_{v-1}\}\end{aligned}$$

with $e > 0$. Clearly, Ψ'_{uv} yields a lower cost due to the assumed marginal production cost structure. Hence, the result ■

From Lemmas 1 and 2, we get the following corollary:

Corollary 1 *In a production series (of an optimal production plan) in which m denotes the last period of production in the series,*

(i) if $x_t = R_t(> Q_t)$ and $0 < x_{t+1} < R_{t+1}$ then $y_t = 0$ for $u \leq t \leq m - 1$.

(ii) if $x_t = R_t(> Q_t)$ and $y_t > 0$ then $x_{t+1} = R_{t+1}$ for $u \leq t \leq m - 1$.

(iii) Let $m-r+1$ denote the first period in which capacitated production is done in a production series of an optimal plan. Then, $m-r+1 \geq \max(j|E_j < R_j$ for $u \leq j \leq m)$ and $x_t = R_t$ for $m-r+1 \leq t \leq m$.

The above corollary provides the basis of the subtle distinction we like to make between "production at capacity" and "capacitated production". The first refers to the case where production at capacity is done solely to satisfy the net demand of the period (i.e. $y_t = 0$) in *(i)*, whereas the latter refers to production again at capacity but to satisfy more than the net demand in that period (i.e. $y_t > 0$) in *(ii)*. Corollary 1 also implies that, if there is a succession of capacitated production periods, the last capacitated production period coincides with the last production period in the series of an optimal production plan! This result is important in that it gives us the structure in which an optimal production series ends.

Lemma 3 *In an optimal production series Ψ_{uv}, (for $u \leq t \leq m-1$), $x_t > \max(Q_t, [D_t - y_{t-1}]^+)$ only if $x_{t+1} = R_{t+1}$ and $y_{t+1} > 0$.*

Proof. We will prove by contradiction. First, suppose a feasible production series Ψ'_{uv} such that $x'_t > \max(Q_t, [D_t - y'_{t-1}]^+)$, $0 < x'_{t+1} < R_{t+1}$ and $y'_{t+1} \geq 0$. Let

$$q_t = \min\left([R_{t+1} - x'_{t+1}], [x'_t - \max(Q_t, [D_t - y'_{t-1}]^+)]\right)$$

Then, there is a feasible series Ψ_{uv} such that $x_t = x'_t - q_t$, $x_{t+1} = x'_{t+1} + q_t$, $y_{t+1} = y'_{t+1}$ and $x'_i = x_i$ for all other i. If $[R_{t+1} - x'_{t+1}] < [x'_t - \max(Q_t, [D_t -$

$y_{t-1}]^+)]$, we have $x_{t+1} = R_{t+1}$ and $x_t > \max(Q_t, [D_t - y_{t-1}]^+)$; otherwise, $x_t = \max(Q_t, [D_t - y_{t-1}]^+)$ and $x_{t+1} < R_{t+1}$. In either case, Ψ_{uv} yields a lower cost due to the assumed marginal production cost structure; therefore, Ψ'_{uv} cannot be optimal. Next, suppose a feasible production series Ψ'_{uv} such that $x'_t > \max(Q_t, [D_t - y'_{t-1}]^+)$, $x'_{t+1} = R_{t+1}$ and $y'_{t+1} = 0$. This implies that $x'_{t+1} = [D_{t+1} - y'_t]^+ = D_{t+1}$ from the demand structure that $D_{t+1} \leq R_{t+1}$. If so, $y'_t = 0$ and, thereby, $x'_t = [D_t - y'_{t-1}]^+$ which, jointly, contradict $x'_t > \max(Q_t, [D_t - y'_{t-1}]^+)$. Hence, the result. ■

Corollary 2 *In a production series Ψ_{uv} (of an optimal production plan) in which production is done last in period m, if $x_m < R_m$ then $x_t = R_t (> Q_t)$ only if $y_t = 0$ for $u \leq t \leq m - 1$.*

We are now in a position to provide the structure of a production series in an optimal production plan.

Theorem 2 *(Optimal Production Quantity Theorem)*

In a production series Ψ_{uv} of an optimal production plan, in which m is the last period where production is done and r (≥ 0) is the number of successive periods with capacitated production,

(i) $x_t = \max(Q_t, [D_t - y_{t-1}]^+)$, for $u \leq t \leq m - r - 1$ and $m - u > r \geq 0$,

(ii) $x_{m-r} = \left[\sum_{i=m-r}^{v-1} D_i - y_{m-r-1} - \sum_{i=m-r+1}^{m} R_i\right]^+ < \min\left(E_{m-r}, R_{m-r}\right)$ and $m - u \geq r \geq 0$,

(iii) $x_t = R_t$ $(\leq E_t)$ *for* $m - r + 1 \leq t \leq m$ *and* $r \geq 1$.

Proof. For convenience, we will prove in reverse order.

(iii) Follows from Lemma 1 and Corollary 1 *(ii)*.

(ii) Note $t = m - r$. Due to no shortages assumption,

$$\sum_{i=m-r}^{m} x_i + y_{m-r-1} = \sum_{i=m-r}^{v-1} D_i.$$

From *(iii)* we have $x_i = R_i$ for $m - r + 1 \leq i \leq m$. Hence,

$$x_{m-r} = \sum_{i=m-r}^{v-1} D_i - y_{m-k-1} - \sum_{i=m-r+1}^{m} R_i$$

Invoking Lemma 1, we have the result.

(i) $u \leq t \leq m - r - 1$. By definition t is an intermediate period; therefore, $x_t \nless \max(Q_t, [D_t - y_{t-1}]^+)$. First consider $t = m - r - 1$. From *(ii)*, $x_{t+1} = \varepsilon < R_{t+1}$. Since $x_{t+1} \neq R_{t+1}$, from Lemma 3 $x_t \ngtr \max(Q_t, [D_t - y_{t-1}]^+)$. Hence the result. Next consider $t = m - r - 2$. If $x_{t+1} < R_{t+1}$, the argumentation for period $m - r - 1$ also holds. If, however, $x_{t+1} = R_{t+1}$, from Corollary 1*(i)* $y_{t+1} = 0$, since $x_{t+2} = \varepsilon$ (from Theorem 2*(ii)* with $t+2 = m-r$). Since $x_{t+1} = R_{t+1}$ and $y_{t+1} = 0$, from Lemma 3, $x_t \ngtr \max(Q_t, [D_t - y_{t-1}]^+)$. Inductively, the argumentation can be carried out for every period until $t = u$. Hence, the result. ■

Theorem 2 gives the values that production quantities in any period may assume in an optimal production plan in the presence of quantity-dependent warm/cold processes. The above theorem gives, as special cases, the results in Wagner and Whitin(1958) (Theorem 2, p.91) when $Q_t (= R_t) \rightarrow \infty$, and those in Florian and Klein(1971) (Corollary, p.16) when $Q_t = R_t = R$ and $k_t = K_t$, for all t. Thus, it enables one to identify the forms of the production series to be considered in solving problem (P) and forms the basis of the solution algorithms we develop in a later section. To that end, we provide the following corollary.

Corollary 3 *In an optimal production plan, the series Ψ_{uv} can only have the following forms:*

(i)

$$x_i = \begin{cases} \sum_{i=u}^{v-1} D_i & \textit{for } i = u \\ 0 & \textit{for } u+1 \le i \le v-1 \end{cases}$$

(ii)

$$x_i = \begin{cases} \max(Q_t, [D_t - y_{t-1}]^+) & \textit{for } u \le i \le m-1 \\ \sum_{i=m}^{v-1} D_i - y_{m-1} & \textit{for } i = m \\ 0 & \textit{for } m+1 \le i \le v-1 \end{cases}$$

(iii)

$$x_i = \begin{cases} R_i & \textit{for } u \le i \le m \\ 0 & \textit{for } m+1 \le i \le v-1 \end{cases}$$

(iv)

$$x_i = \begin{cases} \left[\sum_{i=m-r}^{v-1} D_i - y_{m-r-1} - \sum_{i=m-r+1}^{m} R_i\right]^+ & \textit{for } i = u \\ R_i & \textit{for } u+1 \le i \le m \\ 0 & \textit{for } m+1 \le i \le v-1 \end{cases}$$

(v)

$$x_i = \begin{cases} \max(Q_t, [D_t - y_{t-1}]^+) & \textit{for } u \le i \le m-r-1 \\ \left[\sum_{i=m-r}^{v-1} D_i - y_{m-r-1} - \sum_{i=m-r+1}^{m} R_i\right]^+ & \textit{for } i = m-r \\ R_i & \textit{for } m-r+1 \le i \le m \\ 0 & \textit{for } m+1 \le i \le v-1 \end{cases}$$

Maintaining the definition of a regeneration point given above;

Definition 4 *Define "a production sequence", S_{uv} as a subset of a feasible pro-*

duction plan X such that S_{uv} includes the components of X for all periods between the two consecutive regeneration points u and v; that is,

$$S_{uv} = \{x_i, i = u, \cdots, v-1 | z_u = 1 = z_v \text{ and } y_{u-1} = 0 = y_{v-1}; y_i \geq 0 \text{ for } u < i < v-1\}$$

where $0 \leq u < v-1 \leq N$.

Clearly, any feasible production plan is composed of one or more production sequences and since $y_0 = y_N = 0$, at least one production sequence exists in an N-period problem. Moreover, any production sequence is composed of at least one production series.

In the CLSP setting, a capacity constrained production sequence is defined in Florian and Klein(1971) as a production sequence in which the production level of at most one period is positive but less than capacity, and all other productions are either zero or at their capacity. In the presence of warm/cold processes, we define a *capacity constrained production series* as a production series in which all productions are either zero or at their capacity. That is, we accept only the series of the form given in Corollary 3 *(iii)* as a capacity constrained series.

Theorem 3 *(Capacity Constrained Series Theorem).*

(i) In the presence of warm/cold processes, an optimal production plan consists of production sequences in which at most one series is not a capacity constrained production series.

(ii) Moreover, if there exists a series which is not capacity constrained, then, it is the first series of that sequence.

Proof. (i) First consider a feasible production sequence S_{uv}, in an optimal plan, consisting of two series such that $S_{uv} = \Psi_{u\tau-1|0,y_{\tau-1}} \cup \Psi_{\tau v|y_{\tau-1},0}$ By definition, $y_{\tau-1} > 0$. Furthermore, let m_1 and $m_1 - r_1 + 1$ denote, respectively, the latest period in which production is done and the earliest period in which capacitated production is done in series $\Psi_{u\tau-1|0,y_{\tau-1}}$, and let m_2 and $m_2 - r_2 + 1$ denote, respectively, the latest period in which production is done and the earliest period in which capacitated production is done in series $\Psi_{\tau v|y_{\tau-1},0}$. Suppose that $\Psi_{\tau v|y_{\tau-1},0}$ is an uncapacitated production series, i.e., $\sum_{i=\tau}^{m_2} (R_i - x_i) > 0$. It is easy to show that a new series $\Psi'_{\tau t_2|y'_{\tau-1},0}$ constructed such that $x'_{m_2-r_2} = x_{m_2-r_2} + \epsilon$ and $y'_{\tau-1} = y_{\tau-1} - \epsilon$ while keeping everything the same improves the cost; hence, S_{uv} cannot be optimal. One can carry on this construction until either $\sum_{i=\tau}^{m_2} (R_i - x'_i) = 0$ or $y'_{\tau-1} = 0$.

If with the new series $\Psi'_{\tau v|y'_{\tau-1},0}$, we end up with $y'_{\tau-1} > 0$ under $\sum_{i=\tau}^{m_2} (R_i - x'_i) = 0$, then, the sequence consists of one non-capacitated and one capacitated series. If we end up with $y'_{\tau-1} = 0$ under $\sum_{i=\tau}^{m_2} (R_i - x'_i) = 0$, then period τ is a regeneration period which implies that there are two production sequences such as $S'_{u\tau}$ and $S'_{\tau v}$, and each of these sequences consists of one series where $\Psi'_{u\tau-1|0,0}$ is uncapacitated production series and $\Psi'_{\tau v|0,0}$ is capacity constrained production series. A similar conclusion is drawn if $y'_{\tau-1} = 0$ but $\sum_{i=\tau}^{m_2} (R_i - x'_i) \neq 0$ under the new construction of series, as well, in which both of the series are uncapacitated. Hence, a production sequence which comprises of two uncapacitated series cannot be optimal.

For the production sequences that comprises more than two series, we can apply the above argument to the last uncapacitated production series and the immediate preceding series.

Hence the result.

(ii) Proof follows immediately from the new series construction described in part (i). ■

Corollary 4 *An optimal production plan has the property* $z_t \cdot y_{t-1} \cdot x_t \cdot (R_t - x_t) = 0$ *for* $t = 1, 2, \cdots, N$ *where* z_t*,* x_t *and* y_{t-1}*are as given in (9.2)-(9.6) .*

Proof. Follows from Proposition 2.1 in Bitran and Yanasse (1982) when $z_t = 1$.

■

Note that the above result is another extension of the result on $G/G/NI/G$ (in the notation of Bitran and Yanasse 1982) capacitated lot-sizing problem with warm/cold processes.

9.2 A Digression: If $Q_t < \hat{Q}_t$

The structural results presented so far are based on the assumption that $Q_t \geq \hat{Q}_t$ $\forall t$, where $\hat{Q}_t$, defined as $R_t - \frac{K_{t+1} - k_{t+1}}{\omega_t}$, represents the point of indifference between the costs of keeping the process warm onto the next period and of incurring a cold setup in the next period. As the discussion below reveals, this is the most realistic setting. However, for completeness, we discuss the consequences of relaxing this assumption. When $Q_t < \hat{Q}_t$, for the production quantities such that $Q_t \leq x_t < \hat{Q}_t$, the cost of keeping the process warm onto the next period is $(R_t - x_t)\,\omega_t$. Since $x_t < \hat{Q}_t$, we have $(R_t - x_t)\,\omega_t > K_{t+1} - k_{t+1}$, which implies that keeping the system warm in this period yields a cost higher than that incurred by having a cold setup in the next period. Hence, when the managerially selected value of the warm process threshold is below $\hat{Q}_t$, the DP formulation (P) provided by (9.1) subject to (9.2)-(9.6) does not reveal the optimal schedule and the cost. This is mainly due to (9.6) which is constructed under the assumption that $Q_t \geq \hat{Q}_t$. In the case of $Q_t < \hat{Q}_t$, a new DP formulation must include the warm process indicator z_t as a binary decision variable, and, as such, the state of the system must be redefined to include z_t, as well. Possible to reconstruct the DP formulation as it may be, it is easy to see that, in an optimal solution, no warming would be done if the production quantity is less than $\hat{Q}_t$ regardless of the value of the managerially set warm process threshold. Therefore, $\hat{Q}_t$ acts as a bound on the warm process decision. Therefore, it is possible to slightly modify the formulation provided in

(9.1) subject to (9.2)-(9.6) to allow for arbitrarily set warm process thresholds by redefining Q_t used in our formulation such that $Q_t = \max\left(\bar{Q}_t, Q_t\right)$, where $\bar{Q}_t$ denotes the warm process threshold arbitrarily set by the management. The DP formulation (P) can then be used as is.

From the above argumentation, it also follows that $\hat{Q}_t$ is the threshold value which gives the lowest possible cost for a given problem setting. Hence, if the management is free to choose the warm process threshold, it would always set it at the point of indifference. This observation was validated by our numerical study, as well.

An illustrative numerical example highlighting the key features of the optimal solution series and some other key results presented above is provided in the next chapter.

Chapter 10

A special case $k_t = 0$

A case of theoretical interest and practical significance is when $k_t = 0$. This corresponds to a production setting where setup carry-over is possible at no cost. For example, in glass manufacturing, keeping the furnaces warm essentially ensures that production in the next period starts with no setup. Other practical applications include a production line whose physical layout or a machine whose calibration is maintained for the next period at no or almost no additional fixed cost.

Theorem 4 *(Single Series Theorem). When* $k_t = 0 \ \forall t = 1, 2, \cdots, N$,

(*i*) *Each sequence* S_{uv} *comprises only one production series* Ψ_{uv}. *This series* *is of the form:*

$$x_t = \begin{cases} \max(Q_t, [D_t - y_{t-1}]^+) & \textit{for } u \le t \le m-1 \textit{ and } m-u > 0 \\ \varepsilon = \left[\sum_{i=m}^{v-1} D_i - y_{m-1}\right]^+ < R_m & \textit{for } t = m \textit{ and } m-u \ge 0 \\ 0 & \textit{for } m+1 \le t \le v-1 \end{cases}$$

(ii) *an optimal production plan has the property* $z_t \cdot x_t \cdot y_{t-1} = 0$ *where* z_t*,* x_t*,* y_{t-1} *are as given in (9.2)-(9.6).*

Proof. (i) We will prove by contradiction. Consider the series $\Psi_{uv|y_{u-1},y_{v-1}}$with $y_{u-1} > 0$ and $y_{v-1} = 0$. By choosing $y_{v-1} = 0$, we start our argumentation with the last series in a production sequence by definition. From Theorem 3 this series is a capacitated production series due to positive beginning inventory, y_{u-1}. Let m be the latest period in which the production done in the series and, let j be the latest production period (in an earlier series) before u. Consider constructing another series $\Psi'_{uv|y_{u-1}-\epsilon,y_{v-1}}$, i.e. producing $\epsilon(> 0)$ units in period $m+1$ and decreasing the initial inventory in period u by the same amount. The newly constructed series results in a cost difference (old cost minus new cost) of at least $([c_j + \sum_{i=t}^{m} h_i - c_{m+1}]\epsilon - k_{m+1})$. (Note that this difference is only a conservative lower bound because producing ϵ units less in period j may also result in further cost reductions due to changes within that production series, as well.) Since $k_t = 0$ for all t, due to the assumed cost structure, the difference is positive implying that the change in the series structure yields a lower cost. Hence, the original series cannot be optimal. In a similar fashion, one can continue the cost reduction until one gets $y_{u-1} = 0$. If there are production series prior to the one under consideration in the sequence in which the initial inventory level is positive, starting with the last one of such series, we apply the same argument to them, until we end up with all series in the original sequence with zero beginning and ending inventories. Hence, in an optimal plan, if there is a cold setup in period t, i.e. $(z_t \cdot x_t > 0)$, then $y_{t-1} = 0$; thus, every period in which a cold setup is done is a regeneration point. Thus, every series is a sequence. Moreover, when $k_{m+1} = 0$, E_m in Lemma 1 reduces to

$$E_m = \frac{\max(Q_m, [D_m - y_{t-1}]^+) \cdot (c_m + h_m - c_{m+1} - \omega_m) + R_m \cdot \omega_m}{c_m + h_m - c_{m+1}}$$

which implies that $E_m \leq R_m$. From Corollary 1*(iii)* there is no capacitated production in the given series, and production schedule is the one given in Corollary 3*(ii)*. Hence the remainder of the result.

(ii) Follows immediately from Theorem 1 since we have established above that at a cold setup period, $y_{t-1} = 0$. ■

Another useful property in solving lot sizing problems is that of partition. In the absence of warm/cold processes, it is possible to optimally partition a longer problem if the constraint $y_t = 0$ is imposed in a period within the horizon for both the classical problem (Wagner and Whitin 1958) and the CLSP while ensuring capacity feasibility for the remainder of the decomposed problem (Florian and

Klein 1971). In the presence of quantity-dependent warm/cold processes, however, the state of the system is no longer fully represented by the current inventory level in a period and further conditions are needed for a partition.

In the following theorem, we state such conditions.

Theorem 5 *(Partition Theorem) Suppose that $k_j = 0$ for $\forall j$.*

(i) If $y_{t-1} = 0$ and $x_{t-1} < Q_{t-1}$ in a $t-$period problem then it is optimal to consider periods 1 through $t-1$ by themselves in any feasible t^*-period problem ($t^* \geq t$); that is, a cold partition occurs in period t.

(ii) If $x_{t-1} \geq Q_{t-1}$, $y_{t-1} = 0$, $x_t > 0$ and $E_t < R_t$ in a $t-$period problem, then it is optimal to consider periods 1 through $t-1$ by themselves in any feasible t^*-period problem ($t^* \geq t$); that is, a warm partition occurs in period t.

Proof. (i) If the demand in period t is positive, the solution to the t-period problem implies that a setup be done in that period to satisfy the demand. Since $x_{t-1} < Q_{t-1}$, the setup will be a cold one. If the demand in period t is zero, then there will be no production due to the marginal production cost structure over the horizon. In either case, the production decision is independent of the decisions in periods 1 through $t-1$. Hence, the proposed result.

(ii) The solution to the t-period problem implies that production in period t be done with a warm setup. From $y_{t-1} = 0$, we have that the demand in period t must be supplied by the production in period t. Furthermore, from Corollary

1*(iii)*, $E_t < R_t$ implies that demands in the future for longer horizons can, at the earliest, be supplied by the production in period t. Hence, a longer problem can be partitioned at period t as proposed. (Note that if the warm setup condition is eliminated, it is easy to design problems where the production plan in periods prior to t may change as the horizon of the problem is extended, and, thereby, (ii) no longer holds.) ■

Note that a partition condition exists only for the case when the warm setup cost is zero. Otherwise, as the horizon of the problem is extended, it is possible to encounter optimal solutions that modify the production schedules in periods 1 through t even if the above stated conditions hold. The existence of a partition implied in Theorem 5 is very important in that it also implies the existence of a forward solution algorithm. In Chapter 11, we elaborate more on such algorithms.

Based on the structural properties of the optimal solution presented above, a number of planning horizons can be developed in the presence of quantity-dependent warm/cold processes for the special case of $k_t = 0$ and $\omega_t = 0$. We use the term planning horizon for a $t-$period problem to mean a sequence of periods starting with period 1 for which the production plan does not change for any t^*-problem ($t^* > t$). In the sequel, we introduce four different types of planning horizons that may be encountered in the dynamic lot sizing problem with warm/cold processes.

Theorem 6 *(Cold Wagner-Whitin-type Planning Horizon)*

(i) Let $l(t)$ be the last period in which a cold setup occurs for the optimal production plan associated with a t-period problem. Then, for any problem of length $t^* > t$, it is necessary to consider only periods $l(t) \leq j \leq t^*$ as candidate periods for the last cold setup.

(ii) Furthermore, if $l(t) = t$, the optimal solution to a t^*- period problem requires a cold setup in period t; therefore, periods 1 through $t-1$ constitute a cold Wagner-Whitin-type planning horizon.

Proof. Follows from the arguments in Wagner and Whitin (1958) and Theorem 5. ∎

Theorem 7 *(Warm Planning Horizon Theorem)*

(i) Suppose $w(t)$ is the last period in which a warm setup is done for the optimal production plan associated with a t-period problem where $w(t) > l(t)$ ($l(t)$ is as in Theorem 6 (i)). Then, for any problem of length $t^* > t$, it is necessary to consider only periods $w(t) \leq j \leq t^*$ as candidate periods for the last setup, which may be warm or cold.

(ii) Furthermore, if $w(t) = t$ and there exists $\hat{t}$ $(l(t) \leq \hat{t} < t)$ defined as the latest period such that $x_{\hat{t}} \geq Q_{\hat{t}}$, $y_{\hat{t}} = 0$, $x_{\hat{t}+1} > 0$ and $E_{\hat{t}+1} < R_{\hat{t}+1}$, the optimal solution to a t^*-period problem requires a warm setup in period $\hat{t}+1$. Therefore, periods 1 through $\hat{t}$ constitute a warm planning horizon.

Proof. (i) In this theorem, we consider the last warm setup (if it exists) done after the last cold setup. Clearly, $w(t)$ $(> l(t))$ is the last period in which production is done in a $t-$period problem; therefore, the demands up to (and including) period t must have been produced in the periods up to (and including) period $w(t)$. When a new period is added, it is easy to see that it is not optimal to supply its demand by producing in a period earlier than $w(t)$, since its setup cost is already sunk. However, it may be beneficial to postpone some of the production scheduled in periods $w(t)$ and earlier to later periods and, possibly, combine them with the production for the demand of the newly added period. Hence, the last production would be at or later than period $w(t)$.

(ii) By definition, a production sequence starts in period $l(t)$ and continuously runs through the end of the horizon of the problem since $w(t) = t$. Since $y_{\hat{t}} = 0$ the production in period $\hat{t}$ satisfies some or all of the demand of that period only (and none of $D_{\hat{t}+1}$). Furthermore, since $E_{\hat{t}+1} < R_{\hat{t}+1}$, Corollary 1 ensures that $m - r > \hat{t}$; thereby, future demands cannot be produced in periods prior to $\hat{t} + 1$ for any problem with longer horizons. With $x_{\hat{t}+1} > 0$, invoking Theorem 5 (ii), we have the proposed result. ■

Theorems 6 and 7 provide planning horizons under a rather restrictive condition that there be a setup in the last period (warm or cold, respectively). Lundin and Morton (1975) have illustrated for the classical problem setting that Wagner-Whitin planning horizons exist for only relatively small lot sizes (i.e. small setup costs vis a vis unit holding costs). Although warm planning horizons as in Theorem 7 occur more frequently than cold Wagner-Whitin-type horizons, it may still be desirable to have more commonly occurring planning horizons. Next, we present two such planning horizons defined by cold and warm setups.

Theorem 8 *(Zabel-type Planning Horizons Theorem)*

(i) *If the optimal t-period solution has the last cold setup in period $l(t)$, then at least one optimal solution to any t^*-period problem (where $t^* > t$) will have a cold setup in period $\bar{t}$ such that $l(t) \leq \bar{t} \leq t^*$.*

(ii) *If the optimal $(\bar{t} - 1)$-period solution, for all $\bar{t}$ such that $l(t) \leq \bar{t} < t$, has the last cold setup in period $\breve{t}$ $(< l(t))$, then periods 1 through $\breve{t} - 1$ constitute a cold Zabel-type planning horizon.*

(iii) *Let $\tilde{t}$ denote the latest period, such that $\breve{t} \leq \tilde{t} < l(t) - 1$ and $x_{\tilde{t}} \geq Q_{\tilde{t}}$ and $y_{\tilde{t}} = 0$ and $D_{\tilde{t}+1} > 0$ and $E_{\tilde{t}+1} < R_{\tilde{t}+1}$ (where $\breve{t}$ and $l(t)$ as in (ii)), then periods 1 through $\tilde{t}$ constitute a warm Zabel-type planning horizon.*

Proof. (i), (ii) : The result is identical to that for the classical problem (Theorem 4, p.467 in Zabel, 1964) and follows from similar arguments omitted for brevity.

(iii) : Follows from (ii) that the production plan for periods 1 through $\breve{t} - 1$ does not change for any problem with a longer horizon, and that, for the rest of the problem, any production plan will start with a cold setup in period $\breve{t}$. Clearly, for any problem of a longer length, the total amount of production scheduled for the production sequence starting with period $\breve{t}$ may increase but may not decrease. Also $x_{\tilde{t}} \geq Q_{\tilde{t}}$ and $y_{\tilde{t}} = 0$ with $D_{\tilde{t}+1} > 0$ imply that, for a longer problem, the production in period $\tilde{t}$ will only satisfy the demand in that period and the rest of the demands will be satisfied by the production done in periods afterwards. Furthermore, Corollary 1 ensures that $m - r > \tilde{t}$ for any problem with longer horizons. Hence, the production plan up to $\tilde{t}$ does not change for any $t^* \, (> t)$ −period problem. ■

10.1 An Illustrative Example

We consider the following example setting: $c_t = c$, $h_t = h$, $\omega_t = \omega$, $K_t = K$, $k_t = k$, $R_t = R$ and $Q_t = Q$ for $t = 1, \cdots, N$. We set $N = 25$, $c = 0$, $h = 1$, $\omega = 0.85$, $K = 15$, $k = 0$, $R = 10$ and $Q = 7$. The demand over the problem horizon is given by $D = \{4, 2, 4, 4, 3, 7, 9, 1, 6, 4, 10, 2, 1, 5, 8, 2, 9, 2, 5, 2, 7, 3, 4, 5, 8\}$.

Below, we present every step of a forward solution. At every step, we consider a T-period problem (i.e., a problem with the horizon length of T starting from the very first period), and generate a set of possible production schedules of this T-period problem by imposing the condition $y_{t-1} = 0$ for a period t $(1 \leq t \leq T)$. In the tables below, each row corresponds to such a schedule. We compute the cost of each suggested schedule such that the cost of the periods 1 through $t - 1$ is the optimal cost obtained from the $(t - 1)$-period problem, and the cost of the periods t through T computed afresh. The notation (*) denotes the schedule that yields the lowest cost as the optimal schedule for the T-period problem.

Note that, since $k = 0$, we only need to consider the production sequences that consist of a single production series. Further note that, we directly use Theorem 4 in constructing the alternative schedules that need to be considered in an optimal solution.

1-period problem:				
$\{\mathbf{4}\}$	$f^*_{0,0} = 0$	$f_{1,1} = 15$	$f_{1,1} = 15$	$f^*_{1,1} = 15$

2-period problem:				
$\{\mathbf{6}, \mathbf{0}\}$	$f^*_{0,0} = 0$	$f_{1,2} = 17$	$f_{1,2} = 17$	$f^*_{1,2} = 17$
$\Psi_{(1,1)}, \{2\}$	$f^*_{1,1} = 15$	$f_{2,2} = 15$	$f_{1,2} = 30$	

3-period problem:				
$\{\mathbf{7}, \mathbf{3}, \mathbf{0}\}$	$f^*_{0,0} = 0$	$f_{1,3} = 24.55$	$f_{1,3} = 24.55$	$f^*_{1,3} = 24.55$
$\Psi_{(1,1)}, \{6, 0\}$	$f^*_{1,1} = 15$	$f_{2,3} = 19$	$f_{1,3} = 34$	
$\Psi_{(1,2)}, \{4\}$	$f^*_{1,2} = 17$	$f_{3,3} = 15$	$f_{1,3} = 32$	

4-period problem:				
$\{\mathbf{7}, \mathbf{7}, \mathbf{0}, \mathbf{0}\}$	$f^*_{0,0} = 0$	$f_{1,4} = 32.55$	$f_{1,4} = 32.55$	$f^*_{1,4} = 32.55$
$\Psi_{(1,1)}, \{7, 3, 0\}$	$f^*_{1,1} = 15$	$f_{2,4} = 26.55$	$f_{1,4} = 41.55$	
$\Psi_{(1,2)}, \{7, 1\}$	$f^*_{1,2} = 17$	$f_{3,4} = 20.55$	$f_{1,4} = 37.55$	
$\Psi_{(1,3)}, \{4\}$	$f^*_{1,3} = 24.55$	$f_{4,4} = 15$	$f_{1,4} = 39.55$	

5-period problem:				
$\{7,7,3,0,0\}$	$f^*_{0,0}=0$	$f_{1,5}=41.10$	$f_{1,5}=41.10$	
$\Psi_{(1,1)},\{7,6,0,0\}$	$f^*_{1,1}=15$	$f_{2,5}=32.55$	$f_{1,5}=47.55$	
$\Psi_{(1,2)},\{\mathbf{7},\mathbf{4},\mathbf{0}\}$	$f^*_{1,2}=17$	$f_{3,5}=23.55$	$f_{1,5}=40.55$	$f^*_{1,5}=40.55$
$\Psi_{(1,3)},\{7,0\}$	$f^*_{1,3}=24.55$	$f_{4,5}=18$	$f_{1,5}=42.55$	
$\Psi_{(1,4)},\{3\}$	$f^*_{1,4}=32.55$	$f_{5,5}=15$	$f_{1,5}=47.55$	

6-period problem:				
$\{7,7,7,3,0,0\}$	$f^*_{0,0}=0$	$f_{1,6}=61.65$	$f_{1,6}=61.65$	
$\Psi_{(1,1)},\{7,7,6,0,0\}$	$f^*_{1,1}=15$	$f_{2,6}=50.10$	$f_{1,6}=65.10$	
$\Psi_{(1,2)},\{7,7,4,0\}$	$f^*_{1,2}=17$	$f_{3,6}=36.10$	$f_{1,6}=53.10$	
$\Psi_{(1,3)},\{\mathbf{7},\mathbf{7},\mathbf{0}\}$	$f^*_{1,3}=24.55$	$f_{4,6}=27.55$	$f_{1,6}=52.10$	$f^*_{1,6}=52.10$
$\Psi_{(1,4)},\{7,1\}$	$f^*_{1,4}=32.55$	$f_{5,6}=21.55$	$f_{1,6}=54.10$	
$\Psi_{(1,5)},\{5\}$	$f^*_{1,5}=41.10$	$f_{6,6}=15$	$f_{1,6}=56.10$	

7-period problem:				
$\{7,7,7,7,5,0,0\}$	$f^*_{0,0}=0$	$f_{1,7}=86.20$	$f_{1,7}=86.20$	
$\Psi_{(1,1)},\{7,7,7,7,1,0\}$	$f^*_{1,1}=15$	$f_{2,7}=73.20$	$f_{1,7}=88.20$	
$\Psi_{(1,2)},\{7,7,7,6,0\}$	$f^*_{1,2}=17$	$f_{3,7}=56.65$	$f_{1,7}=67.65$	
$\Psi_{(1,3)},\{7,7,7,2\}$	$f^*_{1,3}=24.55$	$f_{4,7}=39.65$	$f_{1,7}=64.20$	
$\Psi_{(1,4)},\{7,7,5\}$	$f^*_{1,4}=32.55$	$f_{5,7}=28.10$	$f_{1,7}=60.65$	
$\Psi_{(1,5)},\{\mathbf{7},\mathbf{9}\}$	$f^*_{1,5}=40.55$	$f_{6,7}=17.55$	$f_{1,7}=58.10$	$f^*_{1,7}=58.10$
$\Psi_{(1,6)},\{9\}$	$f^*_{1,6}=52.10$	$f_{7,7}=15$	$f_{1,7}=67.10$	

Note that in this illustrative example, in period 7, the partition condition in Theorem 5(ii) is satisfied; hence, a partition occurs in period 7. The production schedule obtained up to period 6 in the 7-period problem remains the same for any problem with a longer horizon length. In this example, in the optimal solution, there are 8 production series. The solution is as follows: $X = \{[6,0]\,;\ [7,4,0]\,;\ [7,9,7,4,0]\,;\ [10,3,0]\,;\ [7,8,0]\,;\ [9,9,0,0]\,;\ [7,7,0]\,;\ [7,6]\}$. The optimal solution to the full problem is also depicted in Figure 7-1 along with the demands indicated by a diamond mark.

Chapter 11

Computational Issues and Numerical Study

The optimal solution to the problem (P) can, theoretically, be obtained by a backward solution algorithm. However, in a backward solution algorithm, the state of the system needs to be described by the number of periods in the horizon, the ending inventory and production quantity in the previous period, and the maximum of the capacity and the total demand for the remainder of the horizon for each period. Even for discrete demand or largely discretized continuous demand scenarios, the size of the state space for reasonable problem settings becomes prohibitively high. Therefore, it is essential to develop forward solution algorithms when available.

For zero warm setup costs, $(k_t = 0\ \forall t)$, by invoking Theorems 3 and 4, one can obtain a forward DP solution algorithm which provides an optimal solution in polynomial time. Below, we provide such an algorithm. (Prior to using the suggested solution algorithm, we assume that the individual demands are smoothed to ensure feasibility of the problem, which can be done in $O(N)$.)

We retain the labels for the corresponding system parameters in the model but switch to a vector notation, such as $D[\cdot]$, etc. Furthermore, we introduce the following new notation: $s[\cdot]$ is the setup cost actually incurred, $Z\,[i]\,[j]$ is defined as $f_1^{j-1}(0,0) + f_1^i(0,0)$ and is an intermediate variable, $Z\,[i]\,[best[i]]$ is defined as $f_1^i(0,0)$ and denotes the cost of an optimal production policy for the i-period problem, and $\pi\,[i]\,[j]\,[t]$ is the production quantity in period t when a cold setup is forced in period j for an i-period problem. The algorithm is as follows:

$Z[\cdot][\cdot] = 0$

$\pi[\cdot][\cdot][\cdot] = 0$

$best[\cdot] = 1$

for $i \longleftarrow 1$ to n {

for $j \longleftarrow 1$ to i {

for $t \longleftarrow 1$ to $j-1$ {

$\pi[i][j][t] = \pi[j-1][best[j-1]][t]$ }

if $(\pi[i][j][j-1] < Q[j-1])$ then $setup = K[j]$;

else $setup = \omega[j-1] * (R\,[j-1] - \pi[i][j][j-1])$

$s[j] = 0$

for $m \longleftarrow j$ to i {

$F = \sum_{r=j}^{i} D[r] - \sum_{r=j}^{m-1} \pi[i][j][r]$

$H = \sum_{r=j}^{m} D[r] - \sum_{r=j}^{m-1} \pi[i][j][r]$

$E[m] = (\max(Q\,[m]\,, H) * (c\,[m] + h\,[m] - c\,[m+1] - \omega\,[m]) + \omega\,[m] * R\,[m])/ (c\,[m] + h\,[m] - c\,[m+1])$

if $(F \leq E[m])$ then $\pi[i][j][m] = F$

else{ if $(H < Q[m])$ then $\pi[i][j][m] = Q[m]$

else $\pi[i][j][m] = H$ }

if $(\pi[i][j][m] \geq Q[m])$ then $s[m+1] = \omega\,[m] * (R\,[m] - \pi[i][j][m])$

else if $(\pi[i][j][m] > 0)$ then $s[m+1] = K[m+1]$

else $s[m+1] = 0$

}

$Z[i][j] = Z[j-1]\,[best[j-1]] + setup + \sum_{r=j}^{i}(s[r] + (c[r] \cdot \pi[i][j][r]) + h[r] \cdot \sum_{w=j}^{r} (\pi[i][j][w] - D[j]))$

}

$best[i] = i$

for $l \longleftarrow 1$ to i {

if $Z[i][j] < Z[i][best[i]]$ then $best[i] = l$ }

}

For the computational complexity of the proposed algorithm, we provide the following brief argumentation: For any given horizon length T, one generates T sub-problems such that the problem over the periods 1 through T with $y_T = 0$, $(P_{1,T})$, can be solved by decomposing as $P_{1,T} = P_{1,t} + P_{t+1,T}$ with the imposed constraint that $y_t = 0$ for $0 \leq t < T$. Thus, for problem (P), one needs to solve a total of $\frac{1}{2}N(N+1)$ sub-problems. Each of these subproblems can be solved in $O(N)$ time. Hence, the algorithm provides an optimal solution in $O(N^3)$ time. The numerical study was conducted via this algorithm.

With additional conditions, it may be possible to obtain solution algorithms with less complexity, as well. In the following theorem, we state that an improved $O(N)$ time solution algorithm exists for one such special case.

Theorem 9 *(Improved $O(N)$ Solution Algorithm Theorem):*

Given an N-period instance of the dynamic lot sizing problem with warm/cold processes such that, $D_t < Q_t$, $k_t = 0$ (thereby, $R_t \geq E_t = \max(Q_t, [D_t - y_{t-1}]^+) = Q_t$) and $\omega_t = 0$ for $1 \leq t \leq N$, an optimal production schedule can be found in $O(N)$ time.

Proof. Following Aggarwal and Park(1993), let $V(j)$ denote the minimum cost of supplying the demands of periods 1 through $j-1$ such that y_{j-1} is zero for $1 < j \leq N+1$ and $V(1) = 0$. This definition implies that $V(N+1)$ is the cost the optimal production plan for periods 1 through N. Now, consider a production sequence

$$\begin{aligned} S_{ij} &= \{x_i = Q_i, x_{i+1} = Q_{i+1}, ..., x_{m-1} = Q_{m-1}, \\ & \qquad x_m = \left(\sum_{t=i}^{j} D_t - \sum_{t=i}^{m-1} Q_t \right), x_{m+1} = 0, ..., x_j = 0\} \end{aligned}$$

Note that the optimality of such a sequence follows from Theorem 2. For $D_t < Q_t$ $(t = 1, \cdots, N)$, define the $N \times (N+1)$ array $A = \{a[i,j]\}$, where

$$a[i,j] = \begin{cases} V(i) + K_i + \sum_{t=i+1}^{j} \eta_t k_t + \sum_{t=i}^{j} c_t x_t & \text{if } i < j \\ \quad + \sum_{t=i}^{j} h_t (\sum_{j=1}^{t} x_j - \sum_{j=1}^{t} D_j) & \\ +\infty & \text{otherwise} \end{cases}$$

Following Theorem 2, we have $x_t = \eta_t \max\{(D_t - y_{t-1}), \min\{Q_t, \sum_{u=t}^{j} D_u - y_{t-1}\}\}$ with $\eta_t = 1$ if $\sum_{u=t}^{j} D_u - y_{t-1} > 0$ and 0, otherwise. Then, for $1 < j \leq N+1$, $V(j) = \min_{1 \leq i \leq n} a[i,j]$ if $D_{j-1} > 0$ and, $V(j-1)$ otherwise. ■

Definition 5 *After Aggarwal and Park(1993) (p. 556), an $p \times q$ two-dimensional array $A = \{a[i,j]\}$ is Monge if for $1 \leq i < p$ and $1 \leq j < q$,*

$$a[i+1, j+1] - a[i+1, j] \leq a[i, j+1] - a[i, j]. \tag{11.1}$$

Consider the production sequence

$$\begin{aligned} S_{ij} &= \{x_i = Q_i, x_{i+1} = Q_{i+1}, \cdots, x_{m-1} = Q_{m-1}, \\ & \quad x_m = (\sum_{t=i}^{j} D_t - \sum_{t=i}^{m-1} Q_t), x_{m+1} = 0, \cdots, x_j = 0\} \end{aligned}$$

The given production sequence will result in a total cost of $a[i, j]$ as defined in (11.1). Clearly, if a new period $(j + 1)$ is added to the horizon, the quantity to satisfy some or all of its demand D_{j+1} can at the earliest be produced in period m; and, the portion of the production sequence up to $m - 1$ remains unchanged. Then, $a[i, j+1]$ is the sum of $a[i, j]$ and the costs of producing D_{j+1} units starting from period m and carrying them in inventory until period $j + 1$. That is, the cost difference as a new period is added is only due to the production and holding costs incurred for the quantity to satisfy D_{j+1}. Now, consider the same demand pattern from period $i + 1$ to period j. The production sequence in this case will be the same for periods $i+1$ through $m-1$, and the quantity $Q_i - D_i$, which was produced in period i previously will now be produced in period m (and in later periods if necessary). Let m' $(\geq m)$ denote the latest period in which production is done for the production sequence starting in period $i+1$. Then, if a new period $(j+1)$ is added to the horizon, some or all of its demand D_{j+1}can at the earliest be produced in period m'; and, the portion of the production sequence up to period $m' - 1$ remains unchanged. The total cost of production $a[i + 1, j + 1]$ is the sum of $a[i + 1, j]$ and the costs of producing D_{j+1} units starting from period m'

and carrying them in inventory until period $j+1$. Again, the cost difference as a new period is added is only due to the production and holding costs incurred for the quantity to satisfy D_{j+1}. We see that the increase in the total costs for a production sequence to satisfy demands over a given horizon as a new period $j+1$ is added to the horizon is equal to the production and holding costs of the quantity to satisfy the demand in period $j+1$. Due to the marginal production cost structure imposed, the production cost of any quantity to satisfy a demand in the future decreases as the quantity is produced in periods closer to the demand period. The holding cost decreases as well, since the number of periods over which inventory is held decreases. Therefore, $a[i,j+1]-a[i,j] \geq a[i+1,j+1]-a[i+1,j]$. Hence, we establish the Mongité of A given below:

Lemma 4 *A is Monge if* $D_t \leq Q_t$, $k_t = 0$ *and* $\omega_t = 0$ *for* $t = 1, 2, \cdots, N$.

Given the Mongité of A and linear preprocessing time, we can apply Eppstein's on-line array-searching algorithm (Eppstein(1990)). Hence, we have Theorem 6. ■

Following the arguments presented in the proof, note that for positive k_t's, there may be additional cost reductions when period $j+1$ is added if $x_m > Q_m$ and/or $x_{m'} > Q_{m'}$ which imply that some of the production may be pushed forward due to Lemma 4. Hence, the Monge condition for A may not hold any more. Likewise, if $D_t \geq Q_t$ for some t, then we no longer have a two-dimensional array to define the costs since it is not guaranteed that $a[i,j]$ will always involve a cold setup as assumed in the above formulation. It may be interesting for future work to investigate similar linear search algorithms for this case using the properties of higher-dimensional Monge arrays (see Aggarwal and Park 1989 and Aggarwal and Park 1993).

11.1 Numerical Study

We conducted our numerical study to investigate three aspects: *(i)* the sensitivity of the optimal production schedule to various system parameters, *(ii)* the impact of managerial policies of keeping processes warm, and *(iii)* optimal capacity determination in the presence of warm/cold processes.

For our numerical study, we considered a problem horizon of 100 periods. A base demand series was developed such that the base demand in period t, D_t^{base}, is equal to 0 with probability 0.20 and, with probability 0.80 it is generated from the distribution $U(1, 40)$. We considered only integer demands in our analysis; hence, we truncated the random demand values generated to ensure integer values. Different tightness levels of capacity was achieved through six demand patterns as a multiple of the base series (i.e. $D_t = M \cdot D_t^{base}$). We considered constant parameters over the horizon of the problem; for all t, we set $h_t = h = 1$, $k_t = 0$, $K_t = K$, $R_t = R$, $Q_t = Q$, $\omega_i = \omega$ and $c_t = c$. Since no shortages are allowed, we ignored the unit production cost (i.e. $c = 0$). The rest of the parameters of the experimental set is as follows:

$$K \in \{75, 50, 25\}$$

$$R \in \{154, 152, \cdots, 54, 52\}$$

$$M \in \{1, 1.75, 2.5, 3.5, 4.5, 5.5\}$$

For warming costs, we used

$$\omega \in \{0, 0.05, 0.1, 0.15, \cdots, 0.85, 0.9, 0.95\}$$

and $\omega > 1$ as a special case ($Q = R$). Note that, since $h = 1$, one can interpret the values of ω used herein as the ratio of unit warming cost to unit holding cost per period, as well. As discussed above, the least cost is achieved when the warm process threshold is set at the point of indifference; therefore, in our numerical study, we used $Q = \hat{Q}$, unless stated explicitly otherwise.

All instances were solved on an IBM Pentium III using the forward DP algorithm provided above of complexity $O(N^3)$ after smoothing the given individual demands to ensure feasibility, which is done in $O(N)$ time.

11.1.1 Sensitivity

As a representative sample of our results, consider the medium demand case ($M = 2.5$) tabulated in Table 8.1; for brevity only the first 25 periods of the optimal solution are presented. In this table, the periods where there are no production and no warming are left blank, italics indicate the periods where the process is kept warm at the end of the previous period.

We notice that for some values of ω, the optimal solution is the lot-for-lot policy; this is actually the case for all parameter combinations where the point of indifference $\hat{Q}$ is found to be less than or equal to zero ($\Rightarrow Q = 0$). This is to be expected since keeping the process warm onto the next period is more beneficial than incurring a cold setup in the next period even if there is no production done in the current period (indicated with 0). Incidentally, in such cases, the process is being kept warm throughout. For positive $\hat{Q}$, batching occurs as expected. As ω increases, batching becomes more beneficial and run sizes increase.

The impact of the cold setup cost, K is similar to that of ω in inducing batching albeit in opposite direction, and it is more pronounced. As K decreases, the point of indifference $\hat{Q}$ increases; thus, the option of keeping the process warm loses its appeal since it would imply too big run sizes resulting in higher carrying costs. Hence, for small K, the optimal production schedule is closer to the lot-for-lot policy with a few big batches in between. Hence, when K decreases, the number of periods in which there is production increases; however, there are more cold setups done than warm setups.

For the unreported cases of low and high demands, we observed less sensitivity of the optimal schedule to the system parameters. We also observed that the impact of capacity R is primarily through $\hat{Q}$ except in cases of very tight capacity levels.

Our results indicate that the warm process threshold, $\hat{Q}$ plays a more critical role in warm/cold process decisions than the individual values of system parameters. This observation motivated us to investigate the special case of the 'warm-only-if-at-capacity' policy, where $Q_t = R_t$ for all t. This policy would give the optimal solution when $\omega_t \geq h_t$ since $\hat{Q}_t \leq 0$ for all t. In other instances, it is a heuristic corresponding to a constrained solution of the problem. This policy is important because it also corresponds to the cases where undertime options are deliberately not used by management even though they are available. It may also be viewed as a 'big bucket version' of the DLSP. The best schedule obtained under the warm-only-if-at-capacity policy for our base problem is also given in Table 8.1. Since it corresponds to the optimal solution when $\omega > 1$, we see most batching in this case. Furthermore, the imposed policy encourages batching in the best solution for large K values; but, for small K, it gives a schedule similar to that obtained for moderate unit warming costs. This tendency was validated for low and high demand scenarios with other values of the cost parameters, as well.

Similarly, we observe that the deviation of the total cost under the $Q = R$

policy from the optimal decreases as ω increases, for all demand levels and setup cost values. (For an instance of R, we refer the reader to Figure 8-1). For smaller demand levels and smaller cold setup costs, the deviation becomes zero at smaller values of ω. The speed of convergence is more sensitive to the changes in K.

11.1.2 Managerial Implications: Capacity Selection

Next, we study capacity issues for warm/cold processes. We report our findings on the medium demand case with $K = 75$ for a broad range of capacity values, $R = [52, 154]$, where $R = 52$ corresponds to the minimum capacity level for which a feasible solution exists under the given demand pattern. Throughout, we assume that $Q = \hat{Q}$, which provides the lowest attainable cost. (Note that as R changes, so does $\hat{Q}$.) We focus on the behavior of the total cost and its components as the capacity of the process changes and report them in Figures 8-1 - 8-2. The reported costs are for the entire problem horizon ($N = 100$).

We observe a non-monotonic behavior in the total cost with respect to capacity. As capacity decreases, the total cost initially decreases; then, there is an increase for all values of ω. For large values of ω (and for the imposed warm-only-if-at-capacity policy), the total cost curve fluctuates and, in some instances, exhibits sudden jumps (see Figure 8-1). This erratic behavior is best explained through individual cost components.

First consider the setup costs depicted in Figure 8-2. As R decreases, the incurred setup cost decreases almost monotonically followed by a sudden downward jump to the value of $K(=75)$ after which it remains flat. That the setup cost equals a value of K implies that there is a single cold setup over the entire horizon of the problem in the optimal solution. This instance corresponds to the capacity level at which $\hat{Q}$ ceases to attain a positive value as R decreases; hence, the process can be kept warm throughout the horizon even if no production is done. Note that this happens at higher capacity levels for smaller ω.

For the warming cost depicted in Figure 8-3, we observe an opposite behavior. As R decreases, the warming cost increases albeit non-monotonically until a sudden upward jump, followed by an almost steady decrease. The jump coincides with the same capacity level observed in the behavior of setup costs. Similarly, the jump in warming costs occurs at higher capacity levels for smaller ω. The behavior of the two cost components visa vis each other illustrates the fundamental trade off in the presence of warm/cold processes. In fact, a closer examination of the numerical results reveals that the two cost components go in tandem. This is intuitive but still important to observe.

The main component that causes the total cost curve to exhibit a bumpy behavior is the inventory holding cost depicted in Figure 8-4. As R decreases, the holding cost tends to decrease, as expected, since with lower R values, less inventory is carried. At the capacity level where $\hat{Q} \leq 0$, the production schedule is the lot-for-lot policy; hence, no inventory is carried in those cases and we observe zero holding costs after this point, as R decreases. However, if R decreases further, we begin to see the effects of prior demand smoothing to ensure feasibility. That is, to ensure feasibility, the solution is forced a priori to carry more and more inventory in advance as the capacity tightness increases. This increasing portion of the inventory cost is what causes the increase in the total cost as R gets smaller.

Although non-monotonic, we observe that there is an overall 'convex' trend in the total cost with respect to the capacity limit. That is, there is an 'optimal' capacity level which minimizes the total costs over the horizon. The optimal capacity level appears to increase as ω decreases. The analysis of the total cost provides further managerial implications regarding capacity selection and use of undertime options. Next, we discuss such issues.

The model and the solution procedures discussed herein provide a manager with the tools to determine the optimal capacity level in the presence of warm/cold processes, as well. For example, from Figure 8-1, it is easy to see that an economically rational manager would choose $R = 72$ as the optimal capacity level when $\omega = 0.35$ for the numerical setting considered. (However, we should point out that this conclusion is based on a single known sample path of demands and cannot be generalized to a more realistic scenario of stochastic demands. For brevity, in our discussion herein, we consider such robustness issues to be outside the scope of our analysis, which can be addressed in a simulation context.) Yet, the question remains: What are the implications of suboptimal capacity decisions? In particular, what happens if the manager ignores the availability of the undertime option?

We consider two such scenarios. In the first case, the manager restricts warm process decisions to only the instances when production quantity in a period is equal to the capacity limit; that is, the manager sets $Q = R$ and chooses the best capacity level accordingly. Note that the manager is aware of the advantages of keeping a process warm but behaves as if ω is prohibitively high ($\Rightarrow \hat{Q} \geq R$). In the second case, the manager totally ignores the possibility of keeping the process warm and bases the capacity selection decision on the solution of the classical (uncapacitated) problem. Specifically, in this case, the best capacity is selected to equal the maximum production quantity obtained in the Wagner-Whitin solution. In Table 8.2, we present a representative sample of our findings where R^*_{opt}, R^*_1 and R^*_2 are, respectively, the optimal and the best capacity levels selected for the first and second cases. We let $\Delta_i\%$ (for $i = 1, 2$) denote the respective percentage deviations in total costs with respect to the total cost under the optimal capacity decision, and compute it as follows:

$$\Delta_i\% = \frac{\left(TC_{R=R^*_i} - TC_{R=R^*_{opt}}\right)}{TC_{R=R^*_{opt}}} \times 100 \tag{11.2}$$

We find that $R^*_1 < R^*_{opt} < R^*_2$ for all ω. This implies that ignoring potential benefits of warm processes results in selecting a capacity level higher than the optimal schedule necessitates, yielding a lower equipment utilization rate and possibly lower rates of return on investment (ROI). On the other hand, imposing the warm-only-if-at-capacity policy results in selecting a capacity level lower than the optimal. Thus, it yields a higher equipment utilization rate and possibly higher ROI. This apparent efficiency may be the reason behind the popularity of this policy among practitioners. However, the ensuing tigthness of capacity, in fact, increases the total operating costs incurred. Operating with R_1^* results in an excessively large cost differential for low unit warming costs; as ω increases, the differential vanishes in the limit, as expected. The cost differential monotonically decreases over ω. When R_2^* is used instead of R_{opt}^*, interestingly, the cost differential exhibits a concave behavior over ω. It increases very steeply for low ω, is concave over a large range of unit warming cost, and decreases slowly for large ω. Thus, the total ignorance of the undertime option results in the worst performance (more than 100% deviation from the optimal) over a broad range of parameter values. Its concave behavior also implies that management would most benefit from the use of the undertime option in capacity selection decisions for moderate values of unit warming cost.

In conclusion, the presence of warm/cold processes impacts total operating costs not only by yielding differently structured production schedules compared to the classical settings, but also through optimal capacity selection decisions taking into account the undertime option.

11.1.3 Planning Horizons

Using the medium demand case, i.e. $M = 2.5$, and $k_t = 0$, $\omega_t = 0$ we have observed the planning horizon occurrences, for each of the planning horizon rule introduced in Chapter 10. In this part of the numerical study the warm process threshold takes on values between 100 and 60 with the increments of 5, i.e. $Q_t \in \{100, 95, \cdots, 65, 60\}$. For each of the experiments we have recorded the periods where each planning horizon occurs. We present the planning horizon occurring periods for each planning horizon rule in Table 8.3, for $K_t = 75$ and $Q_t \in \{100, 80, 60\}$, as an illustration. We observe that both warm and cold Zabel-type planning horizons occur more frequently then the Wagner-Whitin-type planning horizons for all the values of Q_t. When Q_t is large cold planning horizons are more frequent than the warm planning horizons. However, both warm Wagner-Whitin-type and warm Zabel-type planning horizon occurrences increases as the Q_t decreases. When the warm process threshold is large, warm process is rare, therefore the planning horizons are more likely to be cold type. As Q_t decreases, the warm process is more likely, hence the warm type planning horizon occurrences increase.

Chapter 12

Conclusions

In this dissertation we have considered opportunities existing in the quality control chart design and dynamic lot sizing environment.

Firstly, we have provided procedures for economical design of $\bar{X}-$control charts for a single machine facing exogenous stoppages which are opportunities for inspection and restoration of the process at reduced cost. These opportunities arise from the line stoppages due to the alarms from other machines in the line. We have assumed that opportunities arrive due to a Poisson process. We have formulated the problem as a cost rate minimization problem by invoking the renewal reward theorem in order to determine the control chart design parameters. Regeneration points in the renewal reward process correspond to machine stoppage instances. With this setting we have derived the expected cycle time and expected cycle cost functions which are used in the total cost rate function to be minimized. Through a numerical study we have conducted a sensitivity analysis and shown that using exogenous opportunities for inspection and restoration can significantly improve the cost rate, hence provide savings.

Although we consider the design of $\bar{X}-$control charts, the idea of the opportunistic inspections in the economic design can be applied to other variables- and attributes- control charts, as well. The exponentially distributed opportunity arrival assumption can be relaxed in future studies. Hence, models incorporating Weibull or Gamma distributions for the opportunity arrivals can be studied. Moreover, models incorporating different distributions for the assignable cause occurrences and the opportunistic inspections may be some extensions of the model developed in this research. When the distribution of the assignable cause occurrences does not have the memoryless property (i.e., is IFR or DFR), then conjecturally we can state that an opportunity taker machine behaves selectively to utilize the inspection opportunities depending on the timing of the opportunity. Dekker and Dijkstra (1992) consider one-opportunity-look-ahead policy for the opportunistic maintenance. A similar policy may be implemented for the control charts design in the presence of opportunistic inspections.

Secondly, we have provided exact model derivation for the multiple machine environment by employing semi-Markov processes. For the exact derivation, we consider embedded cycles which are defined between two consecutive system restarts. System state at a system restart is given by the status of each machine and the remaining time to the next sampling instance for each machine. We show that system state transition probabilities can be derived from individual machine state transition probabilities. However, due to the tedious expressions and lack of practical applicability, we have developed an approximate model using the single machine model. We have shown that partitioning of the machines in a line as opportunity takers and opportunity non-takers can further improve the costs. We have employed a greedy heuristic to search over the partitioning alternatives. We have conducted a numerical study to show the partitioning and the joint optimization of the control chart parameters. Furthermore, with a simulation study we have tested our approximate analytical cost rates with the actual, and observed that approximations are acceptable.

For future work, additional robust approximations can be developed. One such approximation may be using the optimal values of the sample size and control limits obtained from the classical model ($\mu = 0$) for each machine and optimizing over only the sampling interval. Heuristics for the partitioning problem other than the greedy one used herein can also be implemented for searching the partitioning of the machines.

In the third part, we have considered lot sizing decisions for a process which can be kept warm for the next period at an additional linear cost if the production quantity in the current period is at least a positive threshold amount. We have formulated the problem as a dynamic programming model. We have established the structure of the optimal production schedule and the conditions under which a forward polynomial time solution is possible. As a special case, we also presented a linear time solution and established conditions for the existence of planning horizons. Through a numerical study, we have also investigated the impact of a warm process option on the required capacity for a given stream.

A number of extensions of the model and the solution techniques herein are possible as future work. Although our focus has been to establish the structure of the optimal solution, development of heuristic solutions remains an open research area. We can conjecture that forward heuristics, in general, would perform relatively better for the case when $k_t = 0$, since, in this case, a forward optimal solution is possible. In addition, meta-heuristics such as tabu search and simulated annealing are also interesting venues of research. Furthermore, most real-world problems exhibit demand uncertainty. Hence, the robustness of the solutions to such changes in demands is also important. Our structural results indicate that the option of keeping the process warm enables longer production series *vis a vis* the CLSP setting. One can conjecture that a warm/cold process would be less susceptible to the 'nervousness' phenomenon, since production is kept going over a number of successive periods. However, it is not possible to say *a priori* which type of heuristic would be the best and it is another fertile research topic. Incorporating the lost sales and backordering options into the model would be an interesting extension. Multiple product scheduling can also be studied under this setting. Additionally, ordering policies with rebates can be modeled as warm/cold processes introduces here, such that if the order quantity is more than a certain quantity (or value), a discount, valid during a certain time window, can be offered for the next purchase. Another extension would be modeling the problem with setup times and production rates instead of setup and constant production costs.

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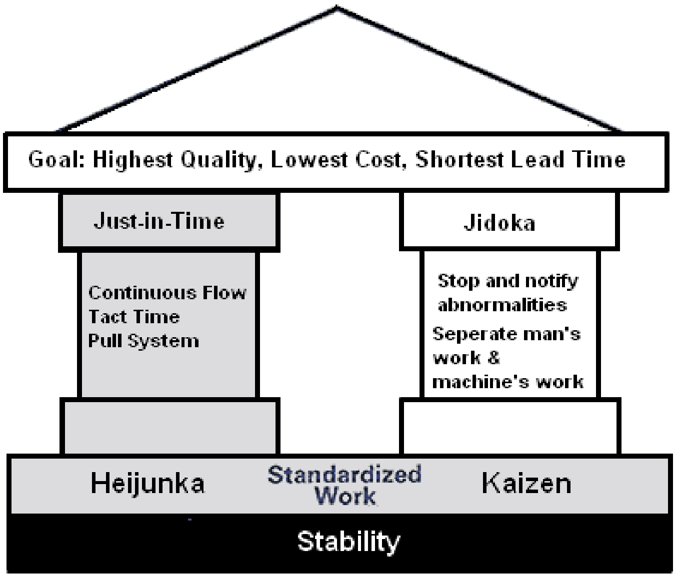

Figure 2-1: The house of Toyota Production System

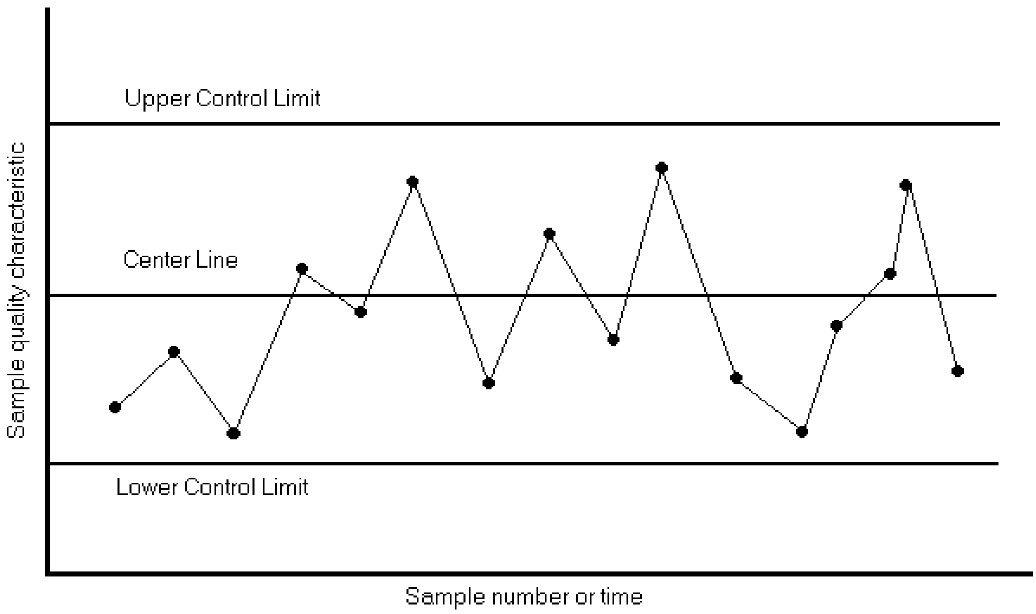

Figure 2-2: An illustration of control charts.

Table 2.1: Categories of quality costs (after Montgomery 2004)

Preventive Costs	*Internal Failure Costs*
Quality planning and engineering	Scrap
New product review	Rework
Product/process design	Retest
Process control	Failure analysis
Burn-in	Downtime
Training	Yield losses
Quality data acquisition and analysis	Downgrading(off-spacing)
Appraisal Costs	*External Failure Costs*
Inspection and test of incoming material	Complaint adjustment
Product inspection and test	Returned product/material
Materials and services consumed	Warranty charges
Maintaining accuracy of test equipment	Liability costs
	Indirect costs

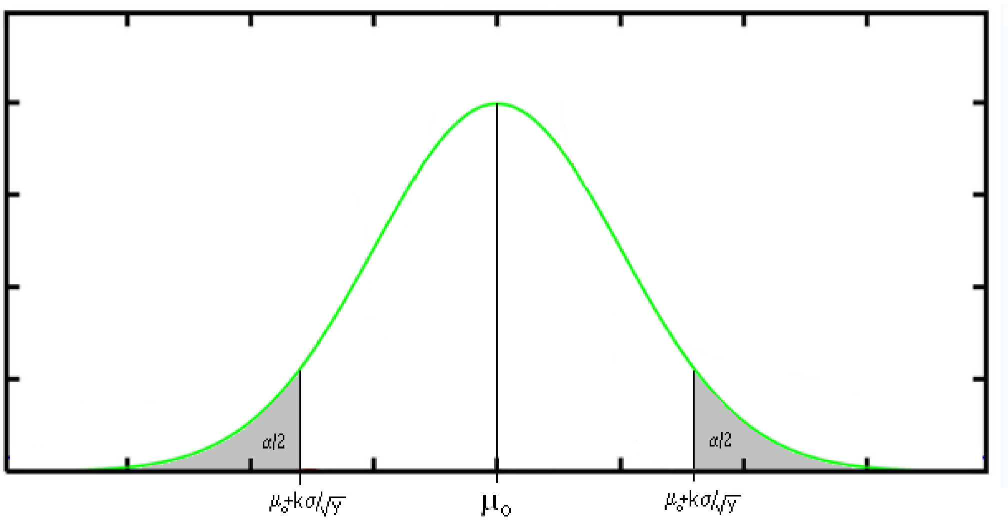

Figure 3-1: Type I error.

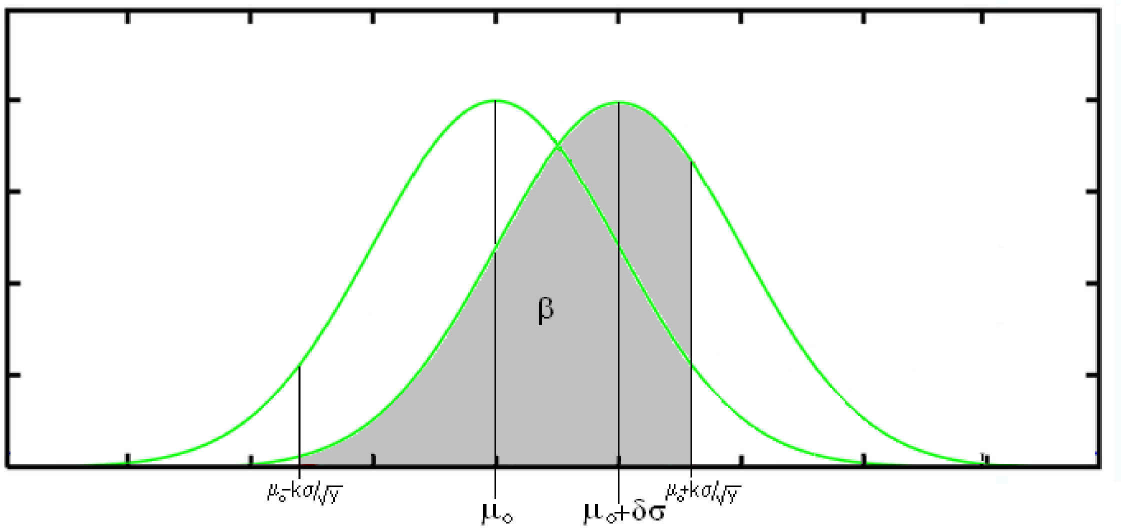

Figure 3-2: Type II error.

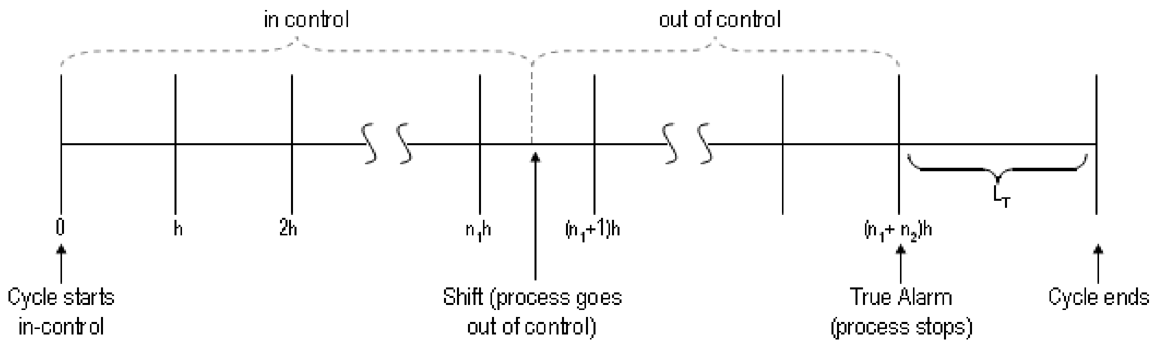

Figure 3-3: Illustration of the cycle type true

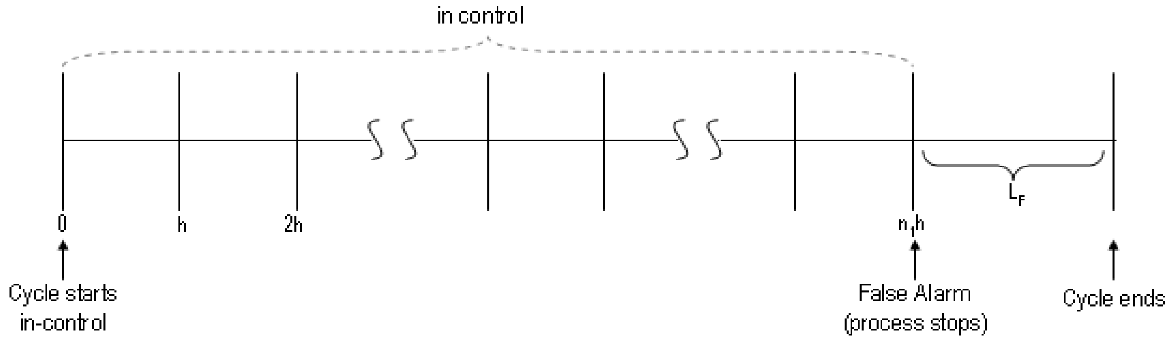

Figure 3-4: Illustration of the cycle type false

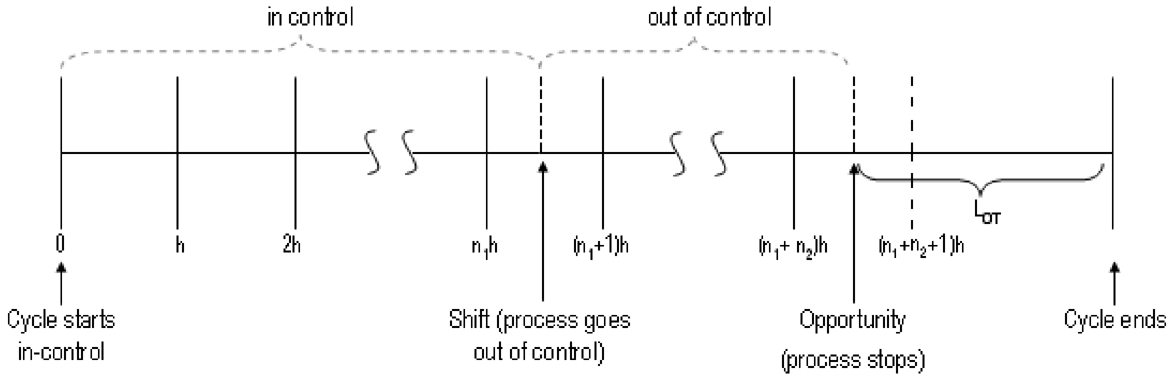

Figure 3-5: Illustration of the cycle type opportunity true

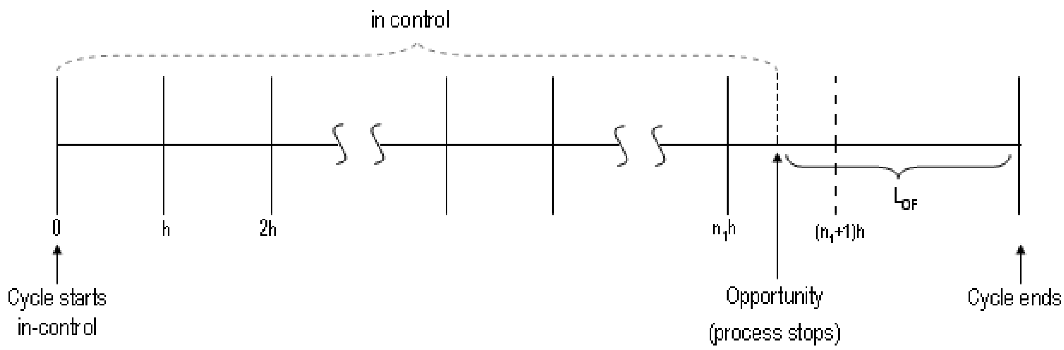

Figure 3-6: Illustration of the cycle type opportunity false

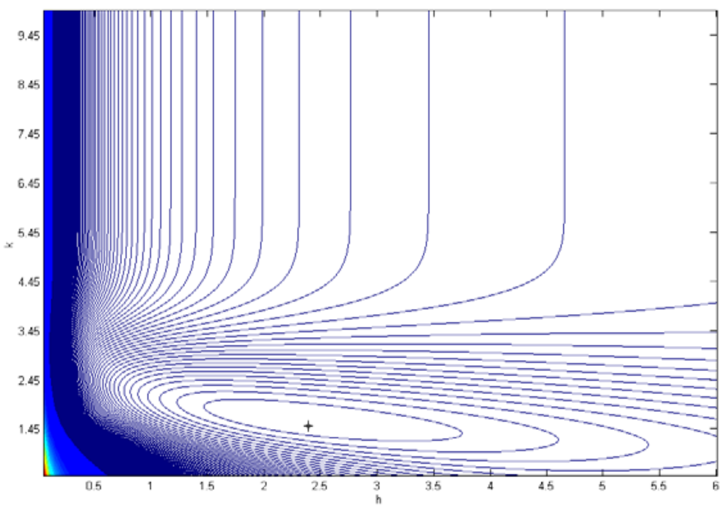

Figure 4-1: Contour plot of the cost rate function over $k-h$ plane for $\pi = 500$, $L_T = L_F = 0.1$, $L_O = 0.1$, $a = 100$, $u = 5$, $b = 0.1$, $\lambda = 0.05$, $\mu = 0.25$ and $y = 1$.

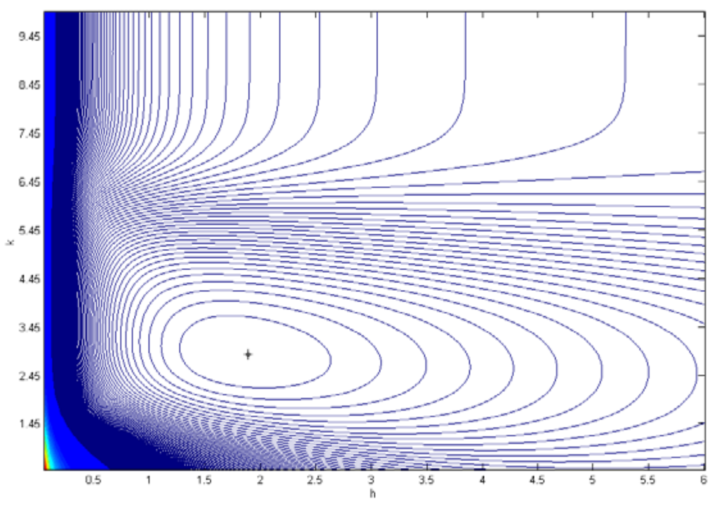

Figure 4-2: Contour plot of the cost rate function over $k-h$ plane for $\pi = 500$, $L_T = L_F = 0.1$, $L_O = 0.1$, $a = 100$, $u = 5$, $b = 0.1$, $\lambda = 0.05$, $\mu = 0.25$ and $y = 6$.

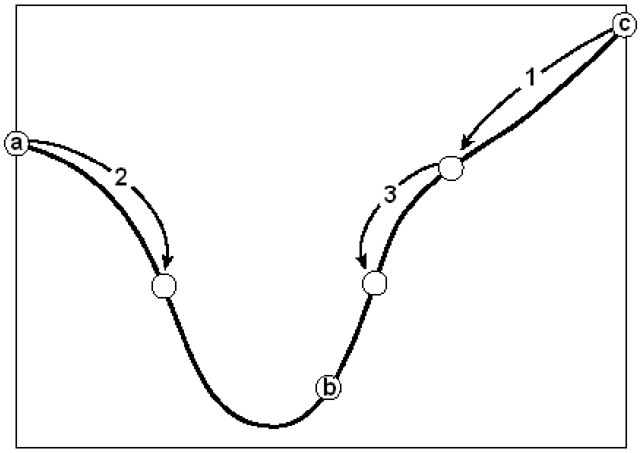

Figure 4-3: Illustration of the Golden Section Search algorithm.

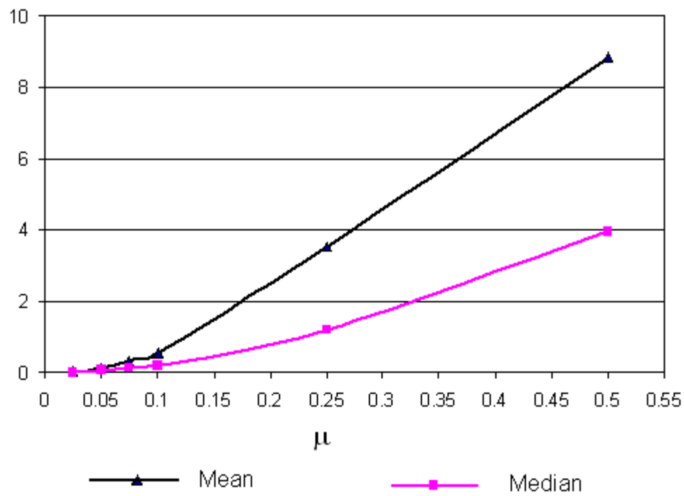

Figure 4-4: Change of the mean and median of the percentage improvement with respect to μ

Table 4.1: The parameter set for the single machine numerical study

Parameter	Values		
λ	0.05		
π	500		
a	50	100	250
b	0.1	0.2	1
u	0	5	10
L_F	0.1	0.25	0.5
L_T	0.1	0.25	0.5
L_o	0.1	0.25	0.5
R_F	0		
R_T	0		

Table 4.2: Sensitivity analysis with respect to u. $L_T = 0.1; L_F = 0.5; L_O = 0.25; a = 100; b = 0.2$

	$u = 0$			$u = 5$			$u = 10$		
	$(y^*; k^*; h^*)$	E[TC]*	%Δ	$(y^*; k^*; h^*)$	E[TC]*	%Δ	$(y^*; k^*; h^*)$	E[TC]*	%Δ
0	(5; 3.43; 0.60)	6.32	0	(8; 3.45; 1.69)	10.5	0	(8; 3.35; 2.27)	13.1	0
0.025	(5; 3.43; 0.62)	9.2	0.02	(8; 3.45; 1.74)	13.2	0.03	(8; 3.35; 2.35)	15.5	0.03
0.05	(5; 3.43; 0.64)	12	0.02	(8; 3.45; 1.80)	15.8	0.09	(8; 3.35; 2.43)	18	0.12
0.075	(5; 3.43; 0.66)	14.8	0.06	(8; 3.45; 1.86)	18.3	0.17	(8; 3.35; 2.52)	20.3	0.23
0.1	(5; 3.43; 0.68)	17.4	0.11	(8; 3.45; 1.93)	20.8	0.27	(8; 3.35; 2.62)	22.6	0.36
0.25	(5; 3.40; 0.87)	32.2	0.42	(7; 3.35; 2.46)	34.3	0.94	(8; 3.35; 3.48)	35.3	1.32
0.5	(5; 3.40; 1.82)	52.8	1.11	(7; 3.35; 6.00)	53.2	2.24	(7; 3.20; 9.96)	53.2	2.79

Table 4.3: Sensitivity analysis with respect to b. $L_T = 0.25; L_F = 0.5; L_O = 0.25; a = 100; u = 5$

	$b = 0.1$			$b = 0.2$			$b = 1$		
	$(y^*; k^*; h^*)$	E[TC]*	%Δ	$(y^*; k^*; h^*)$	E[TC]*	%Δ	$(y^*; k^*; h^*)$	E[TC]*	%Δ
0	(9; 3.60; 1.63)	13.53	0	(8; 3.45; 1.72)	14.01	0	(5; 2.92; 2.11)	16.5	0
0.025	(9; 3.60; 1.70)	16.07	0.04	(8; 3.45; 1.80)	16.52	0.04	(5; 2.92; 2.22)	18.81	0.06
0.05	(9; 3.60; 1.77)	18.55	0.13	(8; 3.45; 1.88)	18.97	0.13	(5; 2.92; 2.34)	21.05	0.20
0.075	(9; 3.60; 1.86)	20.97	0.25	(8; 3.45; 1.97)	21.34	0.27	(5; 2.92; 2.47)	23.24	0.41
0.1	(9; 3.60; 1.96)	23.32	0.40	(8; 3.45; 2.08)	23.66	0.43	(5; 2.92; 2.62)	25.36	0.65
0.25	(9; 3.60; 3.03)	36.15	1.68	(7; 3.29; 3.23)	36.30	1.82	(5; 2.92; 4.39)	36.95	2.66
0.5	(1; 3.70; 50)	53.21	4.78	(1; 5.88; 50)	53.21	4.85	(1; 5.88; 50)	53.21	5.21

Table 4.4: Sensitivity analysis with respect to a. $L_T = 0.25; L_F = 0.1; L_O = 0.1; u = 5; b = 0.2$

	$a = 50$			$a = 100$			$a = 250$		
	$(y^*; k^*; h^*)$	E[TC]*	%Δ	$(y^*; k^*; h^*)$	E[TC]*	%Δ	$(y^*; k^*; h^*)$	E[TC]*	%Δ
0	(6; 2.84; 2.48)	11.4	0	(6; 2.85; 1.67)	13.9	0	(6; 2.85; 1.02)	18.7	0
0.025	(6; 2.84; 2.55)	11.2	0.02	(6; 2.85; 1.69)	13.7	0.01	(6; 2.85; 1.03)	18.5	0
0.05	(6; 2.85; 2.62)	11.0	0.06	(6; 2.85; 1.72)	13.5	0.02	(6; 2.85; 1.04)	18.3	0.01
0.075	(6; 2.85; 2.70)	10.8	0.15	(6; 2.85; 1.75)	13.3	0.06	(6; 2.85; 1.05)	18.1	0.02
0.1	(6; 2.85; 2.78)	10.6	0.26	(6; 2.85; 1.78)	13.1	0.10	(6; 2.85; 1.06)	17.9	0.03
0.25	(6; 2.87; 3.43)	9.33	1.70	(6; 2.87; 1.96)	12	0.63	(6; 2.86; 1.11)	16.8	0.20
0.5	(6; 2.88; 5.92)	7.46	6.53	(6; 2.88; 2.39)	10.3	2.51	(6; 2.87; 1.20)	15.2	0.78

Table 4.5: Sensitivity analysis with respect to L_O. $L_T = 0.1; L_F = 0.5; a = 100; u = 5; b = 0.2$

	$L_o = 0.1$			$L_o = 0.25$			$L_o = 0.5$		
	$(y^*; k^*; h^*)$	E[TC]*	%Δ	$(y^*; k^*; h^*)$	E[TC]*	%Δ	$(y^*; k^*; h^*)$	E[TC]*	%Δ
0	(8; 3.45; 1.69)	10.5	0	(8; 3.45; 1.69)	10.5	0	(8; 3.45; 1.69)	10.5	0
0.025	(8; 3.45; 1.76)	15	0.04	(8; 3.45; 1.74)	13.2	0.03	(8; 3.45; 1.71)	10.3	0.01
0.05	(8; 3.45; 1.84)	19.3	0.13	(8; 3.45; 1.80)	15.8	0.09	(8; 3.45; 1.74)	9.96	0.03
0.075	(8; 3.45; 1.92)	23.5	0.23	(8; 3.45; 1.86)	18.3	0.17	(8; 3.45; 1.77)	9.68	0.07
0.1	(8; 3.45; 2.02)	27.6	0.35	(8; 3.45; 1.93)	20.8	0.27	(8; 3.45; 1.79)	9.41	0.13
0.25	(8; 3.29; 2.97)	50.1	1.18	(7; 3.29; 2.47)	34.3	0.94	(8; 3.45; 1.98)	7.91	0.85
0.5	(1; 6.03; 50)	81.2	2.79	(7; 3.29; 6.00)	53.2	2.24	(7; 3.29; 2.38)	5.88	3.61

Table 4.6: Sensitivity analysis with respect to L_O. $L_T = 0.25; L_F = 0.5; a = 50; u = 5; b = 0.2$

	$L_o = 0.1$			$L_o = 0.25$			$L_o = 0.5$		
	$(y^*; k^*; h^*)$	E[TC]*	%Δ	$(y^*; k^*; h^*)$	E[TC]*	%Δ	$(y^*; k^*; h^*)$	E[TC]*	%Δ
0	(8; 3.44; 2.56)	11.49	0	(8; 3.44; 2.56)	11.49	0	(8; 3.44; 2.56)	11.49	0
0.025	(8; 3.44; 2.80)	15.74	0.12	(8; 3.44; 2.80)	13.91	0.13	(8; 3.45; 2.69)	11.05	0.05
0.05	(8; 3.44; 3.11)	19.86	0.38	(8; 3.44; 3.12)	16.23	0.47	(8; 3.45; 2.84)	10.61	0.23
0.075	(8; 3.44; 3.53)	23.85	0.73	(8; 3.44; 3.55)	18.47	0.97	(8; 3.45; 3.02)	10.17	0.53
0.1	(7; 3.29; 4.08)	27.71	1.17	(7; 3.29; 4.13)	20.61	1.62	(7; 3.29; 3.26)	9.57	2.71
0.25	(7; 6.03; 50)	47.66	5.66	(7; 6.03; 50)	30.84	9.10	(7; 3.29; 6.29)	7.12	8.56
0.5	(1; 6.03; 50)	79.82	3.68	(1; 6.03; 50)	49.54	6.27	(1; 6.03; 50)	3.64	31.08

Table 4.7: Sensitivity analysis with respect to L_O. $L_T = 0.5; L_F = 0.5; a = 50; u = 10; b = 0.2$

	$L_o = 0.1$			$L_o = 0.25$			$L_o = 0.5$		
	$(y^*; k^*; h^*)$	E[TC]*	%Δ	$(y^*; k^*; h^*)$	E[TC]*	%Δ	$(y^*; k^*; h^*)$	E[TC]*	%Δ
0	(8; 3.31; 3.73)	18.58	0	(8; 3.31; 3.73)	18.58	0	(8; 3.31; 3.73)	18.58	0
0.025	(8; 3.31; 4.16)	22.46	0.13	(8; 3.31; 4.16)	20.64	0.15	(8; 3.31; 4.17)	17.62	0.18
0.05	(8; 3.31; 4.74)	26.20	0.48	(8; 3.31; 4.76)	22.62	0.57	(8; 3.31; 4.79)	16.63	0.81
0.075	(8; 3.31; 5.59)	29.81	0.98	(8; 3.31; 5.65)	24.48	1.24	(8; 3.31; 5.74)	15.60	2.05
0.1	(8; 3.31; 7.00)	33.27	1.65	(8; 3.31; 7.16)	26.22	2.19	(8; 3.31; 7.46)	14.48	4.22
0.25	(1; 6.03; 50)	51.85	5.58	(1; 6.03; 50)	35.19	8.49	(1; 6.03; 50)	7.41	32.72
0.5	(1; 5.88; 50)	83.64	2.68	(1; 5.88; 50)	53.64	4.43	(1; 6.03; 50)	3.64	43.48

Table 4.8: Summary of the sensitivity analyses results

	y^*	k^*	h^*	$E[TC^*]$
u	$\nearrow$	$\leftrightarrow$	$\nearrow$	$\nearrow$
b	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$
a	$\leftrightarrow$	$\searrow$	$\searrow$	$\nearrow$
L_O	$\leftrightarrow$	$\leftrightarrow$	$\nearrow$	$\searrow$
L_T	$\leftrightarrow$	$\leftrightarrow$	$\nearrow$	$\nearrow$
L_F	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

Table 4.9: Sensitivity with respect to μ. $L_F = 0.5; L_O = 0.25; a = 100; u = 5; b = 1$

	$L_T = 0.1$				$L_T = 0.5$			
μ	y^*	k^*	h^*	$E[TC^*]$	y^*	k^*	h^*	$E[TC^*]$
0	8	3.45	1.69	10.5	8	3.45	1.78	19.69
0.025	8	3.45	1.74	13.2	8	3.45	1.86	22.09
0.05	8	3.45	1.80	15.8	8	3.45	1.95	24.44
0.075	8	3.45	1.86	18.3	8	3.45	2.06	26.72
0.1	8	3.45	1.93	20.8	8	3.45	2.17	28.94
0.25	7	3.29	2.47	34.3	7	3.29	3.54	41.03
0.5	7	3.29	6.00	53.2	1	5.70	50	57.27

Table 4.10: Sensitivity with respect to μ. $L_F = 0.25; L_O = 0.1; a = 100; u = 5; b = 1$

	$L_T = 0.1$				$L_T = 0.5$			
μ	y^*	k^*	h^*	$E[TC^*]$	y^*	k^*	h^*	$E[TC^*]$
0	7	3.23	1.66	10.5	7	3.23	2.70	17.03
0.025	7	3.23	1.70	13.2	7	3.23	2.86	18.44
0.05	7	3.23	1.74	15.8	7	3.23	3.05	19.82
0.075	7	3.23	1.79	18.3	7	3.23	3.26	21.18
0.1	7	3.23	1.84	20.8	7	3.23	3.52	22.52
0.25	7	3.23	2.21	34.3	7	3.23	8.29	29.98
0.5	7	3.23	3.49	53.2	1	5.88	50	42

Table 4.11: Sensitivity with respect to μ. $L_F = 0.1; L_O = 0.25; a = 100; u = 5; b = 1$

	$L_T = 0.1$				$L_T = 0.5$			
μ	y^*	k^*	h^*	$E[TC^*]$	y^*	k^*	h^*	$E[TC^*]$
0	6	2.85	1.64	10.4	6	2.84	1.73	19.55
0.025	6	2.85	1.66	10.17	6	2.85	1.77	19.20
0.05	6	2.85	1.69	9.95	6	2.85	1.82	18.84
0.075	6	2.85	1.71	9.73	6	2.85	1.88	18.49
0.1	6	2.85	1.74	9.51	6	2.85	1.93	18.15
0.25	6	2.87	1.92	8.29	6	2.86	2.40	16.10
0.5	6	2.88	2.32	6.48	6	2.88	4.56	12.83

Table 4.12: Sensitivity with respect to μ. $L_F = 0.25; L_O = 0.5; a = 100; u = 5; b = 1$

	$L_T = 0.1$				$L_T = 0.5$			
μ	y^*	k^*	h^*	$E[TC^*]$	y^*	k^*	h^*	$E[TC^*]$
0	7	3.23	1.66	10.48	8	3.45	1.78	16.69
0.025	7	3.23	1.68	10.18	8	3.45	1.86	19.08
0.05	7	3.23	1.71	9.90	8	3.45	1.96	18.47
0.075	7	3.23	1.73	9.62	8	3.45	2.07	17.87
0.1	7	3.23	1.76	9.35	8	3.45	2.19	17.27
0.25	7	3.23	1.94	7.87	7	3.29	3.87	13.59
0.5	7	3.23	2.36	5.86	1	6.03	15	7.27

Table 4.13: Cost Breakdown. $L_T = 0.1; L_F = 0.5; L_O = 0.25; a = 100; b = 0.2$

	$b = 0.1$				$b = 0.2$				$b = 1$			
μ	T	F	OT	OF	T	F	OT	OF	T	F	OT	OF
0	99.26	0.74	0	0	97.88	2.12	0	0	93.63	6.37	0	0
0.025	90.90	0.57	0.76	7.7	89.94	1.66	0.82	7.58	86.95	4.90	1.15	7.10
0.05	86.80	0.49	1.39	11.32	85.98	1.41	1.52	11.09	83.16	4.13	2.19	10.53
0.075	84.40	0.43	2.02	13.15	83.64	1.24	2.22	12.91	80.85	3.61	3.23	12.31
0.1	82.84	0.39	2.66	14.10	82.09	1.12	2.94	13.85	79.23	3.22	4.34	13.21
0.25	78.18	0.41	8.27	13.14	77.22	0.65	9.18	12.95	72.47	1.77	14.06	11.70
0.50	57.33	0.11	38.03	4.52	53.78	0.17	41.96	4.10	35.84	0.25	61.70	2.21

Table 4.14: Cost Breakdown. $L_T = 0.1; L_F = 0.5; L_O = 0.25; a = 100; b = 0.2$

	$u = 0$				$u = 5$				$u = 10$			
μ	T	F	OT	OF	T	F	OT	OF	T	F	OT	OF
0	95.67	4.33	0	0	99.05	0.95	0	0	99	1	0	0
0.025	87.24	4.11	0.44	8.22	87.04	0.86	1.10	11	86.38	0.88	1.54	11.21
0.05	83.02	3.97	0.80	12.21	81.05	0.80	1.99	16.16	80.10	0.68	2.80	16.42
0.075	80.51	3.87	1.14	14.49	77.46	0.76	2.82	18.96	76.25	0.63	3.98	19.14
0.1	78.85	3.78	1.47	15.89	75.06	0.73	3.64	20.57	73.62	0.60	5.17	20.61
0.25	74.78	3.38	3.64	18.20	68.38	0.59	9.30	21.74	65.40	0.45	13.70	20.46
0.50	72.27	2.76	8.59	16.39	59.58	0.68	24.43	15.31	50.58	0.19	37.42	11.82

Table 4.15: Cost Breakdown. $L_T = 0.25; L_F = 0.25; L_O = 0.5; u = 5; b = 0.2$

	$a = 50$				$a = 100$				$a = 250$			
μ	T	F	OT	OF	T	F	OT	OF	T	F	OT	OF
0	99.33	0.67	0	0	99.03	0.97	0	0	98.51	1.49	0	0
0.025	92.30	0.57	1.10	6.03	90.46	0.82	0.84	7.88	88.20	1.23	0.61	9.97
0.05	88.93	0.50	2.14	8.44	86.26	0.73	1.56	11.45	83.04	1.09	1.09	14.78
0.075	86.93	0.44	3.29	9.34	83.79	0.67	2.26	13.28	79.96	1.00	1.54	17.51
0.1	85.48	0.39	4.68	9.45	82.17	0.63	2.99	14.21	77.92	0.94	1.97	19.17
0.25	65.05	0.10	31.62	3.23	77.13	0.42	9.31	13.14	72.87	0.74	4.83	21.57
0.50	0	0	100	0	53.04	0.11	42.81	4.04	76.81	12.47	1.17	9.55

Table 4.16: Cost Breakdown. $L_T = 0.5; L_F = 0.25; a = 100; u = 5; b = 0.2$

	$L_O = 0.1$				$L_O = 0.25$				$L_O = 0.5$			
μ	T	F	OT	OF	T	F	OT	OF	T	F	OT	OF
0	99.35	0.65	0	0	99.35	0.65	0	0	99.35	0.65	0	0
0.025	84.82	0.50	1.59	13.09	92.84	0.54	1.32	5.30	93.61	0.53	0.63	5.23
0.05	74.93	0.41	2.90	21.76	89.28	0.48	2.59	7.64	90.91	0.47	1.21	7.41
0.075	67.28	0.34	4.06	28.31	86.87	0.44	3.90	8.79	89.40	0.41	1.85	8.34
0.1	60.99	0.29	5.11	33.60	84.99	0.41	5.28	9.32	88.43	0.37	2.59	8.61
0.25	36.41	0.13	10.49	52.97	75.37	0.26	16.33	8.04	79.67	0.14	15.78	4.41
0.50	12.14	0.02	20.03	67.81	37.79	0.05	60.45	1.72	0	0	100	0

Table 4.17: Cost Breakdown. $L_T = 0.25; L_O = 0.25; a = 100; u = 5; b = 0.2$

	$L_F = 0.1$				$L_F = 0.25$				$L_F = 0.5$			
μ	T	F	OT	OF	T	F	OT	OF	T	F	OT	OF
0	97.88	2.12	0	0	99.03	0.97	0	0	99.31	0.69	0	0
0.025	89.94	1.66	0.82	7.58	90.46	0.82	0.84	7.88	74.71	0.50	0.71	24.08
0.05	85.98	1.41	1.52	11.09	86.26	0.73	1.56	11.45	60.40	0.38	1.14	38.07
0.075	83.64	1.24	2.22	12.91	83.79	0.67	2.26	13.28	50.79	0.30	1.48	47.44
0.1	82.09	1.12	2.94	13.85	82.17	0.63	2.99	14.21	43.77	0.24	1.78	54.21
0.25	77.22	0.65	9.18	12.95	77.12	0.42	9.31	13.15	22.00	0.11	4.56	73.33
0.50	53.78	0.17	41.96	4.10	53.14	0.11	42.68	4.07	0	0	13.79	86.21

Table 4.18: Cost Breakdown. $L_F = 0.25; L_O = 0.25; a = 100; u = 5; b = 0.2$

	$L_T = 0.1$				$L_T = 0.25$				$L_T = 0.5$			
μ	T	F	OT	OF	T	F	OT	OF	T	F	OT	OF
0	98.66	1.34	0	0	99.03	0.97	0	0	99.35	0.65	0	0
0.025	86.86	1.14	1.09	10.90	90.46	0.82	0.84	7.88	92.84	0.54	1.32	5.30
0.05	80.95	1.04	1.98	16.04	86.26	0.73	1.56	11.45	89.28	0.48	2.59	7.64
0.075	77.40	0.96	2.81	18.83	83.79	0.67	2.26	13.28	86.87	0.44	3.90	8.79
0.1	75.02	0.91	3.62	20.45	82.17	0.63	2.99	14.21	84.99	0.41	5.28	9.32
0.25	68.38	0.70	9.25	21.67	77.12	0.42	9.31	13.15	75.37	0.26	16.33	8.04
0.50	59.92	0.45	24	15.63	53.14	0.11	42.68	4.07	37.79	0.05	60.45	1.72

Table 4.19: Summary statistics of the percentage improvement of JPC over the classical SPC.

μ	0.025	0.05	0.075	0.1	0.25	0.5	*Overall*
Mean	0.031	0.12	0.317	0.55	3.527	8.82	1.922
Std.Dev.	0.041	0.167	0.647	1.03	6.443	11.89	5.986
Median	0.014	0.06	0.123	0.21	1.2	3.95	0.12
Min	0	0	0	0	0.03	0.12	0
Max	0.228	1.04	7.69	11.37	34.5	59.77	59.77

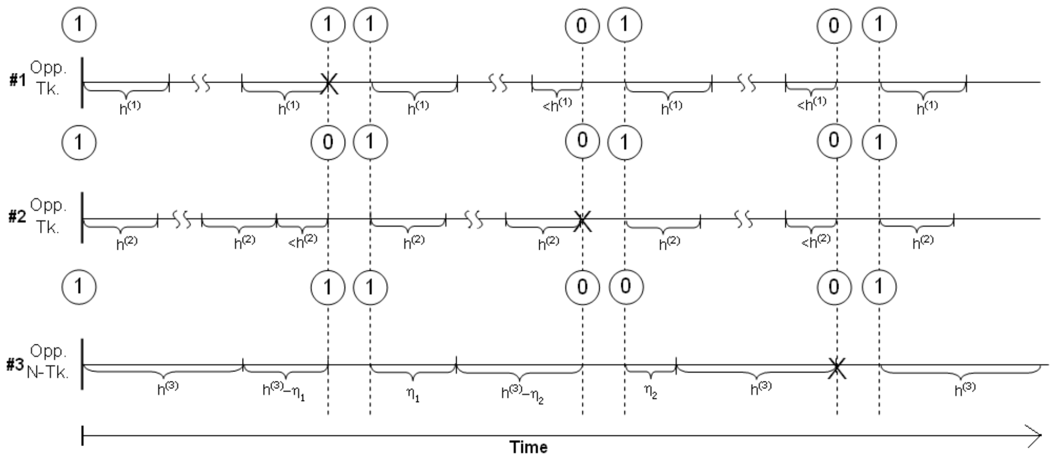

Figure 5-1: An illustration of the three machine system. Machines #1 and #2 are opportunity takers and machine #3 is opportunity non-taker. In-control status denoted by 1, and out-of-control status denoted by 0.

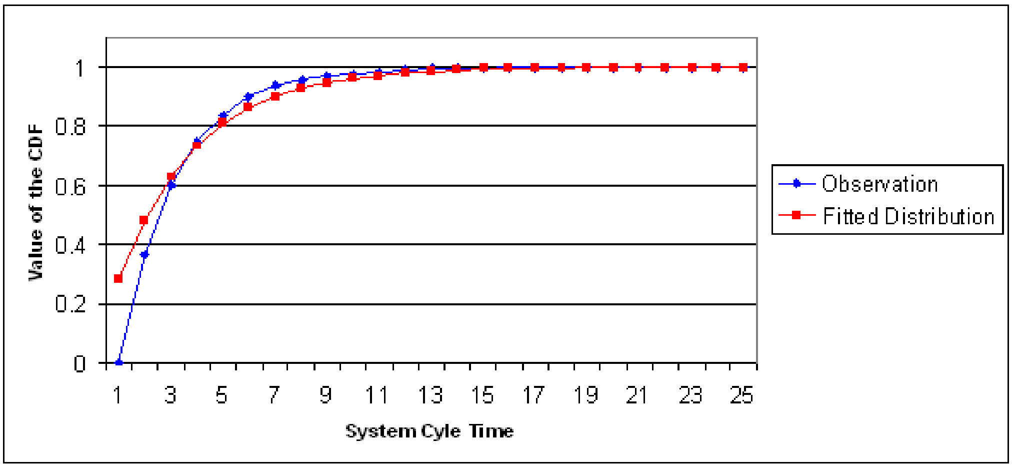

Figure 5-2: Observed frequency vs. Exponential CDF with MLE of the parameter (i.e. the mean of the observed system cycle length) for Experiment #1 and $\pi =$ 500.

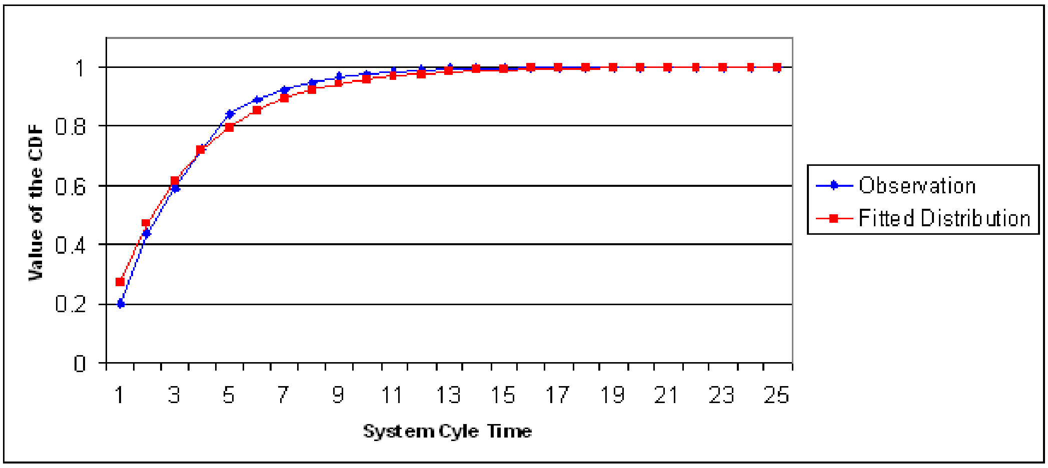

Figure 5-3: Observed frequency vs. Exponential CDF with MLE of the parameter (i.e. the mean of the observed system cycle length) for Experiment #4 and $\pi =$ 500.

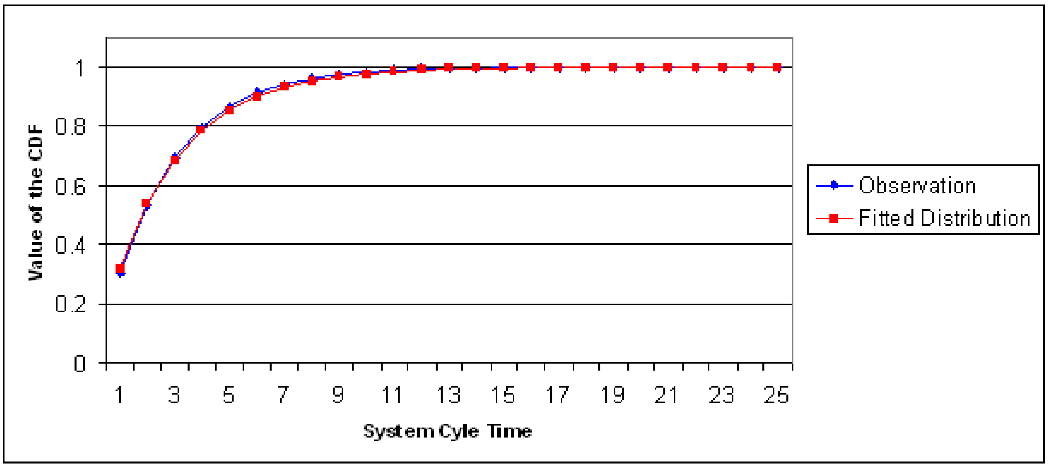

Figure 5-4: Observed frequency vs. Exponential CDF with MLE of the parameter (i.e. the mean of the observed system cycle length) for Experiment #10 and $\pi = 500$.

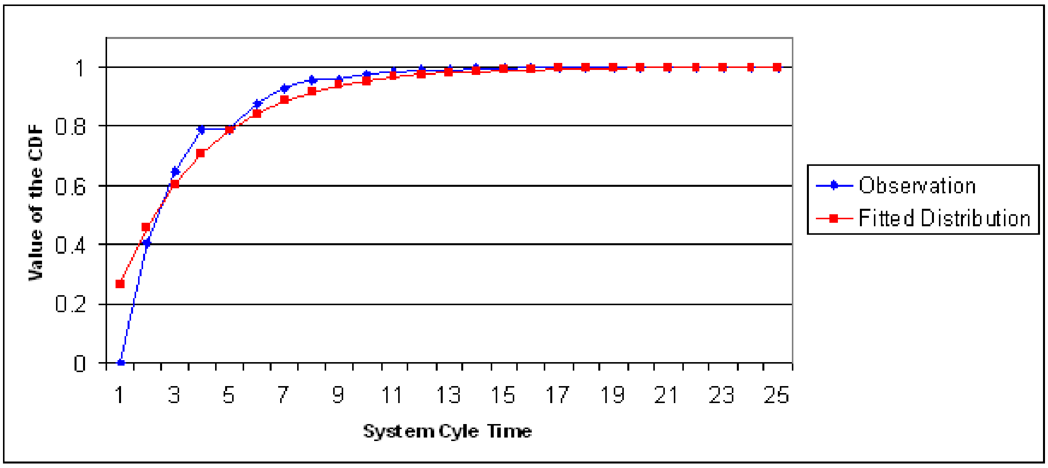

Figure 5-5: Observed frequency vs. Exponential CDF with MLE of the parameter (i.e. the mean of the observed system cycle length) for Experiment #1 and $\pi = 1500$.

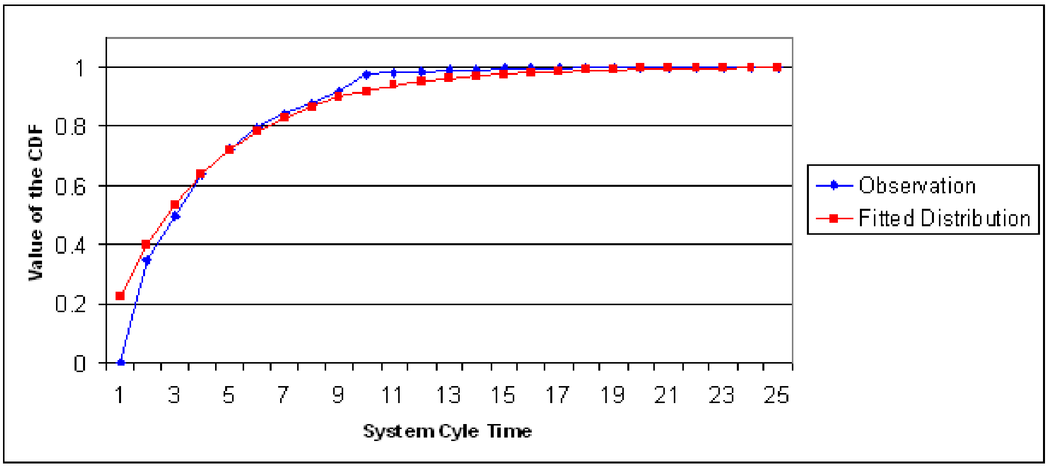

Figure 5-6: Observed frequency vs. Exponential CDF with MLE of the parameter (i.e. the mean of the observed system cycle length) for Experiment #8 and $\pi = 1500$.

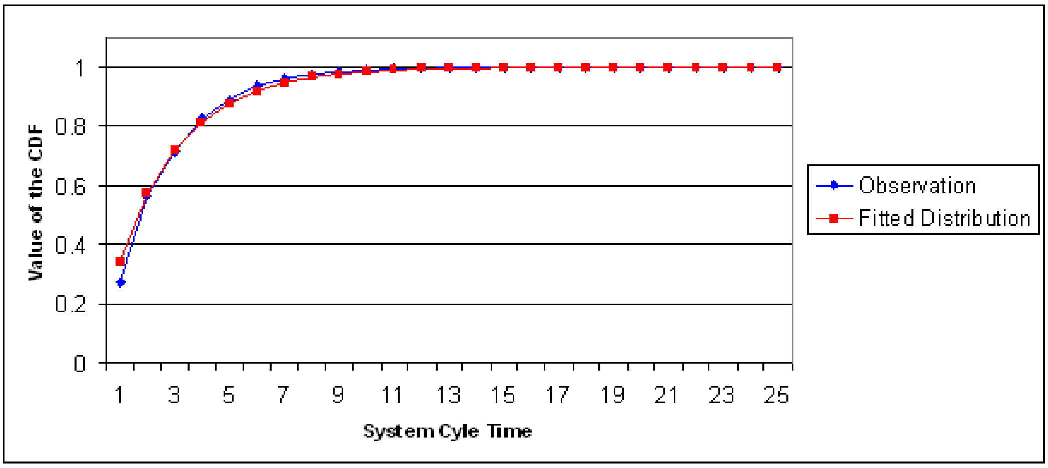

Figure 5-7: Observed frequency vs. Exponential CDF with MLE of the parameter (i.e. the mean of the observed system cycle length) for Experiment #11 and $\pi = 1500$.

Table 5.1: Experiment set for the multiple machine numerical study

Exp #	λ	L	a	Exp #	λ	L	a
1	$\leftrightarrow$	$\leftrightarrow$	$\leftrightarrow$	8	$\nearrow$	$\leftrightarrow$	$\searrow$
2	$\searrow$	$\leftrightarrow$	$\leftrightarrow$	9	$\leftrightarrow$	$\nearrow$	$\nearrow$
3	$\leftrightarrow$	$\nearrow$	$\leftrightarrow$	10	$\leftrightarrow$	$\nearrow$	$\searrow$
4	$\leftrightarrow$	$\leftrightarrow$	$\nearrow$	11	$\nearrow$	$\nearrow$	$\nearrow$
5	$\nearrow$	$\nearrow$	$\leftrightarrow$	12	$\searrow$	$\nearrow$	$\nearrow$
6	$\searrow$	$\nearrow$	$\leftrightarrow$	13	$\searrow$	$\nearrow$	$\searrow$
7	$\nearrow$	$\leftrightarrow$	$\nearrow$	14	$\nearrow$	$\nearrow$	$\searrow$

Table 5.2: Partitioning of the machines as the opportunity taker and opportunity non-taker

	$\pi = 500$ (Machine #)								$\pi = 1500$ (Machine #)							
Exp #	1	2	3	4	5	6	7	8	1	2	3	4	5	6	7	8
1	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*
2	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*
3	*T*	*T*	*T*	**N**	**N**	**N**	**N**	**N**	*T*	*T*	**N**	**N**	**N**	**N**	**N**	**N**
4	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*
5	*T*	*T*	*T*	*T*	*T*	**N**	**N**	**N**	*T*	*T*	*T*	*T*	**N**	**N**	**N**	**N**
6	*T*	*T*	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**
7	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*
8	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*	*T*
9	*T*	*T*	*T*	**N**	**N**	**N**	**N**	**N**	*T*	*T*	**N**	**N**	**N**	**N**	**N**	**N**
10	*T*	*T*	*T*	**N**	**N**	**N**	**N**	**N**	*T*	**N**	**N**	**N**	**N**	**N**	**N**	**N**
11	*T*	*T*	*T*	*T*	*T*	**N**	**N**	**N**	*T*	*T*	*T*	*T*	**N**	**N**	**N**	**N**
12	*T*	*T*	**N**	**N**	**N**	**N**	**N**	**N**	*T*	**N**	**N**	**N**	**N**	**N**	**N**	**N**
13	*T*	*T*	**N**	**N**	**N**	**N**	**N**	**N**	*T*	**N**	**N**	**N**	**N**	**N**	**N**	**N**
14	*T*	*T*	*T*	*T*	*T*	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**	**N**

Table 5.3: Analytical cost rate and deviation from simulation $\pi = 500$

Exp #	All non-taker	All taker	Partitioned
1	$(126.26; -0.10)$	$(110.22; 8.19)$	$(110.22; 8.19)$
2	$(206.86; -0.03)$	$(185.55; 1.22)$	$(185.55; 1.22)$
3	$(120.51; -0.43)$	$(131.44; -0.55)$	$(117.81; 0.04)$
4	$(132.59; 0.03)$	$(116.06; 1.67)$	$(116.06; 1.67)$
5	$(139.47; 0.32)$	$(144.11; -2.58)$	$(132.95; -0.82)$
6	$(123.96; -0.10)$	$(145.85; -3.04)$	$(121.72; 0.63)$
7	$(141.22; 0.28)$	$(121.43; 1.98)$	$(121.43; 1.98)$
8	$(123.65; 0.21)$	$(102.12; 3.04)$	$(102.12; 3.04)$
9	$(114.86; 0.27)$	$(125.42; -4.11)$	$(111.79; -0.43)$
10	$(114.26; -0.61)$	$(122.79; -0.84)$	$(111.77; 0.13)$
11	$(142.08; 0.19)$	$(147.27; -2.62)$	$(135.38; -0.32)$
12	$(109.37; -0.72)$	$(129.71; -4.99)$	$(106.75; 0.48)$
13	$(139.47; -0.60)$	$(161.94; -2.23)$	$(137.35; -0.10)$
14	$(123.86; -0.19)$	$(124.08; -1.45)$	$(117.68; -0.89)$

Table 5.4: Analytical cost rate and deviation from simulation $\pi = 1500$

Exp #	All non-taker	All taker	Partitioned
1	$(185.48; 0.77)$	$(157.9; 5.95)$	$(157.9; 5.95)$
2	$(327.75; 0.30)$	$(288.66; 0.83)$	$(288.66; 0.83)$
3	$(168.75; 0.34)$	$(219.90; -4.96)$	$(167.03; 0.59)$
4	$(191.72; 0.45)$	$(160.43; 2.67)$	$(160.43; 2.67)$
5	$(213.00; 0.44)$	$(248.41; -4.69)$	$(207.73; -0.03)$
6	$(167.05; 0.30)$	$(254.50; -7.10)$	$(167.05; 0.30)$
7	$(213.40; 0.56)$	$(176.74; 2.03)$	$(176.74; 2.03)$
8	$(194.27; 0.22)$	$(148.76; 6.45)$	$(148.76; 6.45)$
9	$(163.10; 0.59)$	$(210.75; -9.72)$	$(160.47; -1.01)$
10	$(161.77; 0.36)$	$(197.49; -0.04)$	$(160.24; -0.57)$
11	$(216.01; -0.10)$	$(253.87; -6.69)$	$(209.51; -1.22)$
12	$(152.11; 0.60)$	$(224.91; -11.72)$	$(149.48; 0.10)$
13	$(182.76; -0.40)$	$(270.10; -4.69)$	$(180.91; -0.06)$
14	$(194.77; -0.05)$	$(200.86; -1.76)$	$(194.77; -0.05)$

Table 5.5: Percentage improvement in the multiple machine model

Exp #	$\pi = 500$ A	$\pi = 500$ S	$\pi = 1500$ A	$\pi = 1500$ S
1	12.71	4.82	14.87	10.18
2	10.30	9.16	11.93	11.48
3	2.24	1.86	1.02	0.78
4	12.47	11.00	16.32	14.42
5	4.68	5.75	2.47	2.93
6	1.81	1.09	0	0
7	14.01	12.53	17.18	15.95
8	17.41	14.98	23.43	18.32
9	2.67	3.34	1.62	3.18
10	2.18	1.46	0.95	1.86
11	4.72	5.20	3.01	4.08
12	2.40	1.22	1.73	2.22
13	1.52	1.03	1.01	0.68
14	5.00	5.66	0	0

Table 5.6: Summary statistics of the improvements in the multiple machine model

	$\pi = 500$ A	$\pi = 500$ S	$\pi = 1500$ A	$\pi = 1500$ S
Mean	6.72	5.65	6.82	6.15
Median	4.69	5.01	2.10	3.05
Min	1.52	1.03	0	0
Max	17.41	14.98	23.43	18.32

Table 5.7: Control parameters of the machines for selected experiments $(y^*; k^*; h^*)$

π	$Mach\#$	$Experiment\#$ 3	6	12	14
500	1	$(7; 2.78; 1.06)$	$(7; 2.78; 0.75)$	$(6; 2.58; 1.97)$	$(7; 2.78; 1.78)$
	2	$(7; 2.91; 1.06)$	$(7; 2.91; 0.73)$	$(7; 2.90; 1.37)$	$(7; 2.91; 1.09)$
	3	$(7; 2.99; 1.07)$	$(7; 2.99; 0.85)$	$(7; 2.98; 1.01)$	$(7; 2.99; 0.98)$
	4	$(8; 3.20; 0.98)$	$(8; 3.20; 0.84)$	$(8; 3.20; 0.94)$	$(8; 3.20; 1.01)$
	5	$(8; 3.24; 0.98)$	$(8; 3.24; 0.90)$	$(8; 3.24; 0.90)$	$(8; 3.24; 0.90)$
	6	$(8; 3.25; 0.99)$	$(8; 3.25; 0.99)$	$(8; 3.25; 0.90)$	$(8; 3.24; 1.09)$
	7	$(8; 3.33; 0.99)$	$(8; 3.33; 1.27)$	$(8; 3.33; 1.00)$	$(8; 3.31; 1.23)$
	8	$(8; 3.29; 0.99)$	$(8; 3.39; 2.16)$	$(8; 3.39; 1.53)$	$(8; 3.34; 1.87)$
1500	1	$(7; 2.99; 1.08)$	$(7; 2.98; 0.70)$	$(7; 2.97; 2.35)$	$(7; 2.99; 1.52)$
	2	$(8; 3.25; 1.12)$	$(8; 3.25; 0.74)$	$(8; 3.25; 1.22)$	$(8; 3.25; 0.99)$
	3	$(8; 3.34; 0.99)$	$(8; 3.34; 0.79)$	$(8; 3.34; 1.04)$	$(7; 2.99; 0.90)$
	4	$(8; 3.39; 0.99)$	$(8; 3.39; 0.85)$	$(8; 3.39; 0.95)$	$(8; 3.20; 0.91)$
	5	$(9; 3.56; 1.01)$	$(9; 3.56; 0.92)$	$(9; 3.56; 0.92)$	$(9; 3.56; 0.97)$
	6	$(9; 3.60; 1.01)$	$(9; 3.60; 1.01)$	$(9; 3.60; 0.92)$	$(9; 3.59; 1.09)$
	7	$(9; 3.64; 1.02)$	$(9; 3.64; 1.29)$	$(9; 3.65; 1.01)$	$(9; 3.63; 1.40)$
	8	$(9; 3.68; 1.03)$	$(9; 3.68; 2.20)$	$(9; 3.68; 1.55)$	$(9; 3.65; 3.45)$

Table 5.8: Least Common Multiples of the sampling intervals for selected experiments

Exp#	$\pi = 500$	$\pi = 1500$
3	1.59×10^{12}	9.25×10^{8}
6	2.76×10^{7}	4.43×10^{9}
12	7.02×10^{11}	9.14×10^{9}
14	5.05×10^{13}	1.68×10^{13}

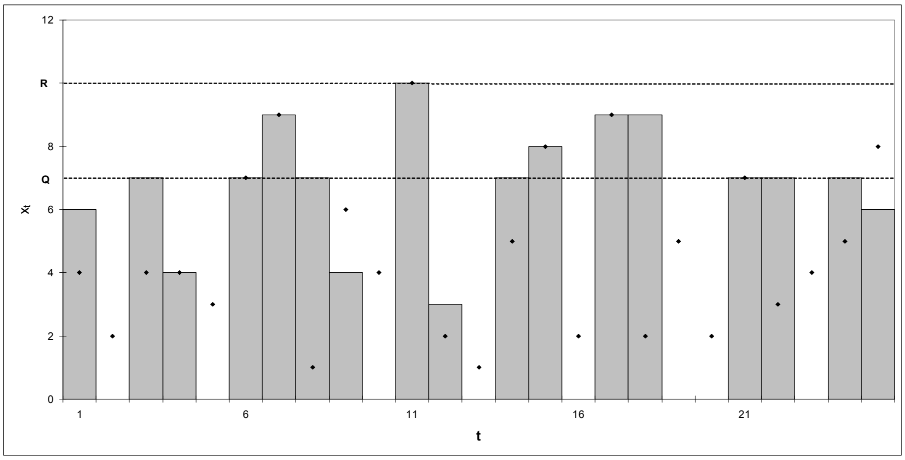

Figure 7-1: Illustrative Example: Demand pattern and optimal production schedule. (Bars indicate the production quantity and diamonds indicate the demand quantity)

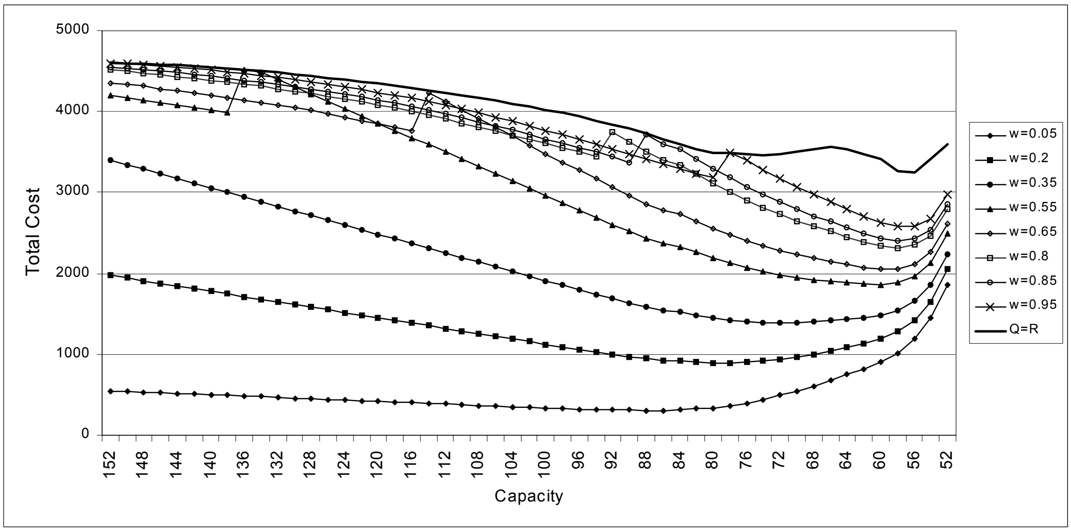

Figure 8-1: Total cost vs. capacity (Medium Demand, K=75)

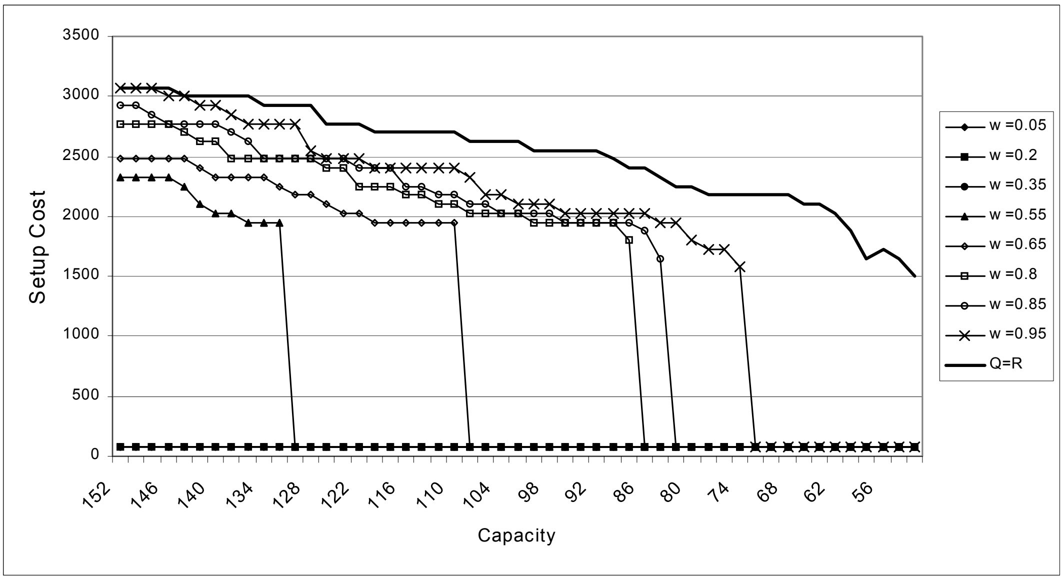

Figure 8-2: Setup cost vs. capacity (Medium Demand, K=75)

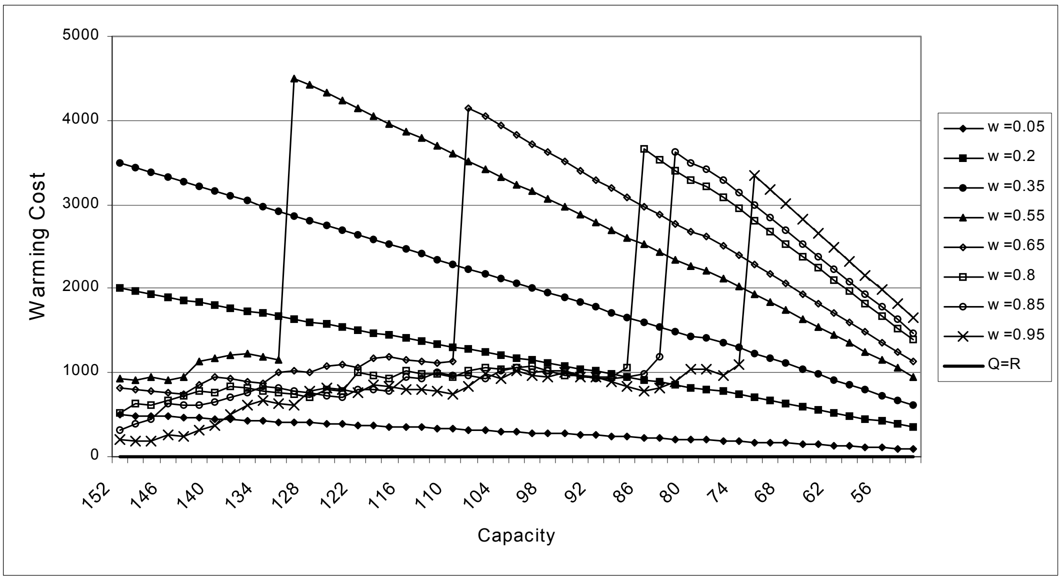

Figure 8-3: Warming cost vs. capacity (Medium Demand, K=75)

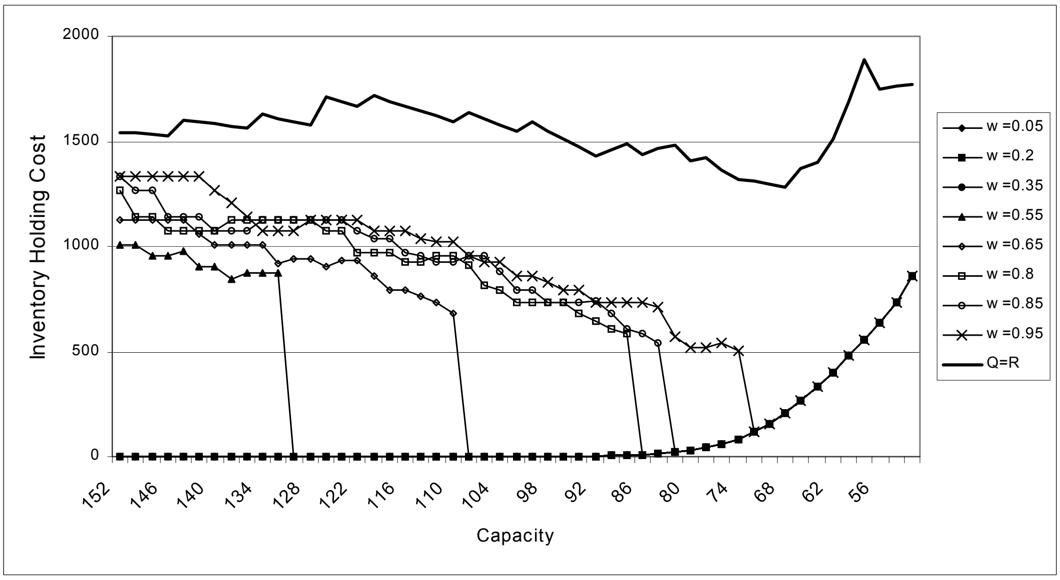

Figure 8-4: Inventory holding cost vs. capacity (Medium Demand, K=75)

Table 8.1: First 25 periods of the optimal production schedules (Medium Demand, R=100)

K		**75**	**50**	**25**	**75**	**50**	**25**	**75**	**50**	**25**	**75**	**50**	**25**
Q		**0**	**0**	**0**	**0**	**9.09**	**54.54**	**21.05**	**47.36**	**73.68**	**100**	**100**	**100**
t	$\mathbf{D}_t$	ω **= 0.05**			ω **= 0.55**			ω **= 0.95**			**Q = R**		
1	25	25	25	25	25	52	25	52	52	25	52	52	25
2	27	*27*	*27*	*27*	*27*		27			27			27
3	57	*57*	*57*	*57*	*57*	57	57	57	57	57	100	100	57
4	92	*92*	*92*	*92*	*92*	*94*	94	*94*	*94*	94	*51*	*51*	94
5	2	*2*	*2*	*2*	*2*								
6	80	*80*	*80*	*80*	*80*	80	80	80	80	80	100	100	80
7	0	*0*	*0*	*0*	*0*	*27*		*27*	*27*		*7*	*7*	
8	27	*27*	*27*	*27*	*27*		27			27			27
9	40	*40*	*40*	*40*	*40*	40	40	75	75	40	75	75	40
10	20	*20*	*20*	*20*	*20*	35	35			35			35
11	15	*15*	*15*	*15*	*15*								
12	42	*42*	*42*	*42*	*42*	42	42	42	42	42	100	42	42
13	45	*45*	*45*	*45*	*45*	*45*	65	*87*	65	65	*7*	65	65
14	20	*20*	*20*	*20*	*20*	*42*							
15	22	*22*	*22*	*22*	*22*		22		64	22	64	64	22
16	42	*42*	*42*	*42*	*42*	42	42	42		42			42
17	92	*92*	*92*	*92*	*92*	*92*	92	*92*	92	92	100	100	100
18	42	*42*	*42*	*42*	*42*	*42*	42	*42*	*84*	*42*	*76*	*76*	*34*
19	42	*42*	*42*	*42*	*42*	*42*	42	*42*		42			42
20	77	*77*	*77*	*77*	*77*	*77*	77	*77*	77	77	100	100	100
21	25	*25*	*25*	*25*	*25*	*27*	27	*27*	*27*	*27*	*4*	*4*	*4*
22	2	*2*	*2*	*2*	*2*								
23	52	*52*	*52*	*52*	*52*	52	52	89	52	52	89	52	52
24	0	*0*	*0*	*0*	*0*								
25	27	*27*	*27*	*27*	*27*	37	37		37	37	0	37	37

Table 8.2: Impact of capacity selection policies on total costs (medium demand, K=75)

ω	0	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
Δ_1	1389.6	287.5	90	40.8	19.5	9.3	5.6	3.3	2	1.1	0.3
Δ_2	0	82.8	116.8	135.5	151.4	169.2	127	113.5	101.5	90.1	78.5

Table 8.3: Period in which planning horizons occur for K=75

100	Cold WW	1-3-6-12-20-23-27-30-33-36-42-46 -50-53-57-59-62-65-69-81-86-88-94
	Warm WW	-
	Cold Z	1-3-6-8-12-15-16-20-23-27-30-33-36-38-42-46 -50-53-57-59-62-65-69-71-75-77-81-86-88-94-98
	Warm Z	6-7-8-12-13-14-15-16-36-38-39-69-70-71-72-73 -74-75-76-77-81-82-88-89-90-94-95-96-97-98
80	Cold WW	1-3-6-9-12-23-27-33-42 -57-59-62-69-81-86-88-94
	Warm WW	29-34-43 54-60-63-90
	Cold Z	1-3-9-12-15-16-19-23-27-33-36-42-46 -50-52-57-62-69-75-77-81-86-88-94-98
	Warm Z	12-13-14-15-16-17-18-19-27-28-29-33 -34-35-36-42-43-44-45-46-47-48-49 -50-51-52-62-63-64-69-70-71-72-73 -74-75-76-77-88-89-90-94-95-96-97-98
60	Cold WW	1-12-16-23-27-33-42 -57-62-69-77-86-94
	Warm WW	5-18-28-29-34-38-39-43-44 -53-58-63-70-71-72-82-89-90-95
	Cold Z	1-3-6-8-12-16-23-33-42-46 -50-52-62-77-78-86-88-94
	Warm Z	1-2-3-4-5-6-7-8-33-34-35-42-43 -44-45-46-47-48-49-50-51-52 -62-63-64-77-78-79-80-86-87 -88-89-90-94-95-96-97-98

Appendix A

Glossary

A.1 Single Machine

μ_0	Process mean when in control
σ	Process standart deviation
δ	Shift magnitude in terms of process standard deviation
λ	Assignable cause (shift) rate
μ	Opportunity arrival rate
s	Cycle class $\{T, F, OT, OF\}$
x	The time of the process shift
z	The arrival time of an opportunity
n_1	The number of sampling instances before the shift has occurred
n_2	The number of sampling instances after the shift

L_O	Process shut down duration when stopped by an opportunity
L_s	Shut down duration of cycle class s
τ	Operating time
τ_s	Operating time of cycle type s
T	Cycle type: True
F	Cycle type: False
OT	Cycle type: Opportunity True
OF	Cycle type: Opportunity False
y	Sample size. (*) indicates the optimum value
h	Sampling interval. (*) indicates the optimum value
k	Control limits in multiples of process standard deviation. (*) indicates the optimum value
α	Type I error probability
β	Type II error probability
u	Fixed cost of sampling
b	Per unit cost of sampling
a	The cost of operating in the out-of-control status
R_s	Fixed cost of inspection and repair of cycle type s
π	The profit (per unit of time)
TC	Total cost per unit produced
CC	Cycle cost
CC_s	Cycle cost of cycle type s

$f(\cdot)$	Joint probability function, takes τ, s, n_1, n_2, x, as parameters
$P_s(\mu)$	The marginal probability function for given μ and cycle class s
$\Omega(s)$	The set of values that the sextuple $(\tau, s, n_1, n_2, x, z)$ can assume for $s \in \{T, F, OT, OF\}$

A.2 Multiple Machine

M	Set of machines
M_{TK}	Set of opportunity taker machines
M_{NTK}	Set of opportunity non-taker machine
m	Number of machines
$y^{(i)}$	Sample size of machine i
$h^{(i)}$	Sampling interval of machine i
$k^{(i)}$	Control limits
$\mu^{(i)}$	Opportunity rate observed by machine i
Γ	System stoppage rate
$\gamma^{(i)}$	Stoppage rate generated by machine i
$\phi^{(i)}$	Machine status of machine i at the system restart
$\boldsymbol{\phi}$	Vector of machine status at the system restart
$\eta^{(i)}$	Time from the system restart to the first sampling instant for machine i
$\boldsymbol{\eta}$	The vector of time from the system restart to the first sampling instant
$E_s^{(j)}$	Event of cycle type s when the system stoppage is triggered by j
$\Psi_s^{(j)}$	Set of machines in cycle class s when the system stoppage is triggered by j
$C_s^{(j)}$	Cost of event $E_s^{(j)}$
$\tilde{\pi}(\cdot)$	Stationary probability matrix

$\varpi^{(i)}$	The status of machine i at a stoppage instant
$\boldsymbol{\varpi}$	The vector of the status of the system at a stoppage instant
$e\left[l^{(i)}, \acute{l}^{(i)}; j\right]$	The state transition probability of machine i from state $l^{(i)}$ to $\acute{l}^{(i)}$ when the system stoppage is triggered by machine j

A.3 Lot Sizing Problem

R_t	Capacity in period t
Q_t	Warm system treshold in period t
N	Problem horizon
D_t	Demand in period t
x_t	Production in period t
c_t	Unit production cost in period t
y_t	The inventory on hand at the end of period t
h_t	Inventory holding cost in period t
ω_t	Per unit warming cost
K_t	Cold setup cost
k_t	Warm setup cost
$f_t^N(x_{t-1}, y_{t-1})$	The minimum total cost under an optimal production schedule for periods t through N
δ_t	Production indicator
z_t	Warm system indicator
Ψ_{uv}	A production series with starting in period u ending in period $v-1$
E_t	Economic bound in period t
S_{uv}	A production sequence
$l(t)$	The last period with cold setup in a t-period problem
$w(t)$	The last period with warm setup in a t-period problem

Appendix B

Derivation of the Expected Operating Time Function

$$E(\tau) = E(\tau_T) + E(\tau_F) + E(\tau_{OT}) + E(\tau_{OF})$$

$$\begin{aligned} E(\tau_T) \quad &= \quad \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \Big\{ \left[(n_1 + n_2 + 1) \, h \right] \left[(1 - \alpha) \, .e^{-(\mu+\lambda).h} \right]^{n_1} \\ &\quad \times \left[\beta.e^{-\mu h} \right]^{n_2} \left[(1 - \beta) \left(1 - e^{-\lambda h} \right) e^{-\mu h} \right] \Big\} \end{aligned}$$

$$E(\tau_F) = \sum_{n_1=0}^{\infty} \left\{ \left[(n_1 + 1) \, h \right] \left[(1 - \alpha) \, e^{-(\mu+\lambda).h} \right]^{n_1} \left[\alpha.e^{-(\mu+\lambda)h} \right] \right\}$$

$$
\begin{aligned}
E\left(\tau_{OT}\right) \;\; = \;\; & \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\int_{x=0}^{h}\Big\{\left[\left(n_1+n_2\right)h+x\right]\left[\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right]^{n_1} \\
& \qquad \times\left[\beta.e^{-\mu h}\right]^{n_2}\left[\left(1-e^{-\lambda h}\right)\mu e^{-\mu x}\right]\Big\}\,dx+ \\
& \sum_{n_1=0}^{\infty}\int_{x=0}^{h}\Big\{\left[\left(n_1.h+x\right)\right]\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1} \\
& \qquad \times\left[\left(1-e^{-\lambda x}\right)\mu e^{-\mu x}\right]\Big\}\,dx+
\end{aligned}
$$

$$
E\left(\tau_{OF}\right)=\sum_{n_1=0}^{\infty}\int_{x=0}^{h}\left\{\left[n_1h+x\right]\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\left[\mu.e^{-(\mu+\lambda)x}\right]\right\}dx
$$

$$
\begin{aligned}
E\left(\tau_T\right) &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Big\{\left[\left(n_1+n_2+1\right)h\right]\left[\left(1-\alpha\right)e^{-(\mu+\lambda)h}\right]^{n_1} \\
&\qquad \times\left[\beta e^{-\mu h}\right]^{n_2}\left[\left(1-\beta\right)\left(1-e^{-\lambda h}\right)e^{-\mu h}\right]\Big\} \\
&= \left(1-\beta\right)\left(1-e^{-\lambda h}\right)he^{-\mu h}\sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right)e^{-(\mu+\lambda)h}\right]^{n_1} \\
&\qquad \times\sum_{n_2=0}^{\infty}\left(n_1+n_2+1\right)\left[\beta e^{-\mu h}\right]^{n_2} \\
&= \left(1-\beta\right)\left(1-e^{-\lambda h}\right)he^{-\mu h}\sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right)e^{-(\mu+\lambda)h}\right]^{n_1} \\
&\qquad \times\left[\frac{n_1}{1-\beta e^{-\mu h}}+\frac{1}{\left(1-\beta e^{-\mu h}\right)^2}\right] \\
&= \frac{\left(1-\beta\right)\left(1-e^{-\lambda h}\right)he^{-\mu h}}{1-\beta e^{-\mu h}}\frac{\left(1-\alpha\right)e^{-(\mu+\lambda)h}}{\left[1-\left\{\left(1-\alpha\right)e^{-(\mu+\lambda)h}\right\}\right]^2} \\
&\qquad +\left(\frac{\left(1-\beta\right)\left(1-e^{-\lambda h}\right)he^{-\mu h}}{\left(1-\beta e^{-\mu h}\right)^2}\right. \\
&\qquad \left.\times\frac{1}{\left[1-\left\{\left(1-\alpha\right)e^{-(\mu+\lambda)h}\right\}\right]}\right) \\
&= \frac{\left(1-\beta\right)\left(1-e^{-\lambda h}\right)he^{-\mu h}}{\left(1-\beta e^{-\mu h}\right)\left(1-\left\{\left(1-\alpha\right)e^{-(\mu+\lambda)h}\right\}\right)} \\
&\qquad \times\left(\frac{\left(1-\alpha\right).e^{-(\mu+\lambda)h}}{\left(1-\left\{\left(1-\alpha\right).e^{-(\mu+\lambda)h}\right\}\right)}+\frac{1}{\left(1-\beta e^{-\mu h}\right)}\right) \\
&= \left(1-\beta\right)\left(1-e^{-\lambda h}\right)he^{-\mu h} \\
&\qquad \times\frac{1-\left\{\left(1-\alpha\right)e^{-(\mu+\lambda)h}\beta e^{-\mu h}\right\}}{\left(1-\beta e^{-\mu h}\right)^2\left(1-\left\{\left(1-\alpha\right)e^{-(\mu+\lambda)h}\right\}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
E\left(\tau_F\right) &= \sum_{n_1=0}^{\infty}\left\{\left[\left(n_1+1\right)h\right]\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\left[\alpha e^{-(\mu+\lambda)h}\right]\right\} \\
&= \alpha.e^{-(\mu+\lambda)h}h\sum_{n_1=0}^{\infty}\left[\left(n_1+1\right)h\right]\left[\left(1-\alpha\right).e^{-(\mu+\lambda)h}\right]^{n_1} \\
&= \frac{\alpha e^{-(\mu+\lambda)h}h}{\left(1-\left\{\left(1-\alpha\right)e^{-(\mu+\lambda)h}\right\}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
E\left(\tau_{OT}\right) &= \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\int_{x=0}^{h}\Big\{\left[\left(n_1+n_2\right)h+x\right]\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1} \\
&\qquad \times\left[\beta e^{-\mu h}\right]^{n_2}\left[\left(1-e^{-\lambda h}\right)\mu e^{-\mu x}\right]\Big\}\,dx+ \\
&\sum_{n_1=0}^{\infty}\int_{x=0}^{h}\Big\{\left[\left(n_1 h+x\right)\right]\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1} \\
&\qquad \times\left[\left(1-e^{-\lambda x}\right)\mu e^{-\mu x}\right]\Big\}\,dx
\end{aligned}
$$

$$\sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\int_{x=0}^{h}\Big\{\left[(n_1+n_2)h+x\right]\left[(1-\alpha)e^{-(\mu+\lambda).h}\right]^{n_1} \times\left[\beta.e^{-\mu h}\right]^{n_2}\left[\left(1-e^{-\lambda h}\right)\mu e^{-\mu x}\right]\Big\}dx$$

$$= \left(1-e^{-\lambda h}\right)\mu\sum_{n_1=0}^{\infty}\left\{\left((1-\alpha)e^{-(\mu+\lambda).h}\right)^{n_1}\sum_{n_2=1}^{\infty}\left[\left(\beta e^{-\mu h}\right)^{n_2} \times\int_{x=0}^{h}\left((n_1+n_2)h+x\right)e^{-\mu x}dx\right]\right\}$$

$$= \left(1-e^{-\lambda h}\right)\mu\sum_{n_1=0}^{\infty}\left\{\left((1-\alpha)e^{-(\mu+\lambda).h}\right)^{n_1}\sum_{n_2=1}^{\infty}\left[\left(\beta.e^{-\mu h}\right)^{n_2} \times\left(\frac{(n_1+n_2)h\left(1-e^{-\mu h}\right)}{\mu}+\frac{1}{\mu^2}-\frac{he^{-\mu h}}{\mu}-\frac{e^{-\mu h}}{\mu^2}\right)\right]\right\}$$

$$= \left(1-e^{-\lambda h}\right)\mu\sum_{n_1=0}^{\infty}\left\{\left((1-\alpha)e^{-(\mu+\lambda).h}\right)^{n_1} \times\left[\frac{n_1h\left(1-e^{-\mu h}\right)\beta e^{-\mu h}}{\mu\left(1-\beta e^{-\mu h}\right)}+\frac{h\left(1-e^{-\mu h}\right)\beta e^{-\mu h}}{\mu\left(1-\beta e^{-\mu h}\right)^2} +\frac{\left(\frac{1}{\mu^2}-\frac{h.e^{-\mu h}}{\mu}-\frac{e^{-\mu h}}{\mu^2}\right)\beta e^{-\mu h}}{1-\beta e^{-\mu h}}\right]\right\}$$

$$= \left(1-e^{-\lambda h}\right)\mu\left\{\frac{h\left(1-e^{-\mu h}\right)\left(\beta.e^{-\mu h}\right)\left((1-\alpha).e^{-(\mu+\lambda).h}\right)}{\mu\left(1-\beta e^{-\mu h}\right)\left[1-\left((1-\alpha)e^{-(\mu+\lambda).h}\right)\right]^2} +\frac{h\left(1-e^{-\mu h}\right)\beta e^{-\mu h}}{\mu\left(1-\beta e^{-\mu h}\right)^2\left[1-\left((1-\alpha)e^{-(\mu+\lambda).h}\right)\right]} +\frac{\left(\frac{1}{\mu^2}-\frac{h.e^{-\mu h}}{\mu}-\frac{e^{-\mu h}}{\mu^2}\right)\beta e^{-\mu h}}{\left(1-\beta e^{-\mu h}\right)\left[1-\left((1-\alpha)e^{-(\mu+\lambda).h}\right)\right]}\right\}$$

$$
\begin{aligned}
= \quad & \left(1-e^{-\lambda h}\right)\mu\left\{\frac{h\left(1-e^{-\mu h}\right)\left(\beta.e^{-\mu h}\right)\left((1-\alpha).e^{-(\mu+\lambda).h}\right)}{\mu\left(1-\beta e^{-\mu h}\right)\left[1-\left((1-\alpha)e^{-(\mu+\lambda).h}\right)\right]^{2}}\right. \\
& +\frac{h\left(1-e^{-\mu h}\right)\beta e^{-\mu h}}{\mu\left(1-\beta e^{-\mu h}\right)^{2}\left[1-\left((1-\alpha)e^{-(\mu+\lambda).h}\right)\right]} \\
& \left.+\frac{\left(\frac{1}{\mu^{2}}-\frac{h.e^{-\mu h}}{\mu}-\frac{e^{-\mu h}}{\mu^{2}}\right)\beta e^{-\mu h}}{\left(1-\beta e^{-\mu h}\right)\left[1-\left((1-\alpha)e^{-(\mu+\lambda).h}\right)\right]}\right\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{\beta.e^{-\mu h}\left(1-e^{-\lambda h}\right)}{\left[1-\beta.e^{-\mu h}\right]\left[1-\left\{(1-\alpha)e^{-(\mu+\lambda)h}\right\}\right]} \\
& \times\left(\frac{h\left(1-e^{-\mu h}\right)\left(1-\left\{\beta e^{-\mu h}(1-\alpha)e^{-(\mu+\lambda)h}\right\}\right)}{\left[1-\beta e^{-\mu h}\right]\left[1-\left\{(1-\alpha)e^{-(\mu+\lambda)h}\right\}\right]}\right. \\
& \left.+\frac{1-e^{-\mu h}}{\mu}-he^{-\mu h}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{n_1=0}^{\infty}\int_{x=0}^{h}\left\{\left[(n_1.h+x)\right]\left[(1-\alpha)e^{-(\mu+\lambda).h}\right]^{n_1}\right. \\
& \qquad \left.\times\left[\left(1-e^{-\lambda x}\right)\mu e^{-\mu x}\right]\right\}dx
\end{aligned}
$$

$$
\begin{aligned}
&= \mu . \sum_{n_1=0}^{\infty} \left[(1-\alpha)\, e^{-(\mu+\lambda).h} \right]^{n_1} \int_{x=0}^{h} (n_1 h + x) \left(1 - e^{-\lambda x}\right) e^{-\mu x} dx \\
&= \mu . \sum_{n_1=0}^{\infty} \left[(1-\alpha)\, e^{-(\mu+\lambda).h} \right]^{n_1} \left\{ \frac{n_1 h \left(1 - e^{-\mu h}\right)}{\mu} + \frac{1 - e^{-\mu h}}{\mu^2} - \frac{h e^{-\mu h}}{\mu} - \right. \\
&\qquad \left. \frac{n_1 . h . \left(1 - e^{-(\mu+\lambda) h}\right)}{\lambda + \mu} - \frac{1 - e^{-(\mu+\lambda) h}}{(\mu+\lambda)^2} + \frac{h e^{-(\mu+\lambda) h}}{(\mu+\lambda)} \right\} \\
&= \mu \left\{ \frac{h . \left(1 - e^{-\mu h}\right)(1-\alpha) . e^{-(\mu+\lambda).h}}{\mu \left[1 - \left((1-\alpha) . e^{-(\mu+\lambda).h}\right)\right]^2} \right. \\
&\quad - \frac{h . \left(1 - e^{-(\mu+\lambda) h}\right)(1-\alpha) . e^{-(\mu+\lambda).h}}{(\lambda+\mu) \left[1 - \left((1-\alpha) . e^{-(\mu+\lambda).h}\right)\right]^2} \\
&\quad + \frac{\left[\frac{1}{\mu^2} \left(1 - e^{-\mu h} (1 + h\mu)\right) \right]}{1 - \left((1-\alpha) . e^{-(\mu+\lambda).h}\right)} \\
&\quad \left. - \frac{\left[\frac{1}{(\mu+\lambda)^2} \left(1 - e^{-(\mu+\lambda) h} \left(1 + h(\mu+\lambda)\right)\right) \right]}{1 - \left((1-\alpha) . e^{-(\mu+\lambda).h}\right)} \right\} \\
&= \frac{1}{1 - \left((1-\alpha) . e^{-(\mu+\lambda).h}\right)} \left\{ \frac{h . \mu . (1-\alpha) . e^{-(\mu+\lambda).h}}{1 - \left((1-\alpha) . e^{-(\mu+\lambda).h}\right)} \right. \\
&\quad \times \left[\frac{1 - e^{-\mu h}}{\mu} - \frac{1 - e^{-(\mu+\lambda) h}}{\lambda + \mu} \right] \\
&\quad \left. + \left[\frac{\left(1 - e^{-\mu h} (1 + h\mu)\right)}{\mu} - \frac{\mu . \left(1 - e^{-(\mu+\lambda) h} \left(1 + h(\mu+\lambda)\right)\right)}{(\mu+\lambda)^2} \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
E\left(\tau_{OF}\right) &= \sum_{n_1=0}^{\infty}\int_{x=0}^{h}\left\{\left(n_1h+x\right)\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\left[\left(1-e^{-\lambda x}\right)\mu e^{-\mu x}\right]\right\}dx \\
&= \mu\sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right]^{n_1}\int_{x=0}^{h}\left(n_1.h+x\right)\left(1-e^{-\lambda x}\right)e^{-\mu x}dx \\
&= \mu\sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\left\{\frac{n_1h\left(1-e^{-\mu h}\right)}{\mu}\right. \\
&\quad +\frac{1}{\mu^2}-\frac{he^{-\mu h}}{\mu}-\frac{e^{-\mu h}}{\mu^2}-\frac{n_1h\left(1-e^{-(\mu+\lambda)h}\right)}{\lambda+\mu} \\
&\quad \left.-\frac{1}{\left(\mu+\lambda\right)^2}+\frac{he^{-(\mu+\lambda)h}}{\left(\mu+\lambda\right)}+\frac{e^{-(\mu+\lambda)h}}{\left(\mu+\lambda\right)^2}\right\} \\
&= \mu.\left\{\frac{h.\left(1-e^{-\mu h}\right)\left(1-\alpha\right).e^{-(\mu+\lambda).h}}{\mu\left[1-\left(\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right)\right]^2}\right. \\
&\quad -\frac{h.\left(1-e^{-(\mu+\lambda)h}\right)\left(1-\alpha\right).e^{-(\mu+\lambda).h}}{\left(\lambda+\mu\right)\left[1-\left(\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right)\right]^2} \\
&\quad +\frac{\left[\frac{1}{\mu^2}\left(1-e^{-\mu h}\left(1+h\mu\right)\right)\right]}{1-\left(\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right)} \\
&\quad \left.-\frac{\left[\frac{1}{(\mu+\lambda)^2}\left(1-e^{-(\mu+\lambda)h}\left(1+h\left(\mu+\lambda\right)\right)\right)\right]}{1-\left(\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right)}\right\} \\
&= \frac{1}{1-\left(\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right)} \\
&\quad \times\left\{\frac{h.\mu.\left(1-\alpha\right).e^{-(\mu+\lambda).h}}{1-\left(\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right)}.\left[\frac{1-e^{-\mu h}}{\mu}-\frac{1-e^{-(\mu+\lambda)h}}{\lambda+\mu}\right]\right. \\
&\quad \left.+\left[\frac{\left(1-e^{-\mu h}\left(1+h\mu\right)\right)}{\mu}-\frac{\mu.\left(1-e^{-(\mu+\lambda)h}\left(1+h\left(\mu+\lambda\right)\right)\right)}{\left(\mu+\lambda\right)^2}\right]\right\}
\end{aligned}
$$

$$\sum_{n_1=0}^{\infty}\int_{x=0}^{h}\left\{[n_1h+x]\left[(1-\alpha)\,e^{-(\mu+\lambda).h}\right]^{n_1}\left[\mu e^{-(\mu+\lambda)x}\right]\right\}dx$$

$$= \sum_{n_1=0}^{\infty}\left\{\left[(1-\alpha)\,e^{-(\mu+\lambda)h}\right]^{n_1}\left[\int_{x=0}^{h}n_1h\mu e^{-(\mu+\lambda)x}dx+\int_{x=0}^{h}x\mu e^{-(\mu+\lambda)x}dx\right]\right\}$$

$$= \sum_{n_1=0}^{\infty}\left\{\left[(1-\alpha)\,e^{-(\mu+\lambda)h}\right]^{n_1}\left[\left(\frac{\left(1-e^{-(\mu+\lambda)h}\right)\mu n_1h}{\mu+\lambda}\right)+\right.\right.$$
$$\left.\left.\left(\frac{\mu}{\mu+\lambda}\left[\frac{1}{\mu+\lambda}-e^{-(\mu+\lambda)h}\left\{h+\frac{1}{\mu+\lambda}\right\}\right]\right)\right]\right\}$$

$$= \left[\frac{\left(1-e^{-(\mu+\lambda)h}\right)\mu h}{\mu+\lambda}\sum_{n_1=0}^{\infty}n_1\left[(1-\alpha)\,e^{-(\mu+\lambda)h}\right]^{n_1}\right]+$$
$$\left[\frac{\mu}{\mu+\lambda}\left(\frac{1}{\mu+\lambda}-e^{-(\mu+\lambda)h}\left\{h+\frac{1}{\mu+\lambda}\right\}\right)\sum_{n_1=0}^{\infty}\left[(1-\alpha)\,e^{-(\mu+\lambda)h}\right]^{n_1}\right]$$

$$= \left[\frac{\left(1-e^{-(\mu+\lambda)h}\right)\mu h}{\mu+\lambda}\frac{(1-\alpha)\,e^{-(\mu+\lambda)h}}{\left[1-\left((1-\alpha)\,e^{-(\mu+\lambda)h}\right)\right]^2}\right]+$$
$$\left[\frac{\mu}{\mu+\lambda}\left(\frac{1}{\mu+\lambda}-e^{-(\mu+\lambda)h}\left\{h+\frac{1}{\mu+\lambda}\right\}\right)\frac{1}{1-\left((1-\alpha)\,e^{-(\mu+\lambda)h}\right)}\right]$$

$$= \frac{\mu}{(\mu+\lambda)\left[1-\left((1-\alpha)\,e^{-(\mu+\lambda)h}\right)\right]}\left[\frac{h\left(1-e^{-(\mu+\lambda)h}\right)(1-\alpha)\,e^{-(\mu+\lambda)h}}{1-\left((1-\alpha)\,e^{-(\mu+\lambda)h}\right)}+\right.$$
$$\left.\frac{1}{\mu+\lambda}-e^{-(\mu+\lambda)h}\left(h+\frac{1}{\mu+\lambda}\right)\right]$$

Appendix C

Derivation of the Expected Cycle Cost Function

$$E(CC) = E(CC_T) + E(CC_F) + E(CC_{OT}) + E(CC_{OF})$$

$$\begin{aligned} E(CC_T) \;\; = \;\; & \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\int_{z=0}^{h}\left[(n_1+n_2+1)(yb+u)+a(n_2h+h-z)+\pi L_T+R_T\right] \\ & \times\left[(1-\alpha)e^{-(\mu+\lambda).h}\right]^{n_1}\left[\beta.e^{-\mu h}\right]^{n_2}\left[(1-\beta)\left(1-e^{-\lambda h}\right)e^{-\mu h}\right]dz \end{aligned}$$

$$E(CC_F) = \sum_{n_1=0}^{\infty}\left[(n_1+1)(yb+u)+\pi L_F+R_F\right]\left[(1-\alpha)e^{-(\mu+\lambda).h}\right]^{n_1}\left[\alpha.e^{-(\mu+\lambda)h}\right]$$

$$
\begin{aligned}
E\left(CC_{OT}\right) \;=\; & \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\int_{x=0}^{h}\int_{z=0}^{h}\left[\left(n_1+n_2\right)\left(yb+u\right)+a\left(n_2h-z+x\right)\right. \\
& \qquad \left.+\pi L_{OT}+R_{OT}\right] \\
& \qquad \times\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\left[\beta e^{-\mu h}\right]^{n_2}\left[\lambda e^{-\lambda h}\mu e^{-\mu x}\right]dzdx \\
& +\sum_{n_1=0}^{\infty}\int_{z=0}^{h}\int_{x=z}^{h}\left[\left(n_1\left(yb+u\right)+a\left(x-z\right)+\pi L_{OT}+R_{OT}\right)\right] \\
& \qquad \times\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\left[\lambda e^{-\lambda x}\mu e^{-\mu x}\right]dzdx
\end{aligned}
$$

$$
\begin{aligned}
E\left(CC_{OF}\right) \;=\; & \sum_{n_1=0}^{\infty}\int_{x=0}^{h}\left[n_1\left(yb+u\right)+\pi L_{OF}+r_{OF}\right] \\
& \times\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\left[\mu e^{-(\mu+\lambda)x}\right]dx
\end{aligned}
$$

$$E\left(CC_T\right) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\int_{z=0}^{h}\left[\left(n_1+n_2+1\right)\left(yb+u\right)+a\left(n_2h+h-z\right)\right.$$
$$\left.+\pi L_T+R_T\right]\left[\left(1-\alpha\right)e^{-(\mu+\lambda)h}\right]^{n_1}\left[\beta e^{-\mu h}\right]^{n_2}\left[\left(1-\beta\right)\lambda e^{-\lambda z}e^{-\mu h}\right]dz$$

$$= \left(1-\beta\right)e^{-\mu h}\sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\sum_{n_2=0}^{\infty}\left[\beta e^{-\mu h}\right]^{n_2}$$
$$\times\int_{z=0}^{h}\left[\left(n_1+1\right)yb+ah-az+\pi L_T+R_T+n_2\left(yb+ah\right)\right]\lambda e^{-\lambda z}dz$$

$$= \left(1-\beta\right)e^{-\mu h}\sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\sum_{n_2=0}^{\infty}\left[\beta e^{-\mu h}\right]^{n_2}$$
$$\times\left\{\left[\left(n_1+1\right)yb+ah-az+\pi L_T+R_T+n_2\left(yb+ah\right)\right]\left[1-e^{-\lambda h}\right]\right.$$
$$\left.-a\left[\frac{1}{\lambda}-he^{-\lambda h}-\frac{e^{-\lambda h}}{\lambda}\right]\right\}$$

$$= \left(1-\beta\right)e^{-\mu h}\sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\left\{\left[\left(n_1+1\right)yb+ah+\pi L_T+R_T\right]\right.$$
$$\left.\times\left[1-e^{-\lambda h}\right]-a\left[\frac{1}{\lambda}-he^{-\lambda h}-\frac{e^{-\lambda h}}{\lambda}\right]\right\}\frac{1}{1-\beta e^{-\mu h}}$$
$$+\frac{\left(1-e^{-\lambda h}\right)\left(yb+ah\right)\beta e^{-\mu h}}{\left(1-\beta e^{-\mu h}\right)^2}$$

$$= \left(1-\beta\right)e^{-\mu h}\left\{\frac{yb\left(1-e^{-\lambda h}\right)}{\left[1-\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^2\left[1-\beta e^{-\mu h}\right]}\right.$$
$$+\frac{\left(ah+\pi L_T+r_T\right)\left(1-e^{-\lambda h}\right)-a\left[\frac{1}{\lambda}-he^{-\lambda h}-\frac{e^{-\lambda h}}{\lambda}\right]}{\left[1-\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]\left[1-\beta e^{-\mu h}\right]}$$
$$\left.+\frac{\left(1-e^{-\lambda h}\right)\left(yb+ah\right)\beta e^{-\mu h}}{\left[1-\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]\left[1-\beta e^{-\mu h}\right]^2}\right\}$$

$$= \frac{(1-\beta)\, e^{-\mu h}\left(1-e^{-\lambda h}\right)}{\left[1-(1-\alpha)\, e^{-(\mu+\lambda).h}\right]\left[1-\beta e^{-\mu h}\right]}\left\{\frac{yb\,(1-\alpha)\, e^{-(\mu+\lambda).h}}{1-(1-\alpha)\, e^{-(\mu+\lambda).h}}+\pi L_T+R_T \right.$$
$$\left. -\frac{a}{\lambda}+\frac{a.h.e^{-\lambda h}}{\left(1-e^{-\lambda h}\right)}+\frac{(y.b+a.h)}{1-\beta.e^{-\mu h}}\right\}$$

$$\begin{aligned}
E\left(CC_F\right) &= \sum_{n_1=0}^{\infty}\left[(n_1+1)(yb+u)+\pi L_F+R_F\right]\left[(1-\alpha)\, e^{-(\mu+\lambda)h}\right]^{n_1}\left[\alpha e^{-(\mu+\lambda)h}\right] \\
&= \left(\alpha e^{-(\mu+\lambda)h}\right)\left(\frac{yb}{\left[1-(1-\alpha)\, e^{-(\mu+\lambda).h}\right]^2}+\frac{\pi L_F+R_F}{\left[1-(1-\alpha)\, e^{-(\mu+\lambda).h}\right]}\right) \\
&= \frac{\alpha e^{-(\mu+\lambda)h}}{1-(1-\alpha)\, e^{-(\mu+\lambda)h}}\left[\pi L_F+R_F+\frac{yb}{1-(1-\alpha)\, e^{-(\mu+\lambda)h}}\right]
\end{aligned}$$

$$
\begin{aligned}
E\left(CC_{OT}\right) &= \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\int_{x=0}^{h}\int_{z=0}^{h}\left[\left(n_1+n_2\right)\left(yb+u\right)+a\left(n_2h-z+x\right)+\pi L_{OT}+R_{OT}\right]\\
&\quad\times\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\left[\beta.e^{-\mu h}\right]^{n_2}\left[\lambda e^{-\lambda h}\mu e^{-\mu x}\right]dzdx
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\sum_{n_2=1}^{\infty}\left[\beta e^{-\mu h}\right]^{n_2}\int_{x=0}^{h}\mu e^{-\mu x}\\
&\quad\times\int_{z=0}^{h}\left[\left(n_1+n_2\right)\left(yb+u\right)+a\left(n_2h-z+x\right)+\pi L_{OT}+R_{OT}\right]\\
&\quad\times\lambda e^{-\lambda z}dzdx
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\sum_{n_2=1}^{\infty}\left[\beta e^{-\mu h}\right]^{n_2}\\
&\quad\times\int_{x=0}^{h}\mu.e^{-\mu x}\left\{\left(n_1\left(yb+u\right)+n_2\left(yb+u+ah\right)+ax+\pi L_{OT}+R_{OT}\right)\right.\\
&\quad\left.\times\left(1-e^{-\lambda h}\right)-a\left(\frac{1}{\lambda}-\left(h+\frac{1}{\lambda}\right)e^{-\lambda h}\right)\right\}dx
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\sum_{n_2=1}^{\infty}\left[\beta e^{-\mu h}\right]^{n_2}\left\{\left(n_1\left(yb+u\right)+n_2\left(yb+u+ah\right)\right.\right.\\
&\quad\left.+\pi L_{OT}+R_{OT}\right)\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)\\
&\quad+a\left(\frac{1}{\mu}-\left(h+\frac{1}{\mu}\right)e^{-\mu h}\right)\left(1-e^{-\lambda h}\right)\\
&\quad\left.-a\left(\frac{1}{\lambda}-\left(h+\frac{1}{\lambda}\right)e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \sum_{n_1=0}^{\infty}\left[(1-\alpha)\,e^{-(\mu+\lambda).h}\right]^{n_1}\left\{\frac{(yb+ah)\,\beta e^{-\mu h}\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)}{\left(1-\beta e^{-\mu h}\right)^2}\right.\\
&\qquad +\frac{(n_1 yb+\pi L_{OT}+r_{OT})\,\beta e^{-\mu h}\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)}{1-\beta e^{-\mu h}}\\
&\qquad +\frac{\left[a\left(\frac{1}{\mu}-\left(h+\frac{1}{\mu}\right)e^{-\mu h}\right)\left(1-e^{-\lambda h}\right)\right]\beta e^{-\mu h}}{1-\beta e^{-\mu h}}\\
&\qquad \left.-\frac{\left[a\left(\frac{1}{\lambda}-\left(h+\frac{1}{\lambda}\right)e^{-\lambda h}\right)\right]\beta e^{-\mu h}}{1-\beta e^{-\mu h}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{(yb+ah)\,\beta e^{-\mu h}\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)}{\left(1-\beta e^{-\mu h}\right)^2\left(1-(1-\alpha)\,e^{-(\mu+\lambda).h}\right)}\\
&\quad +\frac{yb\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)\beta e^{-\mu h}\,(1-\alpha)\,e^{-(\mu+\lambda).h}}{\left(1-\beta e^{-\mu h}\right)\left(1-(1-\alpha)\,e^{-(\mu+\lambda).h}\right)^2}\\
&\quad +\frac{(\pi L_{OT}+R_{OT})\,\beta e^{-\mu h}\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)}{\left(1-\beta e^{-\mu h}\right)\left(1-(1-\alpha)\,e^{-(\mu+\lambda).h}\right)}\\
&\quad +\frac{\left[a\left(\frac{1}{\mu}-\left(h+\frac{1}{\mu}\right)e^{-\mu h}\right)\left(1-e^{-\lambda h}\right)-a\left(\frac{1}{\lambda}-\left(h+\frac{1}{\lambda}\right)e^{-\lambda h}\right)\right]\beta e^{-\mu h}}{\left(1-\beta e^{-\mu h}\right)\left(1-(1-\alpha)\,e^{-(\mu+\lambda).h}\right)}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{\beta e^{-\mu h}\left(1-e^{-\lambda h}\right)\left(1-e^{-\mu h}\right)}{\left(1-\beta e^{-\mu h}\right)\left(1-(1-\alpha)\,e^{-(\mu+\lambda).h}\right)}\\
&\quad \times\left[\frac{yb+ah}{1-\beta e^{-\mu h}}+\frac{yb\,(1-\alpha)\,e^{-(\mu+\lambda).h}}{1-(1-\alpha)\,e^{-(\mu+\lambda).h}}+\pi L_{OT}\right.\\
&\quad \left.+R_{OT}+\frac{a}{\mu}-\frac{a}{\lambda}\right]+\frac{ah\left(e^{-\lambda h}-e^{-\mu h}\right)\beta e^{-\mu h}}{\left(1-\beta e^{-\mu h}\right)\left(1-(1-\alpha)\,e^{-(\mu+\lambda).h}\right)}
\end{aligned}
$$

$$\sum_{n_1=0}^{\infty}\int_{z=0}^{h}\int_{x=z}^{h}\left[\left(n_1\left(yb+u\right)+a\left(x-z\right)+\pi L_{OT}+R_{OT}\right)\right]$$
$$\times\left[\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right]^{n_1}\left[\lambda e^{-\lambda z}\mu e^{-\mu x}\right]dzdx$$

$$= \sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right]^{n_1}\int_{z=0}^{h}\lambda.e^{-\lambda z}$$
$$\times\int_{x=z}^{h}\left[\left(n_1\left(yb+u\right)+a\left(x-z\right)+\pi L_{OT}+R_{OT}\right)\right]\mu e^{-\mu x}dzdx$$

$$= \sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\int_{z=0}^{h}\lambda e^{-\lambda z}\left[\left(n_1\left(yb+u\right)-az+\pi L_{OT}+R_{OT}\right)\right.$$
$$\left.\times\left(e^{-\mu z}-e^{-\mu h}\right)+a.\left(z.e^{-\mu z}+\frac{e^{-\mu z}}{\mu}-h.e^{-\mu h}-\frac{e^{-\mu h}}{\mu}\right)\right]dz$$

$$= \sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\left\{-\left(1-e^{-\lambda h}\right)\left[\left(n_1\left(yb+u\right)+\pi L_{OT}+R_{OT}\right)e^{-\mu h}\right.\right.$$
$$\left.+ahe^{-\mu h}+\frac{ae^{-\mu h}}{\mu}\right]+\int_{z=0}^{h}\left(\lambda e^{-\lambda z}aze^{-\mu h}\right)dz$$
$$+\int_{z=0}^{h}\left(\lambda.e^{-\lambda z}.\frac{a.e^{-\mu z}}{\mu}\right)dz$$
$$\left.+\int_{z=0}^{h}\left(\lambda e^{-\lambda z}\left[n_1\left(yb+u\right)+\pi L_{OT}+r_{OT}\right]e^{-\mu z}\right)dz\right\}$$

$$= \sum_{n_1=0}^{\infty}\left[\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right]^{n_1}\left\{-\left(1-e^{-\lambda h}\right)\left[\left(n_1\left(yb+u\right)+\pi L_{OT}+R_{OT}\right)e^{-\mu h}\right.\right.$$
$$\left.+ahe^{-\mu h}+\frac{ae^{-\mu h}}{\mu}\right]+\left[ae^{-\mu h}\left(\frac{1}{\lambda}-he^{-\lambda h}-\frac{e^{-\lambda h}}{\lambda}\right)\right]$$
$$+\left[\lambda\left(n_1\left(yb+u\right)+\pi L_{OT}+R_{OT}\right)\frac{1-e^{-(\mu+\lambda)h}}{(\mu+\lambda)}\right]$$
$$\left.+\left[\frac{a\lambda}{\mu}\frac{1-e^{-(\mu+\lambda)h}}{(\mu+\lambda)}\right]\right\}$$

$$
\begin{aligned}
= \quad & \frac{-\left(1-\alpha\right)e^{-(\mu+\lambda).h}\left(1-e^{-\lambda h}\right)ybe^{-\mu h}}{\left(1-\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right)^{2}} \\
& \qquad - \frac{\left(1-e^{-\lambda h}\right)e^{-\mu h}\left(\pi L_{OT}+R_{OT}+ah+\frac{a}{\mu}\right)}{\left(1-\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right)} \\
& \qquad + \frac{ae^{-\mu h}\left(\frac{1}{\lambda}-he^{-\lambda h}-\frac{e^{-\lambda h}}{\lambda}\right)}{\left(1-\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right)} + \frac{\left(1-\alpha\right)e^{-(\mu+\lambda).h}\left(1-e^{-(\mu+\lambda)h}\right)\lambda yb}{\left(1-\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right)^{2}\left(\mu+\lambda\right)} \\
& \qquad + \frac{\lambda\left(1-e^{-(\mu+\lambda)h}\right)\left(\pi L_{OT}+R_{OT}\right)}{\left(1-\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right)\left(\mu+\lambda\right)} + \frac{a\lambda\left(1-e^{-(\mu+\lambda)h}\right)}{\mu\left(1-\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right)\left(\mu+\lambda\right)}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{\lambda yb\left(1-\alpha\right)e^{-(\mu+\lambda).h}}{\left(1-\left(1-\alpha\right)e^{-(\mu+\lambda).h}\right)^{2}}\left[\frac{1-e^{-(\mu+\lambda).h}}{\lambda+\mu}-\frac{\left(1-e^{-\lambda h}\right)e^{-\mu h}}{\lambda}\right] \\
& \qquad + \frac{\lambda}{1-\left(1-\alpha\right)e^{-(\mu+\lambda).h}}\left[a.e^{-\mu h}\left(\frac{1-e^{-\lambda h}}{\lambda^{2}}-\frac{h.e^{-\lambda h}}{\lambda}\right)\right. \\
& \qquad - \frac{\left(1-e^{-\lambda h}\right)e^{-\mu h}\left(\pi L_{OT}+R_{OT}+a.h+\frac{a}{\mu}\right)}{\lambda} \\
& \qquad \left. + \frac{\left(\pi L_{OT}+R_{OT}\right)\left(1-e^{-(\mu+\lambda)h}\right)}{\mu+\lambda} + \frac{a\left(1-e^{-(\mu+\lambda)h}\right)}{\mu\left(\mu+\lambda\right)}\right]
\end{aligned}
$$

$$\sum_{n_1=0}^{\infty}\int_{x=0}^{h}\left[n_1.\left(y.b+u\right)+\pi.L_{OF}+R_{OF}\right].\left[\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right]^{n_1}.\left[\mu.e^{-(\mu+\lambda)x}\right].dx$$

$$= \mu.\sum_{n_1=0}^{\infty}\left[n_1.\left(y.b+u\right)+\pi.L_{OF}+R_{OF}\right].\left[\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right]^{n_1}.\int_{x=0}^{h}e^{-(\mu+\lambda)x}dx$$

$$= \frac{\mu.\left(1-e^{-(\mu+\lambda).h}\right)}{\lambda+\mu}.\sum_{n_1=0}^{\infty}\left[n_1.\left(y.b+u\right)+\pi.L_{OF}+R_{OF}\right].\left[\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right]^{n_1}$$

$$= \frac{\mu.\left(1-e^{-(\mu+\lambda).h}\right)}{\lambda+\mu}.\left[\frac{y.b.\left(1-\alpha\right).e^{-(\mu+\lambda).h}}{\left(1-\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right)^2}+\frac{\pi.L_{OF}+R_{OF}}{\left(1-\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right)}\right]$$

$$= \frac{\mu.\left(1-e^{-(\mu+\lambda).h}\right)}{\left(\lambda+\mu\right).\left(1-\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right)}.\left[\pi.L_{OF}+R_{OF}+\frac{y.b.\left(1-\alpha\right).e^{-(\mu+\lambda).h}}{\left(1-\left(1-\alpha\right).e^{-(\mu+\lambda).h}\right)}\right]$$

Appendix D

Multiple Machine Algorithm

Matrix of system states, S, is $m \times 2^m$, where rows correspond to a machine and each element takes 0 and 1 values, indicating the state of the corresponding machine (in-control or out-of-control).
R is an $m \times m \times 2^m$ matrix, the first dimension indicates the stoppage triggering machine, the other two dimensions define the system state. Elements of the matrix are the additional repair times required for each machine when the system is stopped by a particular machine.

begin
for $i = 1 \longrightarrow m\{$
    set $\mu^{(i)} = 0$
    OPTIMIZE to get $n^{(i)}, k^{(i)}, h^{(i)}, E\left[CR^{(i)}\right]$
    compute $E\left(\tau_T^{(i)}\right)$ (Eqn. 3.8) and $E\left(\tau_F^{(i)}\right)$ (Eqn. 3.12)
    set $\gamma^{(i)} = \frac{1}{E\left(\tau_T^{(i)}\right)+E\left(\tau_F^{(i)}\right)}$
$\}$
set $\Gamma = \sum_i \gamma^{(i)}$
set $E\left[TC\right] = \sum_i E\left[CR^{(i)}\right]$
$TCR_{old} = \infty$
for $i = 1 \rightarrow m$ set $i \in M_{TK}$
while $(E\left[TC\right] < TCR_{old})$ or $(M_{TK} \neq \emptyset)$
    for $t = 1 \rightarrow m\{$
        for $i = 1 \longrightarrow m$ $\{$

if ($i \in M_{TK}$) then $M_{TK} = M_{TK} \cup \{i\}$ and $M_{NTK} = M_{NTK} \setminus \{i\}$
}
for $i = 1 \longrightarrow m\{$
set $\mu^{(i)} = 0$
OPTIMIZE to get $n^{(i)}, k^{(i)}, h^{(i)}, E\left[CR^{(i)}\right]$
compute $E\left(\tau_T^{(i)}\right)$ (Eqn. 3.8) and $E\left(\tau_F^{(i)}\right)$ (Eqn. 3.12)
compute $\gamma^{(i)}$(Eqn. 5.67)
}
set $\Gamma = \sum_i \gamma^{(i)}$
for $i = 1 \longrightarrow m$ $\left\{\mu^{(i)} = \Gamma - \gamma^{(i)}\right\}$
Generate matrix S
Generate matrix R
$ExpCst_{new} = 100000$
$ExpCst_{old} = 1000$
while ($\left|\frac{ExpCst_{new} - ExpCst_{old}}{ExpCst_{old}}\right| > 0.0005$){
$ExpCst_{old} = ExpCst_{new}$
$ExpCst_{new} = E\left[TC\right]$ (Eqn. 5.62);
if ($\left|\frac{ExpCst_{new} - ExpCst_{old}}{ExpCst_{old}}\right| > 0.0005$){
for $i = 1 \longrightarrow m\{$
OPTIMIZE to get $n^{(i)}, k^{(i)}, h^{(i)}, E\left[CR^{(i)}\right]$
}
for $i = 1 \longrightarrow m\{$
compute $E\left(\tau_T^{(i)}\right)$ (Eqn. 3.8) and $E\left(\tau_F^{(i)}\right)$ (Eqn. 3.12)
compute $\gamma^{(i)}$(Eqn. 5.67)
}
set $\Gamma = \sum_i \gamma^{(i)}$
for $i = 1 \longrightarrow m$ $\left\{\mu^{(i)} = \Gamma - \gamma^{(i)}\right\}$
}
}
$Cost_t = ExpCst_{new}$
}
$TCR_{old} = E\left[TC\right]$
for $i = 1 \longrightarrow m\{$
if ($E\left[TC\right] > Cost_i$) then $E\left[TC\right] = Cost_i$
}
}
end
Algorithm - $OPTIMIZE$
$n = 1$
$k = 0.01$
$h = 0.01$
$h_{low} = 0.01$

$h_{high} = 20$
$k_{low} = 0.01$
$k_{high} = 20$
while $(n < n_{\max})\{$
    while $(ratio_{old} - ratio > 0.00001)\{$
        $\alpha = 2\Phi(k)$
        $\beta = 1 - \Phi(2\sqrt{n} - k) - \Phi(-2\sqrt{n} - k)$
        $h_\lambda = h_{low} + 0.382 \cdot (h_{high} - h_{low})$
        $h_\mu = h_{low} + 0.618 \cdot (h_{high} - h_{low})$
        set $h = h_\lambda$ and compute $E\left[TC_\lambda\right]$
        set $h = h_\mu$ and compute $E\left[TC_\mu\right]$
        while $(h_{high} - h_{low} \geq 0.005)$
            if $(E\left[TC_\lambda\right] > E\left[TC_\mu\right])$ {
                $h_{low} = h_\lambda$
                $h_\lambda = h_\mu$
                $E\left[TC_\lambda\right] = E\left[TC_\mu\right]$
                $h_\mu = h_{low} + 0.618 \cdot (h_{high} - h_{low})$
                set $h = h_\mu$ and compute $E\left[TC_\mu\right]$
            }
            else{
                $h_{high} = h_\mu$
                $h_\mu = h_\lambda$
                $E\left[TC_\mu\right] = E\left[TC_\lambda\right]$
                $h_\lambda = h_{low} + 0.382 \cdot (h_{high} - h_{low})$
                set $h = h_\lambda$ and compute $E\left[TC_\lambda\right]$
            }
        }
        set $h = (h_{high} + h_{low})/2$
        $k_\lambda = k_{low} + 0.382 \cdot (k_{high} - k_{low})$
        $k_\mu = k_{low} + 0.618 \cdot (k_{high} - k_{low})$
        set $k = k_\lambda$ and compute $\alpha_\lambda, \beta_\lambda$, and $E\left[TC_\lambda\right]$
        set $k = k_\mu$ and compute α_μ, β_μ, and $E\left[TC_\mu\right]$
        while $(k_{high} - k_{low} \geq 0.005)$
            if $(E\left[TC_\lambda\right] > E\left[TC_\mu\right])$ {
                $k_{low} = k_\lambda$
                $k_\lambda = k_\mu$
                $E\left[TC_\lambda\right] = E\left[TC_\mu\right]$
                $k_\mu = k_{low} + 0.618 \cdot (k_{high} - k_{low})$
                set $k = k_\mu$ and compute α_μ, β_μ, and $E\left[TC_\mu\right]$
            }
            else{
                $k_{high} = k_\mu$
                $k_\mu = k_\lambda$
                $E\left[TC_\mu\right] = E\left[TC_\lambda\right]$

$k_\lambda = k_{low} + 0.382 \cdot (k_{high} - k_{low})$

set $k = k_\lambda$ and compute $\alpha_\lambda, \beta_\lambda$, and $E[TC_\lambda]$

}

}

set $k = (k_{high} + k_{low})/2$

compute α, β, and $E[TC]$

$h_{low} = 0.01$

$h_{high} = 20$

$k_{low} = 0.01$

$k_{high} = 20$

$ratio_{old} = ratio$

$ratio = CR$

}

if $(ratio \leq ratio_{opt})${

$ratio_{opt} = ratio$

$h_{opt} = h$

$k_{opt} = k$

$n_{opt} = n$

}

$h_{low} = 0.01$

$h_{high} = 20$

$k_{low} = 0.01$

$k_{high} = 20$

$n = n + 1$

}

Appendix E

Base demand for Chapter 11

t	D_t
1	10
2	11
3	23
4	37
5	1
6	32
7	0
8	11
9	16
10	8
11	6
12	17
13	18
14	8
15	9
16	17
17	37
18	17
19	17
20	31

t	D_t
21	10
22	1
23	21
24	0
25	11
26	4
27	30
28	34
29	14
30	30
31	25
32	0
33	35
34	32
35	0
36	22
37	28
38	24
39	17
40	24

t	D_t
41	3
42	34
43	28
44	32
45	0
46	19
47	15
48	0
49	0
50	9
51	0
52	10
53	33
54	14
55	24
56	0
57	27
58	5
59	32
60	2

t	D_t
61	0
62	34
63	23
64	0
65	16
66	16
67	1
68	3
69	27
70	29
71	32
72	13
73	16
74	0
75	6
76	0
77	8
78	21
79	35
80	0

t	D_t
81	27
82	21
83	23
84	0
85	8
86	28
87	0
88	26
89	40
90	4
91	17
92	16
93	4
94	25
95	7
96	15
97	0
98	4
99	9
100	24