BUNDLE PRICING OF INVENTORIES WITH STOCHASTIC DEMAND

A THESIS

SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL ENGINEERING

AND THE INSTITUTE OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

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September, 2004

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ABSTRACT

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In this study, we consider the single period pricing of two perishable products which are sold individually and as a bundle. Demands come from a Poisson Process with a price-dependent rate. Assuming that the customers' reservation prices follow normal distributions, we determine the optimal product prices that maximize the expected revenue. The performances of three bundling strategies (mixed bundling, pure bundling and unbundling) under different conditions such as different reservation price distributions, different demand arrival rates and different starting inventory levels are compared. Our numerical analysis indicate that, when individual product prices are fixed to high values, the expected revenue is a decreasing function of the correlation coefficient, while for low product prices the expected revenue is an increasing function of the correlation coefficient. We observe that, bundling is least effective in case of limited supply. In addition, our numerical studies show that the mixed bundling strategy outperforms the other two, especially when the customer reservation prices are negatively correlated.

Keywords: Bundling Strategy, Pricing, Stochastic Demand, Revenue Management.

ÖZET

RASSAL TALEP DAĞILIMI ALTINDA PAKET ÜRÜNLERİNİN FİYATLANDIRILMASI

Zümbül Bulut
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Bu çalışmada, bozulabilir iki ürünün tek tek ve paket halinde satıldığı durumda, bir sezondaki fiyatlandırılması incelenmiştir. Talepler, oranı fiyata bağlı Poisson sürecine göre gelmektedir. Müşterilerin ürünlere ödemek istediği en yüksek fiyatların normal dağılımla gösterilebileceği varsayılarak, beklenen geliri en çoklayan fiyatları belirlenmiştir. Farklı rezervasyon fiyatları dağılımı, farklı talep oranları ve farklı başlangıç envanter seviyeleri gibi koşullar altında, üç paketleme stratejisinin (karma paketleme, saf paketleme ve paketlememe) performansları değerlendirilmiştir. Sayısal analizler sonucunda, paketlenmemiş ürün fiyatlarının yüksek değerlere sabitlenmesi durumunda, beklenilen kazancın rezervasyon fiyatlarının korelasyonu ile ters orantılı olduğu görülmüştür. Fiyatlar düşük değerlere sabitlendiğinde, beklenilen kazancın korelasyonla doğru orantılı olarak değiştiği tespit edilmiştir. Paketlemenin, envanterin kısıtlı olduğu durumlarda en verimsiz olduğu gözlemlenmiştir. Sayısal analizler, karma paketleme stratejisinin, özellikle rezervasyon fiyatlarının negatif bağımlı oldukları durumlarda diğer stratejilerden daha iyi sonuçlar verdiğini göstermiştir.

Anahtar sözcükler: Paketleme Stratejisi, Fiyatlandırma, Rassal Talep, Gelir Yönetimi.

Acknowledgement

I would like to express my sincere gratitute to Prof. Dr. Ülkü Gürler and Asst. Prof. Alper Şen for their supervision and encouragement in this thesis. Their vision, guidance and leadership was the driving force behind this work. Their endless patience and understanding let this thesis come to an end.

I am indebted to Asst. Prof. Emre Berk and Asst. Prof. Osman Alp for accepting to read and review this thesis and also for their valuable comments and suggestions.

I would like to express my appreciation to Banu Yüksel Özkaya for her support, guidance and the time she took to answer all of my questions.

I am most thankful to Ayşegül Altın, Mehmet Oğuz Atan, Mustafa Rasim Kılınç and Salih Öztop for being the components of my most favorible bundle.

I would like to take this opportunity to thank Bala Gür not only for breakfasts she prepared each morning to me but also for her support during my desperate times.

I would like to thank to Elvan Alp, Ceren Çıracı, Elif Uz, Emrah Zarifoğlu, Esra Büyüktahtakın, Damla Erdoğan, Güneş Erdoğan, Yusuf Kanık, Emre Karamanoğlu, Kürşad Derinkuyu, Çağrı Latifoğlu and Evren Körpeoğlu for their morale support.

And all my friends I failed to mention, thank you all...

Finally, I would like to express my deepest gratitude to mom, dad and my brother, for their endless love and understanding.

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Chapter 1

INTRODUCTION AND DEFINITIONS

1.1 Introduction

Each organization involved in a production activity or providing services aims to do its best in terms of a performance criteria. Firms may have different objectives such as increasing their profits, their market shares, service levels or reducing operating costs. In order to achieve these goals, companies can follow different strategies. More efficient transportation, marketing and advertisement strategies, more profitable manufacturing methods may be employed to this end.

Most of the firms aim to maximize their profits by either increasing their revenues or cutting their costs down. Although all parties in a supply chain reduce their costs with improved inventory management, lost sales and excess inventories are still unavoidable. This is why, many companies are now looking into the demand side of the supply-demand relation. Retailers are the last party of supply chains. Most of the time, it is easier for retailers to improve profitability by efficient demand management instead of cost reduction. Better demand management via efficient pricing policies becomes an important goal. Temporal price changes are becoming an industry practice to control revenue. However, the price of the product cannot be increased or decreased arbitrarily. There should be some strategy that will dynamically adjust prices. Determining the strategy to use is a very complicated and difficult decision. Another way to achieve high revenues is to sell products to each customer at the best price that the customer is willing to pay, i.e., perfect price discrimination. However, it is almost impossible for a firm to know each individual's valuation for products. Even if it is known, it will be unfair to charge each customer differently. Therefore, to improve revenues, a right price should be set for each product. Determining the right price to charge a customer for a product is a complex task. The company should know not only its supply and operating costs, but also how much the customers value the product and what the future demand will be. The retailer faces a trade-off when setting prices. If the retailer sets the prices too low, he will lose customers' surplus; if he sets the prices too high he will lose the customer and risk having a surplus of goods at the end of the season.

Retail managers always face rapid changes in fashion and customer preferences. Also, products may deteriorate with a rate depending on the age and/or amount of the products. Some items, on the other hand, may display negligible or no loss in quality and value during a fixed lifetime, after which they become useless or obsolete. Such products are called perishable. The "perishability" of the products leads to short selling periods, during which inventory management and pricing strategies are central to success ([2]). Perishable inventories have received considerable attention in recent years. This is a realistic trend since most products such as medicine, dairy products and chemicals start to deteriorate once they are produced. Perishability also applies to services. The inventory of seats on a particular flight, the inventory of rooms at a hotel at a particular night all perish at certain times. Retailers and service providers have the opportunity to enhance their revenues through optimal pricing of their perishable products that must be sold within a fixed period of time. For fashion goods, the selling horizon is usually very short and production/delivery lead times prevent replenishment of inventory. Therefore, the seller has a fixed inventory on hand and must decide on how to price the product over remaining selling horizon.

Revenue management or yield management is concerned with dynamic pricing of perishable products. The main idea behind revenue management is to divide the market into multiple customer classes and to provide different types of products with different prices to each class. Success of yield management practices is closely related with advances in information technologies.

As stated before, determining the right price to charge is a complex task. There are different factors that influence the pricing decision of retailers. Some of these factors are reservation prices of customers, supply availability, intensity of customer arrivals, the length of the planning horizon, the behavior of the competitors and the prices of complementary and substitutable products. In the following paragraphs we explain how each factor affects pricing decisions. The findings provided below are supplemented from the following references: [6], [21] and [22].

Reservation price is defined as the maximum amount that the customer is willing to pay for a product. If the product's price is lower than the reservation price of the customer, the customer buys the product, otherwise she does not. In marketing literature, "value analysis" is used to explain how customers decide whether to buy the product or not by considering "the perceived relative economic value" of the product. Accordingly, the maximum price that can be set is that at which customer disregards the difference between the product and the next best economic alternative. The difference between the maximum amount customers are willing to pay for the product and the amount they actually pay is called customers' surplus (perceived acquisition value). This difference represents the customers' gain from making the purchase (customers' net gain from trade). It is usually assumed that the reservation price is a random variable with a continuous distribution over a population of customers and this distribution may change in time. The reasons for the variance in the reservation prices can be stated as the heterogeneity in the market (difference in income, age etc.) and a lack of information about the customer's tastes and needs. The goal of the seller should be to adjust the prices so that the total expected profit is maximized over the planning horizon, while taking into consideration the heterogeneity of the population of customers in their willingness to pay for the product.

Almost all of the studies in the literature conclude that the profit increases with the level of the initial inventory. In some cases, the initial inventory is taken to be fixed due to some kind of commitments between the supplier and the retailer. The retailer should have an accurate forecasting strategy to determine the amount to order at the beginning of the planning horizon and an efficient strategy for pricing his initial inventory. Even in the case when the demand is not known in advance (and cannot be forecasted accurately) and the retailer has excess initial inventory, the prices should be set low (compared with valuation of customers) to increase the probability that all of the goods on hand will be sold. On the other hand, if the initial inventory is low and demand is known to be higher than the on hand inventory, the prices should be set high (again, compared with valuation of customers). In that case, the products will be sold to those customers with high reservation prices. Low prices with low initial inventory lead to the loss of the customer surplus. The retailer will then experience lost sales, loss of goodwill and decrease in market share since the inventory will be depleted before the horizon ends.

The arrival process of customers is another factor that affects the retailers' pricing decisions. The arrival rate is often a response to their regular purchasing patterns during the selling season rather than a function of individual prices ([3]). The arrival pattern of the customers can be affected by advertisement campaigns. If the arrival intensity is dense, the prices are set high. Due to the high arrival rate, the probability of having customers with high reservation prices increases and high-price products are sold. However, when the arrival rate is small, it is more convenient to set the prices low. Otherwise, the small number of arriving customers will not purchase the product, resulting in increased holding cost, excess inventory on-hand and loss of the customer to the competitors.

Purchasing behavior of the customers also affects the retailer's pricing decisions over time. Customers are divided as myopic and strategic according to their purchasing behavior. The first one makes a purchase immediately if the price is below his valuation, without considering future prices. The second type considers possible future prices of the product when making purchasing decision. Therefore, the seller should consider carefully the effects of his price over customers' current and future decisions.

Another factor that affects the pricing decision of the seller is the length of the planning horizon. With short planning horizon, the initial price should be set low. Low prices will trigger the demand up and the possibility of selling all the units in this short period will increase. On the other hand, if we have a long planning horizon, we can set the initial price high. By setting the initial price high, we can get the customer surplus.

Pricing of a product in a competitive market is more difficult than in a monopolistic one. In the absence of direct competition, one can estimate how a price change will affect sales simply by analyzing buyers' price sensitivity. When, however, there is competition, the competitors can make sales estimates useless by changing their prices. In doing so, competitors change buyers' alternatives and thus manipulate what they are willing to pay for it. For example, a company might reasonably estimate that it could double sales by pricing 20 percent below the competitors. But a 20 percent price cut would not necessarily generate such a result. The competitors may respond with price cuts in their products to eliminate, narrow or even reverse the gain that the company hoped to achieve. The greater the potential for price competition, the more important it is for management to evaluate how competitors are likely to use price in their marketing decisions.

Another factor that affects retailers' pricing decisions is the prices of complementary and substitutable products. Most firms sell multiple products. For example, supermarkets sell products as diverse as meats, packaged goods, furniture, toys and clothing. If one product's sales do not affect the sales of the firm's other products, then it can be priced in isolation. Most often, however, the sales of the different products in the firm are interdependent. To maximize the profit, prices must reflect that interaction. The effect of one product's sales on another's can be either adverse or favorable. If adverse, then the products are "substitutes". Most substitutes are different brands in the same product class. Sometimes, however, substitutes appear in completely different product classes.

For example, the sales of macaroni products may rise whenever price increases reduce the sales of beef. If one product's sales favorably affect the sales of another, then the products are "complements". Complementarity can arise for either of two reasons: (1) the products are consumed together in producing satisfaction. For example, tickets to a movie and popcorn are complements; (2) the products are most efficiently purchased together. Buyers often seek to conserve time and money by purchasing a set of products from a single seller. For example, consumers may get accustomed to a particular supermarket and buy all of their needs from there. Substitutes and complements call for adjustments in pricing when the products are sold by the same company as a part of a product line. To correctly evaluate the effect of a price change, management must examine the changes in revenues and costs not only for the product whose price is changed, but also for the other products affected by the price change.

In addition to the main characteristics discussed above, numerous other factors can influence a dynamic pricing policy, such as business rules, cost of implementing price changes, seasonality of and external shocks to demand.

The common objective of almost all retailers is to maximize profits. There are different ways of achieving profit maximization. As explained in detail above, price adjustments are among the most useful ones. Another common strategy is to make promotions by selling two or more products in a bundle and charge a price less than the total amount that will be paid if products are bought individually.

Bundling is a prevalent marketing strategy that takes considerable attention recently. Despite the increased interest on bundling, there are many questions that are left unanswered. For example, reasons for the profitability of bundling, conditions under which the retailers gain from bundling, customers perspective, legality of bundling are some of the subjects that seek further consideration.

Among many studies in the literature, Stremersch and Tellis [27] provides the clearest definitions of bundling terms and principles. They define bundling as the sale of two or more separate products in one package. Here, separate products refer to products for which separate markets exist. Seasonal tickets of sporting and cultural organizations, fixed-price menus, Internet services are examples of bundling. Bundling can take one of two forms: price bundling or product bundling. In price bundling, the retailer sells two or more separate products in a package, without any physical integration of the products. Here bundling does not create added value to customers. Therefore, a discount must be offered to motivate at least some customers to buy the bundle. Selling tickets for different football matches at a price less than the sum of separate games' ticket prices is an example for price bundling. On the other hand, product bundling is the integration and sale of two or more separate products or services at any price. A multimedia PC is an example for product bundling. The different functions of the individual parts are combined in a single product bundle. The multimedia PC has an integral architecture, in that it integrates functions such as connection, data storage, etc. By this integration, a complete PC can do more than the parts that are combined in can do. As a result, price bundling is a pricing and promotional tool, while product bundling is more strategic in that it creates added value. Price bundling is easier to implement compared with product bundling. The latter requires new design, new manufacturing plan, etc. All departments of a manufacturing company are involved in creation of product bundles. On the contrary, marketing departments are usually the only single decision makers for price bundling.

In order to clarify the concepts of price and product bundling, Stremersch and Tellis [27] give the following example: "Consider strategic options of Dell, which markets to consumers who want to buy a portable computer system consisting of a basic laptop, a modem, and a CD burner. First, it can sell these products as separate items, such that the price of each item is independent of consumers' purchase of the other item. In this case, consumers could easily give up purchasing a modem or CD burner, or they could purchase it from a competitor. Second, Dell can sell the products as a price bundle. For example, it could, without physically changing any of the products, give a discount to consumers if they buy all three products together. This offer would probably motivate at least some consumers to buy all three products from Dell. Third, Dell can sell the three items as a product bundle. To meet the latter classification, Dell must design some integration of the three separate products. For example, it could create an enhanced laptop. Not only could this trigger some consumers to buy all products from Dell, but through the value added they might even do so at a premium price."

Retailers most of the time use price bundling, which is the focus of this study. Therefore, from this point on, bundling refers to price bundling. This strategy can further be divided into two different types: pure and mixed. When retailers prefer pure bundling, they sell only the bundle but not the individual products. Mixed bundling is a strategy in which firm sells both the bundle and the separate products that constitute the bundle. Unbundling is a strategy in which products are sold separately, not as a bundle. Which strategy performs the best depends on many factors. Extensive literature concludes that there does not exist a single strategy that always dominates the other two.

The difficulty of making pricing decisions is mentioned above. Pricing becomes even more difficult when bundling is under consideration. Different customer segments attach different values to each of the products that constitute the bundle. Some customers will want to purchase the bundle while others are interested in a specific product. Retailers should carefully decide on a pricing strategy for the individual products and for the bundle. Prices may encourage customers to select a wide range of offerings, including products that they do not value highly, or prices may encourage purchasing at increased prices. Each individual product as well as the bundle should be priced in a such a way that customers who value individual products highly are still willing to buy individual products and customers who do not want a component of the bundle become willing to buy a bundle.

In this study, pricing policy of a retailer selling two types of products is investigated. He sells both the individual products and a bundle composed of them. Retailer's aim is to maximize the revenue from the sales; cost of bundle formation, cost of pricing software and other costs are ignored. Products are assumed to be perishable. There is a fixed planning horizon, during which replenishment is not possible. Therefore, the retailer has a fixed inventory of both products at the start of the season. He sets the prices at the beginning of the period and these prices remains unchanged until the end of the period. The use of mixed bundling strategy is assumed, but the performances of other two strategies are investigated to form a benchmark for comparison.

1.2 Definitions

Reservation Price: The maximum amount that a customer is willing to pay for a product.

Customer Surplus: The difference between the maximum amount customers are willing to pay for the product and the amount they actually pay. This difference represents the customers' gain from making the purchase.

Complementary Products: If the sale of one product favorably affects the sale of another product, these are called complementary products.

Substitutable Products: If the sale of one product adversely affects the sale of another product, these are called complementary products.

Myopic Customers: A customer who makes a purchase immediately if the price is below her reservation price, without considering future prices.

Strategic Customers: A customer who takes into account the future path of prices when making purchasing decision.

The rest of the thesis is organized as follows. In Chapter 2, an extensive literature review about pricing, bundling and bundle pricing is provided. In Chapter 3, the problem under consideration is defined, the model is introduced and possible realizations during the planning horizon are investigated. Depending on the numbers of individual products and bundles sold, the expected revenue function is obtained. Similar expressions are also obtained for pure bundling and unbundling cases. The performance of three strategies under different conditions, such as reservation price distribution of customers, the intensity of customer arrivals is explained via an intensive numerical study in Chapter 4. Finally, in

Chapter 5 conclusions, general results and extensions of this study are stated.

Chapter 2

LITERATURE REVIEW

In literature, pricing problems have been studied extensively. There are a large number of research papers dealing separately with dynamic pricing or fixed number of price changes of perishable products, timing and optimal duration of price changes, bundling of two or more products and pricing of bundles. Below, we present the literature about the theoretical background of these topics.

We will start with pricing of perishable products in the following section. Then, we will provide the literature on bundling, in which we will focus again on pricing studies. Finally, we will mention the shortcomings of the literature that motivated our study.

2.1 Pricing of Perishable Products

In recent years, pricing of perishable inventories has received considerable attention. Before providing a literature review about the pricing strategies for perishable products, we would like to emphasize the classification of general pricing strategies provided by Noble and Gruca [23]. Noble and Gruca divide the pricing strategies encountered in the industry into four broad categories: New Product Pricing Situation, Competitive Pricing Situation, Product Line Pricing

Situation and Cost-Based Pricing Situation. The conditions that determine when a given strategy should be used are referred to as determinants. Examples of determinants are product differentiation, economies of scale, capacity utilization, demand elasticity and product age.

New product pricing is appropriate in the early life of the product. This category has been divided into three strategies;

1. Price Skimming: In this strategy, the price is set high initially and then it is reduced over time gradually. The main objective is to attract customers who are insensitive to the initial high price. As this segment is saturated, the price is lowered to increase the appeal of the product.
2. Penetration Pricing: In this strategy, the price of the product is set low. The aim is to make customers accustomed to the product initially.
3. Experience Curve Pricing: In this strategy again the initial price is set low. However, the aim is to adopt the producer to this new product by building cumulative volume quickly and driving the unit cost down.

Competitive pricing is appropriate when the price of the product is determined relative to the price of one or more competitors' prices. This situation is categorized into three pricing strategies as follows;

1. Leader Pricing: The price leaders initiate price changes and they expect that others in the industry will follow their way in price adjustments. Generally, the price of an identical product is higher if it is sold by the leader company.
2. Parity Pricing: Firms that follow this strategy either tries to maintain a constant relative price between competitors or it imitates prevailing prices in the market.
3. Low Price Supplier: In this strategy, the firm sets the price lower than its competitors and it aims to have higher demand than the others.

Product line pricing situation corresponds to the situation where the price of the main product is affected by the other related products or services from the same company. There are three pricing strategies that are mentioned under this heading;

1. Complementary Product Pricing: The price of the main product is set low then the other complementary products. This strategy is well illustrated by Gillette's strategy of selling razors cheaply and blades dearly.

2. Price Bundling: The product is offered as a component of a bundle of products. The total price of the bundle is set lower than the total price of the products bundled.

3. Customer Value Pricing: In this strategy one version of the product is offered at a very competitive price level, however the product involves fewer features than the other versions.

The fourth situation is cost-based pricing. The firm decides on how much to charge based on the cost incurred in obtaining the product.

This broad classification is valid for all kinds of products. Depending on the product type and on the market it is sold, a firm may need to use two or more of these strategies at the same time. This complicates the pricing decisions.

In the rest of this section, we will review the most frequently referred studies related with pricing of perishable products in the context of revenue management.

One of the first studies related to dynamic pricing of perishable goods is by Rajan, Rakesh and Steinberg [24]. They investigate the relationship between pricing and ordering decisions for a monopolist retailer facing a known demand function where, over the inventory cycle, the product may exhibit physical decay or decrease in market value. The authors study the linear and nonlinear demand cases and exhibit propositions on the optimal price changes and optimal cycle length.

Gallego and van Ryzin [12] consider the dynamic pricing of inventories for a given stock of items that must be sold by a deadline. The demand is stochastic and price sensitive and the objective is revenue maximization. For exponential demand functions, the authors derive an optimal pricing policy in closed form. However, only the deterministic version of the problem is analyzed for general demand functions and an upper bound is obtained for the revenue. With this upper bound, the authors develop a single price policy, which is asymptotically optimal when either remaining shelflife or inventory volume is large. Gallego and van Ryzin [12] report that their policy provides a revenue that is only 5% to 12% below the optimal revenue when the number of items is fewer than 10 and it is nearly optimal for more than 20 items. This work is criticized by Feng and Xiao [11]. They suggest that for short remaining lives and small inventory volumes, the strategy of Gallego and van Ryzin would not work.

Yildirim, Gurler and Berk [34] consider the dynamic pricing of perishables in an inventory system where items have random lifetimes which follow a general distribution and the unit demands come from a Poisson Process with a price-dependent rate. The objective of their study is to determine the optimal pricing policy and the optimal initial stocking level to maximize the discounted expected profit. They consider the instances at which an item is withdrawn as a decision epoch for setting the sales price. Authors determine the optimal price paths for the discounted expected profit for various combinations of remaining lifetimes. Their study indicates that a single price policy results in significantly lower profits when compared with their formulation.

A dynamic pricing model for selling a given stock of a perishable product over a finite time horizon is considered by Zhao and Zheng [35]. They identify a sufficient condition under which the optimal price decreases over time for a given inventory level. They also illustrate that the optimal price decreases with inventory. According to their numerical analysis, their policy achieves 2.4-7.3% revenue improvement over the optimal single price policy.

Feng and Gallego [9] address the problem of deciding the optimal timing of a single price change from a given initial price to either a given lower or higher second price. It is shown that it is optimal to decrease (resp., to increase) the initial price as soon as the time-to-go falls below (resp., above) a time threshold that depends on the number of yet unsold items. Later, the same authors study a similar problem [10]. They aim to decide again on the optimal timing of price changes within a given menu of allowable price paths each of which is associated with a general Poisson process with Markovian, time dependent, predictable intensities. Feng and Gallego [9] show that a set of variational inequalities characterizes the value functions and the optimal time changes. They develop an algorithm to compute the optimal value functions and the optimal pricing policy.

Bitran and Mondschein [3] study a problem similar to Gallego and van Ryzin [12] but the price is allowed to change only periodically. The price is never allowed to rise. Although, the authors present some empirical analysis for their study, no theoretical results are provided. Similarly, Chatwin [4] analyzes the pricing of perishable products where the set of available prices is finite. He indicates that for this problem as well as the problem in which the price is selected from an interval, the maximum expected revenue function is nondecreasing and concave in the remaining inventory and in the time-to-go. In addition, he shows that the optimal price is nondecreasing in the remaining inventory and nondecreasing in the time-to-go. He concludes that these results hold when prices and corresponding demand rates are functions of time-to-go but not when the demand rates are functions of inventory level.

There exist many studies in the literature that reveals the fact that the pricing decisions must be given in coordination with other managerial decisions. The overall objective of the firm can only be achieved by considering all the important decisions at once. Federgruen and Heching [8]'s address the simultaneous determination of pricing and inventory replenishment strategies under demand uncertainty. They show that a base stock list price policy is optimal for the finite horizon with bi-directional price changes. In a base stock list price policy; if the inventory level is below the base stock level, it is raised to the base stock level and the list price is charged. If inventory level is above the base stock level, then nothing is ordered and price discount is offered. Similarly, Wee and Law [33] develop a replenishment and pricing policy by taking into account the time value of money. The inventory system under consideration is deterministic and demand is price-dependent. They derive a near optimal heuristic to maximize the total net present-value profit. Subrahmanyan and Shoemaker [30] study a pricing model that allows replenishments and incorporates learning about demand through Bayesian updates. The model they use is a dynamic programming model which is solved numerically using backward recursion.

Chun [5] also considers the problem of determining the price for several units of a perishable or seasonal product to be sold over a limited period of time. He assumes that the customer's demand can be represented as a negative binomial distribution and determines the optimal product price based on the demand rate, buyers' preferences and the length of the sales period. Since the seller's average revenue decreases as the number of items for sale increases, Chun [5] also considers the optimal-order-quantity that maximizes the seller's expected profit. He also develops a multi-period pricing model, for the cases where the seller can divide the sales period into several short periods.

Bitran, Caldentey and Mondschein [2] examine the coordination of clearance markdown sales of seasonal products in retailer chains. The authors propose a methodology to set prices of perishable items in the context of a retail chain with coordinated prices among its stores and compare its performance with actual practice in a case study. A stochastic dynamic programming problem is formulated and heuristic solutions that approximate optimal solutions satisfactorily are developed.

Elmaghraby and Keskinocak [7] provide a review of the literature and current practices in dynamic pricing. Their focus is on dynamic (intertemporal) pricing in the presence of inventory considerations. This paper constitutes a good summary for dynamic pricing policies.

In this section we reviewed a limited number of papers related with pricing of perishable products.

2.2 Bundle Pricing

Bundling is a prevalent marketing strategy. Despite its importance, little is known about how to find optimal bundle prices and only a few studies are available in the literature. In this section, we will review the most influential studies about bundling and bundle pricing.

Most of the bundling papers are built on the early study of Stigler [28],where the author represents the demand information by reservation prices for the products. Additivity of reservation prices and production costs is assumed. He concludes that bundling is profitable when reservation prices are negatively correlated.

Adams and Yellen [1] develop a two-product, monopoly bundling model by assuming that the reservation prices for products are additive and negatively correlated. They show that the profitability of commodity bundling can stem from its ability to sort customers into groups with different reservation price characteristics, extracting consumer surplus. They consider a monopolist producing two goods with constant unit costs and facing buyers with diverse tastes. The authors assume a discrete number of customers. The reservation prices for the components of the bundle are negatively correlated. This feature makes it appear that bundling serves much the same purpose as third-degree price discrimination. Authors consider three different sales strategies: unbundling, pure bundling and mixed bundling and compare these strategies in terms of seller profit. Adams and Yellen [1] argue that mixed bundling at least weakly dominates pure bundling. The reason is that, customers with negatively correlated reservation prices prefer individual products, while the others prefer the bundle. The low bundle valuation of the demanders make mixed bundling a more profitable strategy compared to pure bundling. The authors' argument presume that bundling does not lead to any cost savings.

Schmalensee [26] also developes a two-product monopoly bundling model in which he relaxes the assumption that the reservation prices of the individual products are negatively correlated. However, he retained the additivity assumption.

By assuming that reservation prices (for firm's two products) are distributed according to bivariate normal probability law, Schmanlensee constructs a class of examples within which the profitability of bundling can be analyzed as a function of production costs, the mean and variance of the reservation price for each commodity, and the correlation between the two commodities' reservation prices. The author obtains some general results for mixed bundling case and compares them with pure bundling and unbundled sales. After comparing pure bundling with unbundled sales, Schmalensee [26] shows explicitly that pure bundling operates by reducing the effective dispersion in buyers' tastes. This happens simply because as long as reservation prices are not perfectly correlated, the standard deviation of reservation prices for the bundle is less than the sum of the standard deviations for the two components. The greater is the average willingness to pay, measured as the normalized difference between mean reservation price and cost, the more likely it is that such a reduction in diversity will enhance profits by permitting more efficient capture of consumers' surplus. If the average willingness to pay is large enough, the increase in profit caused by pure bundling is apparently larger than the fall in consumers' surplus, so that pure bundling increases net welfare. Besides, Schmalensee [26] provides a comparison of the profitability of mixed bundling and unbundled sales. It is shown that mixed bundling combines the advantages of pure bundling and unbundling sales. This policy enables the seller to reduce effective heterogeneity among those buyers with high reservation prices for both goods, while still selling at a high markup to those buyers willing to pay a high price for only one of the goods. At least in the Guassian case, this makes mixed bundling a very powerful price discrimination device. One of the surprising findings of this paper is that bundling can be profitable when demands are uncorrelated or even positively correlated. To summarize, two major results of his work are a confirmation of the profitability of bundling when there is negative correlation, and the benefits of mixed bundling over a restriction to unbundling or pure bundling. Two comments are written to this paper by Long [18] and Jeuland [17].

Long [18] states that heterogeneity in consumer tastes (especially in relative valuations of the firm's two products) is a necessary condition for profitable bundling. Unfortunately, more specific principles to describe concisely the necessary and/or sufficient conditions for profitable bundling are not so obvious. Different from Schmalensee [26], the author assumes that the distribution of consumer reservation prices has a continuous density without restricting it to any particular form. He states that, if an increase in prices (above the monopoly level) increases the number of consumers who buy only one of the two commodities, the bundling will increase profit. Besides, if the reservation prices for the two commodities are not positively correlated, then bundling increases profit. When bundling does not increase profit, a form of promotional couponing does increase the profit. Long [18] concludes that the most favorable case for bundling as a price discrimination device is the case where the bundle components are substitutes in demand.

Jeuland [17] also comments on the paper of Schmalensee [26]. The author states that, depending on the distribution of reservation price, any ranking- in terms of profitability for the seller- of these three strategies is possible.

Salinger [25] focuses on the graphical analysis of bundling and he deals with two-product case. He assumes additive reservation prices. Salinger explores the implications of the relationship between the bundle and aggregated components demand curves for the profitability and welfare effects of bundling. If it does not lower costs, bundling tends to be profitable when reservation values are negatively correlated and high relative to costs. If bundling lowers costs and costs are high relative to reservation values, positively correlated reservation values increase the incentive to bundle.

None of these papers show how to calculate optimal bundle prices. One important study that draw attention to this topic is the study of Hanson and Martin [14] in which they provide a practical method for calculating optimal bundle prices. The basis of the approach is to formulate the model as a mixed integer linear program using disjunctive programming. The theoretical rationale for this approach is given along with computational results for a set of test problems based on actual survey data. An added benefit of the bundle pricing model solution is stated to be the selection of products to include in a firm's product line. Authors also consider one of the most serious problems facing a product line manager addressing the bundling issue: the exponential growth in possible products which results from increasing the number of components considered. An algorithm for finding optimal solutions is given along with computational results.

The published studies are fuzzy about some basic terms and principles, do not discuss the legality of bundling, and do not provide a comprehensive framework on the economic optimality of bundling. Stremersch and Tellis [29] provide a new synthesis of the field of bundling based on a critical review and extension of the marketing, economics and law literature. This paper clearly and consistently defines bundling terms and principles. Authors identify two key underlying dimensions of bundling that enable a comprehensive classification of bundling strategies and formulate clear rules to evaluate the legality of each strategy. In addition, authors propose a framework of twelve propositions that prescribe the optimal bundling strategy in various contexts. The propositions incorporate all the important factors that influence bundling optimality.

As reviewed before, bundling has received considerable attention in economics and marketing literature. Most research in this area studies the conditions under which bundling is profitable for the seller and/or the customer. The general result is that the profitability of bundling depends on the distribution of reservation prices. The previous research also compares the performance of different strategies such as mixed bundling, pure bundling and unbundling and concludes that no unique strategy dominates the others in all circumstances.

We note that bundling studies in economics and marketing literature make an implicit assumption that there is an ample supply of products that could be acquired at a certain cost. In this thesis, however, we assume that there is a fixed amount of inventory for each product to be sold over a finite horizon, and we study how individual and bundle products should be priced to maximize revenue from this limited inventory. In this respect, our study follows the line of yield management or revenue management research. We should also note that while the existing research in marketing and economics literature study the performance of different bundling strategies, they do not consider optimizing the bundle and the individual product prices explicitly. In this thesis, our focus is on optimizing the bundle price individually or bundle and individual prices jointly. We assess the performance of different bundling strategies given that pricing decisions are optimally taken.

Chapter 3

MODEL and THE ANALYSIS

Based on the studies related with bundle pricing in literature, it is observed that very few researchers consider the determination of prices that maximize the revenue in mixed bundling strategy. In this study, we focus on the expected revenue maximization for the mixed bundling strategy with two products and stochastic demand. Given an initial inventory of two products and a finite selling season, we are concerned with the problem of determining prices of the bundle and the individual products so that the expected revenue over the selling season is maximized. To form a basis of comparison, we also study pure bundling and unbundling strategies.

Before defining the problem under consideration, we elaborate on some of the fundamental assumptions used in our model. We first note that we use reservation prices to predict purchasing behavior of customers arriving to the store. Consumer reservation price is a fundamental concept in understanding consumer purchasing decisions and developing pricing strategies. We refer the reader to Jedidi and Zhang [16] for estimating individual consumer reservation prices and to Jedidi et. al [15] for capturing consumer heterogeneity in the joint distribution of reservation prices in the case of bundling. Other ways to model consumer behavior in the case of differentiated products include multinomial logit (MNL) random utility model; see van Ryzin and Mahajan [31] and Mahajan and van Ryzin [19].

We assume that the reservation price for the bundle is equal to the sum of the individual reservation prices. This reflects the assumption that the products are individually valued. Many of the bundling studies in literature (e.g., Adams and Yellen [1], Schmalensee [26], McAfee [20]) use the same assumption. Guiltinan [13] refers to this assumption as the assumption of strict additivity. Venkatesh and Kamakura [32] relax the strict additivity assumption and allow for substitutability and complementarity. If the products are substitutable, customers want to buy only one of them at a time. Then, a customer's reservation price for the bundle would be subadditive (less than the sum of the reservation prices). Alternatively, customers may tend to consume the two products together. These kind of products are called complementary. When products are complements, a customer's reservation price for the bundle is superadditive (more than the sum of the reservation prices).

We also assume that customers' reservation price pairs follow a bivariate normal distribution. According to Schmalensee [26], the frequency with which normal distributions arise in the social sciences makes the Gaussian family a plausible choice to describe the distribution of tastes in a population of buyers. The bivariate normal has a small number of easily interpreted parameters. Because the sum of two normal distributions is also normal, the distributions of reservation prices for the bundle and the components have the same form, when strict additivity is assumed. In addition, handling correlations between demands for the components is simple when normal distributions are used. A problem with normal is that it entails negative valuations. Salinger [25] states that, while there may be cases where customers would pay not to receive a good, the assumption of negative valuations is not appropriate whenever an undesirable component of a bundle can be disposed of freely. Therefore, we select appropriate parameters for the normal distributions in our numerical study to ensure non-negative valuations.

3.1 Problem Definition

We consider a retailer who sells two perishable products, Product 1 and Product 2. Initially, Q_1 units of Product 1 and Q_2 units of Product 2 are available. There is a fixed planning horizon of length T over which the sales are allowed and the retailer aims to maximize his profit.

It is assumed that the retailer forms a monopoly for the two products.

At the beginning of the planning horizon, the retailer sets the price p_1 for Product 1, and p_2 for Product 2. He also provides a bundle option which implies charging the customers less than the sum of the individual products' prices if they buy both. The individual product prices and the bundle price, p_b are determined so that $p_b \leq p_1 + p_2$. We assume that the initial prices remain unchanged until the end of the season. It is assumed that, the retailer incurs fixed costs before the selling season. We therefore consider maximizing the revenue.

Customers arrive at the store according to a Poisson Process with a fixed arrival rate of λ customers/period. A customer is allowed to purchase a single product or a bundle, not both. She may also choose to leave without any purchase. Customers' preferences are reflected by their reservation prices. The reservation price is defined as the maximum amount that a customer is willing to pay to purchase a product. If the product prices are lower than the reservation prices, she prefers the product which brings her maximum surplus (the reservation price for the product - the price of the product).

Customers' reservation prices are assumed to be random variables with continuous distributions. Specifically, we assume that the reservation prices for the two products, referred to as R_1 and R_2, respectively are normally distributed with parameters (μ_1, σ_1) and (μ_1, σ_1), respectively. The reservation price for the bundle is assumed to be the sum of the individual products' reservation prices, i.e., $R_b = R_1 + R_2$. Therefore, we assume products are independent. We define the correlation coefficient of the joint reservation price distribution as ρ.

Upon arrival, a customer compares her reservation prices for the individual products and the bundle with their corresponding prices. She decides to leave without any purchase, buy Product 1, Product 2 or a bundle, with probabilities, α_0, α_1, α_2 and α_b, respectively. In calculation of these probabilities, it is assumed that at least one unit exists from each product. If at any point during the planning horizon, one of the products is depleted, these probabilities change. Let, $\alpha_1^{'}$ be the probability of buying Product 1, after depletion of Product 2. Similarly, $\alpha_2^{'}$ is defined as the probability that a customer buys Product 2 when Product 1 is not available. A customer may leave without a purchase with probability $\alpha_{01}^{'}$=1-$\alpha_1^{'}$, when Product 2 is depleted and with probability $\alpha_{02}^{'}$=1-$\alpha_2^{'}$, when Product 1 is depleted. Note that no bundle can be purchased when either of Product 1 or Product 2 is depleted.

We assume that the retailer knows the reservation price distributions and that he follows a mixed bundling strategy.

In the following sections we derive expressions to calculate the expected revenue for a given set of bundle and individual product prices. Then, these expressions are used to find the bundle and individual product prices that will maximize the expected revenue.

3.2 Problem Formulation

3.2.1 Preliminaries

Before deriving the purchasing probabilities,we introduce some preliminary concepts. Let R_1 and R_2 denote the reservation prices of the two products and $f_{R_1,R_2}(r_1, r_2)$ denote their joint probability density function, with corresponding marginals $f_{R_1}(x)$ and $f_{R_2}(x)$. Assuming bivariate normal distribution for the joint probability, we have

$$\begin{aligned} f_{R_1,R_2}(r_1, r_2) &= \frac{e^{-\theta(r_1,r_2)/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}, \quad , \\ \text{where} \quad \theta(r_1, r_2) &= \frac{1}{1-\rho^2}[(\frac{r_1-\mu_1}{\sigma_1})^2 - 2\rho(\frac{r_1-\mu_1}{\sigma_1})(\frac{r_2-\mu_2}{\sigma_2}) + (\frac{r_1-\mu_1}{\sigma_1})^2]. \end{aligned}$$

where ρ is the correlation between the reservation prices. Then, for $i = 1, 2$ we have the following marginal distributions for the reservation prices R_i, with mean μ_i and standard deviation σ_i:

$$f_{R_i}(r) = \frac{e^{-(r-\mu_i)^2/2\sigma_i^2}}{\sigma_i\sqrt{2\pi}}$$

Under bivariate normality, it is straightforward to show that the distribution of reservation price, R_b=R_1+R_2 for the bundle (consisting of one unit of each good) is normal with mean $\mu_b = \mu_1 + \mu_2$. The standard deviation of R_b is calculated as:

$$\begin{aligned}
\sigma_b &= (\sigma_1 + \sigma_2)\delta, \\
\text{where } \delta &= [1 - 2(1-\rho)\theta(1-\theta)]^{1/2}, \\
\text{and } \theta &= \sigma_1/(\sigma_1 + \sigma_2).
\end{aligned}$$

3.2.2 Purchasing Probabilities

3.2.2.1 Purchasing probabilities when there is no stockout

Suppose first that both products are available at the store so that the customer buys either a single product, or a bundle or leaves without any purchase. The probabilities of these events are given below:

Probability of No Purchase:

A customer will purchase nothing when her reservation prices for the individual products and the bundle are lower then their corresponding sales prices. Thus the probability of no purchase is given by,

$$\begin{aligned}
\alpha_0 &= P(R_1 < p_1, R_2 < p_2, R_b < p_b) \\
&= P(R_1 < p_1, R_2 < p_2, R_1 + R_2 < p_b) \\
&= P(R_1 < p_1, R_2 < \min\{p_2, p_b - R_1\}) \\
&= \int_{-\infty}^{p_1}\int_{-\infty}^{a_1} f_{R_1,R_2}(r_1, r_2)dr_1dr_2
\end{aligned}$$

where a_1=min$\{p_2, p_b - r_1\}$.

Probability of Purchasing Product 1:

A customer will purchase Product 1 if her surplus (the difference between the reservation price and sales price) is positive and larger than her surplus from Product 2 and the bundle. Thus the probability of purchasing Product 1 is given by,

$$\begin{aligned}
\alpha_1 &= P(R_1 > p_1, R_1 - p_1 > R_2 - p_2, R_1 - p_1 > R_b - p_b) \\
&= P(R_1 > p_1, R_1 - p_1 > R_2 - p_2, R_1 - p_1 > R_1 + R_2 - p_b) \\
&= P(R_1 > p_1, R_2 < \min\{R_1 - p_1 + p_2, p_b - p_1\}) \\
&= \int_{p_1}^{\infty} \int_{-\infty}^{a_2} f_{R_1,R_2}(r_1, r_2) dr_1 dr_2
\end{aligned}$$

where $a_2 = \min\{r_1 - p_1 + p_2, p_b - p_1\}$.

Probability of Purchasing Product 2:

A customer will purchase Product 2 if her surplus is positive and larger than her surplus from Product 1 and the bundle. Thus the probability of purchasing Product 2 is given by,

$$\begin{aligned}
\alpha_2 &= P(R_2 > p_2, R_2 - p_2 > R_1 - p_1, R_2 - p_2 > R_b - p_b) \\
&= P(R_2 > p_2, R_2 - p_2 > R_1 - p_1, R_2 - p_2 > R_1 + R_2 - p_b) \\
&= P(R_2 > p_2, R_1 < \min\{R_2 - p_2 + p_1, p_b - p_2\}) \\
&= \int_{p_2}^{\infty} \int_{-\infty}^{a_3} f_{R_1,R_2}(r_1, r_2) dr_2 dr_1
\end{aligned}$$

where $a_3 = \min\{r_2 - p_2 + p_1, p_b - p_2\}$.

Probability of Purchasing a Bundle:

A customer will purchase the bundle if her surplus is positive and larger than the surplus from Product 1 and Product 2. Thus the probability of the bundle purchase is given by,

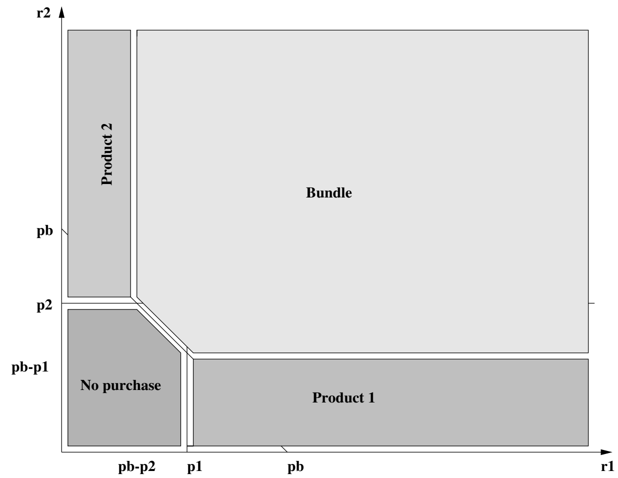

Figure 3.1: Purchasing Probabilities in Mixed Bundling Strategy

$$\begin{aligned}\alpha_b &= P(R_b > p_b, R_b - p_b > R_1 - p_1, R_b - p_b > R_2 - p_2)\\ &= P(R_b > p_b, R_1 + R_2 - p_2 > R_1 - p_1, R_1 + R_2 - p_b > R_2 - p_2)\\ &= P(R_1 > p_b - p_2, R_2 > \max\{p_b - R_1, p_b - p_1\})\\ &= \int_{p_b - p_2}^{\infty}\int_{a_4}^{\infty} f_{R_1,R_2}(r_1, r_2)dr_1dr_2\end{aligned}$$

where a_4=max $\{p_b - r_1, p_b - p_1\}$

Purchasing probabilities are depicted in Figure 3.1.

3.2.2.2 Probabilities when there is a stockout

The above probabilities are valid when both products are available. At any point during the planning horizon one of the products can be depleted. Then, an arriving customer can no longer purchase the bundle. She can either buy one unit from the remaining product or leave the store buying nothing. The probabilities for these cases are found as follows.

Product 1 is depleted before Product 2

Suppose that Product 1 has been depleted. An arriving customer either purchases Product 2 or leave the store without a purchase. If customer's reservation price for Product 2 is less than its price, she buys nothing. The probability for this case is given by,

$$\begin{aligned} \alpha_{02}^{'} &= P(R_2 < p_2) \\ &= \int_{-\infty}^{p_2} f_{R_2}(r_2)dr_2. \end{aligned}$$

If customer's reservation price for Product 2 is larger than its price, she buys Product 2. The probability for this case is given by,

$$\begin{aligned} \alpha_{2}^{'} &= P(R_2 > p_2) \\ &= \int_{p_2}^{\infty} f_{R_2}(r_2)dr_2 \\ &= 1 - \alpha_{02}^{'}. \end{aligned}$$

Product 2 is depleted before Product 1:

Suppose that Product 2 has been depleted. An arriving customer either purchases Product 1 or leave the store without a purchase. If customer's reservation price for Product 1 is less than its price, she buys nothing. The probability for this case is given by,

$$\begin{aligned} \alpha_{01}^{'} &= P(R_1 < p_1) \\ &= \int_{-\infty}^{p_1} f_{R_1}(r_1)dr_1. \end{aligned}$$

If customer's reservation price for Product 1 is larger than its price, she buys Product 1. The probability for this case is given by,

$$\begin{aligned} \alpha_{1}^{'} &= P(R_1 > p_1) \\ &= \int_{p_1}^{\infty} f_{R_1}(r_1)dr_1 \\ &= 1 - \alpha_{01}^{'} \end{aligned}$$

It has already been stated that customers arrive to the store according to a Poisson Process with an arrival rate of λ customers/period. When both products are available, sales of Product 1 (Product 2, bundle) follows a Poisson Process with the sale rate of $\lambda\alpha_1$ ($\lambda\alpha_2$, $\lambda\alpha_b$) products/period. Similarly, when Product 1 (Product 2) is depleted the sales of Product 2 (Product 1) sold follows Poisson Process with rate of $\lambda\alpha_2^{'}$ ($\lambda\alpha_1^{'}$) products/period.

Having calculated the purchasing probabilities, in the following section, we derive expressions to calculate the probabilities for different sale realizations during the planning horizon. Before proceeding with that, note that by assuming $p_1 + p_2 > p_b$, we justify our assumption that in the mixed bundling case the customer buys either Product 1 or Product 2 or bundle since she does not prefer to buy Product 1 and Product 2 together for a price equal to or larger than $p_1 + p_2$.

3.3 Sales Probabilities and the Objective Function

3.3.1 Sales Probabilities for Different Realizations

In order to find the expected revenue at the end of the planning horizon, we need to know how many units of Product 1, Product 2 and bundle are sold. Let N_1, N_2 and N_b be the number of Product 1, Product 2 and the bundle that are sold during the planning horizon and let

$$P(n_1, n_2, n_b) = P(N_1 = n_1, N_2 = n_2, N_b = n_b)$$

be the joint probability function for the sales, where the period starts with Q_1 of Product 1 and Q_2 of Product 2.

Before we write the the objective function we first derive the expressions for $P(n_1, n_2, n_b)$ for different realizations. The derivation of the joint probability function, $P(n_1, n_2, n_b)$, needs some careful consideration. Clearly, there are four possible structures for the period realizations: 1) No stockout in any products (Figure 3.2), 2) Stockout only in Product 2 (Figure 3.3), 3) Stockout only in Product 1 (Figure 3.4), and 4) Stockout in both products (Figure 3.5, Figure 3.6

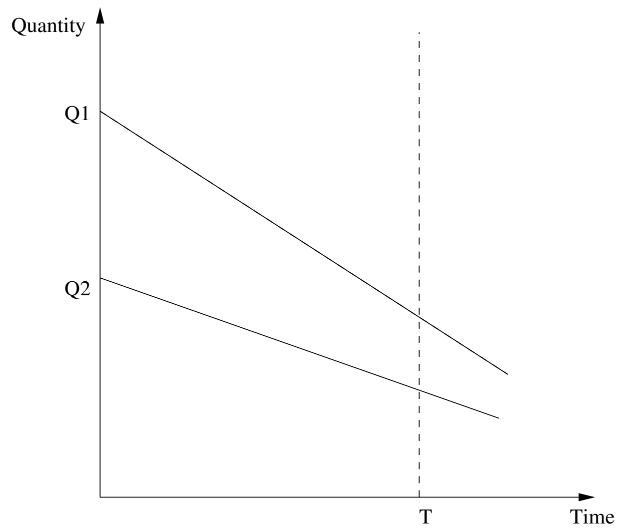

Figure 3.2: No Stockout in both Products

and Figure 3.7). When there is stockout in both products, one should also keep track of the order of the stockout times since the order changes the dynamics of the purchasing behavior of the customers.

Next, the calculation of $P(n_1, n_2, n_b)$ for each of the above cases is illustrated.

3.3.1.1 Case 1: No stockout in both products

No stockout occurs during the planning horizon. Hence, we have both Product 1 and Product 2 left. That is;

$$\begin{aligned} N_1 + N_b &< Q_1 \\ N_2 + N_b &< Q_2. \end{aligned}$$

The probability of a particular realization $N_1 = n_1, N_2 = n_2, N_b = n_b$ in this case can be calculated as,

$$P(n_1, n_2, n_b) = P \left\{ \begin{array}{ll} n_1 & \text{Product 1 purchases in } [0, T] \\ n_2 & \text{Product 2 purchases in } [0, T] \\ n_b & \text{bundle purchases in } [0, T] \end{array} \right\}$$

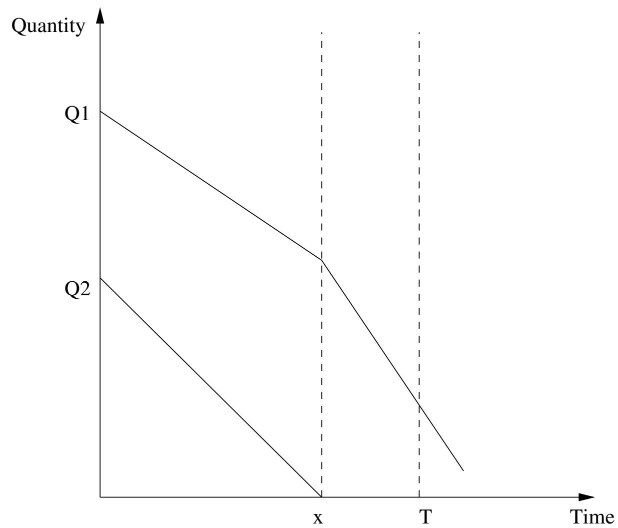

Figure 3.3: No Stockout in Product 1 and Stockout in Product 2

$$P(n_1, n_2, n_b) = \frac{e^{-\lambda\alpha_1 T}(\lambda\alpha_1 T)^{n_1}}{n_1!}\frac{e^{-\lambda\alpha_2 T}(\lambda\alpha_2 T)^{n_2}}{n_2!}\frac{e^{-\lambda\alpha_b T}(\lambda\alpha_b T)^{n_b}}{n_b!}$$

The first (second, third) expression is the probability that n_1 (n_2, n_b) units of Product 1 (Product 2, bundle) is sold in the interval [0,T].

Case 2: No Stockout in Product 1 and Stockout in Product 2

Product 2 is depleted during the planning horizon but at the end, there is at least one unit of Product 1 on hand. That is;

$$\begin{aligned} N_1 + N_b &< Q_1 \\ N_2 + N_b &= Q_2. \end{aligned}$$

In order to calculate the probability of a particular realization, $N_1 = n_1$, $N_2 = n_2$, $N_b = n_b$, we need to condition on the time at which the inventory of Product 2 is depleted. Let x be this time and let $N_{11}(x)$ be the number of Product 1 that is sold in the interval $[0, x]$. The last purchase that depletes the inventory of Product 2 can be either an individual purchase of Product 2 or a bundle purchase. If the last purchase is an individual purchase, n_2^{th} individual purchase of Product 2 is realized at time x and there are n_b bundle purchases in the interval $[0, x)$. If the last purchase is a bundle purchase, then n_b^{th} bundle purchase is realized at time x and there are n_2 individual Product 2 purchases in the interval $[0, x)$. In either case, if $N_{11}(x) = n_{11}$, there are n_{11} Product 1 purchases in the interval $[0, x]$ and $n_1 - n_{11}$ Product 1 purchases in the interval $(x, T]$. Thus, for this stockout situation, the probability of a particular realization n_1, n_2, n_b can be expressed as:

$$
\begin{aligned}
& P(n_1, n_2, n_b) \\
= \; & I(n_2 \geq 1) \int_0^T \sum_{n_{11}=0}^{n_1} P \left\{ \begin{array}{cl} n_{11} & \text{Product 1 purchases in } [0, x] \\ n_1 - n_{11} & \text{Product 1 purchases in } (x, T] \\ n_b & \text{bundle purchases in } [0, x) \end{array} \right\} f_{n_2, \alpha_2 \lambda}(x)\, dx \\
+ \; & I(n_b \geq 1) \int_0^T \sum_{n_{11}=0}^{n_1} P \left\{ \begin{array}{cl} n_{11} & \text{Product 1 purchases in } [0, x] \\ n_1 - n_{11} & \text{Product 1 purchases in } (x, T] \\ n_2 & \text{Product 2 purchases in } [0, x) \end{array} \right\} f_{n_b, \alpha_b \lambda}(x)\, dx
\end{aligned}
$$

where $f_{n_2, \alpha_2 \lambda}$ is the density of an Erlang n_2 random variable with rate $\alpha_2 \lambda$ and $f_{n_b, \alpha_b \lambda}$ is the density of an Erlang n_b random variable with rate $\alpha_b \lambda$. Thus, we have

$$
\begin{aligned}
& P(n_1, n_2, n_b) \\
= \; & I(n_2 \geq 1) \\
& \int_0^T \sum_{n_{11}=0}^{n_1} \frac{e^{-\lambda \alpha_1 x} (\lambda \alpha_1 x)^{n_{11}}}{n_{11}!} \frac{e^{-\lambda \alpha_1' (T-x)} (\lambda \alpha_1' (T-x))^{n_1 - n_{11}}}{(n_1 - n_{11})!} \frac{e^{-\lambda \alpha_b x} (\lambda \alpha_b x)^{n_b}}{n_b!} f_{n_2, \alpha_2 \lambda}(x) dx \\
+ \; & I(n_b \geq 1) \\
& \int_0^T \sum_{n_{11}=0}^{n_1} \frac{e^{-\lambda \alpha_1 x} (\lambda \alpha_1 x)^{n_{11}}}{n_{11}!} \frac{e^{-\lambda \alpha_1' (T-x)} (\lambda \alpha_1' (T-x))^{n_1 - n_{11}}}{(n_1 - n_{11})!} \frac{e^{-\lambda \alpha_2 x} (\lambda \alpha_2 x)^{n_2}}{n_2!} f_{n_b, \alpha_b \lambda}(x) dx
\end{aligned}
$$

where

$$
\begin{aligned}
f_{n_2, \alpha_2 \lambda}(x) &= \frac{(\lambda \alpha_2)^{n_2} x^{n_2 - 1} e^{-\lambda \alpha_2 x}}{(n_2 - 1)!} \\
f_{n_b, \alpha_b \lambda}(x) &= \frac{(\lambda \alpha_b)^{n_b} x^{n_b - 1} e^{-\lambda \alpha_b x}}{(n_b - 1)!}.
\end{aligned}
$$

In the above expression, the first integral corresponds to the probability of a particular realization n_1, n_2, n_b where the last purchase that depletes the inventory of Product 2 is an individual purchase. In order for this to happen, we should have at least one individual Product 2 sold. This is why an indicator function appears before the integral and it equals to 1 if we have at least one Product 2 purchase. The sum inside the integral is over all possible realizations of number of individual Product 1 sold up to x. Note that the time of the n_2^{th} Product 2

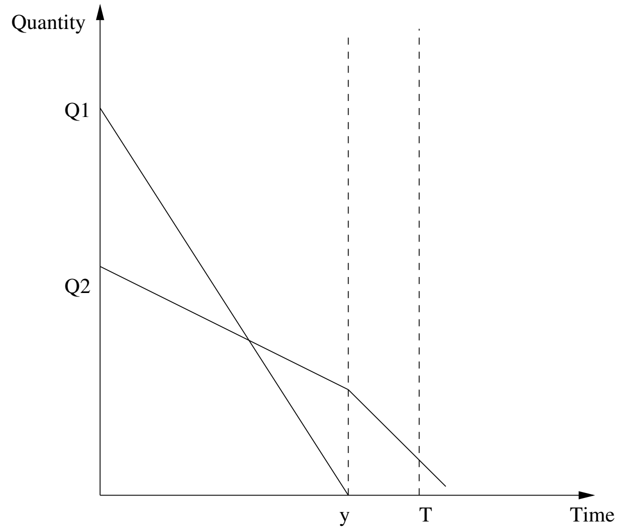

Figure 3.4: No Stockout in Product 2 and Stockout in Product 1

purchase is an n_2 Erlang random variable, since it corresponds to the waiting time until the n_2^{th} Poisson event with rate of $\alpha_2\lambda$.

The second integral corresponds to the probability of a realization where the last purchase that depletes the inventory of Product 2 is a bundle purchase. Note that we need to have at least one bundle purchase for this case to happen. Ww have $I(n_b \geq 1)$=1, if the number of bundles sold is greater than one. On the other hand, if n_b=0, we have $I(n_b \geq 1)$=0 and the second integral is not added to the sale probability.

Case 3: No Stockout in Product 2 and Stockout in Product 1

Product 1 is depleted during the planning horizon but at the end, there is at least one unit of Product 2 on hand. That is;

$$\begin{aligned} N_1 + N_b &= Q_1 \\ N_2 + N_b &< Q_2. \end{aligned}$$

Similar to the previous case, let y be the time at which inventory of Product 1 is depleted and $N_{21}(y)$ be the number of Product 2 that is sold in the interval [0,y]. The last purchase that depletes the inventory of Product 1 can be either an individual purchase of Product 1 or a bundle purchase. If the first event happens,

n_1^{th} individual purchase of Product 1 is realized at time y and there are n_b bundle purchases in the interval [0,y). If the last purchase is a bundle purchase, then n_b^{th} bundle purchase is realized at time y and there are n_1 individual Product 1 purchases in the interval [0,y). In either case, if $N_{21}(y)$=n_{21}, there are n_{21} Product 2 purchases in the interval [0,y] and $n_2 - n_{21}$ Product 2 purchases in the interval (y,T]. Thus, for this particular stockout situation, the probability of a particular realization n_1, n_2, n_b can be expressed as:

$$\begin{aligned} & P(n_1, n_2, n_b) \\ = \; & I(n_1 \geq 1) \int_0^T \sum_{n_{21}=0}^{n_2} P \left\{ \begin{array}{cl} n_{21} & \text{Product 2 purchases in } [0, y] \\ n_2 - n_{21} & \text{Product 2 purchases in } (y, T] \\ n_b & \text{bundle purchases in } [0, y) \end{array} \right\} f_{n_1,\alpha_1\lambda}(y)\; dy \\ + \; & I(n_b \geq 1) \int_0^T \sum_{n_{21}=0}^{n_2} P \left\{ \begin{array}{cl} n_{21} & \text{Product 2 purchases in } [0, y] \\ n_2 - n_{21} & \text{Product 2 purchases in } (y, T] \\ n_1 & \text{Product 1 purchases in } [0, y) \end{array} \right\} f_{n_b,\alpha_b\lambda}(y)\; dy \end{aligned}$$

Thus, we have

$$\begin{aligned} & P(n_1, n_2, n_b) \\ = \; & I(n_1 \geq 1) \\ & \int_0^T \sum_{n_{21}=0}^{n_2} [\frac{e^{-\lambda\alpha_2 y}(\lambda\alpha_2 y)^{n_{21}}}{n_{21}!}][\frac{e^{-\lambda\alpha_2'(T-y)}(\lambda\alpha_2'(T-y))^{n_2-n_{21}}}{(n_2-n_{21})!}][\frac{e^{-\lambda\alpha_b y}(\lambda\alpha_b y)^{n_b}}{n_b!}] f_{n_1,\alpha_1\lambda}(y) dy \\ + \; & I(n_b \geq 1) \\ & \int_0^T \sum_{n_{21}=0}^{n_2} [\frac{e^{-\lambda\alpha_2 y}(\lambda\alpha_2 y)^{n_{21}}}{n_{21}!}][\frac{e^{-\lambda\alpha_2'(T-y)}(\lambda\alpha_2'(T-y))^{n_2-n_{21}}}{(n_2-n_{21})!}][\frac{e^{-\lambda\alpha_1 y}(\lambda\alpha_1 y)^{n_1}}{n_1!}] f_{n_b,\alpha_b\lambda}(y) dy \end{aligned}$$

3.3.1.2 Case 4: Stockout in both Products

In this case, both products are depleted during the planning horizon. That is;

$$\begin{aligned} N_1 + N_b &= Q_1 \\ N_2 + N_b &= Q_2. \end{aligned}$$

The depletion of both products can happen in three different ways: Product 1 can deplete before Product 2, Product 2 can deplete before Product 1 or both products can deplete together. Next, each of these cases are analyzed.

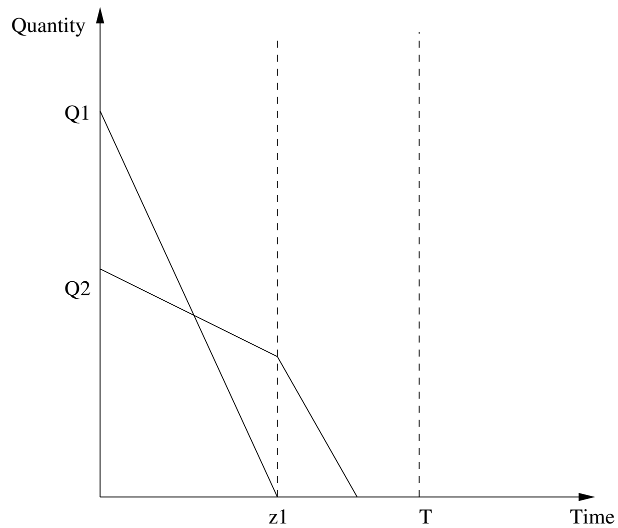

Figure 3.5: Product 1 is Depleted Before Product 2

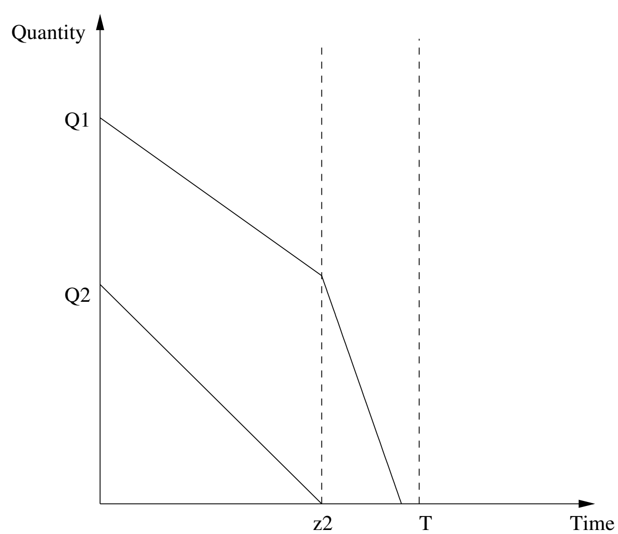

Figure 3.6: Product 2 is Depleted Before Product 1

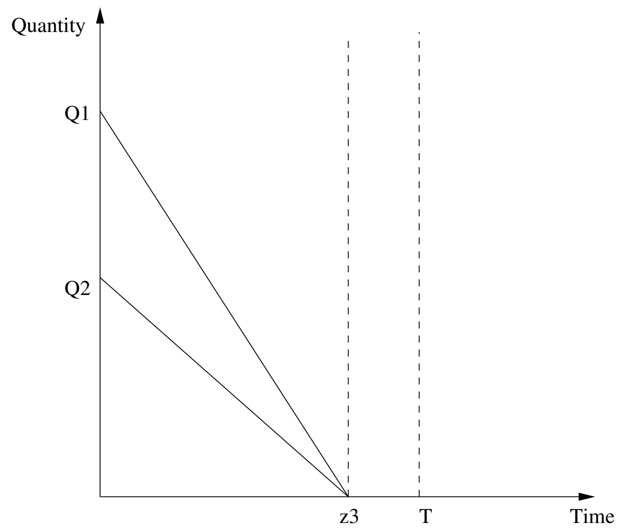

Figure 3.7: Both Products are Depleted Together

Case 4.1: Product 1 is Depleted Before Product 2

For the calculation of the probability of particular realization, $N_1=n_1$, $N_2=n_2$, $N_b=n_b$, we need to condition on the time at which the inventory of Product 1 is depleted. Let z_1 be this time and let $N_{21}^{'}(z_1)$ be the number of Product 2 that is sold in the interval $[0,z_1]$. The last purchase that depleted the inventory of Product 1 can be either an individual purchase of Product 1 or a bundle purchase. If the last purchase is an individual purchase, n_1^{th} individual purchase of Product 1 is realized at time z_1 and there are n_b bundle purchases in the interval $[0,z_1)$. If the last purchase is a bundle purchase, then n_b^{th} bundle purchase is realized at time z_1 and there are n_1 individual Product 1 purchases in the interval $[0,z_1)$. In either case, if $N_{21}^{'}(z_1)=n_{21}^{'}$, there are $n_{21}^{'}$ Product 2 purchases in the interval $[0,z_1]$. The maximum value that $n_{21}^{'}$ can take is n_2-1, since we have to ensure that Product 2 has not depleted before Product 1. Also, in order to make sure that Product 2 is depleted, we must have at least n_2-n_{21} Product 2 purchases from z_1 till the end of the planning horizon. Thus, for this particular stockout situation, the probability of a particular realization n_1, n_2, n_b can be expressed as:

$$
\begin{aligned}
& P_A(n_1, n_2, n_b) \\
= \; & I(n_1 \geq 1) \\
& \int_0^T \sum_{n_{21}^{'}=0}^{n_2-1} P \left\{ \begin{array}{cl} n_{21}^{'} & \text{Product 2 purchases in } [0, z_1] \\ \text{at least } \; n_2 - n_{21}^{'} & \text{Product 2 purchases in } (z_1, T] \\ n_b & \text{bundle purchases in } [0, z_1) \end{array} \right\} f_{n_1,\alpha_1\lambda}(z_1) \; dz_1 \\
+ \; & I(n_b \geq 1) \\
& \int_0^T \sum_{n_{21}^{'}=0}^{n_2-1} P \left\{ \begin{array}{cl} n_{21}^{'} & \text{Product 2 purchases in } [0, z_1] \\ \text{at least } \; n_2 - n_{21}^{'} & \text{Product 2 purchases in } (z_1, T] \\ n_1 & \text{Product 1 purchases in } [0, z_1) \end{array} \right\} f_{n_b,\alpha_b\lambda}(z_1) \; dz_1
\end{aligned}
$$

Thus, we have

$$
\begin{gathered}
P_A(n_1, n_2, n_b) \\
= I(n_1 \geq 1) \\
\int_0^T \sum_{n_{21}^{'}=0}^{max(n_2-1,0)} [\frac{e^{-\lambda\alpha_2 z_1}(\lambda\alpha_2 z_1)^{n_{21}^{'}}}{n_{21}^{'}!}] \left[\sum_{k=n_2-n_{21}^{'}}^{\infty} [\frac{e^{-\lambda\alpha_2^{\prime}(T-z_1)}(\lambda\alpha_2^{\prime}(T-z_1))^k}{k!}]\right] [\frac{e^{-\lambda\alpha_b z_1}(\lambda\alpha_b z_1)^{n_b}}{n_b!}] f_{n_1,\alpha_1\lambda}(z_1) dz_1 \\
+ I(n_b \geq 1) \\
\int_0^T \sum_{n_{21}^{'}=0}^{max(n_2-1,0)} [\frac{e^{-\lambda\alpha_2 z_1}(\lambda\alpha_2 z_1)^{n_{21}^{'}}}{n_{21}^{'}!}] \left[\sum_{k=n_2-n_{21}^{'}}^{\infty} [\frac{e^{-\lambda\alpha_2^{\prime}(T-z_1)}(\lambda\alpha_2^{\prime}(T-z_1))^k}{k!}]\right] [\frac{e^{-\lambda\alpha_1 z_1}(\lambda\alpha_1 z_1)^{n_1}}{n_1!}] f_{n_b,\alpha_b\lambda}(z_1) dz_1
\end{gathered}
$$

In the above expression, the first integral corresponds to the probability of a particular realization n_1, n_2, n_b given that the last purchase that depleted the inventory of Product 1 is an individual product. In order for this to happen, we should have at least one individual Product 1 sold. This fact is reflected by an indicator function before the integral. The integral is taken over all possible values of z_1, i.e., from 0 to T. The sum inside the integral is over all possible realizations of number of individual Product 2 sold up to z_1. The expression inside the first bracket is the probability that $n_{21}^{'}$ units of Product 2 is sold up to z_1. The second bracket is the probability that at least $n_2 - n_{21}^{'}$ units of Product 2 is sold after z_1 until the end of the planning horizon. The third bracket is the probability that n_b units of bundle is sold up to z_1. The time of the n_1^{th} Product 1 purchase is an n_1 Erlang random variable, since it corresponds to the waiting time until the n_1^{th} Poisson event where the rate is $\alpha_1\lambda$.

The second integral corresponds to the probability of a particular realization n_1, n_2, n_b given that the last purchase that depleted the inventory of Product 1 is a bundle purchase. Different from the first integral, the third bracket is the probability that n_1 units of bundle is sold up to z_1. The time of the n_b^{th} Product 1 purchase is an n_b Erlang random variable, since it corresponds to the waiting time until the n_b^{th} Poisson event where the rate is $\alpha_b\lambda$.

Case 4.2: Product 2 is Depleted Before Product 1

For the calculation of the probability of particular realization, $N_1{=}n_1$, $N_2{=}n_2$, $N_b{=}n_b$, we need to condition on the time at which the inventory of Product 2 is depleted. Let z_2 be this time and let $N_{11}^{'}(z_2)$ be the number of Product 1 that is sold in the interval $[0,z_2]$. The last purchase that depleted the inventory of Product 2 can be either an individual purchase of Product 2 or a bundle purchase. If the last purchase is an individual purchase, n_2^{th} individual purchase of Product 2 is realized at time z_2 and there are n_b bundle purchases in the interval $[0,z_2)$. If the last purchase is a bundle purchase, then n_b^{th} bundle purchase is realized at time z_2 and there are n_2 individual Product 2 purchases in the interval $[0,z_2)$. In either case, if $N_{11}^{'}(z_2){=}n_{11}^{'}$, there are $n_{11}^{'}$ Product 1 purchases in the interval $[0,z_2]$. The maximum value that $n_{11}^{'}$ can take is n_1-1, since we have to ensure that Product 1 has not depleted before Product 2. Also, in order to make definite the depletion of Product 1, we must have at least n_1-n_{11} Product 1 purchases from z_2 till the end of the planning horizon. Thus, for this particular stockout situation, the probability of a particular realization n_1, n_2, n_b can be expressed as:

$$
\begin{aligned}
& P_B(n_1, n_2, n_b) \\
= \;& I(n_2 \geq 1) \\
& \int_0^T \sum_{n_{11}^{'}=0}^{n_1-1} P \left\{ \begin{array}{cl} n_{11}^{'} & \text{Product 1 purchases in } [0, z_2] \\ \text{at least } \; n_1 - n_{11}^{'} & \text{Product 1 purchases in } (z_2, T] \\ n_b & \text{bundle purchases in } [0, z_2) \end{array} \right\} f_{n_2,\alpha_2\lambda}(z_2)\, dz_2 \\
+ \;& I(n_b \geq 1) \\
& \int_0^T \sum_{n_{11}^{'}=0}^{n_1-1} P \left\{ \begin{array}{cl} n_{11}^{'} & \text{Product 1 purchases in } [0, z_2] \\ \text{at least } \; n_1 - n_{11}^{'} & \text{Product 1 purchases in } (z_2, T] \\ n_2 & \text{Product 2 purchases in } [0, z_2) \end{array} \right\} f_{n_b,\alpha_b\lambda}(z_2)\, dz_2
\end{aligned}
$$

Thus, we have

$$P_B(n_1, n_2, n_b)$$

$$= I(n_2 \geq 1)$$

$$\int_0^T \sum_{n_{11}^{'}=0}^{max(n_1-1,0)} [\frac{e^{-\lambda\alpha_1 z_2}(\lambda\alpha_1 z_2)^{n_{11}^{'}}}{n_{11}^{'}!}] \left[\sum_{k=n_1-n_{11}^{'}}^{\infty} [\frac{e^{-\lambda\alpha_1^{\prime}(T-z_2)}(\lambda\alpha_1^{\prime}(T-z_2))^k}{k!}]\right] [\frac{e^{-\lambda\alpha_b z_2}(\lambda\alpha_b z_2)^{n_b}}{n_b!}] f_{n_2,\alpha_2\lambda}(z_2) dz_2$$

$$+I(n_b \geq 1)$$

$$\int_0^T \sum_{n_{11}^{'}=0}^{max(n_1-1,0)} [\frac{e^{-\lambda\alpha_1 z_2}(\lambda\alpha_1 z_2)^{n_{11}^{'}}}{n_{11}^{'}!}] \left[\sum_{k=n_1-n_{11}^{'}}^{\infty} [\frac{e^{-\lambda\alpha_1^{\prime}(T-z_2)}(\lambda\alpha_1^{\prime}(T-z_2))^k}{k!}]\right] [\frac{e^{-\lambda\alpha_2 z_2}(\lambda\alpha_2 z_2)^{n_2}}{n_2!}] f_{n_b,\alpha_b\lambda}(z_2) dz_2$$

Case 4.3: Both Products are Depleted Together

The probability of a particular realization, we need to condition on the time at which the inventories of Product 1 and Product 2 are depleted. Let z_3 be this time. In order that this case happens, the retailer must sold the last unit of Product 1 and the last unit of Product 2 in a bundled form. The probability of selling those last units as individual products at the same time is zero. Therefore, we need to have the last customer buying a bundle. Thus, for this particular stockout situation, the probability of a particular realization n_1, n_2, n_b can be expressed as:

$$P_C(n_1, n_2, n_b) = \int_0^T P \left\{ \begin{array}{ll} n_1 & \text{Product 1 purchases in } [0, z_3] \\ n_2 & \text{Product 2 purchases in } [0, z_3] \end{array} \right\} f_{n_b,\alpha_b\lambda}(z_3)\ dz_3$$

Thus, we have

$$P_C(n_1, n_2, n_b) = \int_0^T [\frac{e^{-\lambda\alpha_1 z_3}(\lambda\alpha_1 z_3)^{n_1}}{n_1!}][\frac{e^{-\lambda\alpha_2 z_3}(\lambda\alpha_2 z_3)^{n_2}}{n_2!}] f_{n_b,\alpha_b\lambda}(z_3) dz_3$$

The probabilities that n_1 units of Product 1 and n_2 units of Product 2 are sold during [0,z_3] are shown in the first and second bracket, respectively. Note that the time of the n_b^{th} bundle purchase is an n_b Erlang random variable, since it corresponds to the waiting time until the n_b^{th} Poisson event where the rate is $\alpha_b\lambda$.

When both products deplete, the probability of a particular realization, N_1=n_1, N_2=n_2, N_b=n_b, is the sum of probabilities P_A, P_B and P_C.

$$P(n_1, n_2, n_b) = P_A(n_1, n_2, n_b) + P_B(n_1, n_2, n_b) + P_C(n_1, n_2, n_b)$$

As an example, consider the case Q_1=Q_2=2. When N_1=1, N_2=1 and N_b=1, both products deplete. If the last product sold is a unit of Product 2, $P(1, 1, 1)$ equals

$P_A(1,1,1)$,. Similarly, $P(1,1,1)$ equals $P_B(1,1,1)$, if the last product sold is a unit of Product 1. We have $P(1,1,1)$=$P_C(1,1,1)$, if the last customer that makes a purchase buys the bundle. The overall probability of $P(1,1,1)$ is the sum of these three probabilities.

3.3.2 Objective Function

The expected revenue is:

$$E(R) = \sum_{n_1}\sum_{n_2}\sum_{n_b}(p_1n_1 + p_2n_2 + p_bn_b)P(n_1, n_2, n_b)$$

where $P(n_1, n_2, n_b)$ corresponds to the probability that such a mixture is sold. In the revenue expression, sums are over all possible combinations of n_1, n_2 and n_b. It is obvious that the total number of Product 1 and bundles sold cannot be larger than Q_1 and the total number of Product 2 and bundles sold cannot be larger than Q_2. The total number of (n_1, n_2, n_b) combinations is found via the following formula:

$$\sum_{n_b=0}^{min(Q_1,Q_2)} (Q_1 - n_b + 1)(Q_2 - n_b + 1)$$

Maximum number of bundles that can be sold is $\min(Q_1, Q_2)$. For each possible realization of N_b, there are $Q_1 - n_b + 1$ possible integer values that N_1 can taken. (One is added to account for the possibility of selling zero Product 1 during the season). Similarly, N_2 can take $Q_2 - n_b + 1$ different values for each realization of N_b. As an example, for Q_1=3 and Q_1=4, we have total of 40 (n_1, n_2, n_b) combinations.

For mixed bundling case, we consider the following optimization problem:

$$\begin{array}{ll} \max & \displaystyle\sum_{n_1}\sum_{n_2}\sum_{n_b}(p_1n_1 + p_2n_2 + p_bn_b)P(n_1, n_2, n_b) \\ \text{s.t.} & p_1 + p_2 \geq p_b \end{array}$$

Note that $P(n_1, n_2, n_b)$ is also a function of p_1, p_2 and p_b.

3.3.3 Unbundling and Pure Bundling Cases

In order to compare the performance of the mixed bundling strategy, we derive similar expressions for unbundling and pure bundling strategies.

3.3.3.1 Unbundled Case

If the retailer aims to maximize his revenue by following the unbundling strategy, he sells Product 1 and Product 2 without providing a bundle option. In this case an arriving customer decides buying nothing, Product 1 or Product 2 or both of them at a price equals to p_1+p_2. The assumptions stated in mixed bundling case are still valid.

The derivations for the unbundled case is similar to the mixed bundling case except that purchasing probabilities can now be simplified as below:

$$\begin{aligned}\alpha_0 &= P(R_1 \leq p_1, R_2 \leq p_2),\\ \alpha_1 &= P(R_1 \geq p_1, R_2 \leq p_2),\\ \alpha_2 &= P(R_1 \leq p_1, R_2 \geq p_2),\\ \alpha_b &= P(R_1 \geq p_1, R_2 \geq p_2).\end{aligned}$$

3.3.3.2 Pure Bundling Case

Now consider the case where the retailer follows a pure bundling strategy: he sells Product 1 and Product 2, only in a bundled form. If the retailer follows the pure bundling strategy, the individual products left at the end of the period are useless. Therefore, it would be meaningful to have Q_1=Q_2, at the beginning. Let Q_1=Q_2=Q_b, where Q_b is the number of bundles available for sale at beginning of the planning horizon (If $Q_1 \neq Q_2$, then we can make the following definition min(Q_1, Q_2)=Q_b). By comparing her reservation price for the bundle with the price of the bundle, an arriving customer buys a bundle or leaves the store without buying anything. Reservation price, R_b, for the bundle is Normal with mean μ_b

and standard deviation σ_b. This case simply corresponds to a single product case with reservation price, R_b=R_1+R_2.

The assumptions stated in mixed bundling section are still valid for pure bundling case. A customer buys the bundle if her surplus from the bundle is positive. That is;

$$\begin{aligned}\alpha_b &= P(r_b > p_b) \\ &= \int_{p_b}^{\infty} f_{R_b}(r_b)dr_b\end{aligned}$$

In order to find the expected revenue at the end of the planning horizon, we need to know how many bundles are sold. Let n_b be the number of bundles sold during the planning horizon and $P(n_b)$ be the probability of selling n_b units of bundle during the planning horizon, whose length is T.

During the planning horizon two events are possible: No stockout in bundle and stockout in bundle. Next, for each of these cases, the calculation of $P(n_b)$ is illustrated:

Case 1: No stockout in bundle

At the end of the planning horizon, at least one bundle exist. Hence, no stockout in bundle occurs. That is;

$$N_b < Q_b.$$

The probability for this case is given by,

$$P(n_b) = [\frac{e^{-\lambda\alpha_b T}(\lambda\alpha_b T)^{n_b}}{n_b!}]$$

Case 2: Stockout in bundle

The probability that there are at least Q_b demands for the bundle is given by:

$$P(Q_b) = \sum_{x=Q_b}^{\infty} \frac{e^{-\lambda\alpha_b T}(\lambda\alpha_b T)^{x}}{x!}.$$

The expected revenue for pure bundling case is:

$$E(R) = \sum_{n_b=0}^{Q_b} (p_b n_b) P(n_b).$$

Chapter 4

NUMERICAL RESULTS

In this chapter, we present the results of our numerical study. The purpose of our numerical study is to assess the impact of various factors on pricing decisions in the presence of bundling. Our primary focus is the mixed bundling strategy and the factors that we consider are the correlation between the reservation price distributions, the mean and the variance of the reservation price distributions, starting inventory levels and the customer arrival rate. We particularly study the impact of these factors on expected revenues, bundle prices and individual product prices. We also investigate the conditions under which mixed bundling strategy is most useful. For this purpose, we compare the mixed bundling strategy with pure bundling and unbundling strategies under the same settings.

In Section 4.1, we test the performance of the mixed bundling strategy for fixed values of individual product prices. The performance of this strategy, when the bundle price and the prices of Product 1 and Product 2 are jointly optimized, is explored in Section 4.2. In Section 4.3, we test the performance of unbundling and pure bundling strategies. The chapter ends with the comparison of the three strategies.

4.1 Fixed p_1 and p_2

In order to better understand the impact of price on revenues, we first start with the case of fixed p_1 and p_2. There may be some cases, where the prices of individual products are already determined. The prices may be fixed externally by government or other agencies. In addition, the retailer may decide to offer the bundle, when he has already announced the prices of the individual products (perhaps with some lowest price guarantees). Therefore, he would not be willing to change the prices of individual products. Due to all these reasons, the retailer's only control on sales could be the bundle price. We also note that studying the case where only the bundle prices are optimized may help us understand a more complex case where all prices are jointly optimized.

Determining the optimal bundle price to charge is a complex task. The retailer should take into account several factors such as the distribution of customer reservation prices, the customer arrival rate to the store, the amount of inventory available at the beginning of the planning horizon and the prices of individual products.

4.1.1 Equal Individual Product Prices

In this section, we study the case when both products are identically distributed, i.e., their means and standard deviations are the same. The customer arrival rate equals to the total number of individual products available. The prices of individual products are equal. In our analysis we assume that Q_1=Q_2=10, μ_1=μ_2=15, σ_1=σ_2=2 and λ=20. Throughout the text, we refer to these values as the base case data. The objective is to find the bundle price that maximizes the expected revenue. The optimal value of the bundle price is searched over a fixed set in which prices are taken with 0.25 increments. We study five different values for p_1=p_2, which are 17, 16, 15, 14 and 13 and the results in Table A.1 are obtained.

We first note that for all individual prices, probability of an individual product

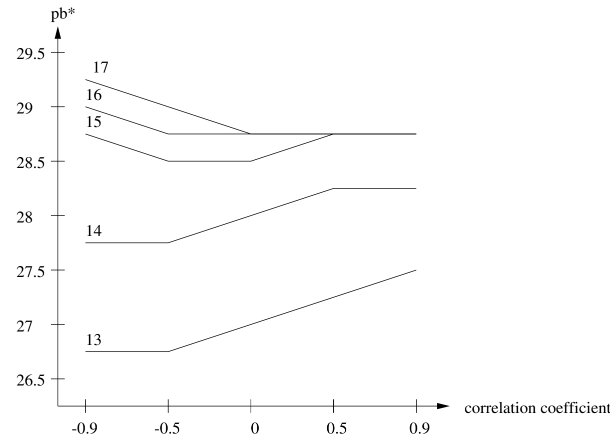

Figure 4.1: Optimal Bundle Price when Individual Products are Fixed

purchase ($\alpha_1 = \alpha_2$) as well as expected individual product sales ($E(n_1) = E(n_2)$) are decreasing functions of the correlation coefficient. This is expected since as the correlation increases, the customers who are already willing to buy one of the products are more willing to buy the other product as well, and thus the bundle becomes an option more attractive than only one of the individual products. This is also reflected in increased bundle sales ($E(n_b)$) as the correlation coefficient increases (despite the fact that the bundle price is increased in some cases).

The way the correlation coefficient impacts the optimal bundle price and the optimal expected revenues depends on the individual product prices (see Figure 4.1 for the impact of correlation coefficient on the optimal bundle price). For high individual product prices, the retailer in fact is having difficulty in selling the products individually. Most customers would not buy the products individually if the bundle option is not offered. The retailer in this case would like to use bundling to increase its sales, and offer a bundle price that will trigger non-buyers to buy the bundle. This can be done best if the variance of the bundle reservation price is smallest. This way, the retailer can improve sales by small reductions in the bundle price. The variance of the bundle reservation decreases, as the correlation coefficient decreases, which explains why the optimal expected revenue and the optimal bundle price are decreasing functions of the correlation coefficient high individual product prices.

For low individual product prices, the retailer is not having any difficulty in selling the products individually. Most customers would buy the products individually if the bundle option is not offered. The retailer in this case would like to move some of these customers from buying individual products to buying the bundle. This is easier when a customer who already intends to buy one of the products values the other product highly as well (positive correlation). This explains why the optimal expected revenue and the optimal bundle price are increasing functions of the correlation coefficient for low individual product prices.

When the individual product prices are moderate, the impact of the correlation coefficient is U-shaped. As correlation coefficient goes from negative to positive, initially we observe a decrease, then an increase in the optimal revenue. For zero correlation coefficient value, we see the smallest revenues and the optimal bundle price equals exactly to the sum of the prices of Product 1 and Product 2. When we increase the correlation coefficient, this helps to find customers who would buy the bundle (rather than individual products) and pay a price even higher price than the sum of individual prices. So we see an increase in revenues. When we decrease the correlation coefficient, this helps to find customer who would buy the bundle now (rather than buying nothing), since the optimal bundle price is smaller than the sum of individual product prices. So, we again see an increase in revenues.

Table A.1 shows the optimal values of the bundle price for different individual prices. Figure 4.2 shows how the expected revenue changes with the bundle price, for three different correlation values (ρ=-0.9, 0.0, 0.9) and $p_1 = p_2 = 15$.

We observe that for high individual product prices, the probability of no purchase, α_0 is high and this probability decreases as individual product prices decrease. In addition, α_0 is an increasing function of correlation coefficient. When reservation prices are positively correlated, customers have similar valuations of both products, therefore they will either buy the bundle or leave the store without any purchase. We note that the probability of individual product purchase is very low for positive ρ values, even in the case of low individual product prices.

We note that for low individual product prices, if given flexibility, it may be

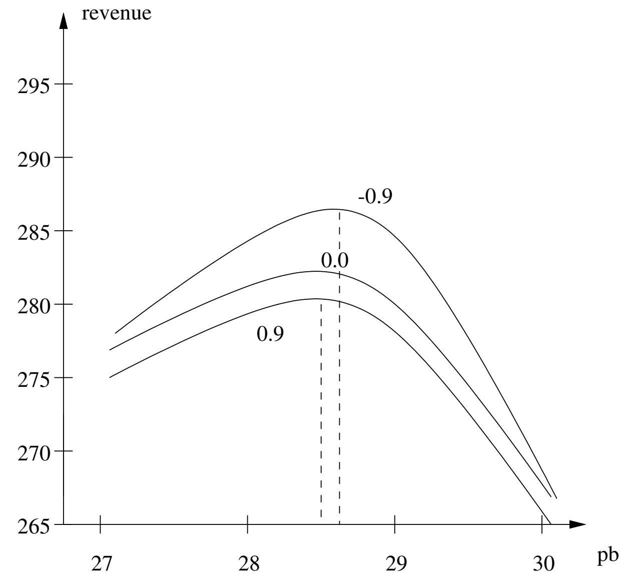

Figure 4.2: Revenue vs. Bundle Price, p_1=p_2

optimal to charge a bundle price more than the sum of individual product prices. The reason is the following. Consider a customer with reservation prices r_1 and r_2. If the retailer is offering prices p_1, p_2 and p_b, it is possible that the utility that this customer gets from the bundle (r_1+r_2-p_b) is more than the utility that she would get from the individual products (r_1-p_1 or r_2-p_2). Therefore, this customer may chose to buy the bundle even though the price of the bundle is more than the sum of individual product prices. Considering such customers, the retailer may find that a bundle price that is higher than sum of the individual product prices optimizes his revenues. Observe that this is possible since we assume that a customer cannot purchase Product 1 and Product 2 separately (outside of the bundle) and the individual product prices are exogenous. In practice, the retailer cannot force the customer to pay more than $p_1 + p_2$ and would simply not offer the bundle if p_1 and p_2 are too low.

4.1.2 Different Individual Product Prices

In this section, we consider the case where the individual product prices are not equal. The base case data is used again. The customer reservation prices for Product 1 and Product 2 are the same. Note that, the product prices are set externally.

First, we analyze the case where one of the individual product prices is set below the mean of the customer reservation price. In the first two parts of Table A.2, p_1 is less than μ_1 and p_2=μ_2. Due to its low price compared to the price of Product 2, the expected number of Product 1 sold is more than that of Product 2. Customers prefer to buy less individual products as correlation coefficient goes from negative to positive. On the other hand, expected number of bundles sold always increases in this direction. The reason is that, for negative correlation, we have customers that value the individual products differently. The bundle is not an attractive option for these customers. Therefore the number of bundles sold is less for negative ρ values. As correlation goes from negative to positive, the optimal bundle price decreases first, then it increases. When ρ=-0.9, the desire to buy a bundle is less. Setting a low bundle price will not encourage the customers to buy a bundle. Therefore, the meaningful action is to set it high, in order at least to extract the surplus of the customers purchasing a bundle. On the other hand, when the reservation prices are highly correlated, i.e. ρ=0.9, customers want both Product 1 and Product 2. Charging high bundle price will not dissuade them from buying it. The increased number of bundles as ρ goes from negative to positive is an evidence of this. For correlation coefficient values in between, the retailer should set the bundle price carefully. He should not lose the customers buying the bundle, although they are not so desirous to do so. At the same time, the retailer should not lose the surplus from the customers who are at least more willing to purchase the bundle. For these ρ values, the optimal bundle price is smaller compared to ρ=-0.9 and ρ=0.9. The revenue is higher for negative correlation coefficient values not only because of the higher number of individual products sold but also due to high bundle price charged.

Consider the case, where one of the individual product prices is set above the mean of the customer reservation price. In the last two parts of Table A.2, $p_1 > \mu_1$ and p_2=μ_2. For the individual product prices, p_1=16 and p_2=15, the results are very similar to previous case; the expected number of individual product purchases decrease, the expected number of bundle purchases increase and the optimal bundle price shows initially decreasing then increasing pattern when correlation coefficient goes from negative to positive. As usual, the expected revenue decreases in this direction. A customer buys an individual product (the bundle) with higher (lower) probabilities for negative values of ρ (these are reflected also by the expected number of purchases). On the other hand, for p_1=17 and p_2=15, the probability of a bundle purchase decrease as correlation coefficient becomes positive, while the expected number of bundles sold incerese. This is an interesting result, since it is expected that the number of bundles sold, $E(n_b)$, and the probability of a bundle purchase, α_b, to show a similar trend. All purchase probabilities, i.e. $\alpha_1, \alpha_2, \alpha_b$ decrease when the correlation coefficient goes from negative to positive. However, the decrease in the probabilities of individual product purchases are sharp, this is why the bigger portion of the products are sold in a bundled form. A customer prefer the bundle when her reservation prices for products are positively correlated. The retailer can make use of this by charging a high price for bundle. However, for p_1=17, p_2=15 case, we observe that the bundle price is not so high compared to the sum of individual product prices (ex. for ρ=0.9, $p_1 + p_2$=32> p_b^*=28.5). As a result, customers opt for bundle.

4.1.3 The Impact of Mean Reservation Price

Next, the case where the means of the customer reservation prices for individual products are different will be explored.

In Table A.3, the prices of Product 1 and Product 2 are 15 and the mean reservation prices are different. When μ_1 increases, the probability of Product 1 purchase and the expected number of Product 1 sold increase. The customers compare their utilities when making purchase ($r_1 - p_1$ versus $r_2 - p_2$ versus $r_1 + r_2 - p_b^*$). As customers willingness toward one product (Product 1) increases, the probability that a customer will choose the other individual product (Product 2) decreases.

Increase in μ_1 implies increase in μ_b, the mean of customer reservation price distribution for bundle. Although, customers are expected to buy more bundles for large μ_b values, the results do not confirm this expectation. It is due to the value of optimal bundle price. As μ_1 increases p_b^* also increases. However, there is not a distinct trend in p_b^* when correlation coefficient changes. As usual, the expected number of bundles sold incerese when we go from negative to positive correlation. The expected revenue increases with μ_1.

In Table A.4, we consider the case when the means of the customer reservation price distributions add up to 30. Both of the individual product prices equal to 15. We analyze three different (μ_1, μ_2) combinations: (13, 17), (14, 16) and (15, 15). For the first two combinations, the average reservation prices are unequal, the retailer faces unequal demands. On the other hand, the initial inventories are equal, i.e., Q_1=Q_2. As in the case of unequal product prices, the individual demand for Product 1 and Product 2 would leave the remaining inventory unbalanced which is of no use for the bundle. Therefore, the retailer would try to sell as much bundle as possible, synchronizing the consumption of individual products. Selling more bundles will help to prevent the imbalance in demand. We see lower bundle prices, as the difference between the means of the reservation prices increases. Having unbalanced demand and balanced supply has also other consequences. Consider the case where ρ=-0.9. For (μ_1, μ_2)=(13, 17), most of the products (80.0%) are sold as bundle. This percenatage decreases as μ_2-μ_1 decrease (71.4% for (14, 16), 56.6% for (15, 15)). For negatively correlated reservation prices, the customers prefer individual products more than the bundle. However, the retailers' effor to balance the supply and demand results in the largest number of bundle sales for the case with maximum mean difference. When the correlation coefficient incerases to positive values, we observe the largest percenatge increase in the number of bundles sold for (15, 15) mean combination. When ρ=0.9 and (μ_1, μ_2)=(15, 15), 94.4% of the products are sold in bundled form (it is 89.1% for (13, 17) and 90% for (14, 16)). For these ρ values, the customers are already willing to buy the bundle. The decreased number of individual products sold is an evidence for this. The retailer does not need to lower his price for the bundle.

4.1.4 The Impact of Standard Deviation

In this section, we explore the case where the standard deviation of the customer reservation prices for individual products are different.

In Table A.5, we analyze the case for $p_1 = p_2$=15. The initial observation is that, the product with smaller standard deviation for reservation price is purchased more than the other product. Increase in σ_1 implies an increase in σ_b. Therefore as σ_1 increases, the probability of bundle purchase decreases. We observe a decrease in the expected revenue as σ_1 increases. The optimal bundle price shows a decreasing trend as σ_1 increases. The retailer needs to set lower prices for the bundle, as the variance for the reservation price of the bundle increases. As correlation coefficient between the reservation prices changes, we observe the usual results. As ρ goes from negative to positive, the probability of individual product purchase as well as the number of individual products sold decreases. We have a decrease in the probability of bundles sold in this direction. However, the expected number of bundles sold increases. This is due to the drastic decrease in α_1 and α_2. As usual, the expected revenue decreases when ρ becomes positive. When we have, $p_1 = p_2$=15, the highest revenue is obtained for (σ_1, σ_2)=(1, 2) pair.

In Table A.6, we study the case where the standard deviations of the reservation price distributions are equal and take values from 1 to 4. The individual prices are both equal to 15. We first note that optimal revenues decline as standard deviations are increased. This is expected since with higher standard deviations, the retailer knows less about the product valuations of customers. We also note that, as the standard deviations increase, the retailer also sells less bundles. This is because, standard deviation of the bundle reservation price also increases in this direction, and bundling is a less powerful strategy when there is more uncertainty.

4.1.5 The Impact of Initial Inventory Levels

All results obtained in the previous sections are for Q_1=Q_2=10. Next, we examine the case where the quantities of Product 1 and Product 2 are different. The means (μ_1=μ_2=15) and the standard deviations (σ_1=σ_2=2) of customer reservation price distributions are equal for Product 1 and Product 2 and they are fixed.

We consider the case where the initial inventories for Product 1 and Product 2 are equal and we analyze five different cases: (Q_1, Q_2)=(5, 5), (8,8), (10, 10), (12, 12) and (15, 15). The results are tabulated in Table A.7. As it is expected, we observe an increase in the revenue as the available number of products increases. When the retailer has limited supply, he sets higher bundle prices in order to sell the products to those customers with high reservation prices. As the supply increases, he sets lower prices in order to avoid the risk of having left with inventories on hand at the end of the planning horizon. When the retailer has limited supply, he sets the bundle price such that, the number of individual purchases are high. The percentage of the products sold in bundled form is small for these cases. We observe an increase in the percentage of the products sold in bundled form as the supply increases. The reason is that, for limited supply the retailer can find customers which value the individual products highly and by selling his products to those customers he achieves high revenues. As the quantities increases, the retailer tries to sell more bundles to accomplish inventory depletion. The results for (Q_1, Q_2)=(15, 15) shows that, he is able sell almost all of his products (for all correlation coefficient values at least 93% of the products are sold) despite the limited number of customer arrivals.

Note that, the bundling is least effective when we have limited supply. When we increase the starting inventory levels, bundling becomes more instrumental. In these cases, having a smaller variance for the bundle reservation price provides a better control. Therefore, we see significant difference between expected revenues for ρ low and ρ high.

4.1.6 The Impact of Customer Arrival Rate

In order to investigate the effect of the number of potential customers on the expected revenue and the optimal bundle price, a numerical analysis is performed for three different λ values (10, 20, 30) and three different price combinations (p_1=p_2=13, 15, 17). Rest of the parameters are the same as in the previous cases: (μ_1, μ_2)=(15, 15), (σ_1, σ_2)=(2, 2), (Q_1, Q_2)=(10, 10). Results are tabulated in Table A.8.

If we allow the bundle price to be higher than the sum, p_1+p_2, we observe that for high arrival rates, 20 and 30 and small individual product prices, 13, $p_b^* > p_1$+p_2. This is not the case when the number of arrivals is small. Since customers value the individual products more than their prices for p_1=p_2=13 and 14, the retailer does not want to forgo the customers surplus. He can freely set high bundle prices, not only due to customers' willingness but also because of high arrival rates. Especially, when the number of arrivals is higher than the number of products available, the retailer is aware of the fact that customers will compete for the products and he aims to sell them to those with higher valuation and he charges high prices. When the rate of customer arrivals increases, the expected revenue increases.

For high individual product prices, 17, the optimal bundle price is much smaller than the sum, p_1+p_2. Due to high individual product prices, customers will not buy Product 1 and Product 2 as much as before. By charging reasonable prices for the bundle, inventory depletion is achieved via bundle sales.

4.2 Optimization of p_1, p_2 and p_b

In this section, we will analyze the case where the retailer decides not only on the price of the bundle but also on the prices of Product 1 and Product 2. Under different reservation price distributions, customer arrival rates and initial quantities, the prices which maximize the revenue will be determined. The base case data is used. We search over a fixed set, in which prices are taken with 0.25 increments and the results are presented in Table A.9 where the value in the third column, d, stands for the difference (p_1^*+p_2^*)-p_b^*.

The optimal prices for Product 1, Product 2 and the bundle decreases when correlation coefficient is increased from negative values to positive values. The decrease in the optimal individual product prices is more than the decrease in the optimal bundle price. When customer reservation prices are negatively correlated, the customers evaluate the individual products differently. If the prices are fairly set, purchasing one of the individual products is a better deal than purchasing the bundle. For example, when all prices are set to the mean of the corresponding reservation price distributions (i.e. p_1=p_2=15, p_b=30), 87.6% of the customers would prefer one of the individual products and 7.3% would prefer the bundle. Capitalizing on customers' willingness to buy individual products, the retailer charges quite high prices for the individual products, and collect the remainder of the revenue through offering a minimal discount on the bundle. The retailer is able to attract many customers through this discount, as the reservation price distribution of the bundle has the smallest variance, when the individual reservation prices are negatively correlated.

When customer reservation prices are positively correlated, the customers have similar valuations for both products. If the prices are fairly set, most customers would prefer the bundle. For example, when all prices are set to the mean of the corresponding reservation price distributions (i.e. p_1=p_2=15, p_b=30) 43.8% of the customers would prefer the bundle, and only 14.7% of the customers would prefer one of the individual products. Note also that in this case, many customers would leave the store without buying anything. In this case, the retailer has to offer substantial discounts on the bundle and the individual products to attract more customers. Yet bundling is not as effective as the case of negatively correlated demand because of the high variance. Therefore, resulting revenues are much smaller as the correlation increases and becomes positive.

4.2.1 The Impact of Mean Reservation Prices

Suppose that one of the individual products is valued more than the other and this is reflected by the difference in the mean of reservation price distributions. Next, we present the case for unequal μ_1 and μ_2 values.

We keep the value of μ_2 constant at 15 and use three different values for μ_1 to 10, 15 and 20. The results are tabulated in Table A.10. All findings explained for the base case are still valid. When the value of the correlation coefficient is changed from negative to positive the optimal prices of all products decrease, the revenue decreases, the expected number of individual products sold and the probabilities of individual product purchases decrease and the expected number of bundles sold increases. The possible reasons behind these results are explained above. When μ_1=10, the optimal price of Product 1 is larger than μ_1 for all correlation coefficient values. The retailer aims to sell the individual products to those customers who value them highly. We observe that p_2^* values for μ_1=10 are either equal or slightly less than for μ_1=15. This small reduction may reflect the effort of balancing the sales of both individual products. Although, we see a decreasing trend (when correlation coefficient goes from negative to positive) in α_b for μ_1=15, for μ_1=10 this probability shows fluctuations; initially decreases then increases. However, this does not affect the general result of increased number of bundle sales as correlation coefficient increases. When μ_1 is raised to 20, the customers are ready to pay larger amounts for Product 1 and the retailer sets high p_1^* values. There is not much difference in the optimal price of Product 2. Since we have an increase in the mean of the customer reservation price for the bundle, the optimal bundle price, p_b^* increases also. We observe that when the means of customer reservation prices are different, the retailer sets prices for individual products and the bundle such that the expected number of individual products sold are close to each other. That is to say, $|E(n_1) - E(n_2)|$ is very close to zero. By balancing the number of individual products sold, the retailer is able to sell more bundles. This is also reflected by high $E(n_b)$ values. Once again, we can conclude that in the mixed bundling strategy, the revenue maximization is achieved via bundle sales.

Next, we analyze the case where the means of the customer reservation prices for Product 1 and Product 2, μ_1 and μ_2, are unequal and $\mu_1+\mu_2=30$. We consider the cases (μ_1, μ_2)=(5, 25), (10, 20) and (15, 15). Results are tabulated in Table A.11. When (μ_1, μ_2)=(5, 25), the average price that customers are willing to pay for Product 2 is much more than the average price they are willing to pay for Product 1. One unit of Product 2 generates much more revenue than one unit of Product 1, therefore we see that the optimal solution ensures higher expected sales for Product 2. The optimal prices reflects these presumptions. The optimal price of Product 1 is very high compared with μ_1. The smallest difference occurs at ρ=0.9 and it is 1/4 of the standard deviation (the largest difference is 7/8). On the other hand, p_2^* is less than μ_2 for positive correlations, and only 1/2 of a standard deviation higher than μ_2 when ρ=-0.9. Note also that, for all correlation coefficient values the optimal bundle price is below μ_b=30. Therefore, the optimal product prices are set such that the sale of more valuable products is ensured. The expected number of Product 1 sales is less than that of Product 2 and again the most of the products are sold in bundled form. When (μ_1, μ_2)=(10, 20), the retailer wants to sell more Product 1 and less Product 2 than the case (μ_1, μ_2)=(5, 25). The optimal prices are adjusted accordingly. The percentage difference between μ_1 and p_1^* is decreased, while that of μ_2 and p_2^* is increased. The results in Table A.11 shows that as the difference between the means of the reservation prices of Product 1 and Product 2 increases, the expected revenue also increases. The reason is that, the retailer sets very high prices for the product with small value and the price of the other product is close to the average reservation price. Since the price of the first product is too high, the retailer can freely set high bundle prices (but smaller than the sum of individual product prices). This is why the optimal bundle prices are the largest in Table A.11. Due to high bundle prices and large number of bundles sold, the retailer achieves higher revenues. In addition, we see that the largest value of the expected number of Product 2 sold occurs, when μ_2=25. By adjusting the prices of individual products and the bundle, the retailer is able to sell the most valuable product in large amounts. This is the other factor that leads to high revenues.

4.2.2 The Impact of Standard Deviation

In order to understand the impact of the standard deviation on the performance of the mixed bundling strategy, we initially change the standard deviation of the reservation price distribution of Product 1. The results are tabulated in Table A.12. As the standard deviation of the reservation price distribution for Product 1 increases, the diversity in customer valuations of Product 1 increases. In order to capture the surplus from the customers who value Product 1 highly, the retailer sets higher p_1^* values when the standard deviation of reservation price distribution is high. Although, p_1^* is high, we have the maximum value of $E(n_1)$ for σ_1=4. Therefore, the retailer is able to sell his Product 1 to customers with high reservation prices for it. Since the retailer wants to balance the sales of individual products, the increase in sales of Product 1 also pulls the sales of Product 2 up (largest values of $E(n_2)$ id for σ_1=4). When the expected revenues are compared, we see that the maximum values are for (σ_1=1, σ_2=2) pair (except for ρ=-0.9). As the variation in the reservation price distributions increases, the revenue decreases.

The results in Table A.13 are obtained when the standard deviations of the reservation prices are the same and equal to 1, 2 and 3. The maximum revenues occur at smaller standard deviation values (σ_1=σ_2=1). For these values, the retailer sets lower prices for the individual products and these prices are very close to the mean of the reservation price distributions. The standard deviation impacts the prices in two ways. The individual prices are above the mean of their corresponding reservation prices, therefore a decrease in standard deviation results in a reduction in these prices. However, the bundle prices are below the mean of the bundle reservation price, therefore a decrease in standard deviation results in an increase in the bundle price.

4.2.3 The Impact of Initial Inventory Levels

Now, we consider the case where starting inventory levels for Product 1 and Product 2 are unequal. The results in Table A.14 are obtained when Q_1 is changed from 5 to 20 and Q_2 is retained at 10. Other parameters are retained as in the base case data. We first note that, the retailer always charges a lower price for a product with higher starting inventory. We observe that as the inventory of one product is increased, its optimal price decreases, while the other product's optimal price increases. We also see a decrease in the bundle price in this direction. The results in Table A.14 shows that the product which is available in larger amounts is priced lower than the other individual product. As the available quantity decreases, the optimal product price increases. By doing so, the retailer aims to sell the products to customers who value them highly.

In order to better understand the impact of initial inventories, we consider two other quantity combinations. The results in Table A.15 are for the case of limited inventories, (Q_1, Q_2)=(5, 5) and the results in and for the case of excess inventories, (Q_1, Q_2)=(15, 15). We first observe that as the initial inventories are higher, the retailer's revenues increase, which is expected. We also see that, the optimal bundle price decreases as the starting inventory levels increase. When the initial quantities are equal to 5, for all correlation coefficient values, the retailer sets all the prices (p_1^*, p_2^*, p_b^*) higher than the average reservation prices (μ_1, μ_2, μ_b). By setting high prices for the products, he is trying to extract the surplus from the customers with high reservation prices. Since the initial inventory level is smaller than the average customer arrival rate, he is able to find customers who can buy the products despite of their high prices. We observe that the optimal values of the individual product prices are higher for negative correlation values. For these values of ρ, the customers want one of the individual products instead of the bundle and the retailer sets a high price for its limited supply. The retailer knows that he can find a number of customers making purchase at these prices at least equal to the number of products available. For positive correlation coefficient values, the customers are willing to buy the bundle.

When we have large initial inventory values, the retailer does not even want to explore the option of selling the products individually. He has an excess supply and he wants to make sure that the customers buy the bundle rather than one of the products. Therefore, he sets an extremely high price for the individual products to zero out all individual purchases.

4.2.4 The Impact of Customer Arrival Rate

In Table A.16 we study the impact of the customer arrival rate. The first part of this table, λ=10, refers to the case where the supply is larger than the demand, while in the last part, λ=30, we have higher demands than the supply. Note that small λ values generate similar results as large starting inventory values. We have excess supply in both cases. Similarly, high λ values generates identical results as small initial inventory values. Both of these implies excess demand. When λ=10, we have a decrease in all product prices when correlation coefficient becomes positive. For negative values of ρ, the retailer sets high prices for individual products and comparatively lower prices for the bundle. The retailer aims to discourage the customers from individual product purchases and makes the bundle a more attractive choice. There are on average 10 customers arriving and he wants to sell all the products in the bundled form (he has 10 bundles) in order to deplete his inventories. For positive correlation coefficient values, the customers are already willing to buy the bundle and the retailer sets lower prices for the individual products. However, we have very low number of individual purchases. The most of the revenue is obtained via bundle sales. For λ=30, the optimal bundle prices are higher than the previous case, λ=10. Here, we have the supply smaller than the demand. Therefore, the retailer sets higher bundle prices in order to sell his products only to those customers with high reservation prices. On the other hand, the individual product prices are comparatively low for negative and high for positive correlation coefficient values. However, these prices never fall below the average reservation prices. By selling an individual product or the bundle, the retailer is able to extract most of the customer surplus.

4.3 Other Strategies

4.3.1 Pure Bundling

In pure bundling strategy, the products are offered for sale only as a part of the bundle. The individual products cannot be purchases independently. In this section, we analyze the numerical results obtained for pure bundling case. First, we briefly report the performance of this strategy under different conditions such as different arrival rates, customer reservation price distributions. The results are compared with the findings obtained from the mixed bundling strategy in Section 4.4.

The results in Table A.17 are obtained when the base case data is used. The probability of bundle purchase, the expected number of bundles sales and the expected revenue are at their maximum, when ρ=-0.9, even though the retailer charges the maximum bundle price at this correlation coefficient value. As ρ increases, p_b^* decreases. Despite this decrease, higher bundle sales and higher revenues cannot be achieved. Therefore, we conclude that bundling is a good practice, when the reservation prices for Product 1 and Product 2 are negatively correlated.

Offering a bundle leads to more homogeneous valuations among customers and thus the retailer can capture more of the customer surplus. In the case of negative correlation, we have customers that evaluate the components of the bundle differently. By charging a bundle price lower than the mean of the customer reservation price distribution, the retailer is able to attract the customers who would otherwise like to buy only one of the products.

The results in Table A.18 shows the impact of the customer arrival rate. Note that the optimal bundle price is an increasing function of the arrival rate. When the customer pool is larger, the retailer is able to charge higher price and still have more revenues. We observe that the correlation coefficient has a substantial effect when the arrival rate is small; i.e., when the supply is much larger than the demand. When the arrival rate is small, negative correlation helps to maintain the bundle price at higher values and still attract customers. With positive correlation, since the variance of the bundle reservation price is high, the retailer has to offer very low prices. Finally, note that when the prices are lower than the mean reservation price a decrease in correlation helps to increase revenues. However, when the price is above the mean (when λ=30, ρ=0, 0.5, 0.9) an increase in correlation helps to increase revenues.

In Table A.19 we study the impact of the mean of the customer reservation price distribution for Product 1. The mean reservation price for Product 2 is retained at 15. Since μ_b=μ_1+μ_2, the above results show nothing but the impact of μ_b on the revenue and the optimal bundle price. As customers have high valuation for the bundle, the retailer sets high prices and obtains larger revenues. In addition, we analyze the case where μ_b=μ_1+μ_2=30. We consider three different (μ_1, μ_2) combinations; (5, 25), (10, 20) and (15, 15). Obviously, for pure bundling, all three cases are exactly the same. Results in Table A.20 are only provided for comparison purposes.

In Table A.21, we study the impact of the standard deviations of the reservation price distributions on pure bundling strategy. Note that σ_b^2=σ_1^2+σ_2^2+$2\rho\sigma_1\sigma_2$ and individual standard deviations impact the solution through the variance of the bundle reservation price distribution only. We observe that with smaller variance, the optimal revenues and the optimal bundle prices are higher.

In Table A.22, we study the impact of starting inventory levels. Starting inventory level has an impact similar to that of arrival rates; one changes supply and the other changes demand and both impact the supply demand balance. When initial inventory level is increased, we see an increase in revenues and decrease in optimal bundle prices. We again note that, when the supply is much more than the demand, correlation impacts the revenues substantially, while the same impact is not visible when supply is much less than demand. Finally, note that negative correlation has a positive impact on revenues when the price charged is below the average bundle reservation price. Otherwise, positive correlation improves sales (e.g. Q_b=5).

4.3.2 Unbundled sales

In the unbundling case, the retailer provides three options to the customers: Product 1, Product 2 and both of them for a price equals to p_1+p_2. In this section, the performance of unbundling strategy under different conditions is reported.

The results in Table A.23 are obtained by using the base case data. The prices of Product 1 and Product 2 are set below the average reservation price. Note that regardless of the correlation value, the optimal product prices are the same. Therefore, for unbundling strategy, ρ does not affect the optimal values of product prices. The correlation coefficient only impacts the proportion of customers who are purchasing both products together. We observe that, for negative correlation values, the customers prefer individual purchases and for positive correlation values they prefer both products. Although, product prices remain the same for all ρ values, the optimal revenue increases when ρ changes form negative to positive and the reason is the increased tendency toward purchase of both products.

The results in Table A.24 show once again that when the intensity of the arrival process increases, the retailer starts to set higher prices for his products and earns higher revenues. As λ increases, the retailer is more likely to find customers who are willing to pay higher prices for each product; therefore he charges higher prices and improves his revenues.

When Product 1 and Product 2 have different means for the customer reservation price distributions, it is observed that the retailer sets higher prices for the product which is valued highly by the customers. As customer willingness to pay increase, the expected revenue increases. Results in Table A.25 confirm these interpretations. In Table A.26, we study the case where the means of individual reservation price distributions add up to 30. We observe that the revenues achieve their maximum when the means have the largest difference (μ_1=5, μ_2=25). This is similar to our observation for the mixed bundling case.

In Table A.27, we study the impact of standard deviations. We observe that, the maximum revenues are achieved when the standard deviations of the customer reservation prices are small. As we increase at least one of them, the revenue starts to decrease. When the expected numbers of individual purchases for (σ_1, σ_1)=(1,1), (2,2) and (3,3) are compared, we observe that in case of large standard deviations we have more individual purchases. Instead of buying both Product 1 and Product 2, the customers prefer to buy them individually. The diversity of customers' evaluation of the products increases and they become less willing to buy both products.

The impact of initial inventories is studied in Table A.28. We see that when the initial inventory increases, the retailer offers lower prices for his products. Although, prices are lower, he earns higher revenues.

4.4 Comparison of the Bundling Strategies

In this section, we compare three bundling strategies; the mixed bundling, the pure bundling and unbundled sales. We analyze the impact of the reservation price distributions, the starting inventory levels and the customer arrival rate on the performances of these strategies and explore the conditions under which bundling is most useful.

Initially, we analyze the results obtained under our base setting. These results are available in Table A.29. For all correlation values except ρ=0.9, the retailer makes the most profit with mixed bundling; followed by pure bundling and then unbundling. For ρ=0.9, the retailer still makes the most profit with mixed bundling, but this time followed by unbundling and then pure bundling.

In order to visualize how well the mixed bundling strategy performs compared with other two, we calculate the percent deviations of their revenues from the revenue of the mixed bundling strategy via the following formula:

$$\%\text{deviation} \quad = \quad ([E(R)_{mix} - E(R)_i]/E(R)_{mix}) * 100 \qquad i \in \ \{\text{pure, unbundling}\}$$

The percent deviations under the base settings are provided in Table A.30. When we compare the results for mixed bundling and pure bundling strategies, we see that expected revenues are very close. In mixed bundling strategy, most of the products are sold in bundled form, which is the only option for pure bundling strategy. The results in Table A.30 show that the maximum percentage deviation between mixed and pure bundling is 0.70%. On the other hand, the expected revenues obtained from the unbundled sales deviate from that of mixed bundling as much as 5.07%. Although, the retailer does not offer the bundle, he provides the opportunity of purchasing both products for a price equal to $p_1^*+p_2^*$. Note that for each correlation coefficient value, the optimal prices for individual products are higher for the mixed bundling case than the optimal prices for the unbundled sales. In mixed bundling strategy, the retailer is able to charge high prices for the individual products and attract customers who value one of the individual products highly, while he is still able to capture the demand for other customers through offering the bundle. Note also that, the difference between the revenues of the mixed bundling strategy and the unbundled sales decreases when the correlation coefficient goes from negative to positive. For negative ρ values, the customers have different valuations for Product 1 and Product 2. By selling the bundle to those customers, the retailer is able to get higher revenues than the unbundling strategy.

Next, we consider the case where the means of the reservation price distributions are unequal. The findings in Table A.31 are obtained for (μ_1, μ_2)=(10, 15) and (μ_1, μ_2)=(20, 15). Instead of including all the detailed results as in Table A.29, we prefer to tabulate only the optimal prices and the percent deviation of the revenues. All other information is available in the previous sections of this chapter. Again, the highest revenues are achieved with mixed bundling strategy. The performance gap between the revenues of pure bundling and mixed bundling is very small, while the performance gap between unbundled sales and mixed bundling is large.

We observe that the revenues obtained from pure bundling strategy and the unbundled sales approach to the revenues acquired from mixed bundling strategy as the mean of the reservation price for Product 1, μ_1, increases. For unbundled sales, this is expected. As one of the products is perceived more valuable by customers, the retailer would like to sell more of these products individually in mixed bundling strategy. This means that less bundles are sold, which makes mixed bundling and unbundling strategies indifferent. For pure bundling strategy, all that matters is the mean of the bundle reservation price, μ_b=μ_1+μ_2. As μ_1 increases, μ_b increases and the retailer finds a larger demand for his limited supply. Consequently, under our settings, we can conclude that as the mean reservation price for an individual product increases the performance of the three strategies approach to each other.

The results in Table A.32 are obtained, when the means of the reservation price distributions add up to 30. We consider two different (μ_1, μ_2) combinations: (5, 25) and (10, 20). The results show that, as the difference between the means increases, the percent deviation of the revenue for the pure bundling strategy increases. If the retailer uses the pure bundling strategy, for all combinations of means that satisfy μ_1+μ_2=μ_b=30, he gets the same optimal bundle price values and the same revenues. Therefore, the increase in the percent deviations as the difference μ_2-μ_1 increases is due higher revenues obtained from the mixed bundling strategy. Counter to the pure bundling strategy, for increased μ_2-μ_1, the percent deviation of the revenue for the unbundled sales decreases. If the customers value one of the products much more than the other, the bundle purchase would not make them better off. These customers are indifferent between the unbundled sales and the mixed bundling. As we further increase the difference between the means of the reservation prices, we would get much closer results with mixed bundling and unbundling strategies.

Next, we analyze the case where the standard deviations of the reservation prices are unequal. The results in Table A.33 are obtained for (σ_1, σ_2)=(1,2) and (4, 2). When we have an increase in the standard deviation of the reservation price distribution of Product 1, we observe that the differences between the revenues of the bundling strategies increase. We have smaller revenues for the mixed bundling strategy but even much smaller ones for the pure and unbundling strategies. As the variation in the customer preferences increases, the mixed bundling strategy outperforms much better than the other strategies, since it can better handle variation through offering more options.

The same conclusion can also be drawn from the results in Table A.34. Here, we have equal standard deviations for the reservation price distributions of the both products. Note that, for negative correlation coefficients and small standard deviations, the performances of pure and mixed bundling strategies are very close to each other. Therefore, by providing only the bundle option, the retailer can be better of when the variation in customer preferences is small. With increased standard deviations, the percentage deviations in the revenues increases both for the pure and the unbundling strategy. As a result, when the dispersion of the customer preferences increases, the retailer can achieve comparatively higher revenues by providing all the options (Product 1, Product 2 and the bundle).

In order to understand the impacts of the limited and excess supplies on the performances of the bundling strategies, we study the following two cases: (Q_1, Q_2)=(5, 5) and (15, 15). The results in Table A.35 show that, the percentage deviation between the mixed bundling and the pure bundling strategies decreases when the supply increases. When the retailer has a supply much larger than the demand, he sets significantly smaller prices for the bundle to achieve more bundle sales. As he sells more bundles and less individual products, the revenues obtained from the mixed bundling and the pure bundling get closer to each other. We assume that inventories of both products are equal. However, when the starting inventories are not the same, the percent deviation between the mixed and pure bundling becomes very large (Table A.36). Therefore, in case of unequal supplies, pure bundling is not a good strategy. Results in Table A.35 shows that we have a decrease in the percentage deviation between the mixed bundling and the unbundling strategies when the supply decreases. When the retailer has a supply much less than the demand, he is already able to command higher prices for the individual products and optimize his revenues without much use of bundling.

Finally, we analyze the effect of the arrival rate on the performances of the bundling strategies. The results for λ=10 and 30 are tabulated in Table A.37. Note that, small λ values corresponds to limited demand and the same conclusions as the excess supply can be drawn. Likely, large λ values mean excess demand and the results are similar to that obtained from limited supply.

Chapter 5

CONCLUSION

In this chapter, we provide a brief summary of the study. In addition, we discuss the limitations of this thesis and address some extension possibilities for future research.

In this study, we consider a retailer who sells two perishable products, which are available in limited quantities at the beginning of a fixed planning horizon. We assume that the selling season is short and there is no replenishment opportunity during the planning horizon. The retailer aims to maximize his revenue by using the mixed bundling strategy. In this strategy, he provides three options to the customers. They can purchase either an individual product or a bundle composed of them. Customers arrive to the store according to a Poisson Process with a fixed arrival rate. Customers' preferences are reflected by their reservation prices, which are assumed to be random variables with Normal Distribution. The retailer aims to determine the prices for the individual products and the bundle that maximize the revenue. At the beginning of the planning horizon, he sets these prices and they remain unchanged until the end of the planning horizon. In order to make the bundle an attractive option, the retailer sets its price lower than the sum of the individual product prices.

We primarily focus on the mixed bundling strategy. He has two other possibilities; the pure bundling strategy and the unbundling strategy. For all of three strategies, we write a FORTRAN code, that give us the maximum revenues and the optimal prices that give rise to these maximum values. In addition, we obtain the expected numbers of Product 1, Product 2 and the bundle sold during the selling season. In order to better understand the impact of the bundle price on the revenue, for the mixed bundling strategy, we fix the values of the individual product prices. Under different conditions the optimal bundle price and the optimal revenue is calculated. By changing the parameters of the customer reservation price distributions, the initial inventory levels and the intensity of the customer arrival process, we evaluate the performances of the mixed bundling, pure bundling and unbundling strategies. Then, we provide a comparison between these three. We explore the conditions under which the mixed bundling strategy outperforms the other two.

For the mixed bundling strategy, we fix the values of individual product prices and optimize the bundle price. It is observed that, the retailer tries to adjust his bundle price such that the most of the products are sold in bundled form. When the individual product prices are high, the retailer faces the problem of selling the products individually and he uses bundling to increase his sales. This is done best if the variance of the bundle reservation price is smallest. The retailer is able to impact the bundle sales with small reductions in the bundle price. On the other hand, when individual product prices are low, most of the customers buy the individual products if the bundle option is not offered. The retailer is able to move some customers to bundle purchase when reservation prices are positively correlated.

From the numerical study performed, we observe that the performances of policies heavily depend on the nature of the demand and hence the parameters of the demand process, i.e. the reservation price distributions and the customer arrival rate. In addition, the initial inventory levels have substantial effects on the performances. Results show that the mixed bundling outperforms other two strategies. Its profitability is even higher for negative correlation values. For these correlation values, the retailer is able to sort customers with different preferences by providing a bundle option.

It is observed that the mixed bundling and pure bundling strategies perform very close when the supply is large. In the case of excess supply (limited demand), the customer pool is small, and the retailer wants to make sure that an arriving customer buys a bundle, therefore most of the products are sold in the form of a bundle. As the number of bundles sold increases, the revenues from two strategies get closer. On the other hand, the revenues obtained from the mixed bundling and unbundling strategies perform very close, when the supply is small (demand is large). In case of mixed bundling strategy, the retailer wants to sell more individual products and less bundles. As the number of individual product sold increases the revenues he earns from the mixed bundling and unbundling strategies get closer.

For the sake of simplicity, we assume that the retailer forms a monopoly for two products and we do not consider any competition. However, this assumption may not be applicable in some of today's markets. Hence, a worthy but complex extension could be the integration of actions of the competitors in the pricing decisions. Another important research area for future studies may include reservation price distributions other than Normal Distribution. Here, we assume that the demand depends on the reservation prices and the product prices. However, the demand rate that also depends on the remaining shelflife may be included in the settings we proposed. The arrival process can also be modelled as a renewal process. In this study, we do not consider any cost component. However, the comparative performances may change in case of charging a cost for bundling the products.

In our study, we assume that the retailer has a very short selling season and we do not allow any price changes during the season. As a further research, one may consider a price change at a time when one of the products depletes. In addition, one can divide the season into two or more periods and allow the retailer to change the product prices at the beginning of each period. In this case, a cost for price changes could also be considered. In addition, our assumption of no replenishment can be converted to one in which the retailer is allowed to replenish product inventories at the beginning of each period. Instead of insisting on the mixed bundling strategy, the retailer may prefer to use pure or unbundling strategies in one or more of the periods during the season.

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Appendix A

TABLES

ρ	$p_1 = p_2$	p_b^*	α_0	$\alpha_1 = \alpha_2$	α_b	$E(n_1) = E(n_2)$	$E(n_b)$	$E(R)$	$\alpha_1' = \alpha_2'$
-0.9	17	29.25	0.14	0.08	0.70	1.21	8.50	289.76	0.16
-0.5	17	29	0.27	0.03	0.67	0.62	9.05	283.50	0.16
0.0	17	28.75	0.30	0.01	0.68	0.18	9.53	280.04	0.16
0.5	17	28.75	0.33	0.00	0.66	0.01	9.64	277.65	0.16
0.9	17	28.75	0.36	0.00	0.64	0.00	9.61	276.16	0.16
-0.9	16	29	0.07	0.15	0.63	2.35	7.41	290.15	0.31
-0.5	16	28.75	0.20	0.08	0.64	1.50	8.23	284.50	0.31
0.0	16	28.75	0.28	0.04	0.64	0.83	8.85	280.75	0.31
0.5	16	28.75	0.33	0.01	0.65	0.22	9.42	277.87	0.31
0.9	16	28.75	0.36	0.00	0.64	0.00	9.61	276.16	0.31
-0.9	15	28.75	0.00	0.27	0.46	4.09	5.66	285.39	0.5
-0.5	15	28.5	0.10	0.18	0.54	2.94	6.81	282.42	0.5
0.0	15	28.5	0.18	0.12	0.58	2.08	7.64	280.12	0.5
0.5	15	28.75	0.28	0.07	0.58	1.35	8.27	278.06	0.5
0.9	15	28.75	0.35	0.01	0.63	0.16	9.44	276.26	0.5
-0.9	14	27.75	0.00	0.26	0.48	4.17	5.68	274.41	0.69
-0.5	14	27.75	0.02	0.24	0.50	3.81	6.03	274.11	0.69
0.0	14	28	0.07	0.22	0.49	3.59	6.19	274.04	0.69
0.5	14	28.25	0.14	0.18	0.50	3.07	6.66	274.08	0.69
0.9	14	28.25	0.23	0.09	0.59	1.72	8.02	274.58	0.69
-0.9	13	26.75	0.00	0.26	0.48	4.20	5.68	261.15	0.84
-0.5	13	26.75	0.00	0.25	0.50	3.99	5.90	261.56	0.84
0.0	13	27	0.01	0.25	0.49	3.99	5.88	262.51	0.84
0.5	13	27.25	0.04	0.23	0.50	3.68	6.17	263.89	0.84
0.9	13	27.5	0.10	0.18	0.54	3.06	6.78	265.95	0.84

Table A.1: Fixed p_1 and p_2: Equal individual product prices

ρ	(p_1, p_2)	p_b^*	α_1	α_2	α_b	$E(n_1)$	$E(n_2)$	$E(n_b)$	$E(R)$	α_1'	α_2'
-0.9	(13,15)	27.25	0.36	0.09	0.57	3.90	3.37	6.09	267.09	0.84	0.5
-0.5	(13,15)	27	0.30	0.06	0.64	3.25	2.84	6.73	266.67	0.84	0.5
0.0	(13,15)	27	0.27	0.03	0.65	2.91	2.50	7.07	266.15	0.84	0.5
0.5	(13,15)	27.25	0.25	0.01	0.63	2.86	2.35	7.11	266.06	0.84	0.5
0.9	(13,15)	27.25	0.21	0.00	0.66	2.36	1.93	7.59	266.61	0.84	0.5
-0.9	(14,15)	28	0.31	0.16	0.54	3.83	3.51	6.12	277.63	0.69	0.5
-0.5	(14,15)	28	0.28	0.13	0.55	3.51	3.15	6.43	276.29	0.69	0.5
0.0	(14,15)	28	0.22	0.08	0.58	2.82	2.47	7.10	275.16	0.69	0.5
0.5	(14,15)	28	0.15	0.03	0.63	1.96	1.64	7.94	274.37	0.69	0.5
0.9	(14,15)	28.25	0.09	0.00	0.64	1.25	0.96	8.59	274.47	0.69	0.5
-0.9	(15,15)	28.75	0.27	0.27	0.46	4.09	4.09	5.66	285.39	0.5	0.5
-0.5	(15,15)	28.5	0.18	0.18	0.54	2.94	2.94	6.81	282.42	0.5	0.5
0.0	(15,15)	28.5	0.12	0.12	0.58	2.08	2.08	7.64	280.12	0.5	0.5
0.5	(15,15)	28.75	0.07	0.07	0.58	1.35	1.35	8.27	278.06	0.5	0.5
0.9	(15,15)	28.75	0.01	0.01	0.63	0.16	0.16	9.44	276.26	0.5	0.5
-0.9	(16,15)	28.75	0.13	0.27	0.59	2.71	3.26	6.68	284.31	0.31	0.5
-0.5	(16,15)	28.5	0.07	0.18	0.62	1.89	2.36	7.55	280.70	0.31	0.5
0.0	(16,15)	28.5	0.03	0.12	0.63	1.23	1.63	8.22	278.45	0.31	0.5
0.5	(16,15)	28.5	0.01	0.05	0.65	0.55	0.78	9.00	277.07	0.31	0.5
0.9	(16,15)	28.75	0.00	0.01	0.64	0.06	0.10	9.52	276.14	0.31	0.5
-0.9	(17,15)	28.25	0.03	0.19	0.78	1.23	2.03	7.96	276.24	0.16	0.5
-0.5	(17,15)	28	0.01	0.13	0.75	0.88	1.50	8.46	274.38	0.16	0.5
0.0	(17,15)	28	0.00	0.08	0.72	0.56	1.01	8.91	274.14	0.16	0.5
0.5	(17,15)	28.25	0.00	0.04	0.69	0.29	0.56	9.26	275.10	0.16	0.5
0.9	(17,15)	28.5	0.00	0.00	0.66	0.02	0.05	9.64	275.93	0.16	0.5

Table A.2: Fixed p_1 and p_2: Different individual product prices

(μ_1, μ_2)	ρ	p_b^*	α_1	α_2	α_b	$E(n_1)$	$E(n_2)$	$E(n_b)$	$E(R)$	α_1'	α_2'
(13,15)	-0.9	26.25	0.03	0.19	0.78	1.23	2.03	7.96	257.85	0.16	0.5
(13,15)	-0.5	26.25	0.02	0.15	0.70	0.99	1.81	8.14	255.81	0.16	0.5
(13,15)	0.0	26.25	0.00	0.10	0.69	0.64	1.25	8.65	255.49	0.16	0.5
(13,15)	0.5	26.5	0.00	0.05	0.65	0.35	0.74	9.05	256.15	0.16	0.5
(13,15)	0.9	26.75	0.00	0.01	0.64	0.05	0.10	9.52	256.79	0.16	0.5
(14,15)	-0.9	27.75	0.13	0.27	0.59	2.71	3.26	6.68	274.92	0.31	0.5
(14,15)	-0.5	27.5	0.07	0.18	0.62	1.89	2.36	7.55	271.26	0.31	0.5
(14,15)	0.0	27.5	0.03	0.12	0.63	1.23	1.63	8.22	269.00	0.31	0.5
(14,15)	0.5	27.5	0.01	0.05	0.65	0.55	0.78	9.00	267.51	0.31	0.5
(14,15)	0.9	27.75	0.00	0.01	0.64	0.06	0.10	9.52	266.56	0.31	0.5
(15,15)	-0.9	28.75	0.27	0.27	0.46	4.09	4.09	5.66	285.39	0.5	0.5
(15,15)	-0.5	28.5	0.18	0.18	0.54	2.94	2.94	6.81	282.42	0.5	0.5
(15,15)	0.0	28.5	0.12	0.12	0.58	2.08	2.08	7.64	280.12	0.5	0.5
(15,15)	0.5	28.75	0.07	0.07	0.58	1.35	1.35	8.27	278.06	0.5	0.5
(15,15)	0.9	28.75	0.01	0.01	0.63	0.16	0.16	9.44	276.26	0.5	0.5
(16,15)	-0.9	29	0.31	0.16	0.54	3.83	3.51	6.12	287.58	0.69	0.5
(16,15)	-0.5	29	0.28	0.13	0.55	3.51	3.15	6.43	286.23	0.69	0.5
(16,15)	0.0	29	0.22	0.08	0.58	2.82	2.47	7.10	285.07	0.69	0.5
(16,15)	0.5	29	0.15	0.03	0.63	1.96	1.64	7.94	284.27	0.69	0.5
(16,15)	0.9	29.25	0.09	0.00	0.64	1.25	0.96	8.59	284.31	0.69	0.5
(17,15)	-0.9	29.25	0.36	0.09	0.57	3.90	3.37	6.09	287.05	0.84	0.5
(17,15)	-0.5	29	0.30	0.06	0.64	3.25	2.84	6.73	286.63	0.84	0.5
(17,15)	0.0	29	0.27	0.03	0.65	2.91	2.50	7.07	286.11	0.84	0.5
(17,15)	0.5	29.25	0.25	0.01	0.63	2.86	2.35	7.11	286.00	0.84	0.5
(17,15)	0.9	29.25	0.21	0.00	0.66	2.36	1.93	7.59	286.53	0.84	0.5

Table A.3: Fixed p_1 and p_2: Different means for reservation price distributions, p_1=p_2=15

(μ_1, μ_2)	ρ	p_b^*	α_1	α_2	α_b	$E(n_1)$	$E(n_2)$	$E(n_b)$	$E(R)$	α_1'	α_2'
(15,15)	-0.9	28.75	0.27	0.27	0.46	4.09	4.09	5.66	285.39	0.5	0.5
(15,15)	-0.5	28.50	0.18	0.18	0.54	2.94	2.94	6.81	282.42	0.5	0.5
(15,15)	0.0	28.50	0.12	0.12	0.58	2.08	2.08	7.63	280.12	0.5	0.5
(15,15)	0.5	28.75	0.07	0.07	0.58	1.35	1.35	8.27	278.06	0.5	0.5
(15,15)	0.9	28.75	0.01	0.01	0.63	0.16	0.16	9.44	276.26	0.5	0.5
(14,16)	-0.9	27.75	0.05	0.27	0.70	2.21	2.85	7.14	273.95	0.31	0.69
(14,16)	-0.5	27.75	0.04	0.24	0.67	2.00	2.70	7.28	272.44	0.31	0.69
(14,16)	0.0	27.75	0.02	0.19	0.67	1.57	2.21	7.75	271.66	0.31	0.69
(14,16)	0.5	27.75	0.00	0.12	0.68	1.06	1.51	8.41	272.00	0.31	0.69
(14,16)	0.9	28	0.00	0.06	0.68	0.59	0.86	9.00	273.57	0.31	0.69
(13,17)	-0.9	26.25	0.00	0.20	0.83	1.06	1.91	8.08	256.64	0.16	0.84
(13,17)	-0.5	26.25	0.00	0.19	0.81	1.03	1.89	8.10	256.51	0.16	0.84
(13,17)	0.0	26.25	0.00	0.16	0.80	0.93	1.70	8.28	256.89	0.16	0.84
(13,17)	0.5	26.5	0.00	0.15	0.76	0.86	1.64	8.34	258.44	0.16	0.84
(13,17)	0.9	26.5	0.00	0.09	0.78	0.60	1.05	8.91	260.89	0.16	0.84

Table A.4: Fixed p_1 and p_2: Means of reservation price distributions add up to 30, p_1=15, p_2=15

(σ_1, σ_2)	ρ	p_b^*	α_1	α_2	α_b	$E(n_1)$	$E(n_2)$	$E(n_b)$	$E(R)$	α_1'	α_2'
(1,2)	-0.9	29	0.30	0.16	0.53	3.78	3.47	6.15	286.98	0.5	0.5
(1,2)	-0.5	28.75	0.20	0.09	0.62	2.63	2.39	7.29	284.88	0.5	0.5
(1,2)	0.0	28.75	0.14	0.05	0.63	1.89	1.68	7.98	283.20	0.5	0.5
(1,2)	0.5	28.75	0.07	0.02	0.66	1.00	0.85	8.83	281.56	0.5	0.5
(1,2)	0.9	28.75	0.01	0.00	0.68	0.09	0.07	9.65	279.82	0.5	0.5
(2,2)	-0.9	28.75	0.27	0.27	0.46	4.09	4.09	5.66	285.39	0.5	0.5
(2,2)	-0.5	28.5	0.18	0.18	0.54	2.94	2.94	6.81	282.42	0.5	0.5
(2,2)	0.0	28.5	0.12	0.12	0.58	2.08	2.08	7.64	280.12	0.5	0.5
(2,2)	0.5	28.75	0.07	0.07	0.58	1.35	1.35	8.27	278.06	0.5	0.5
(2,2)	0.9	28.75	0.01	0.01	0.63	0.16	0.16	9.44	276.26	0.5	0.5
(3,2)	-0.9	28.5	0.23	0.30	0.46	4.02	4.22	5.61	283.54	0.5	0.5
(3,2)	-0.5	28.5	0.18	0.23	0.47	3.35	3.53	6.21	280.23	0.5	0.5
(3,2)	0.0	28.5	0.12	0.16	0.52	2.39	2.55	7.15	277.82	0.5	0.5
(3,2)	0.5	28.5	0.05	0.08	0.58	1.32	1.44	8.22	275.68	0.5	0.5
(3,2)	0.9	28.75	0.01	0.02	0.60	0.27	0.32	9.21	273.61	0.5	0.5
(4,2)	-0.9	28.5	0.23	0.34	0.42	4.32	4.67	5.16	281.96	0.5	0.5
(4,2)	-0.5	28.5	0.18	0.26	0.44	3.56	3.88	5.86	278.55	0.5	0.5
(4,2)	0.0	28.5	0.12	0.18	0.49	2.56	2.84	6.85	276.15	0.5	0.5
(4,2)	0.5	28.5	0.05	0.10	0.55	1.46	1.68	7.96	273.96	0.5	0.5
(4,2)	0.9	28.75	0.01	0.03	0.58	0.35	0.45	9.03	271.73	0.5	0.5

Table A.5: Fixed p_1 and p_2: Different standard deviations for reservation price distributions, p_1=15, p_2=15

(σ_1, σ_2)	ρ	p_b^*	α_1	α_2	α_b	$E(n_1)$	$E(n_2)$	$E(n_b)$	$E(R)$	α_1'	α_2'
(1,1)	-0.9	29.25	0.23	0.23	0.55	3.46	3.46	6.37	289.98	0.5	0.5
(1,1)	-0.5	29	0.13	0.13	0.66	2.12	2.12	7.75	288.40	0.5	0.5
(1,1)	0	29	0.08	0.08	0.67	1.47	1.47	8.37	286.87	0.5	0.5
(1,1)	0.5	29	0.03	0.03	0.70	0.67	0.67	9.15	285.45	0.5	0.5
(1,1)	0.9	29	0.00	0.00	0.71	0.02	0.02	9.78	284.21	0.5	0.5
(2,2)	-0.9	28.75	0.27	0.27	0.46	4.09	4.09	5.66	285.39	0.5	0.5
(2,2)	-0.5	28.5	0.18	0.18	0.54	2.94	2.94	6.81	282.42	0.5	0.5
(2,2)	0.0	28.5	0.12	0.12	0.58	2.08	2.08	7.64	280.12	0.5	0.5
(2,2)	0.5	28.75	0.07	0.07	0.58	1.35	1.35	8.27	278.06	0.5	0.5
(2,2)	0.9	28.75	0.01	0.01	0.63	0.16	0.16	9.44	276.26	0.5	0.5
(3,3)	-0.9	28.25	0.28	0.28	0.44	4.32	4.32	5.40	282.04	0.5	0.5
(3,3)	-0.5	28.25	0.21	0.21	0.45	3.58	3.58	6.05	278.28	0.5	0.5
(3,3)	0	28.25	0.14	0.14	0.51	2.57	2.57	7.03	275.58	0.5	0.5
(3,3)	0.5	28.5	0.08	0.08	0.54	1.65	1.65	7.86	273.41	0.5	0.5
(3,3)	0.9	28.75	0.02	0.02	0.58	0.41	0.41	9.01	271.35	0.5	0.5
(4,4)	-0.9	28	0.30	0.30	0.38	4.79	4.79	4.84	279.43	0.5	0.5
(4,4)	-0.5	28	0.23	0.23	0.41	3.93	3.93	5.62	275.19	0.5	0.5
(4,4)	0	28.25	0.17	0.17	0.44	3.05	3.05	6.40	272.43	0.5	0.5
(4,4)	0.5	28.5	0.10	0.10	0.49	1.99	1.99	7.39	270.23	0.5	0.5
(4,4)	0.9	29	0.03	0.03	0.53	0.75	0.75	8.48	268.33	0.5	0.5

Table A.6: Fixed p_1 and p_2: Equal standard deviations for reservation price distributions, p_1=15, p_2=15

(Q_1, Q_2)	ρ	p_b^*	α_1	α_2	α_b	$E(n_1)$	$E(n_2)$	$E(n_b)$	$E(R)$	$\alpha_1^{'}$	$\alpha_2^{'}$
(5,5)	-0.9	30	0.44	0.44	0.07	4.41	4.41	0.55	148.82	0.5	0.5
(5,5)	-0.5	30.25	0.36	0.36	0.12	3.95	3.95	1.00	148.84	0.5	0.5
(5,5)	0	30.25	0.28	0.28	0.21	3.22	3.22	1.73	148.99	0.5	0.5
(5,5)	0.5	30.5	0.22	0.22	0.25	2.74	2.74	2.20	149.25	0.5	0.5
(5,5)	0.9	30.5	0.12	0.12	0.34	1.73	1.73	3.20	149.63	0.5	0.5
(8,8)	-0.9	29	0.30	0.30	0.38	4.04	4.04	3.84	232.61	0.5	0.5
(8,8)	-0.5	29	0.23	0.23	0.41	3.36	3.36	4.49	231.20	0.5	0.5
(8,8)	0	29.25	0.18	0.18	0.42	2.86	2.86	4.94	230.34	0.5	0.5
(8,8)	0.5	29.25	0.10	0.10	0.49	1.78	1.78	6.03	229.60	0.5	0.5
(8,8)	0.9	29.25	0.02	0.02	0.57	0.44	0.44	7.37	228.76	0.5	0.5
(10,10)	-0.9	28.75	0.27	0.27	0.46	4.09	4.09	5.66	285.39	0.5	0.5
(10,10)	-0.5	28.5	0.18	0.18	0.54	2.94	2.94	6.81	282.42	0.5	0.5
(10,10)	0	28.5	0.12	0.12	0.58	2.08	2.08	7.64	280.12	0.5	0.5
(10,10)	0.5	28.75	0.07	0.07	0.58	1.35	1.35	8.27	278.06	0.5	0.5
(10,10)	0.9	28.75	0.01	0.01	0.63	0.16	0.16	9.44	276.26	0.5	0.5
(12,12)	-0.9	28.25	0.19	0.19	0.63	3.29	3.29	8.37	335.27	0.5	0.5
(12,12)	-0.5	28	0.13	0.13	0.66	2.38	2.38	9.25	330.45	0.5	0.5
(12,12)	0	28	0.08	0.08	0.67	1.61	1.61	9.92	326.22	0.5	0.5
(12,12)	0.5	28	0.03	0.03	0.70	0.72	0.72	10.74	322.40	0.5	0.5
(12,12)	0.9	28	0.00	0.00	0.71	0.02	0.02	11.38	319.23	0.5	0.5
(15,15)	-0.9	27.75	0.13	0.13	0.75	2.56	2.56	11.73	402.32	0.5	0.5
(15,15)	-0.5	27.25	0.07	0.07	0.82	1.52	1.52	12.82	394.90	0.5	0.5
(15,15)	0	27.25	0.04	0.04	0.80	0.96	0.96	13.14	386.85	0.5	0.5
(15,15)	0.5	27	0.01	0.01	0.81	0.23	0.23	13.80	379.64	0.5	0.5
(15,15)	0.9	26.75	0.00	0.00	0.82	0.00	0.00	13.99	374.36	0.5	0.5

Table A.7: Fixed p_1 and p_2: Impact of starting inventory levels, p_1=15, p_2=15

λ	ρ	p_1=p_2	p_b^*	α_1=α_2	α_b	$E(n_1)$=$E(n_2)$	$E(n_b)$	$E(R)$	α_1'=α_2'
10	-0.9	13	25	0.07	0.89	0.75	7.77	213.81	0.84
10	-0.5	13	25	0.07	0.89	0.75	7.77	213.62	0.84
10	0	13	25	0.06	0.89	0.64	7.83	212.47	0.84
10	0.5	13	25	0.04	0.90	0.41	8.01	210.72	0.84
10	0.9	13	25.25	0.01	0.90	0.12	8.15	208.95	0.84
10	-0.9	15	27	0.07	0.89	0.72	7.77	231.29	0.5
10	-0.5	15	26.5	0.04	0.92	0.40	8.10	226.78	0.5
10	0	15	26	0.01	0.93	0.13	8.30	219.70	0.5
10	0.5	15	25.75	0.00	0.91	0.02	8.27	213.46	0.5
10	0.9	15	25.5	0.00	0.90	0.00	8.20	209.22	0.5
10	-0.9	17	28	0.02	0.97	0.23	8.42	243.46	0.16
10	-0.5	17	27	0.00	0.95	0.04	8.46	229.99	0.16
10	0	17	26.25	0.00	0.93	0.00	8.39	220.23	0.16
10	0.5	17	25.75	0.00	0.91	0.00	8.29	213.46	0.16
10	0.9	17	25.5	0.00	0.90	0.00	8.20	209.22	0.16
20	-0.9	13	26.75	0.27	0.48	4.20	5.68	261.15	0.84
20	-0.5	13	26.75	0.26	0.50	3.99	5.90	261.56	0.84
20	0	13	27	0.25	0.49	3.99	5.88	262.51	0.84
20	0.5	13	27.25	0.23	0.50	3.68	6.17	263.89	0.84
20	0.9	13	27.5	0.18	0.54	3.06	6.78	265.95	0.84
20	-0.9	15	28.75	0.27	0.46	4.09	5.66	285.39	0.5
20	-0.5	15	28.5	0.18	0.54	2.94	6.81	282.42	0.5
20	0	15	28.5	0.12	0.58	2.08	7.64	280.12	0.5
20	0.5	15	28.75	0.07	0.58	1.35	8.27	278.06	0.5
20	0.9	15	28.75	0.01	0.63	0.16	9.44	276.26	0.5
20	-0.9	17	29.25	0.08	0.70	1.21	8.50	289.76	0.16
20	-0.5	17	29	0.03	0.67	0.62	9.05	283.50	0.16
20	0	17	28.75	0.01	0.68	0.18	9.53	280.04	0.16
20	0.5	17	28.75	0.00	0.66	0.01	9.64	277.65	0.16
20	0.9	17	28.75	0.00	0.64	0.00	9.61	276.16	0.16
30	-0.9	13	27.25	0.36	0.30	3.97	6.02	264.87	0.84
30	-0.5	13	27.5	0.37	0.29	3.83	6.16	265.64	0.84
30	0	13	27.75	0.34	0.31	4.13	5.87	267.11	0.84
30	0.5	13	28	0.31	0.34	4.61	5.38	269.09	0.84
30	0.9	13	28.25	0.26	0.39	5.33	4.66	271.84	0.84
30	-0.9	15	29.75	0.41	0.13	2.07	7.85	296.86	0.5
30	-0.5	15	29.75	0.31	0.22	3.60	6.31	296.49	0.5
30	0	15	30	0.26	0.26	4.37	5.51	296.44	0.5
30	0.5	15	30	0.17	0.34	5.97	3.91	296.48	0.5
30	0.9	15	30.25	0.10	0.39	7.29	2.54	296.87	0.5
30	-0.9	17	29.75	0.10	0.50	7.67	2.13	300.71	0.16
30	-0.5	17	30	0.06	0.45	8.17	1.56	298.40	0.16
30	0	17	30	0.03	0.49	9.10	0.72	297.36	0.16
30	0.5	17	30.25	0.01	0.48	9.63	0.18	297.19	0.16
30	0.9	17	30.5	0.00	0.46	9.75	0.00	297.37	0.16

Table A.8: Fixed p_1 and p_2: Impact of the arrival rate

ρ	(p_1^*, p_2^*, p_b^*)	d	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_b)$	α_1	α_2	α_b
-0.9	(16.50, 16.50, 29.25)	3.75	290.24	1.90	1.90	7.78	0.12	0.12	0.63
-0.5	(16.00, 16.00, 28.75)	3.25	284.50	1.50	1.50	8.23	0.08	0.08	0.64
0	(15.75, 15.75, 28.75)	2.75	280.88	1.12	1.12	8.54	0.06	0.06	0.62
0.5	(15.25, 15.25, 28.75)	1.75	278.21	0.92	0.92	8.70	0.04	0.04	0.61
0.9	(14.75, 14.75, 28.75)	0.75	276.30	0.46	0.46	9.14	0.01	0.01	0.55

Table A.9: Optimization of p_1, p_2 and p_b: Base case

(μ_1,μ_1)	ρ	(p_1^*, p_2^*, p_b^*)	d	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_b)$	α_1	α_2	α_b
(10,15)	-0.9	(11.50, 16.25, 24.25)	3.50	242.05	2.02	2.20	7.55	0.12	0.15	0.61
(10,15)	-0.5	(11.00, 15.75, 24.00)	2.75	236.29	1.90	2.06	7.62	0.01	0.13	0.57
(10,15)	0	(10.75, 15.50, 24.00)	2.25	232.78	1.48	1.62	7.99	0.07	0.09	0.56
(10,15)	0.5	(10.50, 15.25, 24.00)	1.75	230.25	0.91	1.02	8.54	0.04	0.06	0.58
(10,15)	0.9	(10.25, 14.75, 24.00)	1.00	228.51	0.37	0.51	9.05	0.00	0.03	0.60
(15,15)	-0.9	(16.50, 16.50, 29.25)	3.75	290.24	1.90	1.90	7.78	0.12	0.12	0.63
(15,15)	-0.5	(16.00, 16.00, 28.75)	3.25	284.50	1.50	1.50	8.23	0.08	0.08	0.64
(15,15)	0	(15.75, 15.75, 28.75)	2.75	280.88	1.12	1.12	8.54	0.06	0.06	0.62
(15,15)	0.5	(15.25, 15.25, 28.75)	1.75	278.21	0.92	0.92	8.70	0.04	0.04	0.61
(15,15)	0.9	(14.75, 14.75, 28.75)	0.75	276.30	0.46	0.46	9.14	0.01	0.01	0.55
(20,15)	-0.9	(21.25, 16.25, 34.00)	3.50	339.11	1.93	1.93	7.85	0.13	0.13	0.68
(20,15)	-0.5	(20.75, 16.00, 33.75)	3.00	333.17	1.75	1.63	8.03	0.11	0.08	0.62
(20,15)	0	(20.50, 15.75, 33.50)	2.75	329.43	1.13	1.05	8.65	0.07	0.05	0.65
(20,15)	0.5	(20.00, 15.25, 33.50)	1.75	326.67	0.95	0.88	8.78	0.05	0.03	0.63
(20,15)	0.9	(19.50, 14.75, 33.50)	0.75	324.68	0.53	0.47	9.18	0.03	0.01	0.64

Table A.10: Optimization of p_1, p_2 and p_b: Different means for reservation price distributions

(μ_1,μ_1)	ρ	(p_1^*, p_2^*, p_b^*)	d	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_b)$	α_1	α_2	α_b
(5,25)	-0.9	(6.75, 26.00, 29.25)	3.50	292.83	1.86	2.43	7.42	0.10	0.18	0.61
(5,25)	-0.5	(6.50, 25.50, 29.00)	3.00	286.81	1.55	2.23	7.58	0.06	0.16	0.57
(5,25)	0	(6.00, 25.00, 29.00)	2.00	283.21	1.62	2.31	7.44	0.05	0.16	0.53
(5,25)	0.5	(5.75, 24.50, 29.00)	1.25	280.76	1.47	2.35	7.40	0.02	0.16	0.53
(5,25)	0.9	(5.50, 24.25, 29.00)	0.75	279.22	1.14	1.88	7.84	0.00	0.13	0.55
(10,20)	-0.9	(11.50, 21.25, 29.25)	3.50	290.79	2.02	2.20	7.55	0.01	0.15	0.61
(10,20)	-0.5	(11.00, 20.75, 28.75)	3.00	284.90	1.63	1.75	8.03	0.08	0.11	0.62
(10,20)	0	(10.75, 20.25, 28.75)	2.25	281.34	1.42	1.66	8.09	0.06	0.10	0.59
(10,20)	0.5	(10.50, 20.00, 28.75)	1.75	278.69	0.91	1.10	8.60	0.03	0.07	0.60
(10,20)	0.9	(10.25, 19.50, 28.75)	1.00	276.85	0.51	0.71	8.96	0.00	0.04	0.61
(15,15)	-0.9	(16.50, 16.50, 29.25)	3.75	290.24	1.90	1.90	7.78	0.12	0.12	0.63
(15,15)	-0.5	(16.00, 16.00, 28.75)	3.25	284.50	1.50	1.50	8.23	0.08	0.08	0.64
(15,15)	0	(15.75, 15.75, 28.75)	2.75	280.88	1.12	1.12	8.54	0.06	0.06	0.62
(15,15)	0.5	(15.25, 15.25, 28.75)	1.75	278.21	0.92	0.92	8.70	0.04	0.04	0.61
(15,15)	0.9	(14.75, 14.75, 28.75)	0.75	276.30	0.46	0.46	9.14	0.01	0.01	0.55

Table A.11: Optimization of p_1, p_2 and p_b: Means of reservation price distributions add up to 30

σ_1,σ_2	ρ	(p_1^*, p_2^*, p_b^*)	d	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_b)$	α_1	α_2	α_b
(1,2)	-0.9	(15.75, 15.75, 29.25)	2.25	289.54	2.21	1.90	7.68	0.18	0.07	0.65
(1,2)	-0.5	(15.50, 15.50, 29.00)	2.00	286.08	1.82	1.59	8.05	0.13	0.05	0.64
(1,2)	0	(15.25, 15.25, 28.75)	1.75	283.47	1.32	1.15	8.55	0.09	0.03	0.67
(1,2)	0.5	(15.00, 15.00, 28.75)	1.25	281.56	0.99	0.85	8.83	0.07	0.02	0.66
(1,2)	0.9	(14.75, 14.75, 28.75)	0.75	280.08	0.44	0.36	9.33	0.03	0.00	0.67
(2,2)	-0.9	(16.50, 16.50, 29.25)	3.75	290.24	1.90	1.90	7.78	0.12	0.12	0.63
(2,2)	-0.5	(16.00, 16.00, 28.75)	3.25	284.50	1.50	1.50	8.23	0.08	0.08	0.64
(2,2)	0	(15.75, 15.75, 28.75)	2.75	280.88	1.12	1.12	8.54	0.06	0.06	0.62
(2,2)	0.5	(15.25, 15.25, 28.75)	1.75	278.21	0.92	0.92	8.70	0.04	0.04	0.61
(2,2)	0.9	(14.75, 14.75, 28.75)	0.75	276.30	0.46	0.46	9.14	0.01	0.01	0.55
(4,2)	-0.9	(16.50, 16.50, 29.00)	4.00	288.09	2.35	2.68	7.07	0.11	0.19	0.55
(4,2)	-0.5	(16.50, 16.25, 28.75)	4.00	281.81	1.66	2.00	7.72	0.06	0.13	0.56
(4,2)	0	(16.00, 15.75, 28.75)	3.00	277.54	1.47	1.77	7.86	0.05	0.11	0.55
(4,2)	0.5	(15.75, 15.25, 28.75)	2.25	274.38	1.01	1.35	8.27	0.02	0.09	0.56
(4,2)	0.9	(15.00, 14.50, 28.75)	0.75	272.15	0.89	1.28	8.36	0.01	0.08	0.56

Table A.12: Optimization of p_1, p_2 and p_b: Different standard deviations for reservation price distributions

σ_1,σ_2	ρ	(p_1^*, p_2^*, p_b^*)	d	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_b)$	α_1	α_2	α_b
(1,1)	-0.9	(15.50, 15.50, 29.25)	1.75	291.90	1.60	1.60	8.28	0.11	0.11	0.78
(1,1)	-0.5	(15.50, 15.50, 29.25)	1.75	288.97	1.25	1.25	8.55	0.07	0.07	0.69
(1,1)	0	(15.25, 15.25, 29.00)	1.50	287.05	0.84	0.84	9.01	0.04	0.04	0.72
(1,1)	0.5	(15.00, 15.00, 29.00)	1.00	285.45	0.67	0.67	9.15	0.03	0.03	0.70
(1,1)	0.9	(14.75, 14.75, 29.00)	0.5	284.24	0.26	0.26	9.54	0.01	0.01	0.70
(2,2)	-0.9	(16.50, 16.50, 29.25)	3.75	290.24	1.90	1.90	7.78	0.12	0.12	0.63
(2,2)	-0.5	(16.00, 16.00, 28.75)	3.25	284.50	1.50	1.50	8.23	0.08	0.08	0.64
(2,2)	0	(15.75, 15.75, 28.75)	2.75	280.88	1.12	1.12	8.54	0.06	0.06	0.62
(2,2)	0.5	(15.25, 15.25, 28.75)	1.75	278.21	0.92	0.92	8.70	0.04	0.04	0.61
(2,2)	0.9	(14.75, 14.75, 28.75)	0.75	276.30	0.46	0.46	9.14	0.01	0.01	0.55
(3,3)	-0.9	(17.00, 17.00, 29.00)	5.00	290.44	2.32	2.32	7.30	0.15	0.15	0.58
(3,3)	-0.5	(16.50, 16.50, 28.75)	4.25	282.21	1.96	1.96	7.56	0.11	0.11	0.54
(3,3)	0	(16.00, 16.00, 28.50)	3.50	277.28	1.47	1.47	8.08	0.08	0.08	0.57
(3,3)	0.5	(15.50, 15.50, 28.75)	2.25	273.79	1.17	1.17	8.26	0.06	0.06	0.55
(3,3)	0.9	(15.00, 15.00, 28.75)	1.25	271.35	0.41	0.41	9.01	0.02	0.02	0.58

Table A.13: Optimization of p_1, p_2 and p_b: Equal standard deviations for reservation price distributions

(Q_1,Q_2)	ρ	(p_1^*, p_2^*, p_b^*)	d	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_b)$	α_1	α_2	α_b
(5,10)	-0.9	(17.50, 15.25, 29.50)	3.25	219.18	0.59	5.38	4.30	0.06	0.33	0.46
(5,10)	-0.5	(17.00, 15.00, 29.50)	2.50	216.62	0.56	5.30	4.32	0.05	0.29	0.42
(5,10)	0	(16.75, 14.75, 29.50)	2.00	215.09	0.33	5.07	4.57	0.03	0.25	0.43
(5,10)	0.5	(16.25, 14.50, 29.75)	1.00	213.87	0.29	5.06	4.56	0.02	0.25	0.40
(5,10)	0.9	(15.50, 14.25, 29.75)	0.00	213.08	0.20	5.01	4.66	0.01	0.25	0.40
(10,10)	-0.9	(16.50, 16.50, 29.25)	3.75	290.24	1.90	1.90	7.78	0.12	0.12	0.63
(10,10)	-0.5	(16.00, 16.00, 28.75)	3.25	284.50	1.50	1.50	8.23	0.08	0.08	0.64
(10,10)	0	(15.75, 15.75, 28.75)	2.75	280.88	1.12	1.12	8.54	0.06	0.06	0.62
(10,10)	0.5	(15.25, 15.25, 28.75)	1.75	278.21	0.92	0.92	8.70	0.04	0.04	0.61
(10,10)	0.9	(14.75, 14.75, 28.75)	0.75	276.30	0.46	0.46	9.14	0.01	0.01	0.55
(20,10)	-0.9	(14.25, 23.00*, 28.50)	8.75	386.21	7.97	0.00	9.57	0.36	0.00	0.64
(20,10)	-0.5	(13.75, 22.75, 27.75)	8.75	371.10	7.49	0.00	9.66	0.29	0.00	0.66
(20,10)	0	(13.25, 22.5, 27.25)	8.50	362.34	7.42	0.00	9.69	0.26	0.00	0.67
(20,10)	0.5	(13.00, 22.25, 27.25)	8.00	356.44	7.29	0.00	9.60	0.25	0.00	0.64
(20,10)	0.9	(12.75, 21.75, 27.00)	7.50	353.68	7.26	0.00	9.67	0.23	0.00	0.66

Table A.14: Optimization of p_1, p_2 and p_b: Different starting inventory levels

(Q_1,Q_2)	ρ	(p_1^*, p_2^*, p_b^*)	d	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_b)$	α_1	α_2	α_b
(5,5)	-0.9	(16.00, 16.00, 30.00)	2.00	152.27	2.78	2.78	2.11	0.25	0.25	0.26
(5,5)	-0.5	(16.00, 16.00, 30.25)	1.75	151.62	2.30	2.30	2.58	0.18	0.18	0.29
(5,5)	0	(15.75, 15.75, 30.50)	1.00	151.24	2.17	2.17	2.72	0.16	0.16	0.29
(5,5)	0.5	(15.75, 15.75, 30.75)	0.75	150.98	1.53	1.53	3.34	0.10	0.10	0.33
(5,5)	0.9	(15.75, 15.75, 30.75)	0.75	150.71	0.38	0.38	4.51	0.02	0.02	0.41
(10,10)	-0.9	(16.50, 16.50, 29.25)	3.75	290.24	1.90	1.90	7.78	0.12	0.12	0.63
(10,10)	-0.5	(16.00, 16.00, 28.75)	3.25	284.50	1.50	1.50	8.23	0.08	0.08	0.64
(10,10)	0	(15.75, 15.75, 28.75)	2.75	280.88	1.12	1.12	8.54	0.06	0.06	0.62
(10,10)	0.5	(15.25, 15.25, 28.75)	1.75	278.21	0.92	0.92	8.70	0.04	0.04	0.61
(10,10)	0.9	(14.75, 14.75, 28.75)	0.75	276.30	0.46	0.46	9.14	0.01	0.01	0.55
(15,15)	-0.9	(23.00, 23.00, 28.75)	17.25	419.56	0.00	0.00	14.59	0.00	0.00	0.89
(15,15)	-0.5	(23.00, 23.00, 27.75)	18.25	399.67	0.00	0.00	14.40	0.00	0.00	0.89
(15,15)	0	(23.00, 23.00, 27.25)	18.75	387.62	0.00	0.00	14.22	0.00	0.00	0.85
(15,15)	0.5	(23.00, 22.75, 27.00)	18.75	379.50	0.00	0.00	14.06	0.00	0.00	0.83
(15,15)	0.9	(23.00, 22.25, 26.75)	18.50	374.36	0.00	0.00	13.99	0.00	0.00	0.82

Table A.15: Optimization of p_1, p_2 and p_b: Equal starting inventory levels

λ	ρ	(p_1^*, p_2^*, p_b^*)	d	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_b)$	α_1	α_2	α_b
10	-0.9	(22.75, 23.00, 28.25)	17.50	246.84	0.00	0.00	8.74	0.00	0.00	0.99
10	-0.5	(22.50, 22.25, 27.00)	17.75	230.31	0.00	0.00	8.53	0.00	0.00	0.95
10	0	(18.00, 18.00, 26.25)	9.75	220.24	0.00	0.00	8.39	0.00	0.00	0.93
10	0.5	(15.25, 15.25, 25.75)	4.75	213.46	0.01	0.01	8.28	0.00	0.00	0.91
10	0.9	(13.75, 13.75, 25.50)	2.00	209.22	0.01	0.01	8.20	0.00	0.00	0.90
20	-0.9	(16.50, 16.50, 29.25)	3.75	290.24	1.90	1.90	7.78	0.12	0.12	0.63
20	-0.5	(16.00, 16.00, 28.75)	3.25	284.50	1.50	1.50	8.23	0.08	0.08	0.64
20	0	(15.75, 15.75, 28.75)	2.75	280.88	1.12	1.12	8.54	0.06	0.06	0.62
20	0.5	(15.25, 15.25, 28.75)	1.75	278.21	0.92	0.92	8.70	0.04	0.04	0.61
20	0.9	(14.75, 14.75, 28.75)	0.75	276.30	0.46	0.46	9.14	0.01	0.01	0.55
30	-0.9	(16.00, 16.00, 29.75)	2.25	303.28	4.47	4.47	5.38	0.23	0.23	0.35
30	-0.5	(16.00, 16.00, 30.00)	2.00	301.13	3.75	3.75	6.04	0.17	0.17	0.34
30	0	(15.75, 15.75, 30.25)	1.25	299.63	3.50	3.50	6.27	0.15	0.15	0.34
30	0.5	(15.75, 15.75, 30.25)	1.25	298.64	1.90	1.90	7.89	0.07	0.07	0.41
30	0.9	(15.50, 15.50, 30.50)	0.50	297.71	1.02	1.02	8.72	0.03	0.03	0.43

Table A.16: Optimization of p_1, p_2 and p_b: Impact of the arrival rate

ρ	p_b^*	$E(R)$	$E(n_b)$	α_b
-0.9	29.25	290.10	9.92	0.80
-0.5	29.00	282.91	9.76	0.69
0	28.75	278.93	9.70	0.67
0.5	28.50	276.25	9.69	0.67
0.9	28.50	274.68	9.64	0.65

Table A.17: Pure bundling: Base case

λ	ρ	p_b^*	$E(R)$	$E(n_b)$	α_b
30	-0.9	29.75	296.83	9.98	0.61
30	-0.5	30.00	295.89	9.86	0.50
30	0	30.00	295.89	9.86	0.50
30	0.5	30.25	296.13	9.79	0.47
30	0.9	30.25	296.42	9.80	0.47
20	-0.9	29.25	290.10	9.92	0.80
20	-0.5	29.00	282.91	9.76	0.69
20	0	28.75	278.93	9.70	0.67
20	0.5	28.50	276.25	9.69	0.67
20	0.9	28.50	274.68	9.64	0.65
10	-0.9	28.25	243.78	8.63	0.97
10	-0.5	26.75	227.19	8.49	0.95
10	0	26.25	217.11	8.27	0.91
10	0.5	25.75	210.32	8.17	0.89
10	0.9	25.50	206.06	8.08	0.88
5	-0.9	28.00	137.66	4.92	0.99
5	-0.5	26.25	126.79	4.83	0.97
5	0	25.50	120.01	4.71	0.94
5	0.5	24.75	115.39	4.66	0.94
5	0.9	24.50	112.51	4.59	0.92

Table A.18: Pure bundling: Impact of the arrival rate

(μ_1,μ_2)	ρ	p_b^*	$E(R)$	$E(n_b)$	α_b
(5,15)	-0.9	19.50	191.09	9.80	0.71
(5,15)	-0.5	19.00	185.35	9.76	0.69
(5,15)	0	19.00	182.35	9.60	0.64
(5,15)	0.5	19.00	180.48	9.50	0.61
(5,15)	0.9	19.00	179.40	9.44	0.60
(10,15)	-0.9	24.25	240.51	9.92	0.80
(10,15)	-0.5	24.00	234.13	9.76	0.69
(10,15)	0	23.75	230.42	9.70	0.67
(10,15)	0.5	23.75	228.18	9.61	0.64
(10,15)	0.9	23.75	226.82	9.55	0.63
(15,15)	-0.9	29.25	290.10	9.92	0.80
(15,15)	-0.5	29.00	282.91	9.76	0.69
(15,15)	0	28.75	278.93	9.70	0.67
(15,15)	0.5	28.50	276.25	9.69	0.67
(15,15)	0.9	28.50	274.68	9.64	0.65
(20,15)	-0.9	34.25	339.69	9.92	0.80
(20,15)	-0.5	33.75	332.07	9.84	0.73
(20,15)	0	33.50	327.60	9.78	0.70
(20,15)	0.5	33.50	324.71	9.69	0.67
(20,15)	0.9	33.50	322.87	9.64	0.65

Table A.19: Pure bundling: Different means for reservation price distributions

(μ_1,μ_2)	ρ	p_b^*	$E(R)$	$E(n_b)$	α_b
(5,25)	-0.9	29.25	290.10	9.92	0.80
(5,25)	-0.5	29.00	282.91	9.76	0.69
(5,25)	0	28.75	278.93	9.70	0.67
(5,25)	0.5	28.50	276.25	9.69	0.67
(5,25)	0.9	28.50	274.68	9.64	0.65
(10,20)	-0.9	29.25	290.10	9.92	0.80
(10,20)	-0.5	29.00	282.91	9.76	0.69
(10,20)	0	28.75	278.93	9.70	0.67
(10,20)	0.5	28.50	276.25	9.69	0.67
(10,20)	0.9	28.50	274.68	9.64	0.65
(15,15)	-0.9	29.25	290.10	9.92	0.80
(15,15)	-0.5	29.00	282.91	9.76	0.69
(15,15)	0	28.75	278.93	9.70	0.67
(15,15)	0.5	28.50	276.25	9.69	0.67
(15,15)	0.9	28.50	274.68	9.64	0.65

Table A.20: Pure bundling: Means of reservation price distributions add up to 30

(σ_1, σ_1)	ρ	p_b^*	$E(R)$	$E(n_b)$	α_b
(1, 1)	-0.9	29.50	293.87	9.96	0.87
(1, 1)	-0.5	29.25	289.36	9.89	0.77
(1, 1)	0	29.00	286.43	9.88	0.76
(1, 1)	0.5	29.00	284.53	9.81	0.72
(1, 1)	0.9	29.00	283.21	9.77	0.70
(1, 2)	-0.9	29.25	287.93	9.84	0.74
(1, 2)	-0.5	29.00	284.53	9.81	0.72
(1, 2)	0	28.75	281.74	9.80	0.71
(1, 2)	0.5	28.75	279.78	9.73	0.68
(1, 2)	0.9	28.75	278.46	9.69	0.67
(1, 3)	-0.9	28.75	282.18	9.81	0.72
(1, 3)	-0.5	28.75	279.78	9.73	0.68
(1, 3)	0	28.75	277.46	9.65	0.65
(1, 3)	0.5	28.50	275.73	9.67	0.66
(1, 3)	0.9	28.50	274.60	9.63	0.65
(2, 2)	-0.9	29.25	290.10	9.92	0.80
(2, 2)	-0.5	29.00	282.91	9.76	0.69
(2, 2)	0	28.75	278.93	9.70	0.67
(2, 2)	0.5	28.50	276.25	9.69	0.67
(2, 2)	0.9	28.50	274.68	9.64	0.65
(3, 3)	-0.9	29.00	286.84	9.89	0.77
(3, 3)	-0.5	28.75	278.16	9.68	0.66
(3, 3)	0	28.50	273.53	9.60	0.64
(3, 3)	0.5	28.50	270.73	9.50	0.61
(3, 3)	0.9	28.50	269.11	9.44	0.60
(4, 2)	-0.9	28.75	281.11	9.78	0.70
(4, 2)	-0.5	28.50	276.25	9.69	0.67
(4, 2)	0	28.50	272.80	9.57	0.63
(4, 2)	0.5	28.50	270.47	9.49	0.61
(4, 2)	0.9	28.50	269.07	9.44	0.60

Table A.21: Pure bundling: Different standard deviations for reservation price distributions

Q_b	ρ	p_b^*	$E(R)$	$E(n_b)$	α_b
5	-0.9	30.00	148.71	4.96	0.5
5	-0.5	30.25	148.72	4.92	0.45
5	0	30.50	149.18	4.89	0.43
5	0.5	30.50	149.69	4.91	0.44
5	0.9	30.75	150.13	4.88	0.42
8	-0.9	29.50	234.62	7.95	0.71
8	-0.5	29.25	231.08	7.90	0.65
8	0	29.25	229.37	7.84	0.60
8	0.5	29.25	228.34	7.81	0.59
8	0.9	29.50	227.80	7.72	0.55
10	-0.9	29.25	290.10	9.92	0.80
10	-0.5	29.00	282.91	9.76	0.69
10	0	28.75	278.93	9.70	0.67
10	0.5	28.50	276.25	9.69	0.67
10	0.9	28.50	274.68	9.64	0.65
12	-0.9	29.00	343.58	11.85	0.87
12	-0.5	28.50	331.72	11.64	0.77
12	0	28.00	324.70	11.60	0.76
12	0.5	28.00	319.99	11.43	0.72
12	0.9	27.75	317.12	11.43	0.72
15	-0.9	28.75	417.44	14.52	0.92
15	-0.5	27.75	397.02	14.31	0.87
15	0	27.25	384.54	14.11	0.83
15	0.5	27.00	376.06	13.93	0.81
15	0.9	26.75	370.83	13.86	0.80

Table A.22: Pure bundling: Impact of starting inventory levels

ρ	$(p_1^*, p_2^*, p_{both}^*)$	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_{both})$	α_1	α_2	α_{both}
-0.9	(14.25, 14.25, 28.50)	275.51	5.72	5.72	3.95	0.36	0.36	0.30
-0.5	(14.25, 14.25, 28.50)	275.54	4.96	4.96	4.71	0.30	0.30	0.36
0	(14.25, 14.25, 28.50)	275.58	3.96	3.96	5.71	0.23	0.23	0.43
0.5	(14.25, 14.25, 28.50)	275.62	2.82	2.82	6.85	0.16	0.16	0.50
0.9	(14.25, 14.25, 28.50)	275.67	1.38	1.38	8.30	0.07	0.07	0.59

Table A.23: Unbundling: Base case

λ	ρ	(p_1^*, p_2^*, p_{both}^*)	$E(R)$	$E(n_1)$=$E(n_2)$	$E(n_{both})$	α_1=α_2	α_{both}
30	-0.9	(15.25, 15.25, 30.5)	297.19	9.14	0.60	0.43	0.03
30	-0.5	(15.25, 15.25, 30.5)	297.26	7.47	2.28	0.34	0.12
30	0	(15.25, 15.25, 30.5)	297.31	5.86	3.89	0.25	0.21
30	0.5	(15.25, 15.25, 30.5)	297.37	4.14	5.61	0.17	0.29
30	0.9	(15.25, 15.25, 30.5)	297.45	2.02	7.73	0.07	0.39
20	-0.9	(14.25, 14.25, 28.50)	275.51	5.72	3.95	0.36	0.30
20	-0.5	(14.25, 14.25, 28.50)	275.54	4.96	4.71	0.30	0.36
20	0	(14.25, 14.25, 28.50)	275.58	3.96	5.71	0.23	0.43
20	0.5	(14.25, 14.25, 28.50)	275.62	2.82	6.85	0.16	0.50
20	0.9	(14.25, 14.25, 28.50)	275.67	1.38	8.30	0.07	0.59
10	-0.9	(12.75, 12.75, 25.5)	208.21	1.39	6.77	0.13	0.76
10	-0.5	(12.75, 12.75, 25.5)	208.21	1.37	6.79	0.13	0.76
10	0	(12.75, 12.75, 25.5)	208.22	1.22	6.94	0.12	0.77
10	0.5	(12.75, 12.75, 25.5)	208.23	0.92	7.25	0.09	0.80
10	0.9	(12.75, 12.75, 25.5)	208.25	0.44	7.73	0.04	0.85

Table A.24: Unbundling: Impact of the arrival rate

(μ_1,μ_2)	ρ	(p_1^*, p_2^*, p_{both}^*)	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_{both})$	α_1	α_2	α_{both}
(10,15)	-0.9	(9.5, 14.25, 23.75)	227.87	6.04	6.22	3.45	0.36	0.40	0.26
(10,15)	-0.5	(9.5, 14.25, 23.75)	227.90	5.13	5.31	4.35	0.29	0.34	0.32
(10,15)	0	(9.5, 14.25, 23.75)	227.93	4.05	4.24	5.44	0.22	0.27	0.40
(10,15)	0.5	(9.5, 14.25, 23.75)	227.97	2.86	3.05	6.62	0.14	0.19	0.47
(10,15)	0.9	(9.5, 14.25, 23.75)	228.01	1.41	1.59	8.08	0.05	0.07	0.56
(15,15)	-0.9	(14.25, 14.25, 28.50)	275.51	5.72	5.72	3.95	0.36	0.36	0.30
(15,15)	-0.5	(14.25, 14.25, 28.50)	275.54	4.96	4.96	4.71	0.30	0.30	0.36
(15,15)	0	(14.25, 14.25, 28.50)	275.58	3.96	3.96	5.71	0.23	0.23	0.43
(15,15)	0.5	(14.25, 14.25, 28.50)	275.62	2.82	2.82	6.85	0.16	0.16	0.50
(15,15)	0.9	(14.25, 14.25, 28.50)	275.67	1.38	1.38	8.30	0.07	0.07	0.59
(20,15)	-0.9	(19.25, 14.25, 33.50)	323.85	5.72	5.72	3.95	0.36	0.36	0.30
(20,15)	-0.5	(19.25, 14.25, 33.50)	323.88	4.96	4.96	4.71	0.30	0.30	0.36
(20,15)	0	(19.25, 14.25, 33.50)	323.92	3.96	3.96	5.71	0.23	0.23	0.43
(20,15)	0.5	(19.25, 14.25, 33.50)	323.97	2.82	2.82	6.85	0.16	0.16	0.50
(20,15)	0.9	(19.25, 14.25, 33.50)	324.03	1.38	1.38	8.30	0.07	0.07	0.59
(25,15)	-0.9	(24.00, 14.25, 38.25)	372.61	5.38	5.26	4.41	0.36	0.31	0.35
(25,15)	-0.9	(24.00, 14.25, 38.25)	372.64	4.77	4.65	5.02	0.32	0.27	0.39
(25,15)	-0.9	(24.00, 14.25, 38.25)	372.68	3.86	3.74	5.93	0.25	0.20	0.46
(25,15)	-0.9	(24.00, 14.25, 38.25)	372.72	2.80	2.68	6.99	0.18	0.13	0.53
(25,15)	-0.9	(24.00, 14.25, 38.25)	372.78	1.46	1.34	8.33	0.09	0.05	0.62

Table A.25: Unbundling: Different means for reservation price distributions

(μ_1,μ_2)	ρ	$(p_1^*, p_2^*, p_{both}^*)$	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_{both})$	α_1	α_2	α_{both}
(5,25)	-0.9	(5.00, 24.00, 29.00)	279.06	6.05	7.01	2.78	0.30	0.50	0.21
(5,25)	-0.5	(5.00, 24.00, 29.00)	279.09	5.06	6.02	3.77	0.23	0.43	0.28
(5,25)	0	(5.00, 24.00, 29.00)	279.11	4.01	4.97	4.82	0.16	0.35	0.35
(5,25)	0.5	(5.00, 24.00, 29.00)	279.13	2.95	3.91	5.88	0.08	0.28	0.43
(5,25)	0.9	(5.00, 24.00, 29.00)	279.15	1.92	2.88	6.91	0.12	0.21	0.50
(10,20)	-0.9	(9.50, 19.25, 28.75)	276.21	6.04	6.22	3.45	0.36	0.40	0.26
(10,20)	-0.9	(9.50, 19.25, 28.75)	276.24	5.13	5.31	4.35	0.29	0.34	0.32
(10,20)	-0.9	(9.50, 19.25, 28.75)	276.28	4.05	4.24	5.44	0.22	0.27	0.40
(10,20)	-0.9	(9.50, 19.25, 28.75)	276.33	2.86	3.05	6.62	0.14	0.19	0.47
(10,20)	-0.9	(9.50, 19.25, 28.75)	276.38	1.41	1.59	8.08	0.05	0.10	0.56
(15,15)	-0.9	(14.25, 14.25, 28.50)	275.51	5.72	5.72	3.95	0.36	0.36	0.30
(15,15)	-0.5	(14.25, 14.25, 28.50)	275.54	4.96	4.96	4.71	0.30	0.30	0.36
(15,15)	0	(14.25, 14.25, 28.50)	275.58	3.96	3.96	5.71	0.23	0.23	0.43
(15,15)	0.5	(14.25, 14.25, 28.50)	275.62	2.82	2.82	6.85	0.16	0.16	0.50
(15,15)	0.9	(14.25, 14.25, 28.50)	275.67	1.38	1.38	8.30	0.07	0.07	0.59

Table A.26: Unbundling: Means of reservation price distributions add up to 30

(σ_1,σ_2)	ρ	$(p_1^*, p_2^*, p_{both}^*)$	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_{both})$	α_1	α_2	α_{both}
(1,1)	-0.9	(14.50, 14.50, 29.00)	283.78	4.92	4.92	4.87	0.32	0.32	0.39
(1,1)	-0.5	(14.50, 14.50, 29.00)	283.80	4.43	4.43	5.36	0.28	0.28	0.43
(1,1)	0	(14.50, 14.50, 29.00)	283.83	3.59	3.59	6.19	0.22	0.22	0.49
(1,1)	0.5	(14.50, 14.50, 29.00)	283.86	2.59	2.59	7.20	0.15	0.15	0.56
(1,1)	0.9	(14.50, 14.50, 29.00)	283.90	1.28	1.28	8.51	0.06	0.06	0.64
(1,2)	-0.9	(14.50, 14.25, 28.75)	279.64	5.38	5.26	4.41	0.36	0.31	0.35
(1,2)	-0.5	(14.50, 14.25, 28.75)	279.66	4.77	4.65	5.02	0.32	0.27	0.39
(1,2)	0	(14.50, 14.25, 28.75)	279.69	3.86	3.74	5.93	0.25	0.20	0.46
(1,2)	0.5	(14.50, 14.25, 28.75)	279.73	2.80	2.68	6.99	0.18	0.13	0.53
(1,2)	0.9	(14.50, 14.25, 28.75)	279.77	1.46	1.34	8.33	0.09	0.05	0.62
(1,3)	-0.9	(14.50, 14.25, 28.75)	277.03	5.90	5.59	3.89	0.41	0.31	0.30
(1,3)	-0.5	(14.50, 14.25, 28.75)	277.06	5.16	4.85	4.63	0.35	0.26	0.35
(1,3)	0	(14.50, 14.25, 28.75)	277.09	4.18	3.88	5.61	0.28	0.19	0.42
(1,3)	0.5	(14.50, 14.25, 28.75)	277.13	3.09	2.79	6.69	0.21	0.12	0.50
(1,3)	0.9	(14.50, 14.25, 28.75)	277.18	1.82	1.51	7.97	0.13	0.03	0.58
(2,2)	-0.9	(14.25, 14.25, 28.50)	275.51	5.72	5.72	3.95	0.36	0.36	0.30
(2,2)	-0.5	(14.25, 14.25, 28.50)	275.54	4.96	4.96	4.71	0.30	0.30	0.36
(2,2)	0	(14.25, 14.25, 28.50)	275.58	3.96	3.96	5.71	0.23	0.23	0.43
(2,2)	0.5	(14.25, 14.25, 28.50)	275.62	2.82	2.82	6.85	0.16	0.16	0.50
(2,2)	0.9	(14.25, 14.25, 28.50)	275.67	1.38	1.38	8.30	0.07	0.07	0.59
(3,3)	-0.9	(14.25, 14.25, 28.50)	270.34	6.53	6.53	2.96	0.40	0.40	0.21
(3,3)	-0.5	(14.25, 14.25, 28.50)	270.38	5.45	5.45	4.04	0.32	0.32	0.29
(3,3)	0	(14.25, 14.25, 28.50)	270.43	4.28	4.28	5.21	0.25	0.25	0.37
(3,3)	0.5	(14.25, 14.25, 28.50)	270.48	3.02	3.02	6.47	0.17	0.17	0.45
(3,3)	0.9	(14.25, 14.25, 28.50)	270.55	1.46	1.46	8.04	0.07	0.07	0.54
(4,2)	-0.9	(14.50, 14.25, 28.75)	271.37	6.30	6.76	2.91	0.35	0.45	0.21
(4,2)	-0.9	(14.50, 14.25, 28.75)	271.41	5.25	5.70	3.97	0.28	0.37	0.29
(4,2)	-0.9	(14.50, 14.25, 28.75)	271.46	4.11	4.57	5.10	0.20	0.30	0.36
(4,2)	-0.9	(14.50, 14.25, 28.75)	271.50	2.91	3.37	6.31	0.12	0.22	0.44
(4,2)	-0.9	(14.50, 14.25, 28.75)	271.56	1.53	1.98	7.69	0.03	0.13	0.53

Table A.27: Unbundling: Different standard deviations for reservation price distributions

(Q_1,Q_2)	ρ	$(p_1^*, p_2^*, p_{both}^*)$	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_{both})$	α_1	α_2	α_{both}
(5,5)	-0.9	(15.50, 15.50, 31.00)	150.48	4.74	4.74	0.11	0.40	0.40	0.01
(5,5)	-0.5	(15.50, 15.50, 31.00)	150.51	4.03	4.03	0.82	0.32	0.32	0.09
(5,5)	0	(15.50, 15.50, 31.00)	150.54	3.24	3.24	1.62	0.25	0.25	0.16
(5,5)	0.5	(15.50, 15.50, 31.00)	150.58	2.35	2.35	2.51	0.17	0.17	0.25
(5,5)	0.9	(15.50, 15.50, 31.00)	150.63	1.18	1.18	3.68	0.07	0.07	0.34
(8,8)	-0.9	(14.75, 14.75, 29.50)	228.48	6.16	6.16	1.59	0.43	0.43	0.14
(8,8)	-0.5	(14.75, 14.75, 29.50)	228.52	5.04	5.04	2.71	0.34	0.34	0.23
(8,8)	0	(14.75, 14.75, 29.50)	228.56	3.96	3.96	3.78	0.25	0.25	0.31
(8,8)	0.5	(14.75, 14.75, 29.50)	228.61	2.82	2.82	4.93	0.17	0.17	0.39
(8,8)	0.9	(14.75, 14.75, 29.50)	228.67	1.38	1.38	6.37	0.07	0.07	0.49
(10,10)	-0.9	(14.25, 14.25, 28.50)	275.51	5.72	5.72	3.95	0.36	0.36	0.30
(10,10)	-0.5	(14.25, 14.25, 28.50)	275.54	4.96	4.96	4.71	0.30	0.30	0.36
(10,10)	0	(14.25, 14.25, 28.50)	275.58	3.96	3.96	5.71	0.23	0.23	0.43
(10,10)	0.5	(14.25, 14.25, 28.50)	275.62	2.82	2.82	6.85	0.16	0.16	0.50
(10,10)	0.9	(14.25, 14.25, 28.50)	275.67	1.38	1.38	8.30	0.07	0.07	0.59
(12,12)	-0.9	(14.00, 14.00, 28.00)	318.36	5.55	5.55	5.82	0.31	0.31	0.39
(12,12)	-0.5	(14.00, 14.00, 28.00)	318.38	4.98	4.98	6.39	0.28	0.28	0.43
(12,12)	0	(14.00, 14.00, 28.00)	318.42	4.01	4.01	7.36	0.22	0.22	0.49
(12,12)	0.5	(14.00, 14.00, 28.00)	318.46	2.86	2.86	8.51	0.15	0.15	0.56
(12,12)	0.9	(14.00, 14.00, 28.00)	318.52	1.38	1.38	10.00	0.07	0.06	0.64
(15,15)	-0.9	(13.5, 13.5, 27.00)	372.84	4.49	4.49	9.32	0.23	0.23	0.56
(15,15)	-0.5	(13.5, 13.5, 27.00)	372.84	4.26	4.26	9.55	0.22	0.22	0.57
(15,15)	0	(13.5, 13.5, 27.00)	372.87	3.56	3.56	10.25	0.18	0.18	0.61
(15,15)	0.5	(13.5, 13.5, 27.00)	372.90	2.57	2.57	11.24	0.13	0.13	0.67
(15,15)	0.9	(13.5, 13.5, 27.00)	372.95	1.23	1.23	12.58	0.06	0.06	0.74

Table A.28: Unbundling: Impact of starting inventory levels

Mixed bundling									
ρ	(p_1^*, p_2^*, p_b^*)	d	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_b)$	α_1	α_2	α_b
-0.9	(16.50, 16.50, 29.25)	3.75	290.24	1.90	1.90	7.78	0.12	0.12	0.63
-0.5	(16.00, 16.00, 28.75)	3.25	284.50	1.50	1.50	8.23	0.08	0.08	0.64
0	(15.75, 15.75, 28.75)	2.75	280.88	1.12	1.12	8.54	0.06	0.06	0.62
0.5	(15.25, 15.25, 28.75)	1.75	278.21	0.92	0.92	8.70	0.04	0.04	0.61
0.9	(14.75, 14.75, 28.75)	0.75	276.30	0.46	0.46	9.14	0.01	0.01	0.55
Pure bundling									
ρ	(p_1^*, p_2^*, p_b^*)	d	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_b)$	α_1	α_2	α_b
-0.9	(-, -, 29.25)	-	290.10	-	-	9.92	-	-	0.80
-0.5	(-, -, 29.00)	-	282.91	-	-	9.76	-	-	0.69
0	(-, -, 28.75)	-	278.93	-	-	9.70	-	-	0.67
0.5	(-, -, 28.50)	-	276.25	-	-	9.69	-	-	0.67
0.9	(-, -, 28.50)	-	274.68	-	-	9.64	-	-	0.65
Unbundled sales									
ρ	(p_1^*, p_2^*, p_b^*)	d	$E(R)$	$E(n_1)$	$E(n_2)$	$E(n_b)$	α_1	α_2	α_b
-0.9	(14.25, 14.25, 28.50)	-	275.51	5.72	5.72	3.95	0.36	0.36	0.30
-0.5	(14.25, 14.25, 28.50)	-	275.54	4.96	4.96	4.71	0.30	0.30	0.36
0	(14.25, 14.25, 28.50)	-	275.58	3.96	3.96	5.71	0.23	0.23	0.43
0.5	(14.25, 14.25, 28.50)	-	275.62	2.82	2.82	6.85	0.16	0.16	0.50
0.9	(14.25, 14.25, 28.50)	-	275.67	1.38	1.38	8.30	0.07	0.07	0.59

Table A.29: Comparison: Base cases

ρ	$E(R)_{mix}$	Pure bundling %	Unbundled sales %
-0.9	290.24	0.05	5.07
-0.5	284.50	0.56	3.15
0	280.88	0.69	1.89
0.5	278.21	0.70	0.93
0.9	276.30	0.59	0.23

Table A.30: Comparison: Percentage deviations for base cases

μ_1=10, μ_2=15			**μ_1=20, μ_2=15**		
	Mixed bundling			**Mixed bundling**	
ρ	(p_1^*, p_2^*, p_b^*)	$E(R)$	ρ	(p_1^*, p_2^*, p_b^*)	$E(R)$
-0.9	(11.50, 16.25, 24.25)	242.05	-0.9	(21.25, 16.25, 34.00)	339.11
-0.5	(11.00, 15.75, 24.00)	236.29	-0.5	(20.75, 16.00, 33.75)	333.17
0	(10.75, 15.50, 24.00)	232.78	0	(20.50, 15.75, 33.50)	329.43
0.5	(10.50, 15.25, 24.00)	230.25	0.5	(20.00, 15.25, 33.50)	326.67
0.9	(10.25, 14.75, 24.00)	228.51	0.9	(19.50, 14.75, 33.50)	324.68
	Pure bundling			**Pure bundling**	
ρ	(p_1^*, p_2^*, p_b^*)	%	ρ	(p_1^*, p_2^*, p_b^*)	%
-0.9	(-, -, 24.25)	0.64	-0.9	(-, -, 34.25)	-0.17
-0.5	(-, -, 24.00)	0.92	-0.5	(-, -, 33.75)	0.33
0	(-, -, 23.75)	1.01	0	(-, -, 33.50)	0.55
0.5	(-, -, 23.75)	0.90	0.5	(-, -, 33.50)	0.60
0.9	(-, -, 23.75)	0.74	0.9	(-, -, 33.50)	0.56
	Unbundled sales			**Unbundled sales**	
ρ	(p_1^*, p_2^*, p_b^*)	%	ρ	(p_1^*, p_2^*, p_b^*)	%
-0.9	(9.50, 14.25, 23.75)	5.86	-0.9	(19.25, 14.25, 33.50)	4.50
-0.5	(9.50, 14.25, 23.75)	3.55	-0.5	(19.25, 14.25, 33.50)	2.79
0	(9.50, 14.25, 23.75)	2.08	0	(19.25, 14.25, 33.50)	1.67
0.5	(9.50, 14.25, 23.75)	0.99	0.5	(19.25, 14.25, 33.50)	0.83
0.9	(9.50, 14.25, 23.75)	0.22	0.9	(19.25, 14.25, 33.50)	0.20

Table A.31: Comparison: Different means for reservation price distributions

$\mu_1 = 5, \mu_2 = 25$			$\mu_1 = 10, \mu_2 = 20$		
	Mixed bundling			**Mixed bundling**	
ρ	(p_1^*, p_2^*, p_b^*)	$E(R)$	ρ	(p_1^*, p_2^*, p_b^*)	$E(R)$
-0.9	(6.75, 26.00, 29.25)	292.83	-0.9	(11.50, 21.25, 29.25)	290.79
-0.5	(6.50, 25.50, 29.00)	286.81	-0.5	(11.00, 20.75, 28.75)	284.90
0	(6.00, 25.00, 29.00)	283.21	0	(10.75, 20.25, 28.75)	281.34
0.5	(5.75, 24.50, 29.00)	280.76	0.5	(10.50, 20.00, 28.75)	278.69
0.9	(5.50, 24.25, 29.00)	279.22	0.9	(10.25, 19.50, 28.75)	276.85
	Pure bundling			**Pure bundling**	
ρ	(p_1^*, p_2^*, p_b^*)	%	ρ	(p_1^*, p_2^*, p_b^*)	%
-0.9	(-, -, 29.25)	0.93	-0.9	(-, -, 29.25)	0.24
-0.5	(-, -, 29.00)	1.36	-0.5	(-, -, 29.00)	0.70
0	(-, -, 28.75)	1.51	0	(-, -, 28.75)	0.85
0.5	(-, -, 28.50)	1.61	0.5	(-, -, 28.50)	0.88
0.9	(-, -, 28.50)	1.62	0.9	(-, -, 28.50)	0.78
	Unbundled sales			**Unbundled sales**	
ρ	(p_1^*, p_2^*, p_b^*)	%	ρ	(p_1^*, p_2^*, p_b^*)	%
-0.9	(5.00, 24.00, 29.00)	4.70	-0.9	(9.50, 19.25, 28.75)	5.02
-0.5	(5.00, 24.00, 29.00)	2.69	-0.5	(9.50, 19.25, 28.75)	3.04
0	(5.00, 24.00, 29.00)	1.45	0	(9.50, 19.25, 28.75)	1.80
0.5	(5.00, 24.00, 29.00)	0.58	0.5	(9.50, 19.25, 28.75)	0.85
0.9	(5.00, 24.00, 29.00)	0.02	0.9	(9.50, 19.25, 28.75)	0.17

Table A.32: Comparison: Means of reservation price distributions add up to 30

$\sigma_1 = 1, \sigma_2 = 2$			$\sigma_1 = 4, \sigma_2 = 2$		
	Mixed bundling			**Mixed bundling**	
ρ	(p_1^*, p_2^*, p_b^*)	$E(R)$	ρ	(p_1^*, p_2^*, p_b^*)	$E(R)$
-0.9	(15.75, 15.75, 29.25)	289.54	-0.9	(16.50, 16.50, 29.00)	288.09
-0.5	(15.50, 15.50, 29.00)	286.08	-0.5	(16.50, 16.25, 28.75)	281.81
0	(15.25, 15.25, 28.75)	283.47	0	(16.00, 15.75, 28.75)	277.54
0.5	(15.00, 15.00, 28.75)	281.56	0.5	(15.75, 15.25, 28.75)	274.38
0.9	(14.75, 14.75, 28.75)	280.08	0.9	(15.00, 14.50, 28.75)	272.15
	Pure bundling			**Pure bundling**	
ρ	(p_1^*, p_2^*, p_b^*)	%	ρ	(p_1^*, p_2^*, p_b^*)	%
-0.9	(-, -, 29.25)	0.56	-0.9	(-, -, 28.75)	2.42
-0.5	(-, -, 29.00)	0.54	-0.5	(-, -, 28.50)	1.97
0	(-, -, 28.75)	0.61	0	(-, -, 28.50)	1.71
0.5	(-, -, 28.75)	0.63	0.5	(-, -, 28.50)	1.42
0.9	(-, -, 28.75)	0.58	0.9	(-, -, 28.50)	1.13
	Unbundled sales			**Unbundled sales**	
ρ	(p_1^*, p_2^*, p_b^*)	%	ρ	(p_1^*, p_2^*, p_b^*)	%
-0.9	(14.50, 14.25, 28.75)	3.42	-0.9	(14.50, 14.25, 28.75)	5.81
-0.5	(14.50, 14.25, 28.75)	2.24	-0.5	(14.50, 14.25, 28.75)	3.69
0	(14.50, 14.25, 28.75)	1.33	0	(14.50, 14.25, 28.75)	2.19
0.5	(14.50, 14.25, 28.75)	0.65	0.5	(14.50, 14.25, 28.75)	1.05
0.9	(14.50, 14.25, 28.75)	0.11	0.9	(14.50, 14.25, 28.75)	0.22

Table A.33: Comparison: Different standard deviations for reservation price distributions

$\sigma_1 = 1, \sigma_2 = 1$			$\sigma_1 = 3, \sigma_2 = 3$		
Mixed bundling			**Mixed bundling**		
ρ	(p_1^*, p_2^*, p_b^*)	$E(R)$	ρ	(p_1^*, p_2^*, p_b^*)	$E(R)$
-0.9	(15.50, 15.50, 29.25)	291.90	-0.9	(17.00, 17.00, 29.00)	290.44
-0.5	(15.50, 15.50, 29.25)	288.97	-0.5	(16.50, 16.50, 28.75)	282.21
0	(15.25, 15.25, 29.00)	287.05	0	(16.00, 16.00, 28.50)	277.28
0.5	(15.00, 15.00, 29.00)	285.45	0.5	(15.50, 15.50, 28.75)	273.79
0.9	(14.75, 14.75, 29.00)	284.24	0.9	(15.00, 15.00, 28.75)	271.35
Pure bundling			**Pure bundling**		
ρ	(p_1^*, p_2^*, p_b^*)	%	ρ	(p_1^*, p_2^*, p_b^*)	%
-0.9	(-, -, 29.50)	-0.67	-0.9	(-, -, 29.00)	1.24
-0.5	(-, -, 29.25)	-0.13	-0.5	(-, -, 28.75)	1.43
0	(-, -, 29.00)	0.22	0	(-, -, 28.50)	1.35
0.5	(-, -, 29.00)	0.32	0.5	(-, -, 28.50)	1.12
0.9	(-, -, 29.00)	0.36	0.9	(-, -, 28.50)	0.83
Unbundled sales			**Unbundled sales**		
ρ	(p_1^*, p_2^*, p_b^*)	%	ρ	(p_1^*, p_2^*, p_b^*)	%
-0.9	(14.50, 14.50, 29.00)	2.78	-0.9	(14.25, 14.25, 28.50)	6.92
-0.5	(14.50, 14.50, 29.00)	1.79	-0.5	(14.25, 14.25, 28.50)	4.19
0	(14.50, 14.50, 29.00)	1.12	0	(14.25, 14.25, 28.50)	2.47
0.5	(14.50, 14.50, 29.00)	0.56	0.5	(14.25, 14.25, 28.50)	1.21
0.9	(14.50, 14.50, 29.00)	0.12	0.9	(14.25, 14.25, 28.50)	0.30

Table A.34: Comparison: Impact of the standard deviation

$Q_1 = 5, Q_2 = 5$			$Q_1 = 15, Q_2 = 15$		
Mixed bundling			**Mixed bundling**		
ρ	(p_1^*, p_2^*, p_b^*)	$E(R)$	ρ	(p_1^*, p_2^*, p_b^*)	$E(R)$
-0.9	(16.00, 16.00, 30.00)	152.27	-0.9	(23.00, 23.00, 28.75)	419.56
-0.5	(16.00, 16.00, 30.25)	151.62	-0.5	(23.00, 23.00, 27.75)	399.67
0	(15.75, 15.75, 30.50)	151.24	0	(23.00, 23.00, 27.25)	387.62
0.5	(15.75, 15.75, 30.75)	150.98	0.5	(23.00, 22.75, 27.00)	379.50
0.9	(15.75, 15.75, 30.75)	150.71	0.9	(23.00, 22.25, 26.75)	374.36
Pure bundling			**Pure bundling**		
ρ	(p_1^*, p_2^*, p_b^*)	%	ρ	(p_1^*, p_2^*, p_b^*)	%
-0.9	(-, -, 30.00)	2.34	-0.9	(-, -, 28.75)	0.51
-0.5	(-, -, 30.25)	1.91	-0.5	(-, -, 27.75)	0.66
0	(-, -, 30.50)	1.36	0	(-, -, 27.25)	0.79
0.5	(-, -, 30.50)	0.86	0.5	(-, -, 27.00)	0.91
0.9	(-, -, 30.75)	0.39	0.9	(-, -, 26.75)	0.94
Unbundled			**Unbundled**		
ρ	(p_1^*, p_2^*, p_b^*)	%	ρ	(p_1^*, p_2^*, p_b^*)	%
-0.9	(15.50, 15.50, 31.00)	1.18	-0.9	(13.50, 13.50, 27.00)	11.14
-0.5	(15.50, 15.50, 31.00)	0.73	-0.5	(13.50, 13.50, 27.00)	6.71
0	(15.50, 15.50, 31.00)	0.46	0	(13.50, 13.50, 27.00)	3.81
0.5	(15.50, 15.50, 31.00)	0.27	0.5	(13.50, 13.50, 27.00)	1.74
0.9	(15.50, 15.50, 31.00)	0.05	0.9	(13.50, 13.50, 27.00)	0.38

Table A.35: Comparison: Impact of starting inventory levels

(Q_1, Q_2)	ρ	**Mixed bundling**		**Pure bundling**	
(5,10)	-0.9	(17.50, 15.25, 29.50)	219.18	30.00	32.15
(5,10)	-0.5	(17.00, 15.00, 29.50)	216.62	30.25	31.34
(5,10)	0	(16.75, 14.75, 29.50)	215.09	30.50	30.64
(5,10)	0.5	(16.25, 14.50, 29.75)	213.87	30.50	30.01
(5,10)	0.9	(15.50, 14.25, 29.75)	213.08	30.75	29.54
(10,10)	-0.9	(16.50, 16.50, 29.25)	290.24	29.25	0.05
(10,10)	-0.5	(16.00, 16.00, 28.75)	284.50	29.00	0.56
(10,10)	0	(15.75, 15.75, 28.75)	280.88	28.75	0.69
(10,10)	0.5	(15.25, 15.25, 28.75)	278.21	28.50	0.70
(10,10)	0.9	(14.75, 14.75, 28.75)	276.30	28.50	0.59
(20,10)	-0.9	(14.25, 23.00, 28.50)	386.21	29.25	24.88
(20,10)	-0.5	(13.75, 22.75, 27.75)	371.10	29.00	23.77
(20,10)	0	(13.25, 22.50, 27.25)	362.34	28.75	23.02
(20,10)	0.5	(13.00, 22.25, 27.25)	356.44	28.50	22.50
(20,10)	0.9	(12.75, 21.75, 27.00)	353.68	28.50	22.34

Table A.36: Comparison: The impact of different starting inventory levels on the performances of mixed and pure bundling

$\lambda = 10$			$\lambda = 30$		
	Mixed bundling			**Mixed bundling**	
ρ	(p_1^*, p_2^*, p_b^*)	$E(R)$	ρ	(p_1^*, p_2^*, p_b^*)	$E(R)$
-0.9	(22.75, 23.00, 28.25)	246.84	-0.9	(16.00, 16.00, 29.75)	303.28
-0.5	(22.50, 22.25, 27.00)	230.31	-0.5	(16.00, 16.00, 30.00)	301.13
0	(18.00, 18.00, 26.25)	220.24	0	(15.75, 15.75, 30.25)	299.63
0.5	(15.25, 15.25, 25.75)	213.46	0.5	(15.75, 15.75, 30.25)	298.64
0.9	(13.75, 13.75, 25.50)	209.22	0.9	(15.50, 15.50, 30.50)	297.71
	Pure bundling			**Pure bundling**	
ρ	(p_1^*, p_2^*, p_b^*)	%	ρ	(p_1^*, p_2^*, p_b^*)	%
-0.9	(-, -, 28.25)	1.24	-0.9	(-, -, 29.75)	2.13
-0.5	(-, -, 26.75)	1.35	-0.5	(-, -, 30.00)	1.74
0	(-, -, 26.25)	1.42	0	(-, -, 30.00)	1.25
0.5	(-, -, 25.75)	1.47	0.5	(-, -, 30.25)	0.84
0.9	(-, -, 25.50)	1.51	0.9	(-, -, 30.25)	0.43
	Unbundled			**Unbundled**	
ρ	(p_1^*, p_2^*, p_b^*)	%	ρ	(p_1^*, p_2^*, p_b^*)	%
-0.9	(12.75, 12.75, 25.50)	15.65	-0.9	(15.25, 15.25, 30.50)	2.01
-0.5	(12.75, 12.75, 25.50)	9.59	-0.5	(15.25, 15.25, 30.50)	1.29
0	(12.75, 12.75, 25.50)	5.46	0	(15.25, 15.25, 30.50)	0.77
0.5	(12.75, 12.75, 25.50)	2.45	0.5	(15.25, 15.25, 30.50)	0.42
0.9	(12.75, 12.75, 25.50)	0.46	0.9	(15.25, 15.25, 30.50)	0.09

Table A.37: Comparison: Impact of the arrival rate