# ZERO CURVATURE AND GEL'FAND-DIKII FORMALISMS

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

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September, 2004

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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# ABSTRACT

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In soliton theory, integrable nonlinear partial differential equations play an important role. In that respect such differential equations create great interest in many research areas. There are several ways to obtain these differential equations; among them zero curvature and Gel'fand-Dikii formalisms are more effective. In this thesis, we studied these formalisms and applied them to explicit examples.

*Keywords:* Integrable systems, simple Lie algebra, soliton, zero curvature formalism, Gel'fand-Dikii formalism.

# ÖZET

## SIFIR EĞRİLİK VE GEL'FAND-DIKII FORMULASYONLARI

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Eylül, 2004

İntegre edilebilir doğrusal olmayan kısmi türevli denklemler soliton teorisinde önemli bir rol oynamaktadır. Bu anlamda böyle denklemler çok çeşitli alanlarda ilgi çekmektedir. Bu denklemleri elde etmede değişik yaklaşımlar bulunmaktadır; bunlardan sıfır eğrilik ve Gel'fand-Dikii formulasyonları en geçerli olanlarıdır. Bu tezde, bu formulasyonları çalıştık ve bu formulasyonları bazı örneklere uyguladık.

*Anahtar sözcükler*: İntegre edilebilir sistemler, basit Lie cebir, soliton, sıfır eğrilik formulasyonu, Gel'fand-Dikii formulasyonu.

# Acknowledgement

I would like to express my sincere gratitude to my supervisor Prof. Dr. Metin Gürses to whom to study with is an honor. His efforts, encouragement and tolerance throughout my studies have been a major source of support. Without his valuable suggestions, excellent guidance and comments, this thesis could not be exist.

I would like to express my deep gratitude to Prof. Dr. A. Okay Çelebi for persuading me to be a mathematician. If I have not met him, I would be deprived of a second father who always listens with infinite patience and supports me during my personal and academic life.

I would like to thank Prof. Dr. Atalay Karasu and Asst. Prof. Dr. Konstantyn Zheltukhin who read this thesis during the preparation and gave valuable comments about this thesis.

I am so grateful to have the chance to thank my family for being with me in any situation, their encouragement, support, endless love and trust.

I am also grateful to Dr. Yosum Kurtulmaz for being a deep part of all my life.

I wish to thank Dr. Seçil Gergün for her endless helps about Latex and sharing her experiences with me.

Last but not least with my best feelings I would like to thank my close friends - especially Süleyman Tek, Özden Yurtseven and Fatma Altunbulak- for all their valuable help and for the warm atmosphere they created in the department and at home.

# Contents

# Chapter 1

# Introduction

The theory of solitons and the related theory of integrable nonlinear evolution equations have being studied by a large number of mathematicians and physicists ranging from algebraic geometry to applied hydrodynamics.

The study of solitary waves began with John Scott Russell's observations (1838). These observations inspired Russell to state all water waves in two classes, '*the great wave of translation*'(eventually called as solitary wave) and '*all other waves belong to the second or oscillatory order of waves*'[1]. His studies brought many essential results to the soliton theory:
i. Solitary waves, which are long waves of permanent form, exist.
ii. The speed of a solitary wave is given by

$$\nu = [g(h+\eta)]^{\frac{1}{2}}, \tag{1.1}$$

where $\eta$ is the height of the wave above the plane of the fluid, $h$ is the depth throughout the fluid and $g$ is the measure of the gravity. It is important in equation (1.1) that, the speed of a solitary wave is proportional to its amplitude.

In 1885, Korteweg and de Vries [2] derived the KdV equation describing the propagation of waves on the surface of a shallow channel,

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.2}$$

In 1965, while Zabusky and Kruskal [3] were studying the Fermi-Pasta-Ulam [4] problem of recurrence on a nonlinear lattice, the KdV equation arised. They found out that the periodic boundary conditions (initial data of a cosine function) came across to a series of pulses, each of which developed the solitary wave solution. Since the speed of the wave is directly proportional to the amplitude, the larger pulses travel faster than the smaller ones. When the faster ones catch the slower, they undergo nonlinear interaction but finally they reappear unchanged, retaining their width, height and speed. Because of the particle like nature of these interacting solitary waves, Zabusky and Kruskal gave the name ‘ soliton ’ to describe the pulses.

**Definition 1.1.** *A solution of any nonlinear partial differential equation or a system is called a soliton if*
***i.*** *it represents a wave of permanent form,*
***ii.*** *it is localized, so that it decays or approaches to a constant at infinity,*
***iii.*** *it can interact strongly with other solitons and retain its identity.*

In 1967, Gardner, Greene, Kruskal and Miura [5] used the ideas of direct and inverse scattering and hence derived a method of solution for the KdV equation. In 1968, Lax [6], generalized the results of Gardner, Greene, Kruskal, Miura and introduced the concept of a Lax pair. Lax approach, considers two operators $L$ and $A$, where $L$ is the operator of the spectral problem and $A$ is the operator of an associated time evolution equation,

$$Lv = \lambda v, \tag{1.3}$$

$$v_t = Av. \tag{1.4}$$

If we take time derivative of (1.3), use (1.4) and choose $\lambda_t = 0$, we get

$$L_t = [A, L] \tag{1.5}$$

where $[A, L] = AL - LA$ (the commutator of $A$ and $L$). The equation (1.5) is called *the Lax equation* and the operators $L$ and $A$ are called the *Lax pair.* The Lax equation corresponds to a nonlinear evolution equation if $L$ and $A$ are correctly chosen. Lax proposed a representation for the KdV equation:

**Example 1.1.** A Lax pair for the KdV equation is

$$L = D_x^2 + u, \tag{1.6}$$

$$A = (\gamma + u_x) - (4\lambda + 2u)D_x, \tag{1.7}$$

where $\gamma$ is a constant and $\lambda$ is the eigenvalue of the Sturm-Liouville problem $Lu = \lambda u$ of KdV equation. The KdV equation therefore can be written as

$$L_t + [L, A] = u_t + 6uu_x + u_{xxx}. \tag{1.8}$$

However there are difficulties with the method of Lax. First, one must guess a suitable $A$ for a given $L$ to satisfy (1.3) and (1.4). Second, it is usually hard to work with differential operators.

In 1971, Zakharov and Shabat [7], introduced the Lax pair for the nonlinear Schrödinger equation. Being influenced by the ideas of the Princeton Group (Gardner, Greene, Kruskal and Miura) and by the ideas of Zakharov and Shabat; in 1974 Ablowitz, Kaup, Newell and Segur [8] developed a new method (called as AKNS Scheme) as an alternative to Lax approach. The AKNS scheme includes a wide range of solvable nonlinear evolution equations, such as the sine-Gordon equation and mKdV equation. This technique can be formulated by considering two linear equations;

$$\begin{aligned} \phi_x &= U\phi, \\ \phi_t &= V\phi, \end{aligned} \tag{1.9}$$

where $\phi$ is a 2-dimensional vector and $U$, $V$ are $2\times 2$ matrices. Using compatibility condition $\phi_{xt} = \phi_{tx}$, for (1.9) we find

$$U_t - V_x + [U, V] = 0, \qquad [U, V] \equiv UV - VU, \tag{1.10}$$

which is the *zero curvature condition*[19].

The soliton theory has been applied to many areas of mathematics and physics such as algebraic geometry (the solution of the Schottky problem), group theory (the discovery of quantum groups), topology (the connection of Jones polynomials with integrable models), quantum gravity (the connection of the KdV equation with integrable models).

In Chapter 2, we studied the zero curvature formalism which is the generalization of the AKNS scheme [19]. The AKNS scheme includes the nonlinear Schrödinger hierarchy, the KdV hierarchy and the sine-Gordon equation. We give the nonlinear Schrödinger and KdV hierarchies with the use of recursion operators. In this scheme, the potentials are independent of the spectral parameter. However there are systems where the potentials depend on the spectral parameter, such as Ma-Zhou system [9] and Tam-Zhang system [10]. The zero curvature formalism is based on the Lax equation for $n \times n$ matrix valued functions, which form the basis of a matrix algebra. In AKNS formalism this algebra is $sl(2,\mathbb{R})$ algebra and in Tam-Zhang System it is $su(1,1)$ algebra.

In Chapter 3, we use the matrix representation of Lie algebras. In order to obtain nonlinear partial differential equations on homogeneous spaces, Fordy and Kulish [13] have obtained the nonlinear Schrödinger equations on homogeneous spaces and Fordy [14] has obtained the derivative nonlinear Schrödinger equations. Using similar approach, we use a simple Lie algebra valued soliton connection, introduced by Gürses, Oğuz and Salihoğlu in [12]. In Section 3.2, we first introduce the usual Cartan-Weyl basis which is the standard form of the commutation relations for a semisimple Lie algebra. Let $g$ be the Lie algebra of a Lie group $G$. Then $g$ can be identified as the decomposition of Cartan subalgebra $h$ of $g$ and the complement of the Cartan subalgebra in $g$;

$$g = h \bigoplus h^C. \tag{1.11}$$

This decomposition leads to the usual Cartan-Weyl basis [11],

$$[H_a, H_b] = 0 \quad \text{for all} \quad a, b = 1, 2, ..., p, \tag{1.12a}$$

$$[H_a, E_{\overline{\alpha}}] = \overline{\alpha}_a E_{\overline{\alpha}}, \tag{1.12b}$$

$$[E_{\overline{\alpha}}, E_{\overline{\beta}}] = \sum_{a=1}^{p} C^a_{\overline{\alpha}+\overline{\beta}} H_a \qquad \text{if} \quad \overline{\alpha} + \overline{\beta} = 0, \tag{1.12c}$$

$$[E_{\overline{\alpha}}, E_{\overline{\beta}}] = C^{\overline{\alpha}+\overline{\beta}}_{\overline{\alpha}\overline{\beta}} E_{\overline{\alpha}+\overline{\beta}} \qquad \text{if} \quad \overline{\alpha} + \overline{\beta} \neq 0, \tag{1.12d}$$

where $p$ is the rank of the algebra, $H_a$ ' s are bases of the Cartan subalgebra,

$E_{\overline{\alpha}}$ ' s are the bases of the complement of the Cartan subalgebra, $\overline{\alpha}_a$ and $C$ are the structure constants of the commutation relations.

In Section 3.3, we give the Cartan-Weyl basis on homogeneous spaces, which is constructed from a new decomposition due to the following definitions.

**Definition 1.2.** *Let $G$ be a Lie group. A homogeneous space of $G$, is any differentiable manifold $M$, on which $G$ acts transitively [13].*

**Definition 1.3.** *The subgroup of $G$ which leaves a given point $p_0 \in M$ fixed, is called the isotropy group at $p_0$ and is defined by [13]*

$$K \equiv K_{p_0} = \{g \in G : g \cdot p_0 = p_0\} \tag{1.13}$$

If $K$ is an isotropy group of some $p_0 \in M$, then $M$ can be identified as a coset space $G/K$. Let $g$ and $k$ be the Lie algebras of $G$ and $K$ respectively. Let $m$ be the vector space, complement of $k$ in $g$, then

$$g = k \bigoplus m. \tag{1.14}$$

and $m$ is identified as the tangent space of $G/K$ at $p_0 \in M$ [13]. When $g$ satisfies the conditions

$$g = k \bigoplus m, \qquad [k,k] \subset k, \qquad [k,m] \subset m, \tag{1.15}$$

then $M = G/K$ is called a '*reductive homogeneous space*' [12]. When $g$ satisfies the conditions

$$g = k \bigoplus m, \qquad [k,k] \subset k, \qquad [k,m] \subset m, \qquad [m,m] \subset k, \tag{1.16}$$

then $M = G/K$ is called a '*symmetric space*' [12].

The comparison of the Cartan-Weyl basis on homogeneous spaces with the usual Cartan-Weyl basis is discussed in Section 3.4 . In Section 3.5, to obtain nonlinear partial differential equations, we deal with the simple Lie algebra valued soliton connection which is defined as:

**Definition 1.4.** *The simple Lie algebra valued soliton connection 1-form* $\Omega$*, associated to a reductive homogeneous space* $G/K$ *with the generators* $H_a$*,* $E_D$ *is defined as*

$$\Omega = (ik\lambda H_s + Q^A E_A)dx + (A^a H_a + B^A E_A + C^D E_D)dt, \tag{1.17}$$

*where* $K$ *is the isotropy group of* $G$*,* $\lambda$ *is the spectral parameter,* $k$ *is a constant not depending on* $\lambda$*,* $s \in \{1, 2, ...p\}$ *is a fixed constant,* $Q^A(x,t)$ *is potential,* $A^a$*,* $B^A$ *and* $C^D$ *are arbitrary functions of* $x$*,* $t$ *and* $\lambda$*.*

The Lax equation in differential forms and R curvature 2-form are respectively;

$$d\phi = \Omega\phi, \qquad R = d\Omega + \Omega \wedge \Omega. \tag{1.18}$$

Here $\Omega$ is the flat connection, that is

$$d\Omega + \Omega \wedge \Omega = 0. \tag{1.19}$$

which is the *zero curvature condition.* AKNS scheme and the corresponding zero curvature condition in (1.10) are special cases of (1.18). In this case $g = sl(2, \mathbb{R})$ and

$$H_s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

In Section 3.5.1, we determine the catalogue of the structure constants in the zero curvature condition. In Section 3.5.2, we determine a system of integrable partial differential equations, with the corresponding recursion operator by the use of the catalogue. In Section 3.6, we obtain the integrable evolution equations without using the catalogue. There is another way to obtain integrable systems, so called the Gelf'and-Dikkii formalism. In Chapter 4, we deal with the Gel'fand-Dikii formalism which gives the direct method to determine the function $A$ in the Lax equation (1.5). This formalism gives a construction of all Lax pairs, based on the calculation of fractional powers of operator $L$. On an algebra $\mathcal{G}$, let '*' be a non-commutative, associative binary product and $F$, $G$ be $\mathcal{G}$-valued functions, then define a bracket $\{,\}_{\mathcal{G}}$ as

$$\{F,G\}_{\mathcal{G}} := \frac{1}{2\kappa}(F*G - G*F), \quad \kappa \in \mathbb{R} \tag{1.20}$$

which satisfies skew-symmetry, Jacobi identity and Leibniz rule. Let $L$ be $\mathcal{G}$-valued Lax operator which is a polynomial of some variable. The Lax equation is defined as

$$\frac{\partial L}{\partial t} = \{A, L\}_{\mathcal{G}}, \tag{1.21}$$

for some $\mathcal{G}$- valued function $A$. In order to obtain $A$, we find $\overline{A}$ s.t

$$\{L, \overline{A}\}_{\mathcal{G}} = 0. \tag{1.22}$$

Apart from the matrix algebra we can take $\overline{A} = L^{\frac{n}{m}}$, then (1.22) holds, where $n \neq am$; $a$, $n \in \mathbb{Z}$. We put $A = (\overline{A})_{\geqslant k}$ that is

$$A = (L^{\frac{n}{m}})_{\geqslant k}. \tag{1.23}$$

So we obtain a consistent equation (1.21). Here the restriction of being larger or equal to $k$ is for $A$ to be the polynomial part of $L^{\frac{n}{m}}$ except first $k-1$ terms. For the matrix algebra we find $\overline{A}$ by solving $\{L, \overline{A}\}_{\mathcal{G}} = 0$, then we set

$$A = (\overline{A})_{\geqslant k}. \tag{1.24}$$

The Gel'fand-Dikii formalism makes use of some algebras. In this work we use the pseudo-differential algebra [15], polynomial algebra [16], [20], Moyal [17] and matrix algebras [15]. If $\mathcal{G}$ is the pseudo-differential algebra, then the bracket $\{F,G\}_{\mathcal{G}}$ defined in (1.20) corresponds to the usual commutator provided that '*' is the operational product, $\kappa = \frac{1}{2}$ and $F$, $G$ are two pseudo-differential operators. The Lax operator of the pseudo-differential algebra is a series of a differential operator,

$$L = D_x^m + u_{m-2}D_x^{m-2} + ... + u_1 D_x + u_0, \tag{1.25}$$

where $u_i$, $i = 0, 1, ..., m-2$ are functions of $x$ and $t$. The Lax equation is

$$L_{t_n} = [A_n, L], \tag{1.26}$$

where operators $A_n$ is defined to be

$$A_n := (L^{\frac{n}{m}})_+, \tag{1.27}$$

and $'+'$ means the polynomial part of $L^{\frac{n}{m}}$.

If $\mathcal{G}$ is the polynomial algebra, then the bracket $\{F, G\}_{\mathcal{G}}$ defined in (1.20) corresponds to the standard Poisson bracket with $\kappa = \frac{1}{2}$ and $F$, $G$ are two differentiable functions. The Lax operator of the polynomial algebra is a series of an auxiliary variable (momentum $p$),

$$L = p^{N-1} + \sum_{i=-1}^{N-2} p^i S_i(x, t). \tag{1.28}$$

The Lax equation is (see Section 4.3)

$$\frac{\partial L}{\partial t_n} = \{(L^{\frac{n}{N-1}})_{\geqslant -k+1}; L\}_k, \tag{1.29}$$

where $n = j + l(N-1)$ and $j = 1, 2, .., (N-1)$, $\quad l \in \mathbb{N}$.

If $\mathcal{G}$ is the Moyal algebra, then the bracket $\{F, G\}_{\mathcal{G}}$ defined in (1.20) corresponds to the Moyal bracket provided that '*' is the Moyal product and $F$, $G$ are two differentiable functions. Similar to the case of polynomial algebra, the Lax operator is a series of momentum $p$,

$$L_n = p^n + u_1(x) * p^{n-1} + .... + u_n(x) + u_{n+1}(x) * p^{-1} + ... \tag{1.30}$$

and the Lax equation is (see Section 4.4)

$$\frac{\partial L_n}{\partial t_k} = \{L_n, (L^{\frac{k}{n}})_{\geqslant m}\}_\kappa, \tag{1.31}$$

where $k \neq an$; $k$, $a$ are integers and $m = 0, 1, 2...$

If $\mathcal{G}$ is the matrix algebra, then the bracket $\{F, G\}_\mathcal{G}$ defined in (1.20) corresponds to the usual commutator provided that '*' is the matrix multiplication, $F$, $G$ are $n \times n$ matrices and $\kappa = \frac{1}{2}$. The Lax operator is a series of a spectral constant. For the matrix algebra, the function $A$ in the Lax equation (1.21) takes the form

$$A = (\bar{A})_{\geqslant k}. \tag{1.32}$$

For each integrable equation, we have an infinite hierarchy of symmetries. In order to determine the hierarchies of symmetries of a system of differential equations, there are different approaches. In this work, we will deal with the use of 'recursion operators' defined [18] as:

**Definition 1.5.** *Let*

$$u_t = F(t, x, u, u_x, ...., u_{nx}), \tag{1.33}$$

*be a system of differential equations. A recursion operator for* (1.33) *is a linear operator,* $\mathbb{R} : A^q \longrightarrow A^q$*, in the space of q-tuples of differential function with the property that whenever* $Q$ *is an evolutionary symmetry of* (1.33)*, so is* $\bar{Q}$ *with* $\bar{Q} = \mathbb{R}Q$*. If* (1.33) *admits a nonconstant recursion operator, this system is called integrable.*

Therefore, if we know a recursion operator $\mathbb{R}$ for a system of differential equations, we can generate an infinite family of symmetries at once, by applying the recursion operator successively to an initial symmetry $Q_0$;

$$Q_i = \mathbb{R}^i Q_0, \quad i = 0, 1, 2... \tag{1.34}$$

where each $Q_i$, $i = 0, 1, 2...$ is the symmetries of the partial differential equations. Generally, $\mathbb{R}$ is $q \times q$ matrix of differential operators.

# Chapter 2

# Zero Curvature Formalism

## 2.1 AKNS scheme

AKNS (Ablowitz, Kaup, Newell and Segur) scheme [8] is a generalization of Sturm-Liouville problem to $2 \times 2$ eigenvalue problem. It is a linear eigenvalue problem defined as

$$\begin{aligned} \phi_x &= U\phi, \\ \phi_t &= V\phi, \end{aligned} \tag{2.1}$$

where $\phi$ is a 2-dimensional vector and $U, V$ are $2 \times 2$ matrices. Let

$$U = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix}, \tag{2.2}$$

and

$$V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.3}$$

where $\lambda$ is a spectral parameter; $q(x,t)$, $r(x,t)$ are potentials; $A$, $B$, $C$, $D$ are functions of $q$, $r$, $\lambda$ and the derivatives of $q$, $r$ with respect to $x$ and $t$. Then

$$\phi_x = \begin{pmatrix} \phi_{1,x} \\ \phi_{2,x} \end{pmatrix} = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{2.4}$$

$$\phi_{1,x} = -i\lambda\phi_1 + q\phi_2, \tag{2.5}$$

$$\phi_{2,x} = i\lambda\phi_1 + r\phi_2, \tag{2.6}$$

and

$$\phi_t = \begin{pmatrix} \phi_{1,t} \\ \phi_{2,t} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{2.7}$$

$$\phi_{1,t} = A\phi_1 + B\phi_2, \tag{2.8}$$

$$\phi_{2,t} = C\phi_1 + D\phi_2. \tag{2.9}$$

Using compatibility condition $\phi_{xt} = \phi_{tx}$, for (2.1) we find

$$U_t - V_x + [U, V] = 0, \qquad [U, V] \equiv UV - VU, \tag{2.10}$$

which is the *zero curvature condition.* To express these equations in terms of A, B, C and D; we have the following proposition.

**Proposition 2.1.** *The zero curvature condition* (2.10) *reduces to the following equations for the functions $A$, $B$ and $C$, where $D = -A$*

$$A_x = qC - rB, \tag{2.11a}$$

$$B_x + 2i\lambda B = q_t - 2qA, \tag{2.11b}$$

$$C_x - 2i\lambda C = r_t + 2rA. \tag{2.11c}$$

### 2.1.1 The nonlinear Schrödinger and KdV hierarchies

Since $\lambda$ is a free parameter, we can assume that $A$, $B$, $C$ have Taylor series expansion on $\lambda$.

$$A = \sum_{j=0}^{n} a_j \lambda^{n-j}, \qquad B = \sum_{j=0}^{n} b_j \lambda^{n-j}, \qquad C = \sum_{j=0}^{n} c_j \lambda^{n-j}. \tag{2.12}$$

Then using Proposition (2.1) we have the following proposition

**Proposition 2.2.** *Let the functions $A$, $B$, $C$ in the equations* (2.11a), (2.11b), (2.11c) *have expansions as in* (2.12), *then we obtain the system of equations below*

$$a_{l,x} = qc_l - rb_l, \qquad l = 0, ..., n, \tag{2.13a}$$

$$b_{l,x} + 2ib_{l+1} = -2qa_l \qquad l = 0, .., n-1, \tag{2.13b}$$

$$b_{n,x} = q_t - 2qa_n, \quad b_0 = 0, \tag{2.13c}$$

$$c_{l,x} - 2ic_{l+1} = 2ra_l, \qquad l = 0, .., n-1, \tag{2.13d}$$

$$c_{n,x} = r_t + 2ra_n, \quad c_0 = 0. \tag{2.13e}$$

It is possible to write (2.13b) and (2.13d) in terms of (2.13a) as follows:

$$\begin{aligned} b_{l+1} &= \tfrac{i}{2}[b_{l,x} + 2qa_l] = \tfrac{i}{2}[b_{l,x} + 2qD_x^{-1}(qc_l) - 2qD_x^{-1}(rb_l)], \\ c_{l+1} &= -\tfrac{i}{2}[c_{l,x} - 2ra_l] = \tfrac{i}{2}[2rD_x^{-1}(qc_l) - 2rD_x^{-1}(rb_l) - c_{l,x}]. \end{aligned}$$

Then in a matrix form we have

$$\begin{pmatrix} b_{l+1} \\ c_{l+1} \end{pmatrix} = \frac{i}{2} \begin{pmatrix} D_x - 2qD_x^{-1}r & 2qD_x^{-1}q \\ -2rD_x^{-1}r & -D_x + 2rD_x^{-1}q \end{pmatrix} \begin{pmatrix} b_l \\ c_l \end{pmatrix}. \tag{2.14}$$

Denote

$$\Psi = \frac{i}{2} \begin{pmatrix} D_x - 2qD_x^{-1}r & 2qD_x^{-1}q \\ -2rD_x^{-1}r & -D_x + 2rD_x^{-1}q \end{pmatrix}, \qquad z_l = \begin{pmatrix} b_l \\ c_l \end{pmatrix}. \tag{2.15}$$

Then $z_{l+1} = \Psi z_l$ or

$$z_n = \Psi^{n-1} z_1 = \Psi^n z_0. \tag{2.16}$$

where $z_0 = \begin{pmatrix} b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This leads to the following proposition.

**Proposition 2.3.** *The evolution equations for $q$ and $r$ can be found by writing the equations* (2.13c) *and* (2.13e) *as:*

$$q_t = b_{n,x} + 2qD_x^{-1}(qc_n) - 2qD_x^{-1}(rb_n), \tag{2.17}$$

$$r_t = c_{n,x} - 2rD_x^{-1}(qc_n) + 2rD_x^{-1}(rb_n), \tag{2.18}$$

*or in $2 \times 2$ matrix form:*

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \begin{pmatrix} D_x - 2qD_x^{-1}(r) & 2qD_x^{-1}(q) \\ 2rD_x^{-1}(r) & D_x - 2rD_x^{-1}(q) \end{pmatrix} \begin{pmatrix} b_n \\ c_n \end{pmatrix}, \tag{2.19}$$

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \frac{2}{i}\sigma_3\Psi \begin{pmatrix} b_n \\ c_n \end{pmatrix} = \frac{2}{i}\mathbb{R}^{n+1}\sigma_3 \begin{pmatrix} b_0 \\ c_0 \end{pmatrix} \tag{2.20}$$

*where* $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *and*

$$\mathbb{R} = \sigma_3\Psi\sigma_3. \tag{2.21}$$

*Then*

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \frac{2}{i}\mathbb{R}^{n+1} \begin{pmatrix} b_0 \\ c_0 \end{pmatrix}, \tag{2.22}$$

*which are the evolution equations. Here $\mathbb{R}$ is called the recursion operator. This is the nonlinear Schrödinger hierarchy.*

**Example 2.1.** Case $n = 2$ : The nonlinear Schrödinger equations.

We have by (2.16), $z_1 = \Psi z_0$. Then

$$\begin{pmatrix} b_1 \\ c_1 \end{pmatrix} = \frac{i}{2} \begin{pmatrix} D_x - 2qD_x^{-1}r & 2qD_x^{-1}q \\ -2rD_x^{-1}r & -D_x + 2rD_x^{-1}q \end{pmatrix} \begin{pmatrix} b_0 \\ c_0 \end{pmatrix}.$$

So

$$\begin{aligned} b_1 &= \tfrac{i}{2}[b_{0,x} + 2q(D_x^{-1}(qc_0) - D_x^{-1}(rb_0))] = i\epsilon_1 q \\ c_1 &= \tfrac{i}{2}[-c_{0,x} + 2r(D_x^{-1}(qc_0) - D_x^{-1}(rb_0))] = i\epsilon_1 r \end{aligned}$$

where $\epsilon_1$ is a constant. Similarly $z_2 = \Psi z_1$ gives;

$$b_2 = \tfrac{i}{2}[b_{1,x} + 2qD_x^{-1}(qc_1) - 2qD_x^{-1}(rb_1)] = -\tfrac{1}{2}\epsilon_1 q_x, \tag{2.23}$$

$$c_2 = \tfrac{i}{2}[-c_{1,x} + 2rD_x^{-1}(qc_1) - 2rD_x^{-1}(rb_1)] = \tfrac{1}{2}\epsilon_1 r_x. \tag{2.24}$$

Therefore the corresponding evolution equations directly come from (2.19)

$$\begin{aligned} q_t &= b_{2,x} - 2q[D_x^{-1}(rb_2) - D_x^{-1}(qc_2)] \\ &= -\tfrac{1}{2}\epsilon_1 q_{xx} + \epsilon_1 q D_x^{-1}(r_x q_x) \\ &= -\tfrac{1}{2}\epsilon_1 q_{xx} + \epsilon_1 q^2 r. \end{aligned}$$

By the same procedure $r_t$ can be found. Hence

$$q_t = -\frac{1}{2}\epsilon_1 q_{xx} + \epsilon_1 q^2 r, \tag{2.25a}$$

$$r_t = \frac{1}{2}\epsilon_1 r_{xx} - \epsilon_1 r^2 q. \tag{2.25b}$$

If we set $r = \pm q^*, \epsilon_1 = 2i$, in (2.25a) then we have

$$iq_t - q_{xx} \pm 2q^2 q^* = 0,$$

which is *the nonlinear Schrödinger equation*. Here * is the complex conjugation.

**Example 2.2.** Case $n = 3$ : KdV and mKdV equations. If we assume the integration constant is not equal to zero in (2.14), then we have

$$\begin{aligned} b_2 &= -\tfrac{1}{2}\epsilon_1 q_x + i\epsilon_2 q, \\ c_2 &= \tfrac{1}{2}\epsilon_1 r_x + i\epsilon_2 r. \end{aligned} \tag{2.26}$$

where $\epsilon_2$ is a constant. By $z_3 = \Psi z_2$, we can find $b_3$ and $c_3$ as

$$\begin{aligned} b_3 &= \tfrac{i}{2}(b_{2,x} + 2qD_x^{-1}(qc_2) - 2qD_x^{-1}(rb_2)) \\ &= -\tfrac{i}{4}\epsilon_1 q_{xx} - \tfrac{1}{2}\epsilon_2 q_x + \tfrac{i\epsilon_1}{2} q[D_x^{-1}(rq_x + qr_x)] \\ &= -\tfrac{i}{4}\epsilon_1 q_{xx} - \tfrac{1}{2}\epsilon_2 q_x + \tfrac{i}{2}\epsilon_1 q^2 r + i\epsilon_3 q. \end{aligned} \tag{2.27}$$

where $\epsilon_3$ is an integration constant. Similarly

$$c_3 = -\tfrac{i}{4}\epsilon_1 r_{xx} + \tfrac{1}{2}\epsilon_2 r_x + \tfrac{1}{2} i\epsilon_1 r^2 q + i\epsilon_3 r. \tag{2.28}$$

So the corresponding evolution equations directly come from (2.19)

$$q_t + \frac{1}{4}i\epsilon_1[q_{xxx} - 6rqq_x] + \frac{1}{2}\epsilon_2[q_{xx} - 2q^2 r] - i\epsilon_3 q_x - 2\epsilon_4 q = 0, \tag{2.29a}$$

$$r_t + \frac{1}{4}i\epsilon_1[r_{xxx} - 6qrr_x] + \frac{1}{2}\epsilon_2[2r^2 q - r_{xx}] - i\epsilon_3 r_x - 2\epsilon_4 r = 0. \tag{2.29b}$$

By choosing the constants $\epsilon_i$'s, $i = 1, 2, 3, 4$ properly in (2.29a), we respectively get the KdV, mKdV and nonlinear Schrödinger equations.

$$q_t + q_{xxx} + 6qq_x = 0, \qquad (\epsilon_1 = -4i, \quad \epsilon_2 = \epsilon_3 = \epsilon_4 = 0, \quad r = -1,)$$
$$q_t - 6q^2q_x + q_{xxx} = 0, \qquad (\epsilon_1 = -4i, \quad \epsilon_2 = \epsilon_3 = \epsilon_4 = 0, \quad r = q,)$$
$$iq_t + q_{xx} - 2q^2q^* = 0, \qquad (\epsilon_1 = \epsilon_3 = \epsilon_4 = 0, \quad \epsilon_2 = -2i, \quad r = \pm q^*.)$$

### 2.1.2 The sine-Gordon equation

In this section, we will consider the case when the functions $A$, $B$, $C$ have terms containing inverse powers of $\lambda$. In this case we will obtain different nonlinear partial differential equations.

**Proposition 2.4.** *Let* $A = \frac{a(x,t)}{\lambda}$, $B = \frac{b(x,t)}{\lambda}$, $C = \frac{c(x,t)}{\lambda}$, *where* $a$, $b$, $c$ *are differentiable functions of* $x$ *and* $t$. *Then using compatibility conditions* (2.11a), (2.11b) *and* (2.11c) *we get,*

$$a_x = qc - rb, \tag{2.31a}$$
$$b_x = -2aq, \qquad 2ib = q_t, \tag{2.31b}$$
$$c_x = 2ar, \qquad -2ic = r_t. \tag{2.31c}$$

Using the Proposition (2.4) we have the following Corollary.

**Corollary 2.5.** *Let* $A = \frac{a(x,t)}{\lambda}$, $B = \frac{b(x,t)}{\lambda}$, $C = \frac{c(x,t)}{\lambda}$. *Assume that* $a = \frac{i}{4}\cos u$, $b = c = \frac{i}{4}\sin u$ *provided that* $q = -r = -\frac{1}{2}u_x$. *Then we obtain*

$$\sin u = u_{xt}.$$

*which is the sine-Gordon equation.*

**Proof:** Consider

$$a_x = qc - rb = q(-\frac{r_t}{2i}) - r(\frac{q_t}{2i}) = -\frac{1}{2i}(\frac{\partial(qr)}{\partial t})$$

then

$$a_x = -\frac{1}{4} i u_x u_{xt}. \tag{2.32}$$

On the other hand

$$a_x = \frac{\partial(\frac{1}{4} i \cos u)}{\partial x} = -\frac{1}{4} i u_x \sin u. \tag{2.33}$$

then combining (2.32) and (2.33), we get $\sin u = u_{xt}$. □

## 2.2 Ma-Zhou system

In the AKNS scheme, the potentials were taken as independent of the spectral parameter. In this section we will consider the case where the potentials depend on the spectral parameter. Consider a spectral problem

$$\begin{aligned} \phi_x &= U\phi, \\ \phi_t &= V\phi, \end{aligned} \tag{2.34}$$

where $\phi$ is a 2-dimensional vector and $U$, $V$ are $2 \times 2$ matrices. Let

$$U = \begin{pmatrix} \lambda & q \\ (\alpha + \beta\lambda) r & -\lambda \end{pmatrix}, \tag{2.35}$$

and

$$V = \begin{pmatrix} a & b \\ (\alpha + \beta\lambda) c & -a \end{pmatrix}, \tag{2.36}$$

where $\lambda$ is a spectral parameter; $\alpha$ and $\beta$ are arbitrary constants; $q$, $r$ are functions of $x$ and $t$; $a$, $b$, $c$ are functions of $q$, $r$, $\alpha$, $\beta$, $\lambda$ and the derivatives of $q$, $r$ with respect to $x$ and $t$ [9]. Using compatibility condition $\phi_{xt} = \phi_{tx}$ for (2.34) we have the *zero curvature condition*

$$U_t - V_x + [U, V] = 0. \tag{2.37}$$

**Proposition 2.6.** *The zero curvature condition* (2.37) *reduces to the following equations for the functions a, b and c,*

$$a_x = (\alpha + \beta\lambda)(qc - rb), \tag{2.38a}$$
$$b_x = 2\lambda b - 2aq + q_t, \tag{2.38b}$$
$$c_x = 2ar - 2\lambda c + r_t. \tag{2.38c}$$

Since $\lambda$ is a free parameter, assume $a$, $b$, $c$ are analytic in $\lambda$,

$$a = \sum_{j=0}^{n} a_j \lambda^{n-j}, \qquad b = \sum_{j=0}^{n} b_j \lambda^{n-j}, \qquad c = \sum_{j=0}^{n} c_j \lambda^{n-j}. \tag{2.39}$$

Then using Proposition (2.6) we have the following proposition.

**Proposition 2.7.** *Let the functions a, b, c in the equations* (2.38a), (2.38b), (2.38) *have expansions as in* (2.39) , *then we obtain the system of equations below,*

$$a_{l,x} = \alpha(qc_l - rb_l) + \beta(qc_{l+1} - rb_{l+1}), \quad l = 0, 1, ., n-1, \tag{2.40a}$$
$$\beta(qc_0 - rb_0) = 0, \tag{2.40b}$$
$$a_{n,x} = \alpha(qc_n - rb_n), \tag{2.40c}$$

$$b_{l,x} = 2b_{l+1} - 2\alpha q D_x^{-1}(qc_l - rb_l) - 2\beta q D_x^{-1}(qc_{l+1} - rb_{l+1}), \quad l = 0, 1, ., n-1, \tag{2.41a}$$
$$b_{n,x} = -2\alpha q D_x^{-1}(qc_n - rb_n) + q_t, \qquad b_0 = 0, \tag{2.41b}$$

$$c_{l,x} = 2\alpha r D_x^{-1}(qc_l - rb_l) + 2\beta r D_x^{-1}(qc_{l+1} - rb_{l+1}) - 2c_{l+1}, \quad l = 0, 1, ., n-1, \tag{2.42a}$$
$$c_{n,x} = 2\alpha r D_x^{-1}(qc_n - rb_n) + r_t, \qquad c_0 = 0. \tag{2.42b}$$

Assume $a_0 = 1$, since $b_0 = c_0 = 0$ ; the equations (2.41a) and (2.42a) lead to

$$b_1 = q, \quad c_1 = r. \tag{2.43}$$

For $a_i, b_i$ and $c_i$, $i \geqslant 2$ we will consider the recursion equation. The equations (2.41a) and (2.42a) can be written in matrix form as follows,

$$\begin{pmatrix} 2\alpha q D_x^{-1} q & D_x - 2\alpha q D_x^{-1} r \\ D_x - 2\alpha r D_x^{-1} q & 2\alpha r D_x^{-1} r \end{pmatrix} \begin{pmatrix} c_l \\ b_l \end{pmatrix} = \begin{pmatrix} -2\beta q D_x^{-1} q & 2 + 2\beta q D_x^{-1} r \\ -2 + 2\beta r D_x^{-1} q & -2\beta r D_x^{-1} r \end{pmatrix} \begin{pmatrix} c_{l+1} \\ b_{l+1} \end{pmatrix}. \tag{2.44}$$

This leads to the following proposition.

**Proposition 2.8.** *The relation between $z_{l+1}$ and $z_l$ is given by the operator $\Psi$ as,*

$$z_{l+1} = \Psi z_l, \qquad l = 0, ..n - 1. \tag{2.45}$$

*where*

$$z_l = \begin{pmatrix} c_l \\ b_l \end{pmatrix},$$

$$\Psi = \begin{pmatrix} -\frac{1}{2}D_x + \alpha r D_x^{-1} q - \frac{1}{2}\beta r D_x^{-1} q D_x & -\frac{1}{2}\beta r D_x^{-1} r D_x - \alpha r D_x^{-1} r \\ \alpha q D_x^{-1} q - \frac{1}{2}\beta q D_x^{-1} q D_x & \frac{1}{2}D_x - \alpha q D_x^{-1} r - \frac{1}{2}\beta q D_x^{-1} r D_x \end{pmatrix}.$$

**Proof:** By the equation (2.44), denoting,

$$z_l = \begin{pmatrix} c_l \\ b_l \end{pmatrix},$$

$$M = \begin{pmatrix} 2\alpha q D_x^{-1} q & D_x - 2\alpha q D_x^{-1} r \\ D_x - 2\alpha r D_x^{-1} q & 2\alpha r D_x^{-1} r \end{pmatrix},$$

$$J = \begin{pmatrix} -2\beta q D_x^{-1} q & 2 + 2\beta q D_x^{-1} r \\ -2 + 2\beta r D_x^{-1} q & -2\beta r D_x^{-1} r \end{pmatrix},$$

then we have

$$Mz_l = Jz_{l+1}; \qquad l = 0, .., n-1, \tag{2.48a}$$

$$z_{l+1} = J^{-1}Mz_l = \Psi z_l, \qquad l = 0, ..n-1. \tag{2.48b}$$

Here

$$J^{-1} = \frac{1}{2}\begin{pmatrix} -\beta r D_x^{-1} r & -1 - \beta r D_x^{-1} q \\ 1 - \beta q D_x^{-1} r & -\beta q D_x^{-1} q \end{pmatrix}. \tag{2.49}$$

Hence

$$\Psi = \begin{pmatrix} -\frac{1}{2}D_x + \alpha r D_x^{-1} q - \frac{1}{2}\beta r D_x^{-1} q D_x & -\frac{1}{2}\beta r D_x^{-1} r D_x - \alpha r D_x^{-1} r \\ \alpha q D_x^{-1} q - \frac{1}{2}\beta q D_x^{-1} q D_x & \frac{1}{2}D_x - \alpha q D_x^{-1} r - \frac{1}{2}\beta q D_x^{-1} r D_x \end{pmatrix}.\square \tag{2.50}$$

It should be noted that we always need to select zero constants for integration in deriving $a_j$, $b_j$, $c_j$ , $j = 1, ..n-1$; that is we require that

$$a_j|_{[u_j]=0} = b_j|_{[u_j]=0} = c_j|_{[u_j]=0} = 0, \quad \text{where} \quad u = (q\ \ r)^T, \quad [u] = (u, u_x, ....).$$

For instance

$$z_2 = \begin{pmatrix} c_2 \\ b_2 \end{pmatrix} = \Psi z_1 = \begin{pmatrix} -\frac{1}{2}r_x + \alpha r D_x^{-1}(qr - rq) - \frac{1}{2}\beta r D_x^{-1}(rq_x + qr_x) \\ \alpha q D_x^{-1}(qr - rq) + \frac{1}{2}q_x - \frac{1}{2}\beta q D_x^{-1}(rq_x + qr_x) \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{1}{2}r_x - \frac{1}{2}\beta r^2 q \\ \frac{1}{2}q_x - \frac{1}{2}\beta q^2 r \end{pmatrix}.$$

The evolution equations for $q$ and $r$ can be found by writing the equations (2.41b) and (2.42b) as,

$$q_t = b_{n,x} + 2\alpha q D_x^{-1} q c_n - 2\alpha q D_x^{-1} r b_n,$$
$$r_t = c_{n,x} - 2\alpha r D_x^{-1} q c_n + 2\alpha r D_x^{-1} r b_n.$$

So we have

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \begin{pmatrix} 2\alpha q D_x^{-1} q & D_x - 2\alpha q D_x^{-1} r \\ D_x - 2\alpha r D_x^{-1} q & 2\alpha r D_x^{-1} r \end{pmatrix}\begin{pmatrix} c_n \\ b_n \end{pmatrix}.$$

Hence

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = Mz_n = Jz_{n+1}. \tag{2.52}$$

**Proposition 2.9.** *The evolution equations can be determined as*

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = Jz_{n+1} = \mathbb{R}^n \begin{pmatrix} 2q \\ -2r \end{pmatrix}, \tag{2.53}$$

*where* $\mathbb{R}$ *is the recursion operator;*

$$\mathbb{R} = MJ^{-1} = \Psi^{-1} = \begin{pmatrix} \frac{1}{2}D_x - \alpha q D_x^{-1} r - \frac{1}{2}\beta D_x q D_x^{-1} r & -\alpha q D_x^{-1} q - \frac{1}{2}\beta D_x q D_x^{-1} q \\ \alpha r D_x^{-1} r - \frac{1}{2}\beta D_x r D_x^{-1} r & -\frac{1}{2}D_x + \alpha r D_x^{-1} q - \frac{1}{2}\beta D_x r D_x^{-1} q \end{pmatrix}.$$

**Proof:** The equation (2.48b) gives $Jz_{n+1} = J(J^{-1}M)^n z_1$, where

$$z_1 = \begin{pmatrix} r \\ q \end{pmatrix}.$$

Then we seek for the validity of the equality in the claim as,

$$\begin{aligned} J(J^{-1}M)^n \begin{pmatrix} r \\ q \end{pmatrix} &= (MJ^{-1})^n \begin{pmatrix} 2q \\ -2r \end{pmatrix}, \\ M(J^{-1}M)^{n-1} \begin{pmatrix} r \\ q \end{pmatrix} &= M(J^{-1}M)^{n-1} J^{-1} \begin{pmatrix} 2q \\ -2r \end{pmatrix}, \\ \begin{pmatrix} r \\ q \end{pmatrix} &= J^{-1} \begin{pmatrix} 2q \\ -2r \end{pmatrix}. \end{aligned}$$

Consider

$$J^{-1} \begin{pmatrix} 2q \\ -2r \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\beta r D_x^{-1}(2rq - 2qr) + r \\ \frac{1}{2}\beta q D_x^{-1}(2qr - 2rq) + q \end{pmatrix} = \begin{pmatrix} r \\ q \end{pmatrix}. \tag{2.55}$$

Hence

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = Mz_n = Jz_{n+1} = \mathbb{R}^n \begin{pmatrix} 2q \\ -2r \end{pmatrix}, \tag{2.56}$$

which are the *evolution equations.* □

**Example 2.3.** The first four systems of the hierarchy.

For the case $n = 0$ we have,

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \begin{pmatrix} 2q \\ -2r \end{pmatrix}. \tag{2.57}$$

For the case $n = 1$ we have,

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \begin{pmatrix} q_x \\ r_x \end{pmatrix}.$$

For the case $n = 2$ we have,

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \mathbb{R}^2 \begin{pmatrix} 2q \\ -2r \end{pmatrix} = \begin{pmatrix} \frac{1}{2}q_{xx} - \alpha q^2 r - \beta q q_x r - \frac{1}{2}\beta q^2 r_x \\ -\frac{1}{2}r_{xx} + \alpha r^2 q - \beta r r_x q - \frac{1}{2}\beta r^2 q_x \end{pmatrix}.$$

For the case $n = 3$ we have,

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = \mathbb{R}^3 \begin{pmatrix} 2q \\ -2r \end{pmatrix} = \mathbb{R} \begin{pmatrix} \frac{1}{2}q_{xx} - \alpha q^2 r - \beta q q_x r - \frac{1}{2}\beta q^2 r_x \\ -\frac{1}{2}r_{xx} + \alpha r^2 q - \beta r r_x q - \frac{1}{2}\beta r^2 q_x \end{pmatrix}.$$

Hence

$$\begin{aligned} q_t =& \frac{1}{4}q_{xxx} - \frac{3}{4}\beta r(q_x)^2 - \frac{3}{4}\beta q q_x r_x - \frac{3}{4}\beta q r q_{xx} - \frac{3}{2}\alpha q r q_x + \\ & \frac{3}{4}\alpha\beta q^3 r^2 + \frac{9}{8}\beta^2 r^2 q^2 q_x + \frac{3}{4}\beta^2 q^3 r r_x, \\ r_t =& \frac{1}{4}r_{xxx} + \frac{3}{4}\beta q(r_x)^2 + \frac{3}{4}\beta r r_x q_x + \frac{3}{4}\beta q r r_{xx} - \frac{3}{2}\alpha q r r_x - \\ & \frac{3}{4}\alpha\beta q^2 r^3 + \frac{9}{8}\beta^2 r^2 q^2 r_x + \frac{3}{4}\beta^2 r^3 q q_x. \end{aligned}$$

All systems in the hierarchy (2.56), except the first system (2.57), are exactly the coupled AKNS-Kaup -Newell systems in the hierarchy. Therefore the system (2.56) is another expression for the coupled AKNS-Kaup-Newell hierarchy.

## 2.3 Tam-Zhang system

In this section, we again cover the case where the potentials depend on the spectral parameter. Consider a spectral problem so that $deg(U) = 2$, where $deg(U)$ is the highest degree of $\lambda$. Let

$$\begin{aligned} \phi_x &= U\phi, \\ \phi_t &= V\phi. \end{aligned} \tag{2.60}$$

where $\phi$ is a 2-dimensional vector and $U$, $V$ are as follows.

$$U = \lambda^2 e_3 + \lambda q e_1 + \lambda r e_2, \tag{2.61a}$$

$$V = a e_3 + b e_1 + c e_2, \tag{2.61b}$$

with the commutation relations among the base elements of the $su(1,1)$ algebra

$$[e_1, e_2] = -2e_3, \qquad [e_1, e_3] = -2e_2, \qquad [e_2, e_3] = -2e_1,$$

where $\lambda$ is a spectral parameter; $q$, $r$ are functions of $x$, $t$; $a$, $b$, $c$ are functions of $q$, $r$, $\lambda$ and the derivatives of $q$, $r$ with respect to $x$ and $t$ [10]. The compatibility condition of the system (2.60) gives us the *zero curvature condition*

$$U_t - V_x + [U, V] = 0. \tag{2.62}$$

**Proposition 2.10.** *The zero curvature condition* (2.62) *reduces to the following equations for the functions $a$, $b$ and $c$,*

$$a_x = 2\lambda br - 2\lambda cq, \tag{2.63a}$$

$$b_x = 2\lambda^2 c - 2\lambda ar + \lambda q_t, \tag{2.63b}$$

$$c_x = 2\lambda^2 b - 2\lambda aq + \lambda r_t. \tag{2.63c}$$

Since $\lambda$ is a free parameter, we can assume that $a$, $b$ and $c$ are analytic in $\lambda$. Then

$$a = \sum_{j=0}^{n} a_j \lambda^{n-j}, \qquad b = \sum_{j=0}^{n} b_j \lambda^{n-j}, \qquad c = \sum_{j=0}^{n} c_j \lambda^{n-j}. \tag{2.64}$$

Using Proposition (2.10) we have

**Proposition 2.11.** *Let the functions a, b, c in the equations* (2.63a), (2.63b), (2.63c) *have expansions as in* (2.64)*, then we respectively have the system of equations*

$$a_{l,x} = 2rb_{l+1} - 2qc_{l+1}, \qquad l = 0,..,n-1, \tag{2.65a}$$

$$2rb_0 - 2qc_0 = 0, \qquad a_{n,x} = 0, \tag{2.65b}$$

$$b_{l,x} = 2c_{l+2} - 4rD_x^{-1}(rb_{l+2} - qc_{l+2}), \qquad l = 0,1,..n-2, \tag{2.66a}$$

$$c_0 = 0, \qquad 2c_1 - 4rD_x^{-1}(rb_1 - qc_1) = 0, \tag{2.66b}$$

$$b_{n,x} = 0, \qquad b_{n-1,x} = q_t - 2ra_n, \tag{2.66c}$$

$$c_{l,x} = 2b_{l+2} - 4qD_x^{-1}(rb_{l+2} - qc_{l+2}), \qquad l = 0,1,..n-2, \tag{2.67a}$$

$$b_0 = 0, \qquad 2b_1 - 4qD_x^{-1}(rb_1 - qc_1) = 0, \tag{2.67b}$$

$$c_{n,x} = 0, \qquad c_{n-1,x} = r_t - 2qa_n. \tag{2.67c}$$

Solving the equations (2.66b) and (2.67b) we have

$$b_1 = \epsilon_1 q \quad \text{and} \quad c_1 = \epsilon_1 r. \tag{2.68}$$

where $\epsilon_1$ is constant. We write the equations (2.66a) and (2.67a) in matrix form as follows to find the recursion equation in order to obtain other terms $b_i$ and $c_i$, $i \geqslant 2$;

$$\begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} \begin{pmatrix} c_l \\ b_l \end{pmatrix} = \begin{pmatrix} 2 + 4rD_x^{-1}q & -4rD_x^{-1}r \\ 4qD_x^{-1}q & 2 - 4qD_x^{-1}r \end{pmatrix} \begin{pmatrix} c_{l+2} \\ b_{l+2} \end{pmatrix}. \tag{2.69}$$

This leads to the following proposition.

**Proposition 2.12.** *The relation between $z_{l+2}$ and $z_l$ is given by the operator $\Psi$ as,*

$$z_{l+2} = \Psi z_l, \qquad l = 0, ...n-2 \tag{2.70}$$

*where*

$$z_l = \begin{pmatrix} c_l \\ b_l \end{pmatrix}, \tag{2.71a}$$

$$\Psi = \begin{pmatrix} rD_x^{-1}rD_x & \frac{1}{2}D_x - rD_x^{-1}qD_x \\ \frac{1}{2}D_x + qD_x^{-1}rD_x & -qD_x^{-1}qD_x \end{pmatrix}. \tag{2.71b}$$

**Proof:** By the equation (2.69), denoting

$$z_l = \begin{pmatrix} c_l \\ b_l \end{pmatrix},$$

$$M = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} 2 + 4rD_x^{-1}q & -4rD_x^{-1}r \\ 4qD_x^{-1}q & 2 - 4qD_x^{-1}r \end{pmatrix},$$

then we have

$$\begin{aligned} Mz_l &= Jz_{l+2}, \qquad l = 0, ...n-2, \\ z_{l+2} &= J^{-1}Mz_l = \Psi z_l, \qquad l = 0, ...n-2. \end{aligned}$$

Here

$$J^{-1} = \begin{pmatrix} \frac{1}{2} - rD_x^{-1}q & rD_x^{-1}r \\ -qD_x^{-1}q & \frac{1}{2} + qD_x^{-1}r \end{pmatrix}. \tag{2.74}$$

Hence

$$\Psi = \begin{pmatrix} rD_x^{-1}rD_x & \frac{1}{2}D_x - rD_x^{-1}qD_x \\ \frac{1}{2}D_x + qD_x^{-1}rD_x & -qD_x^{-1}qD_x \end{pmatrix}. \square$$

Therefore $z_i$, $i \geqslant 2$ are as follows:

For $n = 2$,

$$z_2 = \Psi z_0 = \begin{pmatrix} \epsilon_2 r \\ \epsilon_2 q \end{pmatrix}. \tag{2.75}$$

For $n = 3$,

$$z_3 = \Psi z_1 = \Psi \begin{pmatrix} \epsilon_1 r \\ \epsilon_1 q \end{pmatrix} = \begin{pmatrix} \epsilon_1(\ \ \frac{1}{2}r^3 - \frac{1}{2}rq^2 + \frac{1}{2}q_x) + \epsilon_3 r \\ \epsilon_1(-\frac{1}{2}q^3 + \frac{1}{2}qr^2 + \frac{1}{2}r_x) + \epsilon_3 q \end{pmatrix}. \tag{2.76}$$

Since $z_1$ and $z_2$ are the same up to the integration constants, similarly we have for $n = 4$,

$$z_4 = \Psi z_2 = \begin{pmatrix} \epsilon_2(\ \ \frac{1}{2}r^3 - \frac{1}{2}rq^2 + \frac{1}{2}q_x) + \epsilon_4 r \\ \epsilon_2(-\frac{1}{2}q^3 + \frac{1}{2}qr^2 + \frac{1}{2}r_x) + \epsilon_4 q \end{pmatrix}. \tag{2.77}$$

For $n = 5$,

$$z_5 = \Psi z_3 = \Psi \begin{pmatrix} \epsilon_1(\ \ \frac{1}{2}r^3 - \frac{1}{2}rq^2 + \frac{1}{2}q_x) + \epsilon_3 r \\ \epsilon_1(-\frac{1}{2}q^3 + \frac{1}{2}qr^2 + \frac{1}{2}r_x) + \epsilon_3 q. \end{pmatrix},$$

then

$$\begin{aligned} c_5 =& \epsilon_1(\frac{3}{8}r^5 + \frac{3}{8}q^4 r - \frac{3}{4}r^3 q^2 + \frac{3}{4}r^2 q_x - \frac{3}{4}q^2 q_x + \frac{1}{4}r_{xx}) + \\ & \epsilon_3(\ \ \frac{1}{2}r^3 - \frac{1}{2}rq^2 + \frac{1}{2}q_x) + \epsilon_5 r, \end{aligned} \tag{2.78a}$$

$$\begin{aligned} b_5 =& \epsilon_1(\frac{3}{8}q^5 + \frac{3}{8}r^4 q - \frac{3}{4}q^3 r^2 - \frac{3}{4}q^2 r_x + \frac{3}{4}r^2 r_x + \frac{1}{4}q_{xx}) + \\ & \epsilon_3(-\frac{1}{2}q^3 + \frac{1}{2}qr^2 + \frac{1}{2}r_x) + \epsilon_5 q. \end{aligned} \tag{2.78b}$$

Similarly for $n = 6$,

$$z_6 = \Psi z_4 = \Psi \begin{pmatrix} \epsilon_2(\ \ \frac{1}{2}r^3 - \frac{1}{2}rq^2 + \frac{1}{2}q_x) + \epsilon_4 r \\ \epsilon_2(-\frac{1}{2}q^3 + \frac{1}{2}qr^2 + \frac{1}{2}r_x) + \epsilon_4 q \end{pmatrix}.$$

Then

$$
\begin{aligned}
c_6 =&\epsilon_2(\frac{3}{8}r^5+\frac{3}{8}q^4r-\frac{3}{4}r^3q^2+\frac{3}{4}r^2q_x-\frac{3}{4}q^2q_x+\frac{1}{4}r_{xx})+\\
&\epsilon_4(\ \frac{1}{2}r^3-\frac{1}{2}rq^2+\frac{1}{2}q_x)+\epsilon_6 r, \qquad (2.79a)\\
b_6 =&\epsilon_2(\frac{3}{8}q^5+\frac{3}{8}r^4q-\frac{3}{4}q^3r^2-\frac{3}{4}q^2r_x+\frac{3}{4}r^2r_x+\frac{1}{4}q_{xx})+\\
&\epsilon_4(-\frac{1}{2}q^3+\frac{1}{2}qr^2+\frac{1}{2}r_x)+\epsilon_6 q. \qquad (2.79b)
\end{aligned}
$$

Note that $\epsilon_i$ are integration constants for $i \geqslant 1$.

**Remark 2.13.** *$z_{2l} = z_{2l-1}$, $l \geqslant 1$, up to integration constants. Hence we can ignore one of them and the recursion equation for the Tam-Zhang system results as*

$$z_{2l+1} = \Psi^l z_1, \qquad l = 0, 1, 2, .. \qquad (2.80)$$

**Proposition 2.14.** *The evolution equations for $q$ and $r$ can be determined from exactly the equations* (2.66c) *and* (2.67c)*.*

$$q_t = b_{n-1,x} + 2ra_n, \qquad (2.81a)$$

$$r_t = c_{n-1,x} + 2qa_n, \qquad (2.81b)$$

*where $a_n$ is constant for all $n$.*

**Example 2.4.** Substituting the equations (2.66b),(2.67b),(2.68),(2.75),(2.76),(2.77), (2.78a),(2.78b),(2.79a), (2.79b) in (2.81a) and (2.81b), we respectively we find the hierarchies for $n = 1, 2, .., 7$.

For $n = 1$ we have,

$$q_t = b_{0,x} + 2ra_1 = 2ra_1, \qquad r_t = c_{0,x} + 2qa_1 = 2qa_1. \qquad (2.82)$$

For $n = 2$ we have,

$$q_t = b_{1,x} + 2ra_2 = \epsilon_1 q_x + 2ra_2, \qquad r_t = c_{1,x} + 2qa_2 = \epsilon_1 r_x + 2qa_2. \qquad (2.83)$$

For $n = 3$ we have,

$$q_t = b_{2,x} + 2ra_3 = \epsilon_2 q_x + 2ra_3, \qquad r_t = c_{2,x} + 2qa_3 = \epsilon_2 r_x + 2qa_3. \tag{2.84}$$

For $n = 4$ we have,

$$q_t = b_{3,x} + 2ra_4 = \epsilon_1(\frac{1}{2}r_{xx} + \frac{1}{2}r^2 q_x + qrr_x - \frac{3}{2}q^2 q_x) + \epsilon_3 q_x + 2ra_4, \tag{2.85a}$$

$$r_t = c_{3,x} + 2qa_4 = \epsilon_1(\frac{1}{2}q_{xx} - \frac{1}{2}q^2 r_x - rqq_x + \frac{3}{2}r^2 r_x) + \epsilon_3 r_x + 2qa_4. \tag{2.85b}$$

If we assume $\epsilon_3 = a_4 = 0$, the above equations reduce to a generalized Burgers equation. Similarly for $n = 5$ we have,

$$q_t = b_{4,x} + 2ra_5 = \epsilon_2(\frac{1}{2}r_{xx} + \frac{1}{2}r^2 q_x + qrr_x - \frac{3}{2}q^2 q_x) + \epsilon_4 q_x + 2ra_5, \tag{2.86a}$$

$$r_t = c_{4,x} + 2qa_5 = \epsilon_2(\frac{1}{2}q_{xx} - \frac{1}{2}q^2 r_x - rqq_x + \frac{3}{2}r^2 r_x) + \epsilon_4 r_x + 2qa_5. \tag{2.86b}$$

For $n = 6$ we have,

$$\begin{aligned} q_t =& b_{5,x} + 2ra_6 = \epsilon_1(\frac{1}{4}q_{xxx} + \frac{3}{4}r^2 r_{xx} - \frac{3}{4}q^2 r_{xx} + \frac{3}{2}r(r_x)^2 - \frac{3}{2}qq_x r_x - \frac{9}{4}q^2 r^2 q_x + \\ & \frac{3}{8}r^4 q_x + \frac{15}{8}q^4 q_x + \frac{3}{2}qr^3 r_x - \frac{3}{2}q^3 rr_x) + \epsilon_3(\frac{1}{2}r_{xx} + \frac{1}{2}r^2 q_x + qrr_x - \frac{3}{2}q^2 q_x) + \\ & \epsilon_5 q_x + 2ra_6, \end{aligned} \tag{2.87a}$$

$$\begin{aligned} r_t =& c_{5,x} + 2qa_6 = \epsilon_1(\frac{1}{4}r_{xxx} + \frac{3}{4}r^2 q_{xx} - \frac{3}{4}q^2 q_{xx} - \frac{3}{2}q(q_x)^2 + \frac{3}{2}rr_x q_x - \frac{9}{4}q^2 r^2 r_x + \\ & \frac{3}{8}q^4 r_x + \frac{15}{8}r^4 r_x + \frac{3}{2}rq^3 q_x - \frac{3}{2}r^3 qq_x) + \epsilon_3(\frac{1}{2}q_{xx} - \frac{1}{2}q^2 r_x - rqq_x + \frac{3}{2}r^2 r_x) + \\ & \epsilon_5 r_x + 2qa_6. \end{aligned} \tag{2.88a}$$

Similarly for $n = 7$ we have,

$$q_t = b_{6,x} + 2ra_7 = \epsilon_2(\frac{1}{4}q_{xxx} + \frac{3}{4}r^2 r_{xx} - \frac{3}{4}q^2 r_{xx} + \frac{3}{2}r(r_x)^2 - \frac{3}{2}qq_x r_x - \frac{9}{4}q^2 r^2 q_x + \frac{3}{8}r^4 q_x + \frac{15}{8}q^4 q_x + \frac{3}{2}qr^3 r_x - \frac{3}{2}q^3 r r_x) + \epsilon_4(\frac{1}{2}r_{xx} + \frac{1}{2}r^2 q_x + qrr_x - \frac{3}{2}q^2 q_x) + \epsilon_6 q_x + 2ra_7, \tag{2.89a}$$

$$r_t = c_{6,x} + 2qa_7 = \epsilon_2(\frac{1}{4}r_{xxx} + \frac{3}{4}r^2 q_{xx} - \frac{3}{4}q^2 q_{xx} - \frac{3}{2}q(q_x)^2 + \frac{3}{2}rr_x q_x - \frac{9}{4}q^2 r^2 r_x + \frac{3}{8}q^4 r_x + \frac{15}{8}r^4 r_x + \frac{3}{2}rq^3 q_x - \frac{3}{2}r^3 q q_x) + \epsilon_4(\frac{1}{2}q_{xx} - \frac{1}{2}q^2 r_x - rqq_x + \frac{3}{2}r^2 r_x) + \epsilon_6 r_x + 2qa_7. \tag{2.89b}$$

Using proposition (2.14) we have the following proposition.

**Proposition 2.15.** *The evolution equations for $q$ and $r$ can be written in terms of the recursion operator $\mathbb{R}$ as follows,*

$$\begin{pmatrix} r_t \\ q_t \end{pmatrix} = \mathbb{R}^l \begin{pmatrix} \epsilon_1 r_x \\ \epsilon_1 q_x \end{pmatrix}, \tag{2.90}$$

*where $l = 0, 1, 2...,$ and*

$$\mathbb{R} = \begin{pmatrix} D_x r D_x^{-1} r & \frac{1}{2}D_x - D_x r D_x^{-1} q \\ \frac{1}{2}D_x + D_x q D_x^{-1} r & -D_x q D_x^{-1} q \end{pmatrix}. \tag{2.91}$$

**Proof:** If we rewrite the equations (2.81a)and (2.81b) in matrix form we have,

$$\begin{pmatrix} r_t \\ q_t \end{pmatrix} = \begin{pmatrix} c_{n-1,x} \\ b_{n-1,x} \end{pmatrix} + \begin{pmatrix} 2a_n q \\ 2a_n r \end{pmatrix} \tag{2.92a}$$

$$\begin{pmatrix} r_t \\ q_t \end{pmatrix} = D_x z_{n-1} + \begin{pmatrix} 2a_n q \\ 2a_n r \end{pmatrix} \tag{2.92b}$$

Since the odd numbered and the even numbered hierarchies give the same equations, we can ignore the even numbered hierarchies. We can assume $n-1$ is odd. Moreover in the equation (2.92b), $\begin{pmatrix} 2a_n q \\ 2a_n r \end{pmatrix}$ is a symmetry. The summation of symmetries is again a symmetry, so we can ignore the righthandside of the equation (2.92b). By the equation (2.80) we have

$$\begin{pmatrix} r_t \\ q_t \end{pmatrix} = D_x z_{2l+1} = D_x \Psi^l z_1$$

Let $\mathbb{R} = D_x \Psi D_x^{-1}$ where $\Psi$ is defined in (2.71b). Hence

$$\begin{pmatrix} r_t \\ q_t \end{pmatrix} = \mathbb{R}^l D_x z_1 = \mathbb{R}^l \begin{pmatrix} \epsilon_1 r_x \\ \epsilon_1 q_x \end{pmatrix}. \tag{2.94}$$

where $l = 0, 1, 2, ...$ □

# Chapter 3

# Classical Lie Algebras

## 3.1 Introduction

In order to obtain integrable nonlinear partial differential equations, the usual procedure is to use the zero curvature formalism which is based on the Lax equation for $n \times n$ matrix valued functions. These are traceless real matrices which form a basis of a matrix algebra. For the AKNS scheme this algebra is $sl(2, \mathbb{R})$ algebra. To obtain more examples, we will work in simple Lie algebras.

In Section 3.2, we will introduce the usual Cartan-Weyl basis, in Section 3.3, we will give the Cartan-Weyl basis on homogeneous spaces, introduced in [12]. In Section 3.4, we will compare these two bases. In Section 3.5, we use a simple Lie algebra valued soliton connection to obtain some integrable nonlinear partial differential equations on homogeneous spaces, recently introduced in [12]. In Section 3.5.2, we determine the corresponding recursion operator by the use of the catalogue. In Section 3.6, we obtain the integrable evolution equations without using the catalogue.

## 3.2 Cartan-Weyl basis

The Cartan-Weyl basis most frequently used by physicists is the standard form of the commutation relations for a semisimple Lie algebra.

Let $g$ be the Lie algebra of a Lie group $G$ and $h$ be the Cartan subalgebra which is the maximal abelian subalgebra in $g$. Then $g$ can be identified as

$$g = h \bigoplus h^C \tag{3.1}$$

where $h^C$ is the complement of the Cartan subalgebra in $g$.

**Definition 3.1.** *The above decomposition leads to the Cartan-Weyl basis as follows:*

$$[H_a, H_b] = 0 \quad \textit{for all} \quad a, b = 1, 2, ..., p, \tag{3.2a}$$

$$[H_a, E_{\overline{\alpha}}] = \overline{\alpha}_a E_{\overline{\alpha}}, \tag{3.2b}$$

$$[E_{\overline{\alpha}}, E_{\overline{\beta}}] = \sum_{a=1}^{p} C^a_{\overline{\alpha}+\overline{\beta}} H_a \qquad \textit{if} \quad \overline{\alpha} + \overline{\beta} = 0, \tag{3.2c}$$

$$[E_{\overline{\alpha}}, E_{\overline{\beta}}] = C^{\overline{\alpha}+\overline{\beta}}_{\overline{\alpha}\overline{\beta}} E_{\overline{\alpha}+\overline{\beta}} \qquad \textit{if} \quad \overline{\alpha} + \overline{\beta} \neq 0, \tag{3.2d}$$

*where $p$ is the rank of the algebra, $H_a$ ' s are bases of the Cartan subalgebra, $E_{\overline{\alpha}}$ ' s are the bases of the complement of the Cartan subalgebra, $\overline{\alpha}_a$ and $C$ are the structure constants of the commutation relations.[11]*

## 3.3 Cartan-Weyl basis on homogeneous spaces

In this section we will improve the usual Cartan-Weyl basis to the Cartan-Weyl basis on homogeneous spaces. For this purpose let us give the following definitions.

Let $G$ be a Lie group. *A homogeneous space* of $G$, is any differentiable manifold $M$, on which $G$ acts transitively. If $K$ is an isotropy group of some point $p_0 \in M$, then $M$ can be identified as a coset space $G/K$. Let $g$ and $k$ be the Lie algebras of $G$ and $K$ respectively. Let $m$ be the vector space, complement of $k$ in $g$, then

$$g = k \bigoplus m. \tag{3.3}$$

and $m$ is identified as the tangent space of $G/K$ at $p_0 \in M$. When $g$ satisfies the conditions

$$g = k \bigoplus m, \qquad [k,k] \subset k, \qquad [k,m] \subset m, \tag{3.4}$$

then $M = G/K$ is called a '*reductive homogeneous space*'. When $g$ satisfies the conditions

$$g = k \bigoplus m, \qquad [k,k] \subset k, \qquad [k,m] \subset m, \qquad [m,m] \subset k, \tag{3.5}$$

then $M = G/K$ is called a '*symmetric space*'.

In a simple Lie algebra, we denote by $H_a$ the commuting generators where $a = 1, 2..., p$. Here p is the rank of the algebra. We denote by $E_\alpha$ and $E_d$ the step operators where $\alpha$' s and $d$' s are the roots. The $k$ part of the algebra has the generators $H_a$, $E_D$ $(D = d, -d)$, the $m$ part of the algebra consists of the generators $E_A$ $(A = \alpha, -\alpha)$. In Cartan-Weyl basis we can write the commutation relation of the generators as [12] :

$$[H_a, H_b] = 0 \quad \text{for all} \quad a, b = 1, 2....p, \tag{3.6a}$$
$$[H_a, E_A] = f^B_{aA} E_B, \tag{3.6b}$$
$$[H_a, E_D] = f^E_{aD} E_E, \tag{3.6c}$$
$$[E_D, E_E] = f^a_{DE} H_a + f^F_{DE} E_F, \tag{3.6d}$$
$$[E_D, E_A] = f^B_{DA} E_B, \tag{3.6e}$$
$$[E_A, E_B] = f^a_{AB} H_a + f^D_{AB} E_D + f^C_{AB} E_C, \tag{3.6f}$$

where $A$, $B$, $C$ $(\pm\alpha, \pm\beta, \pm\gamma)$ are the indices for the generators in $m$; $D$, $E$, $F$ $(\pm d, \pm e, \pm f)$ are the indices for the generators in $k$. Here note that for the generators $E_A$ of $m$ and for the generators $E_D$ of $k$

$$\mid A \mid > \mid D \mid . \tag{3.7}$$

The structure constants representing the roots can be written as

$$f_{a\pm\alpha}^{\pm\beta} = \alpha_a \delta_{\pm\alpha}^{\pm\beta}, \qquad f_{a\pm d}^{\pm e} = \pm d_a \delta_{\pm d}^{\pm e}. \tag{3.8}$$

We shall now give an example, sl$(3,\mathbb{R})$.

**Example 3.1.** Let $g = sl(3,\mathbb{R})$ with $g = \{H_1, H_2, E_1, E_{-1}, E_2, E_{-2}, E_3, E_{-3}\}$ where the corresponding base elements of Cartan-Weyl basis are as follows,

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We seek for the commutation relations so that $[X,Y] = XY - YX$ for all $X, Y \in g$ :

$$[H_1, H_2] = 0, \tag{3.9}$$

$$\begin{aligned} &[H_1, E_1] = -2E_1, \quad [H_1, E_{-1}] = 2E_{-1}, \\ &[H_2, E_1] = 0, \quad [H_2, E_{-1}] = 0, \end{aligned} \tag{3.10a}$$

$$\begin{aligned} &[H_1, E_2] = E_2, \quad [H_1, E_{-2}] = -E_{-2}, \quad [H_1, E_3] = -E_3, \\ &[H_1, E_{-3}] = E_{-3}, \quad [H_2, E_2] = 3E_2, \quad [H_2, E_{-2}] = -3E_{-2}, \\ &[H_2, E_3] = 3E_3, \quad [H_2, E_{-3}] = -3E_{-3}, \end{aligned} \tag{3.10b}$$

$$\begin{aligned} &[E_1, E_{-1}] = -H_1, \quad [E_2, E_{-2}] = \frac{1}{2}(H_1 + H_2), \\ &[E_3, E_{-3}] = \frac{1}{2}(H_2 - H_1), \end{aligned} \tag{3.11a}$$

$$[E_1, E_2] = E_3, \quad [E_1, E_{-2}] = 0, \quad [E_1, E_3] = 0, \quad [E_1, E_{-3}] = -E_{-2},$$
$$[E_{-1}, E_2] = 0, \quad [E_{-1}, E_{-2}] = -E_{-3},$$
$$[E_{-1}, E_3] = E_2, \quad [E_{-1}, E_{-3}] = 0, \tag{3.12a}$$

$$[E_2, E_3] = 0, \quad [E_2, E_{-3}] = E_1,$$
$$[E_{-2}, E_3] = -E_{-1}, \quad [E_{-2}, E_{-3}] = 0. \tag{3.12b}$$

**Case1:** The Cartan-Weyl basis

According to the decomposition for the usual Cartan-Weyl basis, we have $g = h \bigoplus h^C$ where

$$\begin{aligned} h &= \{H_1, H_2\}, \\ h^C &= \{E_1, E_{-1}, E_2, E_{-2}, E_3, E_{-3}\}. \end{aligned} \tag{3.13}$$

By (3.9), it is clear that the commutation relations among the elements of Cartan subalgebra are zero which corresponds to the equation(3.2a). The structure constants and the roots of the algebra $sl(3, \mathbb{R})$ are found as follows.

By (3.10a) and (3.10b) we have

$$[H_1, E_1] = -2E_1 \Rightarrow \overline{\alpha_1} = -2; \quad [H_1, E_{-1}] = 2E_{-1} \Rightarrow \overline{\alpha_1} = 2,$$
$$[H_2, E_1] = 0 \Rightarrow \overline{\alpha_2} = 0; \quad [H_2, E_{-1}] = 0 \Rightarrow \overline{\alpha_2} = 0,$$
$$[H_1, E_2] = E_2 \Rightarrow \overline{\alpha_1} = 1; \quad [H_1, E_{-2}] = -E_{-2} \Rightarrow \overline{\alpha_1} = -1,$$
$$[H_1, E_3] = -E_3 \Rightarrow \overline{\alpha_1} = -1; \quad [H_1, E_{-3}] = E_{-3} \Rightarrow \overline{\alpha_1} = 1,$$
$$[H_2, E_2] = 3E_2 \Rightarrow \overline{\alpha_2} = 3; \quad [H_2, E_{-2}] = -3E_{-2} \Rightarrow \overline{\alpha_2} = -3,$$
$$[H_2, E_3] = 3E_3 \Rightarrow \overline{\alpha_2} = 3; \quad [H_2, E_{-3}] = -3E_{-3} \Rightarrow \overline{\alpha_2} = -3,$$

Hence

$$[H_a, E_{\overline{\alpha}}] = \overline{\alpha}_a E_{\overline{\alpha}}; \quad a = 1, 2; \qquad \overline{\alpha} = \pm 1, \pm 2, \pm 3,$$

which corresponds to (3.2b). By (3.11a),

$$
\begin{aligned}
&[E_1, E_{-1}] = -H_1, \quad \Rightarrow \quad C^1_{1-1} = -1,\\
&[E_2, E_{-2}] = \frac{1}{2}(H_1 + H_2), \quad \Rightarrow \quad C^1_{2-2} = \frac{1}{2}, \quad C^2_{2-2} = \frac{1}{2},\\
&[E_3, E_{-3}] = \frac{1}{2}(H_2 - H_1) \quad \Rightarrow \quad C^1_{3-3} = -\frac{1}{2}, \quad C^2_{3-3} = \frac{1}{2},
\end{aligned}
$$

Hence

$$
[E_{\overline{\alpha}}, E_{\overline{\beta}}] = \sum_{a=1}^{2} C^a_{\overline{\alpha}+\overline{\beta}} H_a \quad \text{if} \quad \overline{\alpha} + \overline{\beta} = 0; \qquad \overline{\alpha}, \overline{\beta} = \pm 1, \pm 2,
$$

which corresponds to (3.2c). By (3.12a) and (3.12b) we have,

$$
\begin{aligned}
&[E_1, E_2] = E_3 \quad \Rightarrow \quad C^3_{12} = 1; \quad [E_1, E_{-2}] = 0 \quad \Rightarrow \quad C^{-1}_{1-2} = 0,\\
&[E_1, E_3] = 0 \quad \Rightarrow \quad C^{1+3}_{13} = 0; \qquad [E_1, E_{-3}] = -E_{-2} \quad \Rightarrow \quad C^{-2}_{1-3} = -1,\\
&[E_{-1}, E_2] = 0 \quad \Rightarrow \quad C^{-1+2}_{-12} = 0; \qquad [E_{-1}, E_{-2}] = -E_{-3} \quad \Rightarrow \quad C^{-3}_{-1-2} = -1\\
&[E_{-1}, E_3] = E_2 \quad \Rightarrow \quad C^2_{-13} = 1; \quad [E_{-1}, E_{-3}] = 0 \quad \Rightarrow \quad C^{-1+(-3)}_{-1-3} = 0,\\
&[E_2, E_3] = 0 \quad \Rightarrow \quad C^{2+3}_{23} = 0; \qquad [E_2, E_{-3}] = E_{-1} \quad \Rightarrow \quad C^{-1}_{2-3} = 1,\\
&[E_{-2}, E_3] = -E_1 \quad \Rightarrow \quad C^1_{-23} = -1; \qquad [E_{-2}, E_{-3}] = 0 \quad \Rightarrow \quad C^{-2+(-3)}_{-2-3} = 0.
\end{aligned}
$$

Hence

$$
[E_{\overline{\alpha}}, E_{\overline{\beta}}] = C^{\overline{\alpha}+\overline{\beta}}_{\overline{\alpha}\overline{\beta}} E_{\overline{\alpha}+\overline{\beta}} \quad \text{if} \quad \overline{\alpha} + \overline{\beta} \neq 0; \qquad \overline{\alpha}, \overline{\beta} = \pm 1, \pm 2, \pm 3,
$$

which corresponds to (3.2d).

**Case 2:** The Cartan-Weyl basis on homogeneous spaces

According to the decomposition on homogeneous spaces we have $g = k \bigoplus m$ such that

$$
\begin{aligned}
k &= \{H_1, H_2, E_1, E_{-1}\},\\
m &= \{E_2, E_{-2}, E_3, E_{-3}\}.
\end{aligned}
$$

Similar to Case 1, the equation (3.9) corresponds to (3.6a). By (3.10a)

$$[H_1,E_1]=-2E_1 \quad\Rightarrow\quad f_{11}^{1}=-2; \quad [H_1,E_{-1}]=2E_{-1} \quad\Rightarrow\quad f_{1-1}^{-1}=2,$$
$$[H_2,E_1]=0 \quad\Rightarrow\quad f_{21}^{1}=0; \qquad [H_2,E_{-1}]=0 \quad\Rightarrow\quad f_{2-1}^{-1}=0.$$

Then we have,

$$[H_a,E_D]=f_{aD}^{E}E_E \quad a=1,2; \quad D,E=\pm1,$$

which corresponds to (3.6c). By (3.10b),

$$[H_1,E_2]=E_2 \qquad f_{12}^{2}=1; \qquad \Rightarrow [H_1,E_{-2}]=-E_{-2} \quad\Rightarrow\quad f_{1-2}^{-2}=-1,$$
$$[H_1,E_3]=-E_3 \quad\Rightarrow\quad f_{13}^{3}=-1; \quad [H_1,E_{-3}]=E_{-3} \quad\Rightarrow\quad f_{1-3}^{-3}=1,$$
$$[H_2,E_2]=3E_2 \quad\Rightarrow\quad f_{22}^{2}=3; \qquad [H_2,E_{-2}]=-3E_{-2} \quad\Rightarrow\quad f_{2-2}^{-2}=-3,$$
$$[H_2,E_3]=3E_3 \quad\Rightarrow\quad f_{23}^{3}=3; \qquad [H_2,E_{-3}]=-3E_{-3} \quad\Rightarrow\quad f_{2-3}^{-3}=-3,$$

then we have

$$[H_a,E_A]=f_{aA}^{B}E_B \quad a=1,2; \quad A,B=\pm2\pm3,$$

which corresponds to (3.6b). By (3.11a), $[E_1,E_{-1}]=-H_1$ then $f_{1-1}^{1}=-1$. So we have $[E_D,E_E]=f_{DE}^{a}H_a$ where $a=1$; $D,E=\pm1$. But in general,

$$[E_D,E_E]=f_{DE}^{a}H_a+f_{DE}^{F}E_F.$$

which is (3.6d). By (3.12a)

$$[E_1,E_2]=E_3 \quad\Rightarrow\quad f_{12}^{3}=1; \quad [E_1,E_{-2}]=0 \quad\Rightarrow\quad f_{1-2}^{-1}=0,$$
$$[E_1,E_3]=0 \quad\Rightarrow\quad f_{13}^{1+3}=0; \qquad [E_1,E_{-3}]=-E_{-2} \quad\Rightarrow\quad f_{1-3}^{-2}=-1,$$
$$[E_{-1},E_2]=0 \quad\Rightarrow\quad f_{-12}^{1}=0; \qquad [E_{-1},E_{-2}]=-E_{-3} \quad\Rightarrow\quad f_{-1-2}^{-3}=-1,$$
$$[E_{-1},E_3]=E_2 \quad\Rightarrow\quad f_{-13}^{2}=1; \quad [E_{-1},E_{-3}]=0 \quad\Rightarrow\quad f_{-1-3}^{(-1)+(-3)}=0,$$

then

$$[E_D,E_A]=f_{DA}^{B}E_B \quad D=\pm1, \quad A,B=\pm2,\pm3,$$

which corresponds to (3.6e). By (3.11a) and (3.12b),

$$[E_2, E_{-2}] = \frac{1}{2}(H_1 + H_2) \quad \Rightarrow \quad f^1_{2-2} = \frac{1}{2}, \quad f^2_{2-2} = \frac{1}{2},$$
$$[E_3, E_{-3}] = \frac{1}{2}(H_2 - H_1) \quad \Rightarrow \quad f^1_{3-3} = \frac{-1}{2}, \quad f^2_{3-3} = \frac{1}{2},$$
$$[E_2, E_3] = 0 \quad \Rightarrow \quad f^{2+3}_{23} = 0; \qquad [E_2, E_{-3}] = E_{-1} \quad \Rightarrow \quad f^{-1}_{2-3} = 1,$$
$$[E_{-2}, E_3] = -E_1 \quad \Rightarrow \quad f^1_{-23} = -1; \qquad [E_{-2}, E_{-3}] = 0 \quad \Rightarrow \quad f^{(-2)+(-3)}_{-2-3} = 0.$$

So we have,

$$[E_A, E_B] = f^a_{AB}H_a + f^D_{AB}E_D + f^C_{AB}E_C, \quad a = 1, 2; \quad D = \pm 1, \quad A, B, C = \pm 2, \pm 3,$$

which is (3.6f).

## 3.4 Comparison of Cartan -Weyl bases

In this section we will compare the usual Cartan-Weyl basis with the one on homogeneous spaces. Obviously (3.2a) and (3.6a) requires the commuting generators. By (3.1) the usual Cartan-Weyl basis consists of Cartan subalgebra and the complement of the Cartan subalgebra. On the other hand; by (3.4) the Cartan-Weyl basis on homogeneous spaces decomposed into $k$ part and $m$ part. The $k$ part has generators $H_a$, $E_D$ $(D = d, -d)$ and $m$ part has generators $E_A$ $(A = \alpha, -\alpha)$. To emphasize the difference between these two bases, $h^C$ (in the usual Cartan-Weyl basis) has been improved to have a decomposition of two vector spaces having generators $E_D$ and $E_A$. Hence for the derivation of the basis from the usual one; we have to take into account the general behaviour of $\overline{\alpha}_a$ which is now altered to $\overline{\alpha}_a = (\alpha_a, d_a)$. If we assume $\overline{\alpha} = \pm\alpha$ , by (3.2b) we have $[H_a, E_{\pm\alpha}] = \alpha_a E_{\pm\alpha}$. Using (3.8); if we let $\pm\alpha = \pm\beta$ , we conclude that

$$[H_a, E_{\pm\alpha}] = f^{\pm\beta}_{a\pm\alpha}E_{\pm\beta},$$
$$[H_a, E_A] = f^B_{aA}E_B,$$

which is (3.6b). If we assume $\overline{\alpha} = \pm d$, by (3.2b) we have $[H_a, E_{\pm d}] = \pm d_a E_{\pm d}$. Using (3.8); if we let $\pm d = \pm e$ , we conclude that

$$[H_a, E_{\pm d}] = f_{a\pm d}^{\pm e} E_{\pm e},$$
$$[H_a, E_D] = f_{aD}^{E} E_E,$$

which is (3.6c). If we assume $\overline{\alpha} = d$ and $\overline{\beta} = e$, by (3.2c) and (3.2d) we have,

$$[E_d, E_e] = \sum_{a=1}^{p} C_{d+e}^{a} H_a \qquad \text{if} \quad d+e=0,$$
$$[E_d, E_e] = C_{de}^{d+e} E_{d+e} \qquad \text{if} \quad d+e \neq 0.$$

On the other hand; by (3.4) $[E_d, E_e] \subset k$ where $k$ has generators $H_a$ and $E_D$. Then

$$[E_d, E_e] = f_{de}^{a} H_a + f_{de}^{d+e} E_{d+e}. \tag{3.21}$$

Here

$$\text{if} \quad d+e=0 \quad \text{then} \quad f_{de}^{a} = C_{d+e}^{a},$$
$$\text{if} \quad d+e \neq 0 \quad \text{then} \quad f_{de}^{d+e} = C_{de}^{d+e}.$$

Similarly $\overline{\alpha} = -d$, $\overline{\beta} = -e$ implies,

$$[E_{-d}, E_{-e}] = f_{-d-e}^{a} H_a + f_{-d-e}^{(-d)+(-e)} E_{(-d)+(-e)}. \tag{3.22}$$

Let $E_{(\pm d)+(\pm e)} = E_{(\pm f)} \subset k$. Hence (3.21), (3.22) gives straightforwardly,

$$[E_D, E_E] = f_{DE}^{a} H_a + f_{DE}^{F} E_F,$$

which is (3.6d). If we assume $\overline{\alpha} = d$ and $\overline{\beta} = \alpha$, by (3.2c) and (3.2d) we have,

$$[E_d, E_\alpha] = \sum_{a=1}^{p} C_{d+\alpha}^{a} H_a \qquad \text{if} \quad d+\alpha=0,$$
$$[E_d, E_\alpha] = C_{d\alpha}^{d+\alpha} E_{d+\alpha} \qquad \text{if} \quad d+\alpha \neq 0.$$

On the other hand; since $E_d$ and $E_\alpha$ are elements of $k$ and $m$ respectively, we always have $d+\alpha \neq 0$. Also by (3.4) we have $[E_d, E_\alpha] \subset m$. Hence

$$[E_d, E_\alpha] = f_{d\alpha}^{d+\alpha} E_{d+\alpha}. \tag{3.23}$$

Here $C^a_{d+\alpha} = 0$; $\quad f^{d+\alpha}_{d\alpha} = C^{d+\alpha}_{d\alpha}$. Similarly $\overline{\alpha} = -d$ , $\overline{\beta} = -\alpha$ implies,

$$[E_{-d}, E_{-\alpha}] = f^{(-d)+(-\alpha)}_{-d-\alpha} E_{(-d)+(-\alpha)}. \tag{3.24}$$

Let $E_{(\pm d)+(\pm\alpha)} = E_{(\pm\beta)} \subset m$. Hence (3.23), (3.24) gives straightforwardly

$$[E_D, E_A] = f^B_{DA} E_B,$$

which is (3.6e). If we assume $\overline{\alpha} = \alpha$ and $\overline{\beta} = \beta$ by (3.2c) and (3.2d) we have,

$$[E_\alpha, E_\beta] = \sum_{a=1}^{p} C^a_{\alpha+\beta} H_a \qquad \text{if} \quad \alpha + \beta = 0,$$
$$[E_\alpha, E_\beta] = C^{\alpha+\beta}_{\alpha\beta} E_{\alpha+\beta} \qquad \text{if} \quad \alpha + \beta \neq 0.$$

On the other hand; since $E_\alpha$ and $E_\beta$ are elements of $m$, the condition of reductive homogeneous space does not give enough information about the place of the commutation relation of $[E_\alpha, E_\beta]$. So we have either $k$ or $m$ part. Then

$$[E_\alpha, E_\beta] = f^a_{\alpha\beta} H_a + f^{\alpha+\beta}_{1_{\alpha\beta}} E_{\alpha+\beta} + f^{\alpha+\beta}_{2_{\alpha\beta}} E_{\alpha+\beta}.$$

Here

if $\alpha + \beta = 0$ then $C^a_{\alpha+\beta} = f^a_{\alpha\beta}$;
if $\alpha + \beta \neq 0$ and $[E_\alpha, E_\beta] \subset k$, then $C^{\alpha+\beta}_{\alpha\beta} = f^{\alpha+\beta}_{1_{\alpha\beta}} = f^d_{\alpha\beta}$ where $\alpha + \beta = d$;
if $\alpha + \beta \neq 0$ and $[E_\alpha, E_\beta] \subset m$, then $C^{\alpha+\beta}_{\alpha\beta} = f^{\alpha+\beta}_{2_{\alpha\beta}} = f^\gamma_{\alpha\beta}$ where $\alpha + \beta = \gamma$.

Then

$$[E_\alpha, E_\beta] = f^a_{\alpha\beta} H_a + f^d_{\alpha\beta} E_d + f^\gamma_{\alpha\beta} E_\gamma. \tag{3.25}$$

Similarly $\overline{\alpha} = -\alpha$ , $\overline{\beta} = -\beta$ implies,

$$[E_{-\alpha}, E_{-\beta}] = f^a_{-\alpha-\beta} H_a + f^{-d}_{-\alpha-\beta} E_{-d} + f^{-\gamma}_{-\alpha-\beta} E_{-\gamma}. \tag{3.26}$$

Hence (3.25), (3.26) gives straightforwardly,

$$[E_A, E_B] = f^a_{AB} H_a + f^D_{AB} E_D + f^C_{AB} E_C,$$

which is (3.6f).

## 3.5 A simple Lie algebra valued soliton connection

**Definition 3.2.** *The simple Lie algebra valued soliton connection 1-form $\Omega$, associated to a reductive homogeneous space $G/K$ with the generators $H_a$, $E_D$ is defined as*

$$\Omega = (ik\lambda H_s + Q^A E_A)dx + (A^a H_a + B^A E_A + C^D E_D)dt, \tag{3.27}$$

*where $K$ is the isotropy group of $G$, $\lambda$ is the spectral parameter, $k$ is a constant not depending on $\lambda$, $s \in \{1, 2, ...p\}$ is a fixed constant, $Q^A(x,t)$ is potential, $A^a$, $B^A$ and $C^D$ are arbitrary functions of $x$, $t$ and $\lambda$ [12].*

Assume $H_s$ as one of the commuting generators which satisfies the commutation relation

$$[H_s, E_D] = 0, \tag{3.28}$$

Here we note that $\pm d = \pm(\alpha - \beta)$ and $\alpha_s = \beta_s$, for all $\alpha$ 's, $\beta$ 's.

For $\Omega = Tdt + Xdx$, we have

$$d\phi = \Omega\phi \tag{3.29}$$

as the *Lax equation* in differential forms where $\phi \in G$. R curvature 2-form is $R = d\Omega + \Omega \wedge \Omega$. Here $\Omega$ is the flat connection so

$$d\Omega + \Omega \wedge \Omega = 0. \tag{3.30}$$

which is the *zero curvature condition.* Therefore the equation (3.30) becomes,

$$-Q_t^A E_A + A_x^a H_a + B_x^A E_A + C_x^D E_D + ik\lambda A^a [H_s, H_a] + ik\lambda B^C [H_s, E_C] + \\ ik\lambda C^D [H_s, E_D] + Q^A A^a [E_A, H_a] + Q^A B^C [E_A, E_C] + Q^A C^D [E_A, E_D] = 0. \tag{3.31}$$

The equation (3.6a) leads $[H_s, H_a] = 0$. According to our assumption $[H_s, E_D] = 0$. By (3.6b),(3.6e) and (3.6f), we have respectively

$$[H_s, E_C] = f^A_{sC} E_A, \quad [E_A, H_a] = -f^A_{aC} E_A, \quad [E_A, E_D] = f^A_{BD} E_A, \tag{3.32a}$$

$$[E_A, E_C] = f^a_{AC} H_a + f^D_{AC} E_D + f^A_{BC} E_A. \tag{3.32b}$$

Note that summation on indices enables us to change indices.

**Proposition 3.3.** *By using the conditions* (3.32a) *and* (3.32b) *in the equation* (3.31)*, we have*

$$Q^A_t = B^A_x + ik\lambda f^A_{sC} B^C - f^A_{aC} Q^C A^a + f^A_{BC} Q^B B^C + f^A_{BD} Q^B C^D, \tag{3.33a}$$

$$A^a_x + f^a_{AC} Q^A B^C = 0, \tag{3.33b}$$

$$C^D_x + f^D_{AC} Q^A B^C = 0. \tag{3.33c}$$

We expand $A^a$, $B^A$ and $C^D$ in terms of the positive powers of $\lambda$ as,

$$A^a = \sum_{n=0}^{N} a^a_n \lambda^{N-n}, \qquad B^A = \sum_{n=0}^{N} b^A_n \lambda^{N-n}, \qquad C^D = \sum_{n=0}^{N} c^D_n \lambda^{N-n}. \tag{3.34}$$

**Proposition 3.4.** *Let $A^a$, $B^A$, $C^D$ in equations* (3.33a), (3.33b) *and* (3.33c) *have expansions as in* (3.34)*, then we get the following equations respectively,*

$$b^A_{l,x} + ik f^A_{sC} b^C_{l+1} - f^A_{aC} Q^C a^a_l + f^A_{BC} Q^B b^C_l + f^A_{BD} Q^B c^D_l = 0; \quad l = 0,.,N-1, \tag{3.35a}$$

$$Q^A_t = b^A_{N,x} - f^A_{aC} Q^C a^a_N + f^A_{BC} Q^B b^C_N + f^A_{BD} Q^B c^D_N, \tag{3.35b}$$

$$ik f^A_{sC} b^C_0 = 0, \tag{3.35c}$$

$$a^a_{l,x} + f^a_{AC} Q^A b^C_l = 0; \qquad l = 0, 1.., N, \tag{3.35d}$$

$$c^D_{l,x} + f^D_{AC} Q^A b^C_l = 0; \qquad l = 0, 1.., N. \tag{3.35e}$$

Using the Proposition (3.4) we have the following proposition

**Proposition 3.5.** *If we expand the related indices of the equations* (3.35a), (3.35b), (3.35d) *and* (3.35e)*, we have*

$$b^{\pm\alpha}_{l,x} + ikf^{\pm\alpha}_{s\gamma}b^{\gamma}_{l+1} + ikf^{\pm\alpha}_{s-\gamma}b^{-\gamma}_{l+1} - Q^{\gamma}f^{\pm\alpha}_{a\gamma}a^{a}_{l} - Q^{-\gamma}f^{\pm\alpha}_{a-\gamma}a^{a}_{l} + Q^{\beta}f^{\pm\alpha}_{\beta\gamma}b^{\gamma}_{l} +$$
$$Q^{-\beta}f^{\pm\alpha}_{-\beta\gamma}b^{\gamma}_{l} + Q^{\beta}f^{\pm\alpha}_{\beta-\gamma}b^{-\gamma}_{l} + Q^{-\beta}f^{\pm\alpha}_{-\beta-\gamma}b^{-\gamma}_{l} + Q^{\beta}f^{\pm\alpha}_{\beta d}c^{d}_{l} + Q^{\beta}f^{\pm\alpha}_{\beta-d}c^{-d}_{l} +$$
$$Q^{-\beta}f^{\pm\alpha}_{-\beta d}c^{d}_{l} + Q^{-\beta}f^{\pm\alpha}_{-\beta-d}c^{-d}_{l} = 0; \qquad l = 0, .., N-1, \tag{3.36a}$$

$$Q^{\pm\alpha}_{t} = b^{\pm\alpha}_{N,x} - Q^{\gamma}f^{\alpha}_{a\gamma}a^{a}_{N} - Q^{-\gamma}f^{\pm\alpha}_{a-\gamma}a^{a}_{N} + Q^{\beta}f^{\pm\alpha}_{\beta\gamma}b^{\gamma}_{N} + Q^{-\beta}f^{\pm\alpha}_{-\beta\gamma}b^{\gamma}_{N} + Q^{\beta}f^{\pm\alpha}_{\beta-\gamma}b^{-\gamma}_{N} +$$
$$Q^{-\beta}f^{\pm\alpha}_{-\beta-\gamma}b^{-\gamma}_{N} + Q^{\beta}f^{\pm\alpha}_{\beta d}c^{d}_{N} + Q^{-\beta}f^{\pm\alpha}_{-\beta d}c^{d}_{N} + Q^{\beta}f^{\pm\alpha}_{\beta-d}c^{-d}_{N} + Q^{-\beta}f^{\pm\alpha}_{-\beta-d}c^{-d}_{N}, \tag{3.37a}$$

$$a^{a}_{l,x} + Q^{\alpha}f^{a}_{\alpha\gamma}b^{\gamma}_{l} + Q^{-\alpha}f^{a}_{-\alpha\gamma}b^{\gamma}_{l} + Q^{\alpha}f^{a}_{\alpha-\gamma}b^{-\gamma}_{l} + Q^{-\alpha}f^{a}_{-\alpha-\gamma}b^{-\gamma}_{l} = 0; \qquad l = 0, 1.., N, \tag{3.38}$$

$$c^{\pm d}_{l,x} + Q^{\alpha}f^{\pm d}_{\alpha\gamma}b^{\gamma}_{l} + Q^{-\alpha}f^{\pm d}_{-\alpha\gamma}b^{\gamma}_{l} + Q^{\alpha}f^{\pm d}_{\alpha-\gamma}b^{-\gamma}_{l} + Q^{-\alpha}f^{\pm d}_{-\alpha-\gamma}b^{-\gamma}_{l} = 0; \qquad l = 0, 1.., N. \tag{3.39}$$

### 3.5.1 The catalogue

To deduce the occurrance of the structure constants, it is better to determine a catalogue of them. Consider the commutation relations:

$$\begin{aligned} [H_a, E_\gamma] &= \gamma_a E_\gamma \quad \text{by (3.2b)}, \\ &= f^{\alpha}_{a\gamma}E_\alpha + f^{-\alpha}_{a\gamma}E_{-\alpha} \quad \text{by (3.6b)}. \end{aligned}$$

Since $\gamma > 0$ then $f^{-\alpha}_{a\gamma} = 0$ and by (3.8), we have

$$\alpha_a = f^{\alpha}_{a\gamma}. \tag{3.40}$$

Similarly for $[H_a, E_{-\gamma}] = -\gamma_a E_{-\gamma} = f^{\alpha}_{a-\gamma}E_\alpha + f^{-\alpha}_{a-\gamma}E_{-\alpha}$ we have,

$$f^{\alpha}_{a-\gamma} = 0, \quad -\alpha_a = f^{-\alpha}_{a-\gamma}. \tag{3.41}$$

Using (3.2c),(3.2d),(3.6f), the commutation relation becomes,

$$\begin{aligned} [E_\beta, E_\gamma] &= C^{a}_{\beta+\gamma}H_a \quad \text{if} \quad \beta+\gamma = 0, \\ &= C^{\beta+\gamma}_{\beta\gamma}E_{\beta+\gamma} \quad \text{if} \quad \beta+\gamma \neq 0, \\ &= f^{a}_{\beta\gamma}H_a + f^{\alpha}_{\beta\gamma}E_\alpha + f^{-\alpha}_{\beta\gamma}E_{-\alpha} + f^{d}_{\beta\gamma}E_d + f^{-d}_{\beta\gamma}E_{-d}. \end{aligned}$$

Since $\beta, \gamma > 0$, we have

$$f^{a}_{\beta\gamma} = f^{-\alpha}_{\beta\gamma} = f^{\pm d}_{\beta\gamma} = 0 \quad \text{and} \quad C^{\beta+\gamma}_{\beta\gamma} = f^{\alpha}_{\beta\gamma}. \tag{3.43}$$

Similarly for $[E_{-\beta}, E_{-\gamma}]$, we have

$$f^{a}_{-\beta-\gamma} = f^{\alpha}_{-\beta-\gamma} = f^{\pm d}_{-\beta-\gamma} = 0 \quad \text{and} \quad C^{-\beta+(-\gamma)}_{-\beta-\gamma} = f^{-\alpha}_{-\beta-\gamma}. \tag{3.44}$$

For $[E_{\beta}, E_{-\gamma}]$, we have again

$$\begin{aligned}
[E_{\beta}, E_{-\gamma}] &= C^{a}_{\beta+(-\gamma)} H_a \quad \text{if} \quad \beta - \gamma = 0, \\
&= C^{\beta+(-\gamma)}_{\beta-\gamma} E_{\beta+(-\gamma)} \quad \text{if} \quad \beta - \gamma \neq 0, \\
&= f^{a}_{\beta-\gamma} H_a + f^{\alpha}_{\beta-\gamma} E_{\alpha} + f^{-\alpha}_{\beta-\gamma} E_{-\alpha} + f^{d}_{\beta-\gamma} E_d + f^{-d}_{\beta-\gamma} E_{-d}.
\end{aligned}$$

The following cases determine the structure constants of $[E_{\beta}, E_{-\gamma}]$.

**a.** If $\beta - \gamma = 0 \quad \Rightarrow \quad f^{\pm\alpha}_{\beta-\gamma} = f^{\pm d}_{\beta-\gamma} = 0, \quad C^{a}_{\beta+(-\gamma)} = f^{a}_{\beta-\gamma},$ (3.46a)

**b.** if $\beta - \gamma > 0, \quad E_{\beta-\gamma} \subset m \quad \Rightarrow$

$$f^{a}_{\beta-\gamma} = f^{\pm d}_{\beta-\gamma} = f^{-\alpha}_{\beta-\gamma} = 0, \quad C^{\beta+(-\gamma)}_{\beta-\gamma} = f^{\alpha}_{\beta-\gamma}, \tag{3.46b}$$

**c.** if $\beta - \gamma > 0, \quad E_{\beta-\gamma} \subset k \quad \Rightarrow$

$$f^{a}_{\beta-\gamma} = f^{\pm\alpha}_{\beta-\gamma} = f^{-d}_{\beta-\gamma} = 0, \quad C^{\beta+(-\gamma)}_{\beta-\gamma} = f^{d}_{\beta-\gamma},$$

**d.** if $\beta - \gamma < 0, \quad E_{\beta-\gamma} \subset m \quad \Rightarrow$

$$f^{a}_{\beta-\gamma} = f^{\pm d}_{\beta-\gamma} = f^{\alpha}_{\beta-\gamma} = 0, \quad C^{\beta+(-\gamma)}_{\beta-\gamma} = f^{-\alpha}_{\beta-\gamma}, \tag{3.46c}$$

**e.** if $\beta - \gamma < 0, \quad E_{\beta-\gamma} \subset k \quad \Rightarrow$

$$f^{a}_{\beta-\gamma} = f^{\pm\alpha}_{\beta-\gamma} = f^{d}_{\beta-\gamma} = 0, \quad C^{\beta+(-\gamma)}_{\beta-\gamma} = f^{-d}_{\beta-\gamma}. \tag{3.46d}$$

Similarly $[E_{-\beta}, E_{\gamma}]$ can be discussed by substituting $\gamma$ instead of $\beta$ and substituting $\beta$ instead of $\gamma$ which is exactly $-[E_{\beta}, E_{-\gamma}]$. Using (3.2d),(3.6e), the commutation relation becomes,

$$
\begin{aligned}
[E_\beta, E_d] &= C_{\beta d}^{\beta+d} E_{\beta+d}, \\
&= f_{\beta d}^{\alpha} E_\alpha + f_{\beta d}^{-\alpha} E_{-\alpha}.
\end{aligned}
$$

Since $\beta, d > 0$ we have,

$$f_{\beta d}^{-\alpha} = 0 \quad \text{and} \quad C_{\beta d}^{\beta+d} = f_{\beta d}^{\alpha}. \tag{3.47}$$

Similarly for $[E_{-\beta}, E_{-d}]$ we have,

$$f_{-\beta-d}^{\alpha} = 0 \quad \text{and} \quad C_{\beta-d}^{(-\beta)+(-d)} = f_{-\beta-d}^{-\alpha}. \tag{3.48}$$

When it comes to discuss

$$
\begin{aligned}
[E_\beta, E_{-d}] &= C_{\beta-d}^{\beta+(-d)} E_{\beta+(-d)}, \\
&= f_{\beta-d}^{\alpha} E_\alpha + f_{\beta-d}^{-\alpha} E_{-\alpha}.
\end{aligned}
$$

By (3.7), $\beta - d > 0$ then,

$$f_{\beta-d}^{-\alpha} = 0. \tag{3.49}$$

Also (3.6e) leads us, $f_{\beta-d}^{\alpha}$ exists if $E_{\beta-d} \subset m$; otherwise $f_{\beta-d}^{\alpha} = 0$. Hence

$$
\begin{aligned}
C_{\beta-d}^{\beta+(-d)} &= f_{\beta-d}^{\alpha} \quad \text{if} \quad \beta - d > d, \\
C_{\beta-d}^{\beta+(-d)} &= 0 \qquad \text{if} \quad \beta - d \leq d.
\end{aligned} \tag{3.50}
$$

Similarly for $[E_{-\beta}, E_d]$ we have

$$f_{-\beta d}^{\alpha} = 0, \tag{3.51}$$

and

$$
\begin{aligned}
C_{-\beta d}^{(-\beta)+d} &= f_{-\beta d}^{-\alpha} \quad \text{if} \quad -\beta + d > d, \\
C_{-\beta d}^{(-\beta)+d} &= 0 \qquad \text{if} \quad -\beta + d \leq d.
\end{aligned} \tag{3.52}
$$

### 3.5.2 The evolution equations

By the use of catalogue, the equations (3.38) and (3.39) respectively become:

$$a^a_{l,x} + f^a_{\beta-\gamma}(Q^\beta b^{-\gamma}_l - Q^{-\gamma} b^\beta_l) = 0,$$
$$c^{\pm d}_{l,x} + f^{\pm d}_{\beta-\gamma}(Q^\beta b^{-\gamma}_l - Q^{-\gamma} b^\beta_l) = 0,$$

then we have

$$a^a_l = -D^{-1}_x[f^a_{\beta-\gamma}(Q^\beta b^{-\gamma}_l - Q^{-\gamma} b^\beta_l)], \tag{3.54a}$$
$$c^{\pm d}_l = -D^{-1}_x[f^{\pm d}_{\beta-\gamma}(Q^\beta b^{-\gamma}_l - Q^{-\gamma} b^\beta_l)], \tag{3.54b}$$

where $l = 0, ..., N$. Again by catalogue the equation (3.36a) become:

$$b^\alpha_{l,x} + ik\alpha_s b^\alpha_{l+1} - \alpha_a Q^\alpha a^a_l + Q^\beta f^\alpha_{\beta\gamma} b^\gamma_l + f^\alpha_{\beta-\gamma}(Q^\beta b^{-\gamma}_l - Q^{-\gamma} b^\beta_l) + Q^\beta f^\alpha_{\beta d} c^d_l +$$
$$Q^\beta f^\alpha_{\beta-d} c^{-d}_l = 0, \tag{3.55a}$$

$$b^{-\alpha}_{l,x} - ik\alpha_s b^{-\alpha}_{l+1} + \alpha_a Q^{-\alpha} a^a_l + Q^{-\beta} f^{-\alpha}_{-\beta-\gamma} b^{-\gamma}_l + f^{-\alpha}_{\beta-\gamma}(Q^\beta b^{-\gamma}_l - Q^{-\gamma} b^\beta_l) +$$
$$Q^{-\beta} f^{-\alpha}_{-\beta-d} c^{-d}_l + Q^{-\beta} f^{-\alpha}_{-\beta d} c^d_l = 0, \tag{3.55b}$$

where $l = 0, ..., N-1$. Substituting (3.54a) and (3.54b) in (3.55a) we obtain,

$$b^\alpha_{l+1} = \xi(R^\alpha_{1_\beta} b^\beta_l + R^\alpha_{2_{-\beta}} b^{-\beta}_l), \tag{3.56}$$

where $l = 0, ..., N-1$, $\xi = \frac{i}{k\alpha_s}$ and

$$R^\alpha_{1_\beta} = D_x \delta^\alpha_\beta - \alpha_a Q^\alpha f^a_{\beta-\gamma} D^{-1}_x Q^{-\gamma} + Q^\beta f^\alpha_{\beta\gamma} \delta^\gamma_\beta - f^\alpha_{\beta-\gamma} Q^{-\gamma} + Q^\beta f^\alpha_{\beta d} f^d_{\beta-\gamma} D^{-1}_x Q^{-\gamma} +$$
$$Q^\beta f^\alpha_{\beta-d} f^{-d}_{\beta-\gamma} D^{-1}_x Q^{-\gamma}, \tag{3.57a}$$

$$R^\alpha_{2_{-\beta}} = \alpha_a Q^\alpha f^a_{\beta-\gamma} D^{-1}_x Q^\beta \delta^{-\gamma}_{-\beta} + f^\alpha_{\beta-\gamma} Q^\beta \delta^{-\gamma}_{-\beta} - Q^\beta f^\alpha_{\beta d} f^d_{\beta-\gamma} D^{-1}_x Q^\beta \delta^{-\gamma}_{-\beta} -$$
$$Q^\beta f^\alpha_{\beta-d} f^{-d}_{\beta-\gamma} D^{-1}_x Q^\beta \delta^{-\gamma}_{-\beta}. \tag{3.57b}$$

Similarly the substitution of (3.54a),(3.54b) in (3.55b) gives,

$$b^{-\alpha}_{l+1} = -\xi(R^{-\alpha}_{3_\beta} b^\beta_l + R^{-\alpha}_{4_{-\beta}} b^{-\beta}_l), \tag{3.58}$$

where $l = 0, ..., N-1$ and

$$R_{3_\beta}^{-\alpha} = \alpha_a Q^{-\alpha} f^a_{\beta-\gamma} D_x^{-1} Q^{-\gamma} - f^{-\alpha}_{\beta-\gamma} Q^{-\gamma} + Q^{-\beta} f^{-\alpha}_{-\beta-d} f^{-d}_{\beta-\gamma} D_x^{-1} Q^{-\gamma} + Q^{-\beta} f^{-\alpha}_{-\beta d} f^{d}_{\beta-\gamma} D_x^{-1} Q^{-\gamma}, \tag{3.59a}$$

$$R_{4_{-\beta}}^{-\alpha} = D_x \delta^{-\alpha}_{-\beta} - \alpha_a Q^{-\alpha} f^a_{\beta-\gamma} D_x^{-1} Q^{\beta} \delta^{-\gamma}_{-\beta} + Q^{-\beta} f^{-\alpha}_{-\beta-\gamma} \delta^{-\gamma}_{-\beta} + f^{-\alpha}_{\beta-\gamma} Q^{\beta} \delta^{-\gamma}_{-\beta} - Q^{-\beta} f^{-\alpha}_{-\beta-d} f^{-d}_{\beta-\gamma} D_x^{-1} Q^{\beta} \delta^{-\gamma}_{-\beta} - Q^{-\beta} f^{-\alpha}_{-\beta d} f^{d}_{\beta-\gamma} D_x^{-1} Q^{\beta} \delta^{-\gamma}_{-\beta}. \tag{3.59b}$$

**Proposition 3.6.** *The equations* (3.56), (3.58) *give us the following recursion equation and the corresponding recursion operator;* $\mathbb{R}^A_B$

$$B^A_{l+1} = \mathbb{R}^A_B B^B_l \quad l = 0, 1.., N-1, \tag{3.60}$$

*where*

$$B^A_{l+1} = \begin{pmatrix} b^{\alpha}_{l+1} \\ b^{-\alpha}_{l+1} \end{pmatrix},$$
$$\mathbb{R}^A_B = \xi \begin{pmatrix} R^{\alpha}_{1_\beta} & R^{\alpha}_{2_{-\beta}} \\ -R^{-\alpha}_{3_\beta} & -R^{-\alpha}_{4_{-\beta}} \end{pmatrix},$$
$$B^B_l = \begin{pmatrix} b^{\beta}_{l} \\ b^{-\beta}_{l} \end{pmatrix}.$$

*and* $\xi = \frac{i}{k\alpha_s}$ .

When it comes to the evolution equations, by catalogue the equation (3.37a) becomes,

$$Q^{\alpha}_t = b^{\alpha}_{N,x} - \alpha_a Q^{\alpha} a^a_N + Q^{\beta} f^{\alpha}_{\beta\gamma} b^{\gamma}_N + f^{\alpha}_{\beta-\gamma}(Q^{\beta} b^{-\gamma}_N - Q^{-\gamma} b^{\beta}_N) + Q^{\beta} f^{\alpha}_{\beta d} c^d_N + Q^{\beta} f^{\alpha}_{\beta-d} c^{-d}_N, \tag{3.61a}$$

$$Q^{-\alpha}_t = b^{-\alpha}_{N,x} + \alpha_a Q^{-\alpha} a^a_N + Q^{-\beta} f^{-\alpha}_{-\beta-\gamma} b^{-\gamma}_N + f^{-\alpha}_{\beta-\gamma}(Q^{\beta} b^{-\gamma}_N - Q^{-\gamma} b^{\beta}_N) + Q^{-\beta} f^{-\alpha}_{-\beta-d} c^{-d}_N + Q^{-\beta} f^{-\alpha}_{-\beta d} c^d_N. \tag{3.61b}$$

Similar to the argument for recursion equations, after substituting (3.54a), (3.54b) in (3.61a) and (3.61b) we have,

$$\begin{pmatrix} Q_t^{\alpha} \\ Q_t^{-\alpha} \end{pmatrix} = \begin{pmatrix} C^{\alpha}_{1_\beta} & C^{\alpha}_{2_{-\beta}} \\ C^{-\alpha}_{3_\beta} & C^{-\alpha}_{4_{-\beta}} \end{pmatrix} \begin{pmatrix} b^{\beta}_N \\ b^{-\beta}_N \end{pmatrix}. \tag{3.62}$$

Here exactly

$$\begin{aligned} R^{\alpha}_{1_\beta} &= C^{\alpha}_{1_\beta}, \quad R^{\alpha}_{2_{-\beta}} = C^{\alpha}_{2_{-\beta}}, \\ R^{-\alpha}_{3_\beta} &= C^{-\alpha}_{3_\beta}, \quad R^{-\alpha}_{4_{-\beta}} = C^{-\alpha}_{4_{-\beta}}. \end{aligned} \tag{3.63}$$

**Proposition 3.7.** *The evolution equations can be determined as*

$$Q_t^A = \frac{1}{\xi}(\overline{\mathbb{R}}^A_B)^N \sigma_3 B_1^C. \tag{3.64}$$

*where* $\xi = \frac{i}{k\alpha_s}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $\overline{\mathbb{R}}^A_B = \xi\sigma_3 \begin{pmatrix} R^{\alpha}_{1_\beta} & R^{\alpha}_{2_{-\beta}} \\ -R^{-\alpha}_{3_\beta} & -R^{-\alpha}_{4_{-\beta}} \end{pmatrix} \sigma_3.$

**Proof:** The equation (3.62) becomes

$$Q_t^A = \frac{1}{\xi}\sigma_3 \mathbb{R}^A_B B^B_N.$$

By (3.60), we have

$$Q_t^A = \frac{1}{\xi}(\overline{\mathbb{R}}^A_B)^N \sigma_3 B_1^C. \quad \square \tag{3.65}$$

## 3.6 Unified equations

Our approach can be improved without expanding the indices. For this case, the catalogue will not be needed. By equations (3.35d) and (3.35e) we obtain,

$$a^a_l = -D^{-1} f^a_{AC} Q^A b^C_l, \quad c^D_l = -D^{-1} f^D_{AC} Q^A b^C_l, \quad \text{where} \quad l = 0, 1..., N. \tag{3.66}$$

By (3.35a) we have,

$$-ikf^A_{sC} b^C_{l+1} = b^A_{l,x} - f^A_{aC} Q^C a^a_l + f^A_{BC} Q^B b^C_l + f^A_{BD} Q^B c^D_l. \tag{3.67}$$

Let $f^A_{sC} = X^A_C$ be a non-degenerate matrix. Here summation on indices enables us to change the indices. We change $C$ to $B$ and $B$ to $C$ in the above equations (3.66) and (3.67). Then equation (3.67) becomes,

$$b^B_{l+1} = \frac{i}{k}(X^{-1})^B_A \;\; [b^A_{l,x} - f^A_{aB}Q^B a^a_l + f^A_{CB}Q^C b^B_l + f^A_{CD}Q^C c^D_l]. \tag{3.68}$$

Substitution $a^a_l$ and $c^D_l$ for all $l = 0, 1..., N$ from (3.66) in equation (3.68) gives,

$$b^B_{l+1} = \frac{i}{k}(X^{-1})^B_A \;\; [D\delta^A_B + f^A_{aB}Q^B D^{-1} f^a_{AB}Q^A + f^A_{CB}Q^C - f^A_{CD}Q^C D^{-1} f^D_{AB}Q^A] \;\; b^B_l.$$

Hence

$$b^B_{l+1} = \mathbb{R}^A_B b^B_l \tag{3.69a}$$

$$b^B_N = (\mathbb{R}^A_B)^{N-1} b^B_1, \tag{3.69b}$$

where

$$\mathbb{R}^A_B = \frac{i}{k}(X^{-1})^B_A \;\; [D\delta^A_B + f^A_{aB}Q^B D^{-1} f^a_{AB}Q^A + f^A_{CB}Q^C - f^A_{CD}Q^C D^{-1} f^D_{AB}Q^A] \tag{3.70}$$

is the *recursion operator* and (3.69b) is the *recursion equation*. On the other hand by (3.35b) we have,

$$Q^A_t = b^A_{N,x} - f^A_{aB}Q^B a^a_N + f^A_{CB}Q^C b^B_N + f^A_{CD}Q^c c^D_N. \tag{3.71}$$

**Proposition 3.8.** *The evolution equations can be written as*

$$Q^A_t = -ikX^B_A \;\; (\mathbb{R}^A_B)^N \;\; b^B_1, \tag{3.72}$$

*where*

$$\mathbb{R}^A_B = \frac{i}{k}(X^{-1})^B_A \;\; [D\delta^A_B + f^A_{aB}Q^B D^{-1} f^a_{AB}Q^A + f^A_{CB}Q^C - f^A_{CD}Q^C D^{-1} f^D_{AB}Q^A].$$

**Proof:** Substituting the terms in (3.66) into the equation (3.71) we have,

$$Q_t^A = [D\delta_B^A + f_{aB}^A Q^B D^{-1} f_{AB}^a Q^A + f_{CB}^A Q^C - f_{CD}^A Q^C D^{-1} f_{AB}^D Q^A] \ \ b_N^B.$$

So we have

$$Q_t^A = -ikX_A^B \quad \mathbb{R}_B^A \quad b_N^B. \tag{3.73}$$

Hence by (3.69b) the equation (3.73) results as;

$$Q_t^A = -ikX_A^B \quad (\mathbb{R}_B^A)^N \quad b_1^B. \tag{3.74}$$

which is the evolution equation. □

# Chapter 4

# Gel'fand-Dikii Formalism

In the Lax formalism, the main problem is to determine the operator $A$ in the Lax equation (1.5). By the use of Gel'fand- Dikii formalism, we determine such operators straightforwardly. This formalism gives a construction of all Lax pairs, based on the calculation of fractional powers of the operator $L$. Further the Gel'fand- Dikii formalism makes use of some algebras; among such algebras, we can give the pseudo-differential, matrix, polynomial and Moyal algebras. In each Lax representation, the Lax operator $L$, is a polyomial. For the pseudo-differential algebra $L$ is a polynomial of a differential operator $D_x$, for the matrix algebra it is a polynomial of a spectral constant and for the polynomial and Moyal algebras, it is a polynoimal of the auxiliary variable (momentum p).

Let $\mathcal{G}$ be an algebra and '*' be a non-commutative, associative binary product. Let $F$, $G$ and $H$ be $\mathcal{G}$-valued functions, then define a bracket $\{,\}_{\mathcal{G}}$ as

$$\{F,G\}_{\mathcal{G}} := \frac{1}{2\kappa}(F * G - G * F), \quad \kappa \in \mathbb{R} \tag{4.1}$$

which satisfies the properties;

**i.** Skew-symmetry: $\{F,G\}_{\mathcal{G}} = -\{G,F\}_{\mathcal{G}}$,

**ii.** Leibniz rule: $\{FG,H\}_{\mathcal{G}} = F.\{G,H\}_{\mathcal{G}} + \{F,H\}_{\mathcal{G}}.G$,

**iii.** Jacobi identity: $\{\{F,G\}_{\mathcal{G}},H\}_{\mathcal{G}}+\{\{H,F\}_{\mathcal{G}},G\}_{\mathcal{G}}+\{\{G,H\}_{\mathcal{G}},F\}_{\mathcal{G}}=0.$

Let $L$ be $\mathcal{G}$ valued Lax operator. Then the Lax equation is defined as;

$$\frac{\partial L}{\partial t}=\{A,L\}_{\mathcal{G}} \tag{4.2}$$

for some $\mathcal{G}$ valued function $A$. In order to obtain $A$, we find $\overline{A}$ s.t

$$\{L,\overline{A}\}_{\mathcal{G}}=0. \tag{4.3}$$

Apart from the matrix algebra we take $\overline{A}=L^{\frac{n}{m}}$, then (4.3) holds, where $n\neq am$; $a$, $n\in\mathbb{Z}$. We put $A=(\overline{A})_{\geqslant k}$ that is

$$A=(L^{\frac{n}{m}})_{\geqslant k}. \tag{4.4}$$

So we obtain a consistent equation (4.2). Here the restriction of being bigger or equal to $k$ is for $A$ to be the polynomial part of $L^{\frac{n}{m}}$ except first $k-1$ terms. For the matrix algebra we find $\overline{A}$ by solving $\{L,\overline{A}\}_{\mathcal{G}}=0$, then we set

$$A=(\overline{A})_{\geqslant k}. \tag{4.5}$$

In the following sections, we shall present our method and give examples for each Lax representation. In Section 4.1, we consider Lax operator taking values in a pseudo-differential algebra. In Section 4.2, we consider the case of matrix algebra. In Section 4.3, we deal with polynomial algebra. In Section 4.4, we work on the case of Moyal algebra.

## 4.1 Pseudo-differential algebra

As a first example of an algebra $\mathcal{G}$, we consider the pseudo-differential algebra. Let $F$, $G\in\mathcal{G}$ be two pseudo-differential operators;

$$F=f_n(u)D_x^n+...+f_0(u)+f_{-1}(u)D_x^{-1}+.....,$$
$$G=g_m(u)D_x^m+...+g_0(u)+g_{-1}(u)D_x^{-1}+.....,$$

where $f_i, g_j$, $i = .., -1, 0, 1, .., n$; $j = .., -1, 0, 1, .., m$ are differentiable functions. The bracket $\{,\}_{\mathcal{G}}$ defined in (4.1) corresponds to the usual commutator if '*' is the operational product and $\kappa = \frac{1}{2}$.

**Definition 4.1.** *A differential operator of order n is a finite sum*

$$\mathcal{D} = \sum_{i=0}^{n} P_i[u] D_x^i, \tag{4.6}$$

*where the coefficients $P_i[u]$ are differentiable functions.[18]*

The multiplication of differential operators is described by the formula

$$D_x^i . D_x^j = D_x^{i+j}, \tag{4.7}$$

valid for $i, j \geqslant 0$. The derivational property of $D_x$ is given by the Leibniz rule

$$D_x . Q = Q_x + Q D_x, \tag{4.8}$$

where Q is a differentiable function.

**Definition 4.2.** *A pseudo-differential operator of order n is an infinite series*

$$\mathcal{D} = \sum_{i=-\infty}^{n} P_i[u] D_x^i, \tag{4.9}$$

*where $P_i[u]$ are differentiable functions. The operator $D_x^{-1}$ is the formal inverse of $D_x$ ($D_x . D_x^{-1} = D_x^{-1} . D_x = 1$).[18]*

The operator $D_x^{-1}$ of any differentiable function $Q$ is formulated as

$$D_x^{-1} . Q = \sum_{k=0}^{\infty} (-1)^k Q^k D_x^{-k-1}. \tag{4.10}$$

The advantage of introduction of a pseudo-differential operator is that now we can take roots of any pseudo-differential operator.

**Lemma 4.3.** *Every nonzero pseudo-differential operator of order $n > 0$ has an $n-th$ root.[18]*

**Proof:** Suppose $\mathcal{D}$ is a pseudo-differential operator of the form (4.9), with nonzero leading coefficient $P_n$. The n-th root $\varepsilon = \mathcal{D}^{\frac{1}{n}}$ will be a first order pseudo-differential operator of the form

$$\varepsilon = (P_n)^{\frac{1}{n}} D_x + Q_0 + Q_{-1} D_x^{-1} + Q_{-2} D_x^{-2} + ...$$

Substituting into the equation $\varepsilon^n = \mathcal{D}$, leads to a system of equations for the coefficients $Q_k$ , $k = ..., -2, -1, 0$ of $\varepsilon$ which can be solved for the $Q_k$, $k = ..., -2, -1, 0$ in terms of $P_i$'s, $i = ..., -1, 0, 1.., n$. □

**Example 4.1.** Consider the operator $\mathcal{D} = D_x^2 + u$, which corresponds to the KdV equation. Then the square root of $\mathcal{D}$ is

$$\mathcal{D}^{\frac{1}{2}} = D_x + \frac{1}{2} u D_x^{-1} - \frac{1}{4} u_x D_x^{-2} + \frac{1}{8}(u_{xx} - u^2) D_x^{-3} + ... \tag{4.11}$$

We consider equations with the Lax representations of the form

$$L_t = [A, L].$$

where L is differential operator of order $m$ and $A$ is a differential operator whose coefficients are functions of $x$ and $t$. Let

$$L = D_x^m + u_{m-2} D_x^{m-2} + ... + u_1 D_x + u_0, \tag{4.12}$$

where $u_i$, $i = 0, 1, ..., m-2$ are functions of $x$ and $t$. $L^{\frac{1}{m}}$ exists by the Lemma (4.3), so we can consider any fractional power of L; $L^{\frac{n}{m}}$. Set

$$L^{\frac{n}{m}} = (L^{\frac{n}{m}})_+ + (L^{\frac{n}{m}})_-, \tag{4.13}$$

where $(L^{\frac{n}{m}})_+$ is the differential part of the series $L^{\frac{n}{m}}$; $(L^{\frac{n}{m}})_-$ is a series of order less or equal to $-1$ and $n \in \mathbb{Z}$, $n \neq am$, $a \in \mathbb{Z}$. Since $[L, L^{\frac{n}{m}}] = 0$ we have

$$[L,(L^{\frac{n}{m}})_+] = -[L,(L^{\frac{n}{m}})_-]. \tag{4.14}$$

The left-hand side of(4.14) is a differential operator of order$\leqslant n+m-1$, but the right hand side is a series of order$\leqslant m-1$. Hence there are $n$ number of terms cancelling each other which give us a system of evolution equations for the dependent variables $u_i$, $\quad i=0,1,...,m-2 \quad$ by comparing the coefficients of $D_x^i$. Different choices of operator $A$ for given $L$ , leads to a hierarchy of nonlinear systems of differential equations. To have a hierarchy consider

$$L_{t_n} = [A_n, L], \tag{4.15}$$

where $A_n$ can be defined as

$$A_n := (L^{\frac{n}{m}})_+, \tag{4.16}$$

where $'+'$ means the polynomial part of $L^{\frac{n}{m}}$. For such a hierarchy, we can construct a recursion operator.

**Proposition 4.4.** *For any $n$*

$$A_{n+m} = L.A_n + R_n, \tag{4.17}$$

*where $R_n$ is a differential operator of order$\leqslant m-1$.[15]*

**Proof:** By (4.16) and (4.13)

$$A_{n+m} = (L.L^{\frac{n}{m}})_+ = (L.[(L^{\frac{n}{m}})_+ + (L^{\frac{n}{m}})_-])_+.$$

Note that, since $(L.(L^{\frac{n}{m}})_+)_+$ has only positive powers, then

$$(L.(L^{\frac{n}{m}})_+)_+ = L.(L^{\frac{n}{m}})_+.$$

Hence

$$A_{n+m} = L.(L^{\frac{n}{m}})_+ + (L.(L^{\frac{n}{m}})_-)_+ = L.A_n + R_n,$$

by substituting $R_n = (L.(L^{\frac{n}{m}})_-)_+$. Here since $(L^{\frac{n}{m}})_-$ is a series of order less or equal to $-1$ then $ord(R_n) \leqslant m-1$. $\square$

The result obtained from the last Proposition leads to

$$L_{t_{n+m}} = [A_{n+m}, L] = [L.A_n + R_n, L] = L.[A_n, L] + [R_n, L] = L.L_{t_n} + [R_n, L].$$

Hence

$$L_{t_{n+m}} = L.L_{t_n} + [R_n, L]. \tag{4.18}$$

The equation (4.18) is called *the recursion relation* and $R_n$ is called *the remainder.*[15]

**Remark 4.5.** *It follows from the formula*

$$A_{n+m} = (L^{\frac{n}{m}}.L)_+ = (L^{\frac{n}{m}})_+.L + ((L^{\frac{n}{m}})_-.L)_+, \tag{4.19}$$

*that*

$$A_{n+m} = A_n L + \overline{R_n}, \tag{4.20}$$

*and*

$$L_{t_{n+m}} = L_{t_n}.L + [\overline{R_n}, L], \tag{4.21}$$

*where* $\overline{R_n} = ((L^{\frac{n}{m}})_- L)_+$ *is a differential operator and* $ord(\overline{R_n}) \leqslant m-1$.

To find the recursion operator we equate the coefficients of different powers of $D_x$ in (4.18). The comparison of the coefficients of $D_x^i$, $i = 2m-2, ..., m-1$ enables us to determine $R_n$ in terms of the coefficients of operators $L$ and $L_{t_n}$. It is essential that the resulting formulas become linear in the coefficients of $L_{t_n}$. The remaining coefficients of $D_x^i$, $i = m-2, ..., 0$ in (4.18) give us the relation

$$\begin{pmatrix} u_o \\ u_1 \\ . \\ . \\ u_{m-2} \end{pmatrix}_{t_{n+m}} = \mathbb{R} \begin{pmatrix} u_o \\ u_1 \\ . \\ . \\ u_{m-2} \end{pmatrix}_{t_n}, \tag{4.22}$$

where $\mathbb{R}$ is *the recursion operator*. This is indeed the definition of the recursion operator. Instead of the equation (4.18) we can use (4.21) as well, the corresponding recursion operators coincide.

**Example 4.2.** The KdV equation, $u_t = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x$ has a Lax representation with

$$L = D_x^2 + u, \qquad A = (L^{\frac{3}{2}})_+. \tag{4.23}$$

By the equation (4.18) the recursion relation is of the form

$$L_{t_{n+2}} = L.L_{t_n} + [R_n, L].$$

Since $L_{t_n} = u_{t_n}$, the main relation takes the form

$$u_{t_{n+2}} = (D_x^2 + u).u_{t_n} + [R_n, D_x^2 + u], \tag{4.24}$$

with $R_n = a_n D_x + c_n$. ( Note that $R_n = (L.(L^{\frac{3}{2}})_-)_+$ and $ord(R_n) \leqslant 1$) Comparing the coefficients of $D_x^0$, $D_x$ and $D_x^2$ in the equation (4.24) we obtain respectively

$$a_n = \frac{1}{2}D_x^{-1}(u_{t_n}), \qquad c_n = \frac{3}{4}u_{t_n},$$

and

$$u_{t_{n+2}} = (\frac{1}{4}D_x^2 + u + \frac{1}{2}u_x D_x^{-1}).u_{t_n},$$

that gives the standard recursion operator for the KdV equation,

$$\mathbb{R} = \frac{1}{4}D_x^2 + u + \frac{1}{2}u_x D_x^{-1}. \tag{4.25}$$

If we use the recursion relation (4.21) we obtain $\overline{a_n} = \frac{1}{2}D_x^{-1}(u_{t_n}), \overline{c_n} = -\frac{1}{4}u_{t_n}$ for $\overline{R_n} = \overline{a_n}D_x + \overline{c_n}$. The corresponding recursion operator is exactly the one in (4.25).

### 4.1.1 Symmetric and skew-symmetric reductions of a differential Lax operator

The standard reductions of the Gel'fand-Dikii systems are given by the conditions $L^* = L$ or $L^* = -L$. Here * denotes the adjoint operation defined as follows.[15]

**Definition 4.6.** *Let L be a differential operator, $L = \sum_{i=0}^{m} a_i.D_x^i$, then its adjoint $L^*$ is given by*

$$L^* = \sum_{i=0}^{m} (-D_x)^i.a_i \tag{4.26}$$

*where the coefficients $a_i$ are differentiable functions.*

If $L^* = L$, then $ord(L) = m$ must be an even integer. If $L^* = -L$, then $ord(L)$ must be an odd integer. For both cases; if $L^* = L$ or $L^* = -L$, the compatibility condition of (4.15) implies that $(A_n)^* = -A_n$. So all possible $A_n$ defined by (4.16), where $n$ takes odd integer values.

If $L^* = L$, the formula $A_{n+m} = (L.L^{\frac{n}{m}})_+ = L.A_n + R_n$ gives a correct $A_n$ operator, since $n+m$ is an odd integer. Hence in this case Proposition (4.4) remains valid and the recursion operator can be found form (4.18) or (4.21). On the other hand, if $L^* = -L$ then both integers $m$ and $n$ are odd and hence their sum $m+n$ is an even integer. This means that $(L^{\frac{n+m}{m}})_+$ cannot be taken as an $A_n$ operator. In this skew-adjoint case we must take

$$A_{n+2m} = (L^{\frac{n+2m}{m}})_+ = (L^2.L^{\frac{n}{m}})_+$$

to find the recursion operator. Following the proof of Proposition (4.4), we state Proposition (4.7).

**Proposition 4.7.** *If $L^* = -L$ then*

$$A_{n+2m} = L^2 A_n + R_n \tag{4.27}$$

*where $ord(R_n) < 2.ord(L) - 1$. Also the corresponding recursion relation is*

$$L_{t_{n+2m}} = L^2.L_{t_n} + [R_n, L]. \tag{4.28}$$

*Here note that* $R_n = (L^2.(L^{\frac{n}{m}})_-)_+$ *[15].*

**Remark 4.8.** *Instead of* (4.27)*, we can use*

$$A_{n+2m} = L.A_n.L + \breve{R}_n, \tag{4.29}$$

*or*

$$A_{n+2m} = A_n.L^2 + \tilde{R}_n, \tag{4.30}$$

*where* $\breve{R}_n$ *and* $\tilde{R}_n$ *are differential operators and* $ord(\breve{R}_n) = ord(\tilde{R}_n) \leqslant 2ord(L) - 1$*. Then the recursion relations become respectively*

$$L_{t_{n+2m}} = L.L_{t_n}.L + [\breve{R}_n, L], \tag{4.31a}$$

$$L_{t_{n+2m}} = L_{t_n}.L^2 + [\tilde{R}_n, L]. \tag{4.31b}$$

Note that the recursion operators obtained by (4.28), (4.31a) and (4.31b) are all coincide. We now generalize our scheme to the case, where the Lax operator is a pseudo-differential operator. For the skew-symmetric case $A_n$ is defined as either (4.27),(4.29) or (4.30). In the pseudo-differential case, they are not equivalent. Let us consider $L = M.D_x^{-1}$, where $M$ is a differential operator and define $L^\dagger = D_x.L^*.D_x^{-1}$.

**Lemma 4.9.** *Let* $L^\dagger = \epsilon L$*, where* $\epsilon = \pm 1$*. Then*

$$R_n = a_{m-1}D_x^{m-1} + ... + a_0, \qquad for \qquad \epsilon = 1, \tag{4.32}$$

*where* $R_n$ *is defined by* $A_{n+m} = L.A_n + R_n$ *and*

$$\hat{R}_n = a_{2m-1}D_x^{2m-1} + ... + a_{-1}D_x^{-1} \qquad for \qquad \epsilon = -1, \tag{4.33}$$

*where* $\hat{R}_n$ *is defined by* $A_{n+2m} = L.A_n.L + \hat{R}_n$*.[15]*

**Example 4.3.** The KdV equation has, besides the standard Lax pair, the following Lax pair:

$$L = (D_x^2 + u).D_x^{-1}, \qquad A = (L^3)_+. \tag{4.34}$$

Here

$$L^\dagger = D_x.L^*.D_x^{-1} = D_x.(-D_x - D_x^{-1}u).D_x^{-1} = -L.$$

So $\epsilon = -1$ and according to the formula (4.33) we have

$$\hat{R}_n = a_n D_x + b_n + c_n D_x^{-1}.$$

Therefore the corresponding recursion relation is

$$L_{t_{n+2}} = L.L_{t_n}.L + [\hat{R}_n, L]. \tag{4.35}$$

Since in this case $L_{t_n} = u_{t_n} D_x^{-1}$ and $L_{t_{n+2}} = u_{t_{n+2}} D_x^{-1}$, the equation (4.35) becomes

$$u_{t_{n+2}} D_x^{-1} = L.(u_{t_n} D_x^{-1}).L + [a_n D_x + b_n + c_n D_x^{-1}, L]. \tag{4.36}$$

Comparing the coefficients of $D_x^i, \quad i = -3, .., 2$ in the equation (4.36) we obtain

$$a_n = D_x^{-1}(u_{t_n}), \qquad b_n = u_{t_n}, \qquad c_n = -u_{t_n,x} - uD_x^{-1}(u_{t_n}),$$

and

$$u_{t_{n+2}} = (D_x^2 + 4u + 2u_x D_x^{-1}).u_{t_n}.$$

That gives the recursion operator

$$\mathbb{R} = D_x^2 + 4u + 2u_x D_x^{-1}. \tag{4.37}$$

**Example 4.4.** The DSIII (Drinfeld-Sokolov III) system introduced in [21],[22], is given as

$$\begin{aligned} u_t &= -u_{3x} + 6uu_x + 6v_x, \\ v_t &= \quad 2v_{3x} - 6uv_x. \end{aligned} \; . \tag{4.38}$$

The nonlocal Lax representation for this system is

$$\begin{aligned} L &= (D_x^5 - 2uD_x^3 - 2D_x^3 u - 2D_x w - 2wD_x)D_x^{-1}, \\ A &= (L^{\frac{3}{4}})_+, \end{aligned} \tag{4.39}$$

where $w = v - u_{2x}$. Here

$$\begin{aligned} L^* &= D_x^{-1}(D_x^5 - 2uD_x^3 - 2D_x^3 u - 2D_x w - 2wD_x), \\ L^\dagger &= (D_x^5 - 2uD_x^3 - 2D_x^3 u - 2D_x w - 2wD_x)D_x^{-1} = L. \end{aligned}$$

So $\epsilon = 1$ and according to the formula (4.32) we have

$$R_n = a_n D_x^3 + b_n D_x^2 + c_n D_x + d_n.$$

Therefore the corresponding recursion relation is

$$L_{t_{n+4}} = L.L_{t_n} + [R_n, L]. \tag{4.40}$$

Since in this case

$$L_{t_n} = -4u_{t_n} D_x^2 - 2u_{2xt_n} - 6u_{xt_n} D_x - 2v_{xt_n} D_x^{-1} - 4v_{t_n}. \tag{4.41}$$

By equating the coefficients of the powers of $D_x^i \quad i = -1, ..7$ in (4.40), we obtain

$$\begin{aligned} a_n &= -D_x^{-1}(u_{t_n}), \quad b_n = -4u_{t_n}, \\ c_n &= \frac{1}{2}(6uD_x^{-1}(u_{t_n}) - 11u_{t_n,x} - 2D_x^{-1}(uu_{t_n}) - 2D_x^{-1}(v_{t_n})), \\ d_{n,x} &= \frac{1}{2}(6u_{2x}D_x^{-1}(u_{t_n}) + 10u_x u_{t_n} - 5u_{t_n,3x} + 4uu_{t_n,x} - 6v_{t_n,x}), \end{aligned}$$

and

$$\begin{pmatrix} u_{t_{n+4}} \\ v_{t_{n+4}} \end{pmatrix} = \mathbb{R} \begin{pmatrix} u_{t_n} \\ v_{t_n} \end{pmatrix}.$$

That gives the recursion operator of the DSIII system

$$\mathbb{R} = \begin{pmatrix} R_0^0 & R_1^0 \\ R_0^1 & R_1^1 \end{pmatrix}, \tag{4.42}$$

where

$$\begin{aligned}
R_0^0 &= D_x^4 - 8uD_x^2 - 8u_{2x} + 16v - 12u_xD_x + 16u^2 + (12uu_x - 2u_{3x} + 12v_x)D_x^{-1} + 4u_xD_x^{-1}u \\
R_1^0 &= -10D_x^2 + 8u + 4u_xD_x^{-1} \\
R_0^1 &= 12v_{2x} + 10v_xD_x + (4v_{3x} - 12uv_x)D_x^{-1} + 4v_xD_x^{-1}u \\
R_1^1 &= -4D_x^4 + 16uD_x^2 + 8u_xD_x + 16v + 4v_xD_x^{-1}.
\end{aligned} \tag{4.43}$$

## 4.2 Matrix algebra

In this section, we consider $\mathcal{G}$ as the matrix algebra. Let $F$, $G \in \mathcal{G}$ be $n \times n$ matrices. The bracket $\{,\}_{\mathcal{G}}$ defined in (4.1) corresponds to the usual commutator if '*' is the matrix multiplication and $\kappa = \frac{1}{2}$. Let $L$ be a matrix operator of the form

$$L = D_x - (-\lambda a - q), \tag{4.44}$$

where $\lambda$ is the spectral parameter; $q$ is a $\mathcal{G}$ valued function of $x$, $t$ and $a$ belong to the Lie algebra $\mathcal{G}$.

**Proposition 4.10.** *Let $L$ be a matrix operator of the form* (4.44) *then the corresponding recursion relation is*

$$L_{t_{n+1}} = \lambda L_{t_n} + [R_n, L], \tag{4.45}$$

*where $R_n$ is a matrix operator and $ord(R_n) = ord(L)$.(see the paper [15] for the proof)*

**Example 4.5.** The nonlinear Schrödinger equation is equivalent to the system

$$\begin{aligned}
u_t &= -\tfrac{1}{2}u_{xx} + u^2v \\
v_t &= \ \tfrac{1}{2}v_{xx} - v^2u,
\end{aligned} \tag{4.46}$$

has a Lax operator

$$L = D_x + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda + \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}. \tag{4.47}$$

Here the Lie algebra $\mathcal{G}$ is $sl(2)$ and $R_n = \begin{pmatrix} a_n & b_n \\ c_n & -a_n \end{pmatrix}$, since

$$L_{t_n} = \begin{pmatrix} 0 & u_{t_n} \\ v_{t_n} & 0 \end{pmatrix}.$$

Substitution of the related terms in the recursion relation (4.45) enables us to compare the coefficients of the powers of $\lambda^i$, $i = 0, 1$. Hence we obtain

$$\begin{aligned} a_n &= \tfrac{1}{2} D_x^{-1}(u_{t_n} v + v_{t_n} u), \\ b_n &= \tfrac{1}{2} u_{t_n}, \quad c_n = -\tfrac{1}{2} v_{t_n}, \end{aligned}$$

and the recursion operator of the system (4.46) is given by

$$\mathbb{R} = \begin{pmatrix} uD_x^{-1} - \frac{1}{2}D_x & uD_x^{-1}u \\ -vD_x^{-1}v & -vD_x^{-1}u + \frac{1}{2}D_x. \end{pmatrix} \tag{4.48}$$

## 4.3 Polynomial algebra

In this Section, we consider $\mathcal{G}$ as the polynomial algebra. Let $F$, $G \in \mathcal{G}$ be two arbitrary differentiable functions of $x$ and $t$. The bracket $\{,\}_{\mathcal{G}}$ defined in (4.1) corresponds to the Poisson bracket and $\kappa = \frac{1}{2}$

$$\{F, G\}_{\mathcal{G}} = \{F, G\}_k = p^k (\frac{\partial F}{\partial p}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p}).$$

In previous sections, we introduced a direct method to determine a recursion operator of a system of evolution equations when its Lax representation is known. We have considered the cases where the Lax operator is a differential operator or it is a pseudo-differential operator. Such representations are called as standard

Lax representations. On the other hand there are some systems of evolution equations obtained by the nonstandard Lax representations. For this purpose in this section we deal with the nonstandard Lax representations.

**Definition 4.11.** *Let $M$ be an $n$-dimensional manifold. Let $U$ be a function space on $M$. $M$ is called a Poisson manifold if there exists a bracket $\{f,g\}$ (called as standard Poisson bracket) satisfying the following properties, for all $c, c' \in \mathbb{R}$ and $f, g, h \in U$*

***i.*** *Bi-linearity:*

$$\{cf + c'g, h\} = c\{f,h\} + c'\{g,h\},$$
$$\{f, cg + c'h\} = c\{f,g\} + c'\{f,h\},$$

***ii.*** *Skew-symmetry:* $\{f,g\} = -\{g,f\}$,

***iii.*** *Jacobi identity:* $\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$,

***iv.*** *Leibniz rule:* $\{fg,h\} = f.\{g,h\} + \{f,h\}.g$.

*Here '.' denotes the ordinary multiplication of real valued functions. [18]*

Define a *modified Poisson bracket* as

$$\{f,g\}_k = p^k\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}\right), \tag{4.49}$$

where $k$ is an integer and $f,\ g,\ h \in U$ [16]. For $k = 0$, we have the standard Poisson bracket,

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}. \tag{4.50}$$

**Lemma 4.12.** *For any $k$ in $\mathbb{Z}$,* (4.49) *is a Poisson bracket.*

**Proof:** We should check only the Jacobi-Identity. Other properties can be verified easily by the definition of the Poisson bracket. Let us show that

$$\{\{f,g\}_k,h\}_k + \{\{h,f\}_k,g\}_k + \{\{g,h\}_k,f\}_k = 0. \tag{4.51}$$

Note that

$$\{\{f,g\}_k,h\}_k = p^k\{\{f,g\}_k,h\} = p^k\{p^k\{f,g\},h\}\}.$$

Now by Leibnitz' rule

$$\{p^k\{f,g\},h\} = p^k\{\{f,g\},h\} + \{p^k,h\}.\{f,g\} = p^k\{\{f,g\},h\} + kp^{k-1}\{f,g\}.h_x.$$

Then

$$\{\{f,g\}_k,h\}_k = p^k\{p^k\{f,g\},h\}\} = p^{2k}\{\{f,g\},h\} + kp^{2k-1}\{f,g\}h_x.$$

Therefore we have

$$\begin{aligned}\{\{f,g\}_k,h\}_k + \{\{h,f\}_k,g\}_k + \{\{g,h\}_k,f\}_k &= p^{2k}[\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\}]\\ &\quad + kp^{2k-1}[\{f,g\}.h_x + \{h,f\}.g_x + \{g,h\}.f_x].\end{aligned}$$

Since

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0,$$

and applying the Poisson brackets we have

$$\{f,g\}.h_x + \{h,f\}.g_x + \{g,h\}.f_x = 0.$$

Hence the formula (4.51) defines a Poisson bracket. □

**Remark 4.13.** *Although the modified Poisson bracket is equal to the standard Poisson bracket under* $p^k\frac{d}{dp} = \frac{d}{dq}$*, where* $q$ *is a new variable, we will use the modified one.*

For any integer $k$ we can consider hierarchies of equations of hydrodynamic type, defined in terms of the Lax function [16],

$$L = p^{N-1} + \sum_{i=-1}^{N-2} p^i S_i(x,t). \qquad (4.53)$$

By the Lax equation [16]

$$\frac{\partial L}{\partial t_n} = \{(L^{\frac{n}{N-1}})_{\geqslant -k+1}; L\}_k, \tag{4.54}$$

where $n = j+l(N-1)$ and $j = 1,2,..,(N-1)$, $\quad l \in \mathbb{N}$. So we have a hierarchy for each $k$ and $j = 1,2,...,(N-1)$. Also we require $n \geqslant -k+1$ so that $(L^{\frac{n}{N-1}})_{\geqslant -k+1}$ is not zero. With the choice of Poisson brackets $\{,\}_k$, we must take $(L^{\frac{n}{N-1}})_{\geqslant -k+1}$ part of the series expansion of $L^{\frac{n}{N-1}}$ to get the consistent equation (4.54). Since the Lax function a polynomial of order $N-1$,

$$p^{N-1} + \sum_{i=-1}^{N-2} p^i S_i(x,t) = p^{N-1} + S_{N-2}p^{N-2} + .... + S_{-1}p^{-1},$$

it can also be written in terms of the roots of the polynomial, $u_1, ....u_N$ as;

$$L = \frac{1}{p}\prod_{j=1}^{N}(p-u_j). \tag{4.55}$$

### 4.3.1 Recursion operators

For each hierarchy of the equations (4.54), depending on the pair $(N,k)$, we can find a recursion operator.

**Lemma 4.14.** *For any $n$,*

$$L_{t_n} = LL_{t_{n-(N-1)}} + \{R_n; L\}_k, \tag{4.56}$$

*where function $R_n$ has a form*

$$R_n = \sum_{i=0}^{N-1} p^{i-k} A_i(S_{-1}....S_{N-2}, \frac{\partial S_{-1}}{\partial t_{n-(N-1)}} ...... \frac{\partial S_{N-2}}{\partial t_{n-(N-1)}}).[16] \tag{4.57}$$

**Proof:** Consider

$$(L^{\frac{n}{N-1}})_{\geqslant -k+1} = (LL^{\frac{n}{n-1}-1})_{\geqslant -k+1}.$$

Since

$$L^{\frac{n}{N-1}-1} = (L^{\frac{n}{N-1}-1})_{\geqslant -k+1} + (L^{\frac{n}{N-1}-1})_{<-k+1}.$$

Then

$$(L^{\frac{n}{N-1}})_{\geqslant -k+1} = [L(L^{\frac{n}{N-1}-1})_{\geqslant -k+1} + L(L^{\frac{n}{N-1}-1})_{<-k+1}]_{\geqslant -k+1}.$$

So

$$\begin{array}{ll} (L^{\frac{n}{N-1}})_{\geqslant -k+1} & = L(L^{\frac{n}{N-1}-1})_{\geqslant -k+1} + (L(L^{\frac{n}{N-1}-1})_{<-k+1})_{\geqslant -k+1} \\ & -(L(L^{\frac{n}{N-1}-1})_{\geqslant -k+1})_{<-k+1} \end{array}. \tag{4.58}$$

Note that

$$(L(L^{\frac{n}{N-1}-1})_{\geqslant -k+1})_{\geqslant -k+1} = L(L^{\frac{n}{N-1}-1})_{\geqslant -k+1} - (L(L^{\frac{n}{N-1}-1})_{\geqslant -k+1})_{<-k+1}$$

Substituting

$$R_n = (L(L^{\frac{n}{N-1}-1})_{<-k+1})_{\geqslant -k+1} - (L(L^{\frac{n}{N-1}-1})_{\geqslant -k+1})_{<-k+1}, \tag{4.59}$$

into (4.58) we obtain

$$(L^{\frac{n}{N-1}})_{\geqslant -k+1} = L(L^{\frac{n}{N-1}-1})_{\geqslant -k+1} + R_n.$$

Therefore

$$\begin{array}{ll} L_{t_n} & = \{(L^{\frac{n}{N-1}})_{\geqslant -k+1}; L\}_k = \{L(L^{\frac{n}{N-1}-1})_{\geqslant -k+1} + R_n; L\}_k \\ & = L\{(L^{\frac{n}{N-1}-1})_{\geqslant -k+1}, L\}_k + \{R_n, L\}_k = LL_{t_{n-(N-1)}} + \{R_n; L\}_k. \end{array} \tag{4.60}$$

The equation (4.56) is satisfied. Evaluating the powers of $(L(L^{\frac{n}{N-1}-1})_{<-k+1})_{\geqslant -k+1}$ and $(L(L^{\frac{n}{N-1}-1})_{\geqslant -k+1})_{<-k+1}$ we get that $R_n$ has form (4.57). $\square$

### 4.3.2 An integrable system

Multi-component hierarchy containing the shallow water wave equations corresponds to the case $k = 0$. Let us give the first equation of the hierarchy and a recursion operator for $N = 2$.

**Proposition 4.15.** *In the case $N = 2$ we have the Lax function,*

$$L = p + S + Pp^{-1}, \tag{4.61}$$

*and the Lax equation for $n = 2$*

$$\begin{array}{l} \frac{1}{2}S_t = SS_x + P_x \\ \frac{1}{2}P_t = SP_x + PS_x \end{array}, \tag{4.62}$$

*and the corresponding recursion operator*

$$\mathbb{R} = \begin{pmatrix} S + S_x D_x^{-1} & 2 \\ 2P + P_x D_x^{-1} & S \end{pmatrix}. \tag{4.63}$$

**Proof:** Since $k = 0$ and $N = 2$ the recursion relation is

$$L_{t_n} = LL_{t_{n-1}} + \{R_n, L\}_0, \tag{4.64}$$

and $R_n$ is of the form $R_n = a_n + b_n p$. Then by comparing the coefficients of the powers of $p^i$ ,$i = -2, .., 1$ we obtain

$$a_n = -D_x^{-1}(P_{t_n}), \qquad b_n = D_x^{-1}(S_{t_n}),$$

and

$$\begin{pmatrix} S_{t_{n+1}} \\ P_{t_{n+1}} \end{pmatrix} = \begin{pmatrix} S + S_x D_x^{-1} & 2 \\ 2P + P_x D_x^{-1} & S \end{pmatrix} \begin{pmatrix} S_{t_n} \\ P_{t_n} \end{pmatrix},$$

that gives the recursion operator $\mathbb{R}$. □

These equations known as the shallow water wave equations or as the equations of polytropic gas dynamics for $\gamma = 2$.

## 4.4 Moyal algebra

In this section, we consider $\mathcal{G}$ as the Moyal algebra recently introduced in [17]. Let $F$, $G \in \mathcal{G}$ be two differentiable functions. The bracket $\{,\}_{\mathcal{G}}$ defined in (4.1) corresponds to the Moyal bracket if '*' is the Moyal product

$$\{F,G\}_{\mathcal{G}} = \{F,G\}_{\kappa},$$

where $\kappa$ is a real parameter.

**Definition 4.16.** *Let $\mathcal{G}$ be the Moyal algebra and '*' be the Moyal product defined for all $A, B \in \mathcal{G}$ as follows*

$$A(x,p) * B(x,p) = e^{\kappa(\partial x \partial \tilde{p} - \partial p \partial \tilde{x})} A(x,p) B(\tilde{x},\tilde{p})|_{\tilde{x}=x,\tilde{p}=p}, \qquad (4.65)$$

*where $\kappa$ is the parameter of non-commutativity.[17]*

**Proposition 4.17.** *The Moyal product, for arbitrary integers $m$, $n$ and $f$, $h \in \mathcal{G}$ satisfies the following properties:*

***i.*** $p^n * p^m = p^{n+m}$,

***ii.*** $f(p) * h(p) = f(p).h(p)$,

***iii.*** $p * f(x) = f(x) * p - 2\kappa f_x$,

***iv.*** $p^2 * f(x) = f(x) * p^2 - 4\kappa f_x * p + 4\kappa^2 f_{xx}$,

***v.*** *For all $n$,*

$$p^n * f(x) = \sum_{m=0} \begin{pmatrix} n \\ m \end{pmatrix} (-2\kappa)^m \frac{\partial^m f}{\partial x^m} * p^{n-m},$$

*where*

$$\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n(n-1)...(n-m+1)}{m!}.$$

**Definition 4.18.** *For all $A, B \in \mathcal{G}$ the Moyal bracket is defined to be*

$$\{A(x,p), B(x,p)\}_\kappa = \frac{1}{2\kappa}(A * B - B * A). \tag{4.66}$$

**Proposition 4.19.**

$$\lim_{\kappa \to 0}\{A, B\}_\kappa = \{A, B\}. \tag{4.67}$$

*where $\{A, B\}$ is the standard Poisson bracket.*

**Proof:** Since

$$\lim_{\kappa \to 0}\{A, B\}_\kappa = \lim_{\kappa \to 0}\frac{1}{2\kappa}(A * B - B * A),$$

we need

$$A * B = AB + \kappa(\tfrac{\partial A}{\partial x}\tfrac{\partial B}{\partial p} - \tfrac{\partial A}{\partial p}\tfrac{\partial B}{\partial x}) + \tfrac{\kappa^2}{2}(\tfrac{\partial^2 A}{\partial x^2}\tfrac{\partial^2 B}{\partial p^2} + \tfrac{\partial^2 A}{\partial p^2}\tfrac{\partial^2 B}{\partial x^2}) + ....,$$
$$B * A = BA + \kappa(\tfrac{\partial B}{\partial x}\tfrac{\partial A}{\partial p} - \tfrac{\partial B}{\partial p}\tfrac{\partial A}{\partial x}) + \tfrac{\kappa^2}{2}(\tfrac{\partial^2 B}{\partial x^2}\tfrac{\partial^2 A}{\partial p^2} + \tfrac{\partial^2 B}{\partial p^2}\tfrac{\partial^2 A}{\partial x^2}) + ....,$$

then

$$\lim_{\kappa \to 0}\{A, B\}_\kappa = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} = \{A, B\}. \qquad \square$$

We define Lax operator on the Moyal algebra as

$$L_n = p^n + u_1(x) * p^{n-1} + .... + u_n(x) + u_{n+1}(x) * p^{-1} + ... \tag{4.68}$$

**Proposition 4.20.** *The Moyal-Lax representation is*

$$\frac{\partial L_n}{\partial t_k} = \{L_n, (L^{\frac{k}{n}})_{\geqslant m}\}_\kappa, \tag{4.69}$$

*where $k \neq an$; $k$, $a$ are integers and $m = 0, 1, 2...$*

**Remark 4.21.** *Rational powers of the Lax operator $L_n$ can be found by Moyal product*

$$L_n^{\frac{k}{n}} = L_n^{\frac{1}{n}} * L_n^{\frac{1}{n}} * ... * L_n^{\frac{1}{n}}, \tag{4.70}$$

*where there are k such factors and n-th root can also be determined by equating the Moyal product of n-th roots to the Lax operator itself.*

**Remark 4.22.** *If* $m = 0$, *we denote*

$$(L^{\frac{k}{n}})_+ := (L^{\frac{k}{n}})_{\geqslant 0},$$

*and the corresponding Lax representation is called the standard Moyal-Lax representation.*

**Definition 4.23.** *The coefficient of the* $p^{-1}$ *term with respect to the Moyal product is called the residue.*

**Proposition 4.24.** *Let* $A, B \in \mathcal{G}$ *be two arbitrary operators, the residue of the Moyal bracket is a total derivative*

$$Res\{A, B\}_\kappa = \frac{\partial}{\partial x} C. \tag{4.71}$$

We can uniquely define

$$TrA := \int dx Res(A), \tag{4.72}$$

and by proposition (4.24) we consequently have the following proposition.

**Proposition 4.25.** $Tr(L_n)$ *is a conserved quantity.*

**Proof:** If we take the trace of both sides of the Moyal-Lax representation in (4.69), we have

$$\frac{d}{dt_k}(TrL_n) = \int_{-\infty}^{\infty} dx Res\{L_n, (L^{\frac{k}{n}})_{\geqslant m}\}_\kappa = \int_{-\infty}^{\infty} dx (\frac{\partial}{\partial x} C) = 0.$$

Hence $Tr(L_n)$ is a conserved quantity for all $k$. $\square$

**Proposition 4.26.** *Let* $C$, $D$ *be two functions on the Moyal algebra. Then*

$$\lim_{\kappa \to 0} (C * D)_{\geqslant m} = (CD)_{\geqslant m}. \tag{4.73}$$

Following the proof of the proposition(4.19) and proposition (4.26) we have the following corollary.

**Corollary 4.27.** *If we take the limit of the Moyal-Lax representation* (4.69), *we have*

$$\frac{dL_n}{dt_k} = \{L_n, (L^{\frac{k}{n}})_{\geqslant m}\}, \tag{4.74}$$

*where the bracket on the right hand side is the standard Poisson bracket.*

**Example 4.6.** KdV hierarchy: Let us consider the Lax operator

$$L = p^2 + u(x,t). \tag{4.75}$$

If we consider $L^{\frac{3}{2}}$, we have

$$(L^{\frac{3}{2}})_+ = (L^{\frac{1}{2}} * L^{\frac{1}{2}} * L^{\frac{1}{2}})_+, \tag{4.76}$$

where

$$L^{\frac{1}{2}} = p + \frac{1}{2}u * p^{-1} + \frac{1}{2}\kappa u_x * p^{-2} + .... \quad . \tag{4.77}$$

and

$$(L^{\frac{3}{2}})_+ = p^3 + \frac{3}{2}u * p - \frac{3}{2}\kappa u_x. \tag{4.78}$$

Hence the Moyal-Lax representation

$$\frac{\partial L}{\partial t} = \{L, (L^{\frac{3}{2}})_+\}_\kappa, \tag{4.79}$$

gives

$$u_t = \kappa^2 u_{xxx} + \frac{3}{2}uu_x. \tag{4.80}$$

which is the KdV equation.

# Chapter 5

# Conclusion

In this thesis, we have studied the zero curvature and the Gelf'and-Dikii formalisms to obtain integrable nonlinear partial differential equations. The zero curvature formalism is a generalization of the AKNS scheme. We have covered the AKNS scheme including the sine-Gordon equation, nonlinear Schrödinger and KdV hierarchies where the potentials are independent of the spectral parameter. To cover the cases where the potentials depend on the spectral parameter, we have studied Ma-Zhou system and Tam-Zhang system. We have used matrix representation of Lie algebras to determine integrable evolution equations via a simple Lie algebra.

We have studied the Gelf'and-Dikii formalism which gives a construction of all Lax pairs based on the calculation of fractional powers of the Lax operator. We have introduced a bracket on an algebra satisfying skew-symmetry, associativity and Leibniz rule due to a non-commutative, associative binary product. We have covered pseudo-differential, matrix, polynomial and Moyal algebras. We showed that the Gelf'and-Dikii formalism is more effective than the other methods to obtain integrable nonlinear partial differential equations. If a nonlinear partial differential equation is obtained through the Gelf'and-Dikii formalism, then it is straightforward to obtain infinite number of symmetries and conserved quantities.

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