A REVIEW OF NETWORK LOCATION THEORY AND MODELS

A THESIS

SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL ENGINEERING

AND THE INSTITUTE OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

by

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June 2004

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Abstract

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In this study, we review the existing literature on network location problems. The study has a broad scope that includes problems featuring desirable and undesirable facilities, point facilities and extensive facilities, monopolistic and competitive markets, and single or multiple objectives. Deterministic and stochastic models as well as robust models are covered. Demand data aggregation is also discussed. More than 500 papers in this area are reviewed and critical issues, research directions, and problem extensions are emphasized.
Keywords: Survey, Network, Location.

Özet

SERİM ÜZERİNDE YERLEŞTİRME TEORİSİ VE MODELLERİ

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Haziran 2004

Bu çalışmada serim üzerindeki yerleştirme problemlerini eleştirel bir bakış açısıyla inceledik. Çalışmanın kapsamı geniş olup tek ya da çok amaçlı, istenen ve istenmeyen tesisleri, nokta tesislerini ve alanlı tesisleri, tekelci ve rekabetci modelleri içermektedir. Belirli ve rastsal modellere ek olarak sağlam (robust) modeller kapsanmıştır. Talep verilerinin indirgenmesi de tartışılmıştır. Bu alanda 500'den fazla makale gözden geçirilmiş ve kritik konular, araştırma yönleri ve problem uzantıları öne çıkarılmıştır.

Anahtar Kelimeler: Literatür Taraması, Serim, Tesis Yerleştirme

To my devoted mother SANİYE

And my sweetheart LEYLA

Whom I love most on this planet

Ackowledgement

I would like to express my sincere gratitude to Prof. Barbaros Tansel for all the encouragement and trust during my graduate study and, of course, for his unique guidance. I feel lucky to have worked with a supervisor like him who has introduced me into a new way of thinking, which I believe will guide me through all my academic life.

I am indebted to members of my dissertation committee: Asst. Prof. Bahar Yetiş Kara and Asst. Prof. Hande Yaman for showing keen interest in the subject matter and accepting to read and review this thesis. Their remarks and recommendations have been very helpful.

I am mostly indebted to my family. To my mother Saniye Tanaydın for her encouragements, understanding, and confidence, to my aunt Leyla Tanaydın for standing with me in all my life and giving all her love and support to me, to my grandmother Kadriye Tanaydin, my aunts Şaziye Tanaydın and Filiz Karabulut, Şenol Erdoğan, Meral-Ertan Kutluözen, and Fitnat Kızıltan for their prayers and altruism. Their efforts in making me who I am now, and knowing that they are always there for me was a great asset.

I would like to thank to my lovely friends who made the last few years very joyful, I will miss all of them and every one of them has a very special place in my hearth: Çiğdem Ökten, Günnur Ertong, Rasim Volga Ovalı, Murat Güler, Edibe Özmen, Özgür Kutluözen, Esra-İlker Yağlıdere, Mustafa Rasim Kılınç, Zümbül Bulut, Ayşegül Altın, Oğuz Atan, Salih Öztop, Caner Dokuzlar, Sibel Alumur and Sinan-Yeşim Gürel.

Finally, I would like to express my deepest gratitude to my husband, my teacher, my best friend, and my only Güneş Erdoğan for his love, morale support, encouragement and for everything he brought to my life. Whenever I needed help, he was just behind me, holding my hands, helping me, protecting me with his endless love. I would run away crying months ago if I did not have him with me.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

Chapter 1

INTRODUCTION

Location problems began receiving the attention of scientists with the work of Weber (1909) who studied the problem of locating a warehouse in the plane on which the customers are spatially distributed with the objective of minimizing the total walking distance of customers to the facility. The network version of the problem in which customers and the facility are located on an underlying network, which usually represents a real world transportation system with arcs corresponding to the roads and nodes corresponding to intersections of roads has become popular with the seminal work of Hakimi (1964). Location problems attracted many researchers and thousands of papers and hundreds of books are published in this area. Actually, location problems and closely related layout and routing problems honestly deserve this extensive interest because there exist many real world problems that can be modeled as location problems.

Location problems have many variants. Fifty four variants have been defined in the overview paper of Brandeau and Chiu (1989). There exist other problems not considered in this work but can be found in other sources. The variants arise according to the type of the objective function, nature of the demand for the service, nature of the supply of the service, type of the underlying structure such as plane, network, or some special structure, number of facilities to be placed, etc. There exist valuable surveys, books and bibliographic studies related to the location problems such as Francis and White (1974), Lea (1978), Handler and Mirchandani (1979), Francis, McGinnis, and White (1983), Tansel, Francis, and Lowe (1983), Daskin (1985), Domschke and Drexl (1985), Brandeau and Chiu (1989), Mirchandani and Francis (1990), and Hamacher and Nickel (1998). The last 40 years of the location research has been summarized by Francis (1997) who has contributed valuable work in this area. This work is recommended for those who are new to this area and need to be inspired by a story of success.

We believe, regardless of the fact that the first author is the chair of our faculty and the supervisor of this thesis, that the paper by Tansel, Francis and Lowe (1983) is a benchmark in the literature on network location problems that covers all of the previous work before 1983. This paper has given the idea that a newer version of the work may be beneficial for researchers who would like to get access to a comprehensive summary of the work in the area. This thesis is a literature survey on network location problems to accomplish this goal. We first considered writing a survey of all problems in the location area including both the planar and network problems but we only had a limited time of 10 months. Thus, we decided to restrict ourselves to network problems but mentioned the work on planar problems whenever the concepts became hard to explain without doing so. We have reviewed more than 500 papers written between 1909 and 2004 on network location problems. Unfortunately, we are far from being complete because there exist more than a thousand papers in the area and many of which are presented in conferences or available only as technical reports. Nevertheless, we have made our best to cover the most important part in this thesis. In this chapter, we will provide the notation used in the rest of thesis and briefly introduce the problems covered in this work.

1.1 Definitions and Notation:

In all of the problems in this thesis, we are given an embedded network N, which represents the transportation system at hand. Network $N = (V, E)$ consists of the node set $V = \{v_1, v_2, ..., v_n\}$ and the edge set $E = \{e_1, e_2, ..., e_m\}$. Associated with each edge e_i there is a positive number l_i, called the length of edge e_i. A distance function $d(.,.)$ is defined on pairs of points of the network with $d(x,y)$ denoting the length of a shortest path from point x to point y. The function $d(.,.)$ satisfies the properties of nonnegativity, symmetry, and triangular inequality: i.e. $d(x,y) \geq 0$ with $d(x,y) = 0$ *iff* $x=y$, $d(x,y) = d(y,x)$, $d(x,y) + d(y,z) \geq d(x,z)$ $\forall x,y,z \in N$. Consequently, the network N with distance function $d(.,.)$ constitutes a well-defined metric space and the function $d(x, y)$ is a continous function of x on N for a fixed point y.

An n by n distance matrix $\mathbb{D} = [d_{ij}]$ is associated with each network N where $d_{ij}=d(v_i, v_j)$ $\forall$ i, j. The distance matrix can be computed in $O(n^3)$ for general networks and in $O(n^2)$ for tree networks. We assume that the distance matrix has already been computed and present the results on computational complexity without considering the computational time for the distance matrix.

Let X denote a compact subset of N (finite or infinite). The distance from a point y to a set $X \subseteq N$ is defined to be the length of a shortest path prom y to a nearest element in X, and is denoted by $D(y,X)=\min_{x\in X} d(x,y)$. When we aim to locate p facilities on N, we use the set $X=\{x_1,\ldots,x_p\}\subseteq N$ if it is the case that the facilities are indistinguishable from each other in their service characteristics and that there exist no capacity restrictions on the facilities. Otherwise, we use the notation $X=[x_1,\ldots,x_p]\subseteq N^p$ to denote a vector of n distinguishable facilities and refer to it as a location vector. In most of the problems, the facilities may be located on any point on the network, whereas sometimes the facilities are restricted to a subset of the network, which is called the candidate set or the supply set and is denoted by S.

We are also given a set of customers or existing facilities, denoted by Δ, that are distributed along the network and require service from the facilities at some cost. Usually the customers are located on the nodes of the network, whereas in some problems the customers are located along the edges of the network in which case the demand is called continuous.

We will denote general networks by N, tree networks by T, and path networks by P in order to save space.

Additional notation and definitions will be provided in each chapter in the light of the problem at hand. Although it is possible to design a notation system, which can be used through the entire study, we prefer not doing so for the sake of simplicity and clarity.

1.2 Network Location Problems:

We consider locating a number of facilities on a network with different objective functions. Each objective function gives rise to a new problem and each problem is studied in a specific chapter. Namely, Chapter 2 considers the network location problems with minimax objective. We locate p facilities on the network, so that the maximum of the distances between facilities and customers are minimized. The minimax problems are related to the location of emergency services such as ambulance terminals, fire stations, police centers, etc. These facilities are usually public facilities and the service quality is much more important than the total cost of the system. The infamous p-center problem and its variants are studied in this chapter. Chapter 3 considers the network location problems with minisum objective. We locate p facilities on the network, so that the sum of the weighted distances between facilities and customers is minimized. The minisum problems are related to the location of repetitive distibution services such as warehouses, postal services etc. These facilities are usually private organizations and the organization pays for the transportation expenses. The notorious p-median problem and its variants are studied in this chapter. Chapter 4 includes the network location problem in which the distances between the facilities and the customers and the facilities themselves are restricted. The minisum and minimax problems with distance constraints are included. Chapter 5 considers the multiobjective network location problems specifically the centdian problem in which the minisum and minimax objectives are simultaneously considered. These problems are used for facilities in which both the quality of the service and the total cost of the system are important as in, for example, pizza delivery systems. Chapter 6 considers the location of undesirable facilities such as waste disposal sites, nuclear reactors, power stations, etc. In these problems, customers desire to be as far away from facilities as possible. The dispersion and defense problems are among the problems mentioned in this chapter. Chapter 7 is different from the previous chapters in the sense that the facilities considered in this chapter are not single point facilities but network structures such as edges, paths, cycles, subnetworks, etc. The problems in this chapter are closely related to routing problems but the location aspect is emphasized in our work. Chapter 8 deals with the location of facilities when there exist competition between the facilities. These models are used by firms that enter a market in which more than one organization provides service and the customers are willing to be served by any organization such as restaurants, supermarkets, etc. Chapter 9 considers the robust network location problems in which the data is not known deterministically or statistically but only interval or set estimates of the parameters or discrete scenarios are provided. The minimax regret approach is introduced in detail in this chapter. The errors introduced into the previous models because of demand point aggregation is studied in Chapter 10 together with methods to eliminate aggregation errors. This chapter is important because real world data is available in huge data sets and aggregation is a must for tractable analysis. We conclude with a summary chapter in which the literature on every problem considered in previous chapters is summarized in tables and important facts are restressed.

Chapter 2

MINIMAX FACILITY LOCATION ON NETWORKS:

In this chapter we deal with facility location problems on networks in which the cost of providing service to customers is the maximum of transportation costs from the facilities to customers. Facilities are to be located so as to minimize the cost of providing service. Such location problems usually arise in location of emergency service facilities such as hospitals, police stations, and fire stations.

Assume the facilities are identical and uncapacitated. Let S and Δ be subsets of the network N with S denoting the set of points on which new facilities can be located and Δ denoting the set of customers that require service from facilities. Let $X=\{x_1,..,x_p\} \subseteq S$ be the set of new facility locations. Let $f_\delta(.)$ be a nondecreasing function defined on nonnegative reals for each $\delta \in \Delta$ and define the function $F(.)$ by: $F(X) = \max_{\delta \in \Delta} f_\delta(D(\delta, X))$ The minimax multifacility location problem (MMLP) on network N is stated as follows: Find $X^* \subseteq S$ such that $|X^*| = p$ and $F(X^*) \leq F(X) \;\; \forall X \subseteq S$ for which $|X| = p$.

When facilities are not identical (nonhomogenous), they provide different services and a customer may require service from some or all new facilities. The nonhomogenous MMLP may be stated as follows: Find $X^* = (x_1^*, x_2^*, \ldots, x_p^*) \in S^p$ (where S^p is the p-fold Cartesian product of S by itself) such that $G(X^*) \leq G(X)$ $\forall X \in S^p$ where $G(X) = \max_{1\leq i\leq p} \max_{\delta\in\Delta} g_{\delta i}(d(\delta, x_i))$ and $g_{\delta i}(d(\delta, x_i))$ is a nondecreasing function defined for each $\delta \in \Delta$ and $i \in \{1,2,\ldots,p\}$. This problem is equivalent to p independent single facility problems because each facility may be optimally located without considering other facilities.

When there exist an interaction between facilities, the problem is called the MMLP with facility interactions or MMLP with mutual communication. It may be stated as follows: Find $X^* = (x_1^*, x_2^*, \ldots, x_p^*) \in S^p$ such that $H(X^*) \leq H(X)$ $\forall X \in S^p$ where $H(X) = \max\{\max_{1\leq i\leq p, \delta\in\Delta} h^1_{\delta i}(d(x_i, \delta)), \max_{1\leq i<j\leq p} h^2_{ij}(d(x_i, x_j))\}$ and $h^1_{\delta i}(d(x_i, \delta)), h^2_{ij}(d(x_i, x_j))$ are nondecreasing functions $\forall\, i, j,$ and δ.

2.1 Problem variations:

The cost functions used in MMLP are usually linear. The p facility linear minimax facility location problems are specifically called the p-center problems. In the *p-center* problem: $f_\delta(D(X,\delta)) = w_\delta D(X,\delta) + a_\delta$ where w_δ is the associated weight of demand point δ and a_δ is the addend associated with that point. When $w_\delta = c$ for some constant $c \in \Re$ for all δ, the problem is referred to as "unweighted" while it is referred to as "weighted" otherwise. When $a_\delta = 0$ for all δ, the problem is called the p-center "without addends" while it is the problem "with addends" otherwise.

Nonlinear functions have also been used as cost functions and some of the analysis is similar to the linear case.

When the demand set Δ is restricted to a finite set (e.g. the vertex set V), the problem is referred to as “discrete” whereas when the demands are generated by all the points on the network, the problem is referred to as “continuous”. Similarly, when the facilities are to be located only at the vertices of network, i.e. $S=V$, the problem is referred to as “vertex-restricted” whereas it is referred to as “absolute” when the facilities can be located on any point of the network.

Solution procedures and theoretical results differ with respect to the particular choice of the sets S and Δ. Furthermore, for special values of the parameter p, especially for $p = 1$ and $p = 2$, efficient polynomial algorithms and notable theoretical results are provided. These results have formed useful starting points for other values of p. The problem has been widely studied for special networks, especially acyclic networks, and some of the problems which are *NP*-complete on general networks have been solved in polynomial time by exploiting the special network structure. So, the methodology of approaching and solving MMLP depends on the supply and demand sets, the value of the parameter p, and the structure of the network, suggesting a *4*-entry classification that will be used in this chapter. This classification may also be extended to cover the problems with linear / nonlinear cost functions, with / without addends, homogenous / nonhomogenous facilities, with / without mutual communication.

The problem as we have shown has many variants and additional cases may be defined by using probabilistic demands, probabilistic edge lengths, dynamic networks, and so on. In this chapter, we will survey the relevant literature by focusing on complexity issues, algorithms and solvability of many problems. Some extensions and research directions will also be introduced throughout the chapter.

2.2 Inverse problems to MMLP:

In MMLP, we are given a specified number p and aim to find optimal locations of p facilities in order to minimize some objective function value. The related feasibility (recognition) version of the problem can be defined as follows: Given a value r, determine whether or not there exists a feasible location of p facilities with objective function value less than or equal to r. This latter problem is polynomially equivalent to the original problem, i.e. if a polynomial algorithm is devised for the feasibility problem, a polynomial algorithm can also be devised for the original problem. Researchers usually use the feasibility version of the problem to devise polynomial algorithms to minimax multifacility problems. In fact, the minimum value of r such that the corresponding feasibility problem has a solution is the optimal objective value of the original MMLP.

From a different view, given a real number r, the following problem is the inverse problem (called the related cover problem) of a standard homogeneous MMLP:

$$\begin{array}{ll} \text{Min} & p \\ \text{s.t.} & |X| = p \\ & X \subseteq S \end{array}$$

$$F(X) \leq r$$

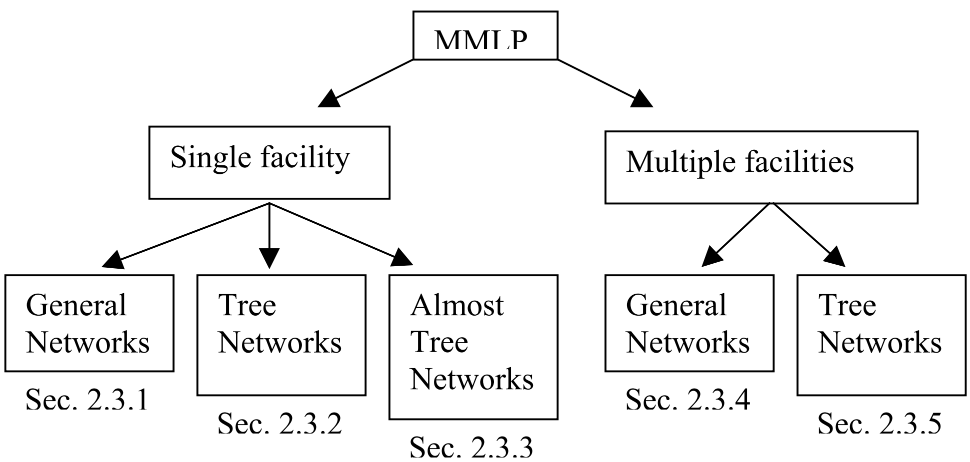
Let the value of the optimal solution of the p-center problem be r_p and that of the inverse problem be $q(r)$ for a given real r. If we are trying to solve a p facility MMLP, and $q(r) > p$ then r_p is greater than r; otherwise, it is smaller than or equal to r.

It is obvious that if we can restrict the possible objective function values to a discrete set R and have a polynomial algorithm for the inverse problem then we can find the optimal objective function value to the homogeneous MMLP by applying a standard search method on the set R in polynomial time.

2.3 The Literature:

In this section we present a review of the literature. We find it appropriate to classify the problems according to the number of facilities and underlying network structure (the last two entries of the classification). The sections are organized as follows:

Figure 1: Organization of Chapter2



Each section presents the results on vertex-restricted discrete problems, absolute discrete problems and continuous problems, in the stated order.

2.3.1 1-facility | General Networks: (*S*/*Δ*/1/*N*)

Although the first minimax facility location problem was proposed more than a century ago for *V*/*V*/1/*N* in a graph theoretical context (Jordan, 1869), the facility location problems have not taken much attention until Hakimi's seminal paper (Hakimi, 1964). Hakimi (1964) precisely defined the (vertex-restricted) center and absolute center of a network and opened a new era of research in operations research. The vertex-restricted 1-center problem (*V*/*V*/1/*N*) is not very interesting for the researchers because one can always solve it by using the distance matrix $\mathbb{D}$. In fact for any set of nonlinear cost functions $f_i(d(v_i,.))$, a matrix $\mathbb{D}' = \{d_{ij}' = f_j(d(v_i,v_j))\}$ can be constructed in $O(\sum_{i=1}^{n} nt_i(n))$ time where $t_i(n)$ is the complexity of evaluating function f_i. For example if $f_i(d(v_i,.))$ is the distance function itself or the weighted distance function, $\mathbb{D}'$ can be computed in $O(n^2)$ time. Then, using $\mathbb{D}'$, the maximum entry in each column can be found and the column with the minimum maximal entry is the optimal location of the facility for *V*/*V*/1/*N*. Once $\mathbb{D}'$ is at hand, the center location can be found in $O(n^2)$ time, so the *V*/*V*/1/*N* problem is solvable in $O(\{n\sum_{i=1}^{n} t_i(n)\} + n^2)$ time for any cost function.

Following Hakimi (1964), the absolute center of a graph is a point x_0 of N such that $\min_{x \in N} \max_{1 \le i \le n} w_i d(v_i, x) = \max_{1 \le i \le n} w_i d(v_i, x_0) = F(x_0)$. According to our notation, the absolute center problem is an *N*/*V*/1/*N* problem with weighted linear cost functions. The absolute center of a graph is found by localizing the search to edges of the graph. A local absolute center of an edge is a point that minimizes the objective function value when the candidate location set is restricted to the specific edge. Once a local optimum is found for every edge, the local center with minimum objective function can be selected as the global optimal center. In fact, the search space for the entire network can be reduced to local centers, which are finite in number. The local center on an edge can be found by observing the function $F(x)$ on the edge. This function is piecewise linear with at most $n(n-1)/2$ breakpoints. The candidate locations are at the breakpoints or on the endpoints of an edge. Hakimi (1964) suggested an enumeration technique for all edges and all candidate points of each edge. The complexity of complete enumeration is $O(n^3m)$ where m is the number of edges. Later, Hakimi, Schmeichel, and Pierce (1978) proposed an $O(mn^2logn)$ algorithm which implements Hakimi's algorithm more efficiently. The complexity of the algorithm is improved by a factor of n for the unweighted case. Kariv and Hakimi (1979) improved the complexity bound on the problem by searching a subset of breakpoints, namely the "suspected points" on each edge. Suspected points are breakpoints where linear functions of opposite signs intersect. Their algorithm solves weighted 1-center on general graphs in $O(mnlogn)$ and unweighted 1-center in $O(mn)$ time. The computational improvement is due to effective search of local optima for each edge. For the unweighted problem Minieka (1981) suggests an $O(n^3)$ algorithm, which is different in nature than previous algorithms. The algorithm does not make use of the point to vertex cost functions but make use of the distance matrix only.

When the cost function is not linear but it is a nonlinear convex function, Hooker (1986) proposes a general-purpose algorithm. Hooker (1986) divides each edge of a general graph into treelike segments. Treelike segments are partial maximal arcs on which each distance function $d(v_i, .)$ is linear. Any two points x and y lie on a treelike segment if and only if every shortest path from these two points to any vertex v_i are the same except the portion of the segment between x and y. The cost function $F(x)$ is convex on each treelike segment. $F(x)$ has a local minimum on every edge which can be found by solving a convex program on a line. It is proven that there exists at most $O(n)$ treelike segments on each arc. Further refinements can be made using some segment elimination techniques. Shier and Dearing (1983) studied a nonlinear unified model that includes the weighted 1-center and 1-median problems as special cases. Directional derivatives are defined for these problems on networks. A directional derivative is the amount of change in the objective function value when the location of the facility if shifted by a small amount. The locally optimal solutions are identified using directional derivatives on both general and tree networks. Although for the linear cases the results are not very surprising but a repetition of some well-known results, the paper is valuable because it presents a completely different point of view to the nonlinear problems.

Some very simple edge elimination techniques are used in absolute center problems. In fact lower bounds on local centers is devised for each edge and edges with lower bounds greater than known feasible solutions are eliminated. Christofides (1975), Handler (1974), Odoni (1974) and Halpern (1979) make use of elimination techniques and achieve computational improvements in terms of CPU times. Halpern (1979) generates a stronger bound than previous ones. Sforza (1990) also proposes a very efficient algorithm for absolute 1-center problem which makes use of an edge elimination technique and solves the problem in $O(mnlogn)$ and $O(kmnlogn)$ time for unweighted and weighted networks, respectively, where k is a factor depending on the precision level and weight distribution for weighted networks. Although Sforza's algorithm does not improve the complexity of the algorithm by Kariv and Hakimi (1979), which is the best known bound for the problem, it is more effective in CPU time, since it eliminates more than 80% of the edges.

The terms general center and continuous center are being used for the $N/N/1/N$ problem. Although there exists definitional differences between general and continuous centers, it can be proven that they are equivalent for the single facility case. The general center of a network N is a point whose maximum distance to a farthest point on each edge is minimized. So, in a sense, general centers serve edges of a network instead of individual points of the network. A continuous center of a network is a point whose maximum distance to any point on the network is minimized. Minieka (1977) showed that Hakimi's algorithm for the absolute1-center could be modified to find the general absolute 1-center by making a change in the definition of the distance function. The distance function $d(x,y)$ is changed with a new edge distance function $d'(x,e_i)$ which is the distance between point x and the farthest point on edge e_i. It is also shown in Frank (1967a) that $N/N/1/N$ problem can be solved using Hakimi's algorithm from a different point of view.

The problem $V/N/1/N$ received very little attention. Minieka (1977) proves that this problem can be solved by constructing a new distance matrix $D' = [d_{ij}^{'} = d_e^{'}(v_i, e_j)]$ and finding the vertex which minimizes the objective function value using the same technique for *V*/*V*/1/*N*. Thus, the demand set although stated as continuous can be reduced to a finite set consisting of the most distant points of each edge from each vertex.

2.3.2 1-facility | Tree Networks: (./ ./1/T)

The distance function *d(x,a)* has a special property on tree networks which provides the opportunity to devise very simple and elegant algorithms. The function *d(x,a)* is convex for any point *a* on a network *N* if and only if the network *N* is a tree (Dearing, Francis, and Lowe; 1976). The convexity of the function *d(x,a)* means that for any fixed point *a* on *T* and *x* on a path joining points *y* and *z*; $d(x,a) \leq \lambda d(y,a) + (1-\lambda)d(z,a)$ $\forall \lambda \in [0,1]$. Because of the convexity of the distance function, any local optimum is a global optimum in *T* for linear cost functions. This property and the algorithms devised for tree networks are very important in location theory because they provide insight for more general networks. Moreover, it's known that for some single-facility location problems (including absolute center problem), there exist equivalent spanning tree problems (Dearing and Francis, 1974). Solution procedures for tree networks play a crucial role in such general network problems.

Goldman (1972a) proposed a decomposition algorithm for networks involving bridges (a bridge is an edge whose removal divides the network into two components). This procedure divides a network into two components by removing a bridge and finds which of the two components involves the absolute center. When the network is a tree, an $O(n^2)$ algorithm is proposed. When the network has cyclic components but also at least one bridge it either decomposes the network into a cyclic component, which contains the weighted absolute center, or finds the optimum weighted absolute center of the network. Handler (1973) proved that the absolute center of an unweighted tree is the midpoint of a longest path in the tree. The absolute and vertex-restricted 1-center of a tree is found in $O(n)$ time. Halfin (1974) modified Goldman's algorithm for unweighted tree networks with addends and found the absolute and vertex restricted 1-center of a tree in $O(n)$ time. In fact, Lin (1975) showed that addends could be incorporated into a network by adding artificial nodes connected to each node by an edge of length equal to the addend. Therefore, addends do not increase the complexity of the algorithms. The weighted absolute 1-center problem with addends on tree networks was solved by Dearing and Francis (1974). They proved that the optimum objective function value of an absolute center problem on a weighted tree network is a value α_{st} defined as follows:

$$\alpha_{st} = \max\left\{ \alpha_{ij} = \frac{d(v_i, v_j) + (a_i / w_i) + (a_j / w_j)}{(1/w_i + 1/w_j)} : v_i, v_j \in V \right\}$$

The absolute center of a tree network occurs at a point x on the path joining some two vertices s and t where $w_s d(v_s,x) + a_s = w_t d(v_t,x) + a_t$. In fact, α_{st} is a lower bound for the weighted absolute center value for general networks. The computation of α_{st} together with critical vertices s and t takes $O(n^2)$ time. Hakimi, Schmeichel, and Pierce (1978) proposed an algorithm which has a time complexity of $O(n(r+1))$ where r is an integer and $r \leq n - 1$ for this problem. Kariv and Hakimi (1979) proposed an algorithm, which reduces the search of the absolute 1-center and the vertex restricted 1-center to subtrees of a tree until a single edge remains. The algorithm is similar to Goldman (1969), but because it selects the centroid of the remaining part of the tree as the break point, it does not enumerate all edges of the tree. Once the subtree, which involves the center, is reduced to a single edge, local center on that edge is found for absolute center or vertices of the edge are compared for vertex-restricted center. The algorithm takes $O(nlogn)$ time for weighted trees. Hedetniemi, Cockayne, and Hedetniemi (1981) suggests an $O(n)$ algorithm for vertex-restricted 1-center on unweighted tree networks using a canonical representation of tree networks. In this representation, the nodes of the tree are labeled so that each node is connected exactly with one node with a smaller index. This labeling is useful for implementation of algorithms. Although, the time complexity of the algorithm does not suggest an improvement on Handler's algorithm the data structure suggested is useful in terms of tractability. Megiddo (1983) solved the weighted absolute center problem in tree networks in $O(n)$ time. The algorithm first selects the centroid of the tree and evaluates the cost function for every vertex adjacent to the centroid. Then because the function is convex on any path, it finds which subtree contains the weighted center of the tree among all the subtrees identified by the centroid. These steps are similar to the algorithm by Kariv and Hakimi (1979) but Megiddo (1983) disregards some of the vertices from further consideration in objective function by using the following observation: Given a real number t and a point of the tree, whether the center lies within a distance t or not of this point can be found in linear time.

The nonlinear version of the problem with strictly increasing cost functions f_i is considered by Dearing (1977) and Francis (1977). Similar results to Dearing and Francis (1974) is obtained. Let b_{st} be defined as follows:

$$b_{st} = \max_{1 \le i < j \le n} \left\{ b_{ij} = (f_i^{-1} + f_j^{-1})^{-1} [d(v_i, v_j)] \right\}$$

b_{st} is a lower bound on the objective function value for general networks and it is attainable for tree networks. The calculation of b_{st} together with definition of the range and domain of the function $(f_i^{-1} + f_j^{-1})^{-1}$ may be quite cumbersome for some functions.

N/*N*/1/*T* problems are equivalent to *N*/*V*/1/T problems on unweighted trees because once the vertices of an edge are covered; the interior points of the edge are necessarily covered. Nevertheless, problems with continuous demand on each edge with associated demand weight functions may be an area of research.

2.3.3 Exploiting the Block Structure: 1-facility | Special graphs other than trees:

For the networks, which are more general than trees, Goldman's reduction algorithm finds the cyclic component in which absolute center lies. Similarly, Chen, Francis, and Lowe (1988) proposed an algorithm for linear and nonlinear cost functions. The algorithm constructs the block diagram of the graph (a block is a maximal subgraph that cannot be disconnected by removing a vertex together with its adjacent edges and the block diagram of a graph is a graph with additional vertices representing each block and edges between each block and its vertices. A block diagram is always a tree). The algorithm directs the edges one by one from nodes of the block diagram to the central node or to the block that consists of the central point. The algorithm is useful for graphs containing more than one block. Assuming evaluating $f_i(.)$ is $O(n)$, the complexity of the algorithm is $O(n.min\{b, \alpha logb\})$ where α is the maximum number of cut points in any block and b is the number of blocks. If the algorithm ends with a block, algorithm of Kariv and Hakimi (1979) may be used for example to locate the absolute center in the block for the linear cost function. Otherwise Hooker (1986)'s treelike segments may be used for increasing convex functions as stated by the authors. Nevertheless, if we have other nonlinear cost functions, finding the location of the single facility in the block is still a hard problem to be solved.

The algorithms exploiting the block structure may be very useful for cactus graphs, which are graphs in which every block with three or more vertices is a cycle. For example, the complexity bound in Chen, Francis, and Lowe (1988) is $O(nlogn)$ for cactus networks. A polynomial time algorithm of complexity bound $O(n)$ is devised for special cactus networks: the cactus networks which are homeorphic to a 3-cactus (Kincaid and Lowe, 1990). The algorithm transforms these special graphs to trees in which point to point distances are preserved. The paper is insightful although it solves a very special class of problems.

2.3.4 *p*>1 | General Networks

Most of the MMLP's are hard problems on general networks although they are relatively easy on tree networks. Kariv and Hakimi (1979) proved that the absolute and vertex-restricted *p*-center problems are *NP*-Complete even if the network is a planar unweighted network with unit edge lengths and maximum vertex degree of three. Although the problem is *NP*-Complete for general p, it is polynomial when p is given. The vertex-restricted unweighted p-center problem is solvable by means of solving a finite number of set-cover problems, because the objective function value of the p-center problem must be one of the $O(n^2)$ vertex-to-vertex distances. Similarly, it is shown that there exists "finite dominating sets", finite sets that include all candidate facility locations, for many network location problems, including some members of the MMLP. Hooker, Garfinkel, and Chen (1991) suggested a unified technique to identify these finite dominating sets for many problems. Specifically, for p-absolute center problem, it is shown that "edge bottleneck points" (the unique points on each edge for which two distance functions $d(v_i,.)$ and $d(v_j,.)$ are equal and not both decreasing in the same direction) together with vertices of the network form a finite dominating set for unweighted networks, (Minieka, 1970). For the weighted problem, Kariv and Hakimi (1979) identified "suspected points" on each edge (where the weighted distance functions of opposite signs intersect). With each dominating set identified for each problem, a finite set of numbers R (which consists of the distances between each candidate location point and each vertex) is also identified. Then set-cover problems with radius $r \in R$ can be solved. Unfortunately the possible number of candidate points is $O(n^2m)$ and set-cover problems involving $O(n^2m)$ variables may be hard to solve with known Integer Programming (IP) techniques.

Minieka (1970) suggested an algorithm that relies on solving a number of set-covering algorithms for increasing values of r. Garfinkel, Neebe, and Rao (1977) solved p-center problems using Minieka's ideas but they reduced the search space by first using a heuristic to find an upper bound on r. They disregarded the candidate points with relative radius greater than r, effectively decreasing the number of variables of the associated set-cover problem. Christofides and Viola (1971) gave an iterative algorithm for finding absolute p-centers on weighted and unweighted graphs. They did not identify finite dominating sets. Instead, for each r, feasible regions on each edge are constructed to cover all vertices and a minimal set of locations is chosen in these feasible regions by solving a set-covering problem. This approach may be useful for problems with distance constraints since the feasible regions may be changed without changing the entire procedure. Toregas, Swain, Revelle, and Bergman (1971) solved the vertex restricted p-center problem by solving a series of set-covering problems. They also added cuts to each set cover problem to resolve fractional solutions when needed.

Kariv and Hakimi (1979) provided an $O(m^p n^{2p-1} \log n)$ algorithm for p-center on general graphs. They used the fact that facilities must be chosen from a finite dominating set and in an optimal solution each facility is associated with a subnetwork for which it is the 1-center. They choose p-1 arbitrary candidate locations and solve for the p^{th} one. The algorithm is improved by a factor of $\log n$ for the unweighted case. Moreno (1986) provided a $O(m^p n^{p+1} \log n)$ bound for the p-center problem. Tamir (1988) improved previous bounds by combining the algorithms of Kariv and Hakimi (1979) and Moreno (1986) and obtained bounds of $O(m^p n^p \log^2 n)$, and $O(m^p n^p \log n)$ for the weighted and unweighted cases, respectively. Further improvement is made by using dynamic data structures and the unweighted p-center problem can be solved in $O(m^p n^{p-1} \log^3 n)$ time. Tamir's algorithm obtains the objective function value dynamically as it passes from one candidate set of locations to another and it uses the fact that in an optimal solution each facility may be associated with a unique edge. As we have mentioned before, the algorithms devised for the p-center problem usually rely on solution of a series of set covering problems. This fact is used by Elloumi, Labbé, and Pochet (2004) in devising a new IP formulation for the problem. The LP-relaxation of this formulation generates better lower bounds for the problem than previous models. The paper includes polynomial algorithms to generate lower and upper bounds for the problem and solves instances up to 1817 using Cplex 7.0. Although the model has been developed for discrete spaces, it can be used for both absolute and vertex-restricted problems by using finite dominating sets.

When the cost function is a nonlinear convex function, Hooker (1989) proposes an algorithm, which is practical for small values of p. The algorithm divides the edges into treelike segments and for each combination of p treelike segments, locates the p facilities optimally via solving a number of easy linear and nonlinear programs on these segments. It enumerates all possible combinations of p segments. It also introduces an upper bounding technique to eliminate some of the combinations. Although the algorithm becomes intractable when the number of facilities exceeds four, the ideas introduced may be useful in further research.

The continuous p-center problem ($N/N/p/N$) is NP-Hard on general graphs even if the graph is a bipartite planar graph of maximum degree 4 with unit edge lengths (Reduction from minimum dominating set problem in Megiddo and Tamir, 1983). Tamir (1985) showed that the objective function value r_p of a continuous p-center problem with positive integer edge lengths is a rational $r_p = p_1/p_2$ where p_1, p_2 are positive integers and

$\log p_i = O(m^5 \log(\sum_{i=1}^{m} l_i) + m^6 \log p)$ for i = 1,2 where l_i is the length of link i. Using this result, he showed that the continuous r-covering problem is equivalent to the continuous p-center problem, and a finite algorithm may be devised for the continuous p-center problem that requires the solution of a series of continuous r-covering problems. A continuous r-covering problem is to locate p centers on a network so that every point of the network is within a distance r of a center. Handler and Rozman (1982) also suggested an approximation algorithm for the continuous p-center problem via solving some discrete problems and approximating the continuous p-center value. As Tamir (1985) stated the number of continuous r-covering problems may be quite large to solve the problem optimally so one can be satisfied with approximation algorithms instead. Tamir (1987) proved that r_p is of the form $T/2q$ where T is the length of a Eulerian tour of a subnetwork of N which is from a special set of subnetworks and $q \in \{1,..,2p\}$. It is shown that the continuous p-center problem can be solved via solving $O(logp+logd)$ continuous r-covering problems where d is the total of edge lengths. Gurevich, Stockmeyer, and Vishkin (1984) state that the r-covering problem is solvable in $O(nlogn)$ time for a class of graphs which have the property that each nontrivial biconnected component is homeorphic to either a cycle or a cycle with a chord (a chord is an edge joining two nonconsecutive vertices of a path or cycle and two graphs are homeomorphic if one can be obtained from the other by inserting new nodes along existing edges). So, continuous p-center problem is solvable in this class of graphs.

Minieka (1977) introduced the general p-center problem in which every edge is a demand point and must be covered by a single center. He shows that the number of candidate locations can be reduced to a finite set when the ordinary distance functions are replaced with edge distance functions as in the single facility case.

The *p*-center problems are very hard problems on general graphs, so it would be appropriate to make use of heuristics. Unfortunately, it is impossible to have a worst-case bound tighter than twice the optimum for any *p*-center problem. Hochbaum and Shmoys (1985) devised a "best possible" 2-approximation algorithm for the problem and showed that finding a better algorithm is *NP*-Hard. In fact, their heuristic has been useful in many other problems. The heuristic solutions were successfully used in branch-and-bound algorithms for some instances of the problem. Hsu and Nemhauser (1979) also proved that finding any approximation algorithm with a performance guarantee better than 2 implies *P*=*NP* for general networks.

Dyer and Frieze (1985) devised an *O(nm)* heuristic for the vertex-restricted *p*-center problem which generates solutions no more than min(3,1+α) times the optimum, where $\alpha = \dfrac{\max_{1 \le i \le p} w_i}{\min_{1 \le i \le p} w_i}$. When the graph is unweighted, the algorithm is a 2-approximation algorithm that is the best possible for vertex-restricted *p*-center problems. Plesnik (1987) modified this algorithm for absolute *p*-center problems.

Handler and Rozman (1985) suggested an approximation algorithm for the absolute and continuous *p*-center problems which may be viewed as a column and row generation generating algorithm. Although the procedure converges to the optimal solution in a finite number of steps for the absolute p-center problem, it lacks this property for the continuous problem.

A vertex-closing approach to the vertex-restricted problem was proposed by Martinich (1988). In this method, instead of choosing vertices which have facilities, vertices which do not have facilities are chosen. Although a worse case bound is not proposed for the heuristic it is shown that in most of the test instances it finds the optimum solution. The paper includes two polynomial algorithms which have complexity bounds of $O(m^2)$ and $O(mlogm)$. The first method is faster in the average in contrast with the higher complexity bound. Lower and upper bounds and theorems to prove optimality of the solutions are included in the analysis. Although Hochbaum and Shmoys (1985) have a worse case guarantee for the same problem, Martinich (1988) finds better solutions for most of the instances.

Bozkaya and Tansel (1998) provided a heuristic, which is different from above approximation algorithms. They prove that for any network N, there exists a spanning tree T^* of N, such that the absolute p-center of T^* is also an absolute center of N. It is shown that finding a finite subset of spanning trees which involve T^* is as hard as the original problem. They used two special classes of spanning trees and made an experimental study on these trees.

2.3.5 $p > 1$ | Tree Networks:

Handler (1978) considered the absolute 2-center ($N/V/2/T$) and continuous 2-center ($N/N/2/T$) problems and devised $O(n)$ algorithms for both problems. He solved three 1-center problems instead of solving a 2-

center problem. Although his algorithm is elegant for the 2-center case it does not seem possible to extend the algorithm to $p > 2$. Hakimi, Schmeichel, and Pierce (1978) devised an $O(n^{p-1})$ algorithm for the absolute p-center problem on unweighted tree networks. Kariv and Hakimi (1979) presented an $O(n^2 logn)$ algorithm for absolute and vertex restricted p-center problems on trees (weighted or unweighted). For unweighted trees they also presented an $O(nlog^{p-2}n)$ algorithm for absolute p-center ($3 \leq p < n$) and an $O(nlog^{p-1}n)$ algorithm for vertex-weighted p-center ($2 \leq p < n$). Kariv and Hakimi's algorithms use the fact that the objective function value r_p must be one of the following $O(n^2)$ values:

$$\alpha_{ij} = \frac{w_i w_j}{w_i + w_j} d(v_i, v_j) \qquad 1 \leq i < j \leq n$$

For each value α_{ij}, a covering algorithm devised in the same paper is used. The covering algorithm is $O(n)$ and a binary search is performed on the set of possible values of r_p. Megiddo and Tamir (1983) presented an $O(n\ log^2 n\ loglogn)$ algorithm for the weighted absolute p-center problem. An $O(nlogn)$ algorithm is presented by Frederickson and Johnson (1983) for unweighted case. Megiddo, Tamir, Zemel, and Chandrasekaran (1981) solved the vertex-restricted p-center in $O(nlog^2n)$ time. For relatively small values of p, Jaeger and Kariv (1985) devised an algorithm of $O(pnlogn)$ for the vertex-restricted and absolute p-center problems on weighted tree networks. When $p<logn$ for the vertex-restricted p-center problem and $p<lognloglogn$ for the absolute p-center problem, this algorithm performs better than previous algorithms.

Shaw (1999) presented a unified column generation approach for a class of facility location problems on trees and presented a multipurpose algorithm for these problems. This algorithm gives a complexity bound of $O(n^2 logn)$ for p-center problems on weighted tree networks.

The nonlinear version of the problem with strictly increasing and continuous cost functions was considered in Tansel, Francis, Lowe, and Chen (1982). The model also included upper bounds on the distances between customers and facilities. They provided an $O(n^4 logn)$ algorithm based on solving a series of $O(n^2)$ covering problems. The dual of the center problem and dual of the covering problem is presented and solved in this paper.

For the continuous p-center problems ($N/N/p/T$); Chandrasekaran and Daughety (1978) showed the problem is polynomially solvable. They provided an $O(n)$ algorithm for solving the r-cover problem on the tree networks, but did not specify a polynomial algorithm for continuous p-center. Chandrasekaran and Tamir (1980) proved that the objective function value r_p of continuous p-center problem belongs to the following set:

$$R = \left\{ \frac{d(v_i, v_j)}{2l} : i, j \in \{1,..,n\}, 1 \leq l \leq p \right\}$$

The cardinality of possible values of r_p is $O(n^2p)$. Chandrasekaran and Tamir (1980) proposed an $O(min(n^2log^2p,\ n^2logn + plog^2n))$ algorithm for continuous p-centers on trees via exploiting the special structure of the set R. Chandrasekaran and Daughety (1981) improved this bound to $O(n^2p)$.

This bound is further improved by Megiddo, Tamir, Zemel, and Chandrasekaran (1981) to an algorithm of complexity $O(min(n^2 logp, pnlog^2 n))$. Frederickson and Johnson (1983) devised an algorithm of $O(n.min(p,n)log(max(p/n,n/p)))$ time complexity. Megiddo and Tamir (1983) devised an algorithm of $O(nlog^3 n)$ complexity. Their algorithm is the first algorithm which has a complexity bound independent from parameter p. They find the objective function value of r_p by constructing an interval (α,β), which includes r_p and contracting this interval to r_p by solving a number of covering problems.

The p-center problem is also extended to the conditional case. Conditional p-center problem arises when p_1 facilities are already located on the network and $p - p_1$ facilities are to be located, or more formally: Given a set $Y \subseteq N$, $|Y| = p_1$, find a set Z^*: $Z^* \subseteq N$, $|Z^*| = p_2$, $p = p_1+p_2$ and $Z^* \in \arg\min_{Z\subseteq N, |Z|=p_2} \left\{ \max_{1\le i\le n} w_i D(v_i, Y \cup Z) \right\}$. Minieka (1977) first considered the conditional 1-center problem. He considered the vertex-restricted, absolute and continuous versions of the problem and showed that the algorithms for unconditional problems can also be used for the conditional problems. For the multiple conditional centers, Kariv and Hakimi (1979) can be used. Drezner (1989) also solved the conditional p-center problem with an algorithm that requires the solution of $O(log\ n)$ unconditional p-center problems. Drezner's observation is true for problems on the Euclidean plane, for rectilinear distances in the plane and for problems on general or tree networks. Berman and Simchi-Levi (1990) showed that the conditional p-center problem could be solved by solving an unconditional (p+1)-center problem.

2.4 Problem Extensions:

2.4.1. Directed Networks

Most of the minimax facility location problems have been solved on undirected networks. Although this assumption simplifies the analysis, it may be unrealistic for many transportation networks. Handler (1984) considered the absolute *p*-center problem on directed networks. It is shown that there exists an optimal solution, which is a subset of the vertices of the directed network. So the *p*-center problem on directed networks can be solved by methods that solve vertex-restricted problems on undirected networks.

2.4.2. Capacitated facilities

In minimax problems we have considered so far, it is assumed that the facilities are uncapacitated. This assumption is not restrictive in most of the situations because these problems usually deal with locations of emergency centers and not many emergency cases occur at the same time. Nevertheless all these facilities have well defined capacity restrictions and there may be cases when the capacity restrictions are tight such as war situations, disasters, and etc. Jaeger and Goldberg (1994) are the first to consider capacity restrictions on the *p*-center problem. They proposed an algorithm for the *p*-center problem on tree networks when the facility capacities are identical. The algorithm, similar to the uncapacitated versions, solves a series of capacitated covering problems in order to solve the capacitated center problem. Since the capacitated covering problem can be solved in $O(n^2)$ time, it is shown that for both vertex-restricted and absolute centers the algorithm requires $O(n)$ times more effort than the uncapacitated version. Any algorithm for an uncapaciated p-center problem combined with the algorithm for the capacitated covering problem would yield a polynomial algorithm for the capacitated version of that problem. Problems with unequal facility capacities and on more general graphs than trees may be studied as stated by the authors.

2.4.3. Round-trip problems

The round trip p-center problem can be defined as follows: Given a network N and a finite set of pairs of existing facilities, minimize the maximum transportation cost where costs are linear or nonlinear increasing functions of the round-trip distance from a nearest new facility. The round-trip distance is the distance traveled by a vehicle that departs from its depot, visits a pair of customers and returns to its depot. The problem first solved by Chan and Francis (1976) for the single facility case on tree networks. The analysis was similar to Dearing and Francis (1974). They proved a lower bound on the objective function value for general networks which is attainable for tree graphs. Kolen (1985) solved the problem for multiple facilities by solving a series of round-trip covering problems. The round-trip covering problem is the problem of finding the minimum number of depots such that each round trip cost is less than or equal to a specified number. The covering problem is solved in $O(nm)$ time where m is number of existing customer pairs. The center problem can be solved in polynomial time using the solutions of a set of covering problems. The details of the algorithms together with duality results may be found in Kolen and Tamir (1984) and in the book *Discrete Location Theory* by Francis and Mirchandani (1985).

2.5 New Problems:

Four new 1-center problems are introduced in Peeters (1998). These problems do not minimize the distance between the facility and a farthest demand point but minimizes the distance of a facility to the k^{th} farthest or nearest demand point. Let $\min^k$ denote the k^{th} smallest element in a set. Let Δ denote the demand set as usual. Note that Δ is restricted to a subset of vertices for these problems. The vertex-restricted lower-k 1-center problem is to find a vertex $v^* \in V$ such that v^* solves

$$\min_{1 \le i \le n} \min_{\delta \in \Delta}{}^{k} w_\delta d(\delta, v_i).$$

The absolute lower-k 1-center problem is to find a vertex $x^* \in N$ such that x^* solves

$$\min_{x \in N} \min_{\delta \in \Delta}{}^{k} w_\delta d(\delta, x).$$

Two other problems, the vertex-restricted and absolute upper-k 1-center problems introduced are equal to lower-($|\Delta|$-k) 1-center problems, so their definitions are omitted. Peeters (1998) introduces an algorithm of $O(n|\Delta|\log n + |\Delta|m)$ for the weighted vertex-restricted problems and unweighted absolute center problems. The algorithm solves the problems when distance matrix is being calculated and finds the optimum before all entries in $\mathbb{D}$ is found. When $\Delta = V$ and $k = 1$, the upper-k 1-center problem is identical to the 1-center problem. Thus, 1-center problems are solvable in $O(n^2\log n + n^2m)$ time without calculating the distance matrix using this algorithm. The ideas in this paper may be extended to the case with multiple facilities.

Chaudhuri, Garg, and Ravi (1998) defined the (vertex-restricted) *k*-neighbour *p*-center problem as follows: Find a subset X^* of V such that $|X^*| = p$ and $\min_{\substack{X \subseteq V \\ |X|=p}} \max_{v \in V-X} d_p(v, X) = \max_{v \in V-X^*} d_p(v, X^*)$, where $d_p(v, X)$ is the distance between v and its p^{th} nearest center in X. This model may be useful when facilities are subject to failures and at most *k* facilities fail at the same time. A best possible 2-approximation algorithm based on an extension of Hochbaum and Shmoys (1985) is presented.

Hochbaum and Pathria (1997) generalized the vertex restricted *p*-center problem to the *Set p-Center* problem. In this problem, the nodes from which the *p* servers are to be selected are partitioned into *k* sets and the number of servers selected from each set must be within a specified range. When there exist 2 vertices in each partition, the problem is called the *p-Pair Center* problem and is introduced by Hudec (1991). Hochbaum and Pathria (1997) proved that the problem is *NP*-Complete. Furthermore, finding an ε-approximation algorithm with $\varepsilon < 2$ is not possible unless *P=NP*. They also provided a 3-approximation algorithm for the problem.

Hochbaum and Patria (1998) introduced the (vertex-restricted) *k*-network *p*-center problem, which is defined as follows: Given *k* sets of weights on a complete network *N*, let $N_j=(V, E_j)$ represent j^{th} network, for $j=1,..,k$, with edge *e* having length l_e^j in N_j. Find a set $X^* \subseteq V$ such that X^* minimizes $\max_{1 \le i \le n} \max_{1 \le j \le k} D_j(v_i, X)$ where $D_j(v_i, X) = \min_{1 \le r \le p} d_j(v_i, x_r)$ with $d_j(v_i,x_r)$ denoting the length of a shortest path between v_i and x_r computed relative to $N_j=(V, E_j)$. The problem is *NP*-complete and a 2-approximation algorithm is provided for k=2 in the paper. This problem may be appropriate to model some situations in which the network structure changes in time. For example, if the network is a city transportation system, the time spent on different edges of the network may change with the time of the day and a city planner may want to consider all possible instances of the network.

A summary of the literature on minimax facility location problems can be found in the following tables:

Table 1: Literature on Single Facility Minimax Location Problems on General Networks

Author	***Year***	***Problem***	***Summary***
Hakimi	1964	Vertex-restricted 1-center	Definition $O(n^2)$ algorithm using D
Hakimi	1964	Absolute 1-center	Definition $O(n^3m)$ algorithm
Hakimi, Schemeichel, and Pierce	1978	Absolute 1-center	$O(mn^2logn)$ algorithm for weighted problem $O(mnlogn)$ algorithm for unweighted problem
Kariv and Hakimi	1979	Absolute 1-center	$O(mnlogn)$ algorithm for weighted problem $O(mn)$ algorithm for unweighted problem
Minieka	1981	Absolute 1-center	$O(n^3)$ algorithm
Hooker	1986	Absolute 1-center with nonlinear cost function (convex)	Treelike segments
Christofides Handler Odoni Halpern Sforza	1975 1974 1974 1979 1990	Absolute 1-center	Edge elimination tech. $O(mnlogn)$ for unweighted, $O(kmnlogn)$ for weighted

Minieka	1977	Vertex-restricted, absolute general (continuous) 1-center	Intoduce edge-to-point distance function and modify Hakimi's algorithm
Frank	1967	Continuous 1-center	Modify Hakimi's algorithm

Table 2: Literature on Single Facility Minimax Location Problems on Tree Networks

Author	***Year***	***Problem***	***Summary***
Goldman	1972	Weighted absolute 1-center	$O(n^2)$ algorithm
Handler	1973	Unweighted absolute and vertex-restricted 1-center	$O(n)$ algorithm that locates the center on the midpoint of the longest path
Halfin	1974	Unweighted absolute 1-center	$O(n)$ algorithm
Dearing and Francis	1974	Weighted absolute 1-center	$O(n^2)$ algorithm Find α_{st}
Hakimi, Schemeichel, and Pierce	1978	Weighted absolute 1-center	$O(n(r+1))$ algorithm
Kariv and Hakimi	1979	Weighted absolute 1-center	$O(nlogn)$ algorithm
Hedetniemi, Cockayne, and Hedetniemi	1981	Unweighted vertex-restricted 1-center	$O(n)$ algorithm Canonical representation
Megiddo	1983	Weighted absolute 1-center	$O(n)$ algorithm
Dearing Francis	1977 1977	Nonlinear 1-center	Find b_{st}

Table 3: Literature on Single Facility Minimax Location Problems on Special Networks

Author	***Year***	***Problem***	***Summary***
Goldman	1972	Graphs with more than 1 block	Decomposition algorithm
Chen, Francis, and Lowe	1988	Graphs with more than 1 block	Block diagram, $O(n\min\{b, \alpha\log b\})$ alg.
Chen, Francis, and Lowe	1988	Cacti	$O(n\log n)$
Kincaid and Lowe	1990	Cacti homeorphic to a 3-cactus	$O(n)$ algorithm

Table 4: Literature on *p*-Facility Minimax Location Problems on General Networks

Author	***Year***	***Problem***	***Summary***
Kariv and Hakimi	1979	Absolute and vertex-restricted *p*-center	*NP*-Complete for general *p*
Minieka	1970	Unweighted absolute *p*-center	Identification of a finite dominating set Solved via solving of set-cover problems
Garfinkel, Neebe, and Rao	1977	Unweighted absolute *p*-center	Solution of a reduced number of set-cover problems
Christofides and Viola	1971	Unweighted and weighted absolute *p*-center	Feasible regions on each edge Solution of set-cover prob.
Toregas, Swain, Revelle, and Bergman	1971	Vertex-restricted *p*-center	Solution of set-cover problems Cutting planes
Kariv and Hakimi	1979	Weighted absolute *p*-center	Finite dominating set $O(m^p n^{2p-1} \log n)$

Moreno	1986	Weighted absolute *p*-center	$O(m^p n^{p+1} logn)$
Tamir	1988	Unweighted and weighted absolute *p*-center	$O(m^p n^p logn)$ – unweighted $O(m^p n^p log^2 n)$ - weighted
Hooker	1989	Nonlinear convex objective function	Trrelike segments
Megiddo and Tamir	1983	Continuous *p*-center	*NP*-Complete for general *p*
Tamir	1985	Continuous *p*-center, integer link lengths	Rational objective function value Solution of finite number of continuous *r*-cover prob.
Tamir	1987	Continuous *p*-center	Solution of $O(logp+logd)$ continuous *r*-cover prob
Minieka	1977	General *p*-center	Edge distance functions
Handler and Rozman	1982	Continuous *p*-center	Approximation algorithm using discrete problems
Hochbaum and Shmoys	1985	*p*-center	2-approximation algorithm
Hsu and Nemhauser	1979	*p*-center	2-approximation algorithm
Dyer and Frieze	1985	Vertex-restricted *p*-center	$O(nm)$ heuristic min(3, 1+α)-approximation
Handler and Rozman	1985	Absolute and continuous *p*-center	Approximation algorithm Finite convergence for the absolute problem
Martinich	1888	Vertex-restricted *p*-center	Vertex-closing heuristic
Bozkaya and Tansel	1998	Absolute *p*-center	Spanning trees Experimental study

Table 5: Literature on *p*-Facility Minimax Location Problems on Tree Networks

Author	***Year***	***Problem***	***Summary***
Handler	1978	Absolute and continuous 2-center	$O(n)$ algorithm

Hakimi, Schmeichel, and Pierce	1978	Absolute *p*-center	$O(n^{p-1})$ algorithm
Kariv and Hakimi	1979	Absolute and vertex-restricted *p*-center	$O(n^2 logn)$ algorithm for weighted trees $O(nlog^{p-2})$, $O(nlog^{p-1})$ for unweighted trees for absolute and restricted cases, resp.
Megiddo and Tamir	1983	Weighted absolute *p*-center	$O(n\ log^2 n\ loglogn)$ algorithm
Frederickson and Johnson	1983	Unweighted absolute *p*-center	$O(nlogn)$ algorithm
Megiddo, Tamir, Zemel, Chandrasekaran	1983	Vertex-restricted *p*-center	$O(nlog^2 n)$ algorithm
Jaeger and Kariv	1985	Absolute and vertex-restricted *p*-center	$O(pnlogn)$ algorithm
Shaw	1999	Weighted *p*-center	$O(n^2 logn)$ column generation algorithm
Tansel, Francis, Lowe, Chen	1982	Strictly increasing nonlinear cost function and distance constraints	$O(n^4 logn)$ algorithm
Chandrasekaran and Daughety	1978	Continuous *p*-center	Polynomially solvable
Chandrasekaran and Tamir	1980	Continuous *p*-center	$O(n^2 p)$ possible values of objective function
Chandrasekaran and Tamir	1980	Continuous *p*-center	$O(min(n^2 log^2 p,\ n^2 logn + plog^2 n))$ algorithm
Chandrasekaran and Daughety	1981	Continuous *p*-center	$O(n^2 p)$ algorithm
Megiddo, Tamir, Zemel, Chandrasekaran	1981	Continuous *p*-center	$O(n.min(p,n)log(max(p/n,n/p)))$ algorithm
Megiddo and Tamir	1983	Continuous *p*-center	$O(nlog^3 n)$ algorithm
Minieka	1977	Conditional *p*-center	Unconditional algorithms can be used

Drezner	1989	Conditional *p*-center	Solution of *O(log n)* unconditional *p*-center problems
Berman and Simchi-Levi	1990	Conditional *p*-center	Solution of an unconditional (*p*+1)-center problem.

Chapter 3

MINISUM FACILITY LOCATION ON NETWORKS:

In this chapter we deal with facility location problems on networks in which the cost function is the total cost of servicing every customer. This type of objective function is usually referred to as the minisum objective. As stated by ReVelle, Marks, and Liebman (1970), most of the private facility problems involve cost functions that are easily measurable in monetary values and the minisum type objective is widely used in private sector problems. We assume that the number of facilities to be located is a priori known. This results in a budget constraint that can be expressed in terms of the number of facilities instead of monetary units under the assumption that the facility establishment costs are essentially identical for all facilities.

In classical network location problems, customers are assumed to be located at discrete points of a network (usually on the nodes). If there are demand points on the links, the node set can be expanded to include such points. Nevertheless, in real world distribution systems such as postal services, traffic highway service systems, and household services, the customers may be continuously distributed on some or all links of the network. Replacing link demands with aggregated demands at discrete points of the network may be an oversimplification of the real problem. In this chapter, we deal with both discrete and continuous demands on networks. Moreover, the demand configuration of the network as well as the lengths of the links of the network may change over time. These types of networks are time varying or stochastic in nature and we deal with both time varying, deterministic and stochastic networks.

If the demand is generated by the nodes of the network, let $f_i(.)$ be a nondecreasing function defined on nonnegative reals for each $v_i \in V$. Let $F(X) = \sum_i f_i(D(v_i, X))$, we define the *multifacility minisum location* problem as follows: Find $X^* \subseteq N$ such that $|X^*| = p$ and $F(X^*) \leq F(X) \quad \forall X \subseteq N$ for which $|X| = p$. If a nonnegative weight w_i is associated with each demand node v_i and the cost function $f_i(.)$ is a linear cost function with slope w_i, the problem is the well-known absolute p-median problem of Hakimi (1965). The weight w_i can be interpreted as the product of the volume of demand at v_i and the unit transportation cost.

3.1 Nodal Optimality Results:

Although absolute p-medians of a network can be located on the interior points of the links as well as on the nodes of the network, it is known that there exists at least one optimal solution which locates all facilities on the nodes of the network for certain cost functions. Hakimi (1964) showed that there exists an absolute median of a network on the vertices. He also proposed an algorithm to find the absolute 1-median of a network using the distance matrix based on the nodal optimality result. His algorithm is a complete enumeration technique with a time complexity of $O(n^2)$. The nodal optimality result is very easy to grasp because the distance function from a fixed point in the network to a variable point in a link is a piecewise linear concave function with at most two pieces. The total distance from finitely many demand points to a facility on an edge is again concave which attains its minimum on one of the end points, i.e. the nodes of the network. The same result is extended to the absolute p-median problem by Hakimi (1965). The result is again easy to grasp because every facility in an optimal p-median of a network is a 1-median of a subnetwork. A complete enumeration algorithm, which searches all the p-cardinality subsets of the node set, is provided by Hakimi (1965). The algorithm becomes intractable for large values of p and n. The complete enumeration has time complexity of $O(n^{p+1}p)$. Goldman and Meyers (1965) have proved the same result with a more general concave cost function. Levy (1967) extended the nodal optimality result to the case when p facilities are to be located which have capacity restrictions independent of the specific facility location. The cost functions are concave with respect to the distance from the nearest facility. The result follows from the concavity of the cost and the distance functions. The results can further be extended to the cases with variable number of facilities, concave establishment and processing costs. The concavity assumptions are not too restrictive because the cost functions are actually concave in most of the real world problems. It is assumed that more than one facility can be placed on a node of the network otherwise the nodal optimality results are not valid. Goldman (1969) used a more general objective function. In Goldman's model, there is a material flow between pairs of nodes via facilities and the cost assigned to each flow is dependent on the particular source and destination pair and the direction of the shipment. All the transportation costs are assumed to be concave with respect to the distance and nodal optimality is proved for the cases when the materials can flow through only one facility or more than one facility. Also multiple commodities, which are processed at different facilities during the transportation process, are considered and nodal optimality results are conjectured to be extended to these problems. When the commodities can be processed at two identical facilities, the problem is a primitive version of the hub-location problem that is extensively studied in the recent years (Campbell, 1996; O'Kelly, Bryan, Skorin-Kapov, and Skorin-Kapov, 1996; Tansel and Kara, 2000). Hakimi and Maheshwari (1971) followed Goldman (1969) and proved the conjectured nodal optimality result for multiple commodities and multiple processing stages. They further proved that the result holds when the facilities are capacitated and multiple facilities are allowed on a single node. Independent from Hakimi and Maheshwari (1971), Wendell and Hurter (1973) generalized the results in Goldman (1969). Wendell and Hurter (1973) studied a more generalized problem, which involves multiple commodities, directed arcs, multiple facilities on a point and multiple processing steps. Nodal optimality results are proven. Also conditions that allow nonnodal facility locations and relations that restrict the optimal locations to nodes (conditions that do not allow nonnodal locations) are discussed. This problem is very general and flexible which can be used to represent real world problems.

For the 1-median problem on stochastic networks in which demands are probabilistic, Frank (1966) defined the maximum probability absolute R-median of a graph to be a point such that the total weighted distance stays within an allowable limit R with maximum probability. It is shown that there exist graphs for which none of the maximum probability medians of the graph is on one of its nodes, so the nodal optimality results fail to hold for stochastic networks with probabilistic demands. Frank (1966) has presented methods to find the local maximum probability medians of a graph on an edge. Methods to handle the case, when the probability distributions for demands are not known a priori but sample data is available for the demands, are also provided. The solution techniques may be quite cumbersome if the probability distribution at hand is complex. Moreover, if the random variables representing the demands are not independently distributed the analysis becomes intractable. Nevertheless, Frank (1967b) studied problems with dependent probabilistic demands, which follow a joint normal distribution. It is shown that the medians are not necessarily on the nodes and methods for finding the local absolute maximum probability medians are provided. The extension of the results by Frank (1966, 1967) to the multiple facilities case is not straightforward.

Mirchandani and Odoni (1979a) extended the nodal optimality results of Hakimi (1965) and Levy (1967) for stochastic networks in which the arc lengths are random variables with known discrete probability distributions. It is shown that when the cost function is concave there exists an optimal solution on the nodes. The result is valid for directed stochastic networks and for three different types of facilities such as inward facilities (facilities to which customers arrive), outward facilities (from which servers travel to customers) and facilities that are both inward and outward. Representing each state of the network with a deterministic network and taking the expected value of the total travel time over all states handle the stochasticity. When the number of states is not large, the problem is easily solved.

Mirchandani and Odoni (1979b) introduced the supporting medians, which are new facilities to be located on the network to support the existing facilities. These new facilities behave like hubs but the destination is always an old facility. The customers either directly go to the old facility where they will receive the service or go to the new facility and are transferred to the old facility with a reduced cost. It is shown that there is always one set of optimal supporting medians on the nodes of the network. Also the conditional p-median problem, which is the problem of locating new facilities identical to the old facilities, is considered in this paper and it is also proven that there exists an optimal set of conditional medians on the nodes of the network. When only one facility that is either a supporting median or a conditional median is to be located simple algorithms are provided by Mirchandani and Odoni (1979b).

In most of the median models, it is assumed that there exists at least one server at the nearest facility at the time when a service request arises. Nevertheless, when the demands and service times are random, and the facilities have limited number of servers, this assumption may fail to hold. When all of the servers at a facility are busy and a request arises, the request may be directed to another facility that has an available server or it may be placed in a queue, which is depleted with respect to some queuing principle. This type of networks is referred to as congested networks and the median problem on this type of netwoks is referred to as the median problem with congestion (Berman and Larson, 1982). The objective function in this problem is to minimize the expected response time (instead of expected travel time) associated with a random service request, where response time is the sum of travel time and queuing delay. It is shown in Berman, Larson, and Chiu (1985) that when one facility with exactly one server working as an M/G/1 queue is to be located, the nodal optimality results fail and the facility may be placed on an interior point of a link and a vertex solution for this problem may not exist. The result is valid for both general networks and tree networks. Only for a very restricted case in which service time is very large compared to the travel time and demands arise with a Poisson distribution, the nodal optimality results hold for multiple facilities and different server preference relations (Berman and Larson, 1982).

3.2 Absolute *p*-Median on General Networks:

It is shown that finding an absolute *p*-median of a network is *NP*-hard even when the network is a planar graph of maximum vertex degree 3 (Kariv and Hakimi, 1979). Based on the nodal optimality results we have discussed in the previous section, the absolute *p*-median has been formulated as the following ILP:

$$Min \sum_i \sum_j c_{ij} X_{ij}$$

$$s.t.$$

$$\sum_j X_{ij} = 1 \qquad \forall i \qquad (1)$$

$$X_{ij} - X_{jj} \leq 0 \qquad \forall i, j \qquad (2)$$

$$\sum_j X_{jj} = p \qquad (3)$$

$$X_{ij} = 0,1 \qquad \forall i, j \qquad (4)$$

In the formulation above, X_{ij} is equal 1 if demand at node *i* is serviced by a facility at node *j* and X_{jj} is equal to 1 if there exists a facility at node *j*. The cost coefficient c_{ij} is the cost of serving node *i* from the facility at node *j*. This formulation is an adaptation (ReVelle and Swain, 1970) of the ILP for Fixed Charge Location Problem by Balinski (1961). It is also a constrained version of the Uncapacitated Facility Location Problem (UFLP), which has been widely investigated in the literature. The interested reader is referred to Francis and Goldstein (1974) for an extensive survey of the problem. The L*P*-relaxation of the formulation is very widely used to provide lower bounds in branch-and-bound algorithms. It generates strong lower bounds for Euclidean, network and tree models and provides results within 0.3 percent of the optimal objective function value almost surely when the number of nodes goes to infinity (Ahn, Cooper, Cornuejols, and Frieze, 1988). Many integer linear programming techniques, especially branch and bound, Lagrangean relaxation and dual ascent procedures are provided for the *p*-median problem based on this formulation. The interested reader is referred to Fisher (1981) for an extensive survey of Lagrangean relaxation technique. There also exists a large number of heuristics available in the literature. We will first go over the exact algorithms in the literature in the following subsection and then briefly survey the available heuristics.

3.2.1 Exact Algorithms for *p*-Median on General Networks:

ReVelle and Swain (1970) solved the problem using L*P*-relaxation and branch-and-bound. Järvinen, Rajala, and Sinervo (1972) also presented a branch-and-bound algorithm for the problem. Their branching rule is first opening *n* facilities on all nodes, and taking vertices away from the facility set one at a time. At most *n*-*p* vertices are removed. Lower bounds are calculated for each facility set with more than *p* facilities and any feasible solution with *p* facilities constitute an upper bound. Garfinkel, Neebe, and Rao (1974) presented another algorithm, which is based on solving the L*P*-relaxation of the ILP formulation of the problem. The L*P*-

relaxation is solved using a decomposition algorithm and in case of noninteger termination, integrality is achieved using group theoretics and dynamic recursion. This approach has some advantages over the classical branch-and-bound algorithms when there are degenerate cases and many alternative solutions of the L*P*-relaxation of the problem. Narula, Ogbu, and Samuelsson (1977) provided a very simple algorithm based on lower bounding via Lagrangean relaxation and subgradient optimization methods. The bounding procedure finds the optimal solution for nearly all practical problems. An important theoretical and algorithmic study related to the Uncapacitated Facility problem is the exceptional paper by Cornuejols, Fisher, and Nemhauser (1977). The problem is solved using a three-step procedure. A greedy heuristic is first used to obtain an upper bound followed by generation of lower bounds by means of a Lagrangean dual. If needed, a third phase that is a classical branch-and-bound procedure is used to solve the problem to optimality. Upper bounds on the deviation of (upper and lower) bounds from the optimal objective value are presented in the paper. This algorithm is very insightful and successful compared to the previous methods and the study has been very useful for solving *p*-median problems. A generalized *p*-median problem in which facilities have different establishment costs is solved by Mavrides (1979). This problem is again a UFLP with the constraint on the number of facilities to be located. A Lagrangean relaxation of the problem with relaxing the constraint on the number of facilities is solved using available UFLP techniques. The algorithm may be useful if the corresponding UFLP can be solved efficiently. Very successful methods exist for UFLP, among which the dual ascent procedures initiated by Erlenkotter's famous algorithm stand out. Galvão (1980) proposed another branch-and-bound algorithm, which uses a heuristic to solve the dual of the L*P*-relaxation of the problem. This dual solution provides a lower bound to the *p*-median problem and it is easily embedded into the branch-and-bound procedure. Medium-sized problems are solved using this algorithm.

Boffey and Karkazis (1984) have reported that they have solved a *p*-median problem with n=206 and p=45. They solved the *p*-median problem by solving a series of UFLPs with varying fixed facility costs. At each iteration, a UFLP is solved and if the number of medians is larger (smaller) than *p*, fixed cost for facility establishment is increased (decreased). If the solution of the UFLP does not result in exactly *p* facilities after adjustment of fixed costs, a branch-and-bound algorithm is used to reach optimality. The *p*-median problem is extended to the case in which the facilities have different types, each providing a different service. The problem cannot be decomposed into independent subproblems because there can be at most one facility at each node. Nevertheless, solving each subproblem and integrating the subproblems as needed handles the general *p*-median problem. Christofides and Beasley (1982) solved *p*-median problems involving up to 200 vertices using Lagrangean relaxation and subgradient optimization. Beasley (1985) improved the algorithm by using a powerful supercomputer. He solved large *p*-median problems with up to 900 vertices and 90 facilities. This is the largest problem solved to optimality in the literature to the best of our knowledge. Mirchandani, Oudjit, and Wong (1985) provided a very successful exact algorithm for the *p*-median problem, which is called the Nested Dual Approach. This is a Lagrangean Dual based solution technique which utilizes Erlenkotter's dual ascent procedure and dual simplex algorithm as subroutines. They have reported that problems up to 200 vertices have been solved using this technique. The paper also extends the problem into a multidimensional one, in which travel times may be stochastic, multiple services and multiple commodities are allowed and multiple minisum objectives are considered. In fact these multidimensional models may be expressed as *p*-median models with a larger set of nodes. The transformation is insightful and easy to grasp.

The exact algorithms provided above depend on the assumption that the cost function is a linear or concave function of distance. When the cost function is a convex function of distance, then the nodal optimality results fail to hold and the algorithms above are invalid for minisum facility location problems. For this type of cost functions, Hooker (1986) proposed a general-purpose algorithm for single facility problems including *p*-median with convex nonlinear cost functions. Hooker divides each edge into treelike segments on which distance functions to each node are linear. The minisum problem is solved on each treelike segment and methods to eliminate some of the segments are also presented. The algorithm is extended to multiple facilities case in subsequent work by Hooker (1989), which again makes use of treelike segments and solves the problem on each set of *p* treelike segments.

3.2.2 Heuristics for *p*-Median on General Networks:

Maranzana (1964) provided a very fast heuristic. The heuristic selects *p* nodes to be facility nodes, assigns the remaining nodes to the nearest facilities, finds the 1-median of the subnetwork composed of nodes assigned to the same facility, and finally finds a new set of facility locations. The procedure iterates until no improvement is possible. Although the heuristic is dependent on the initial solution and optimality is not guaranteed, this heuristic is the fastest heuristic devised for the *p*-median problem. While it does not provide very good solutions, it can be used with many initial solutions to have a reasonable upper bound.

Teitz and Bart (1968) provided a 1-opt heuristic. This heuristic starts with an initial set of *p* facilities and then relocates one of the facilities to another vertex not in the facility set, which provides the best improvement in the objective function. This heuristic is very fast and has been often used in order to obtain initial feasible solutions in exact techniques. Goodchild and Noronha (1983) provided another 1-opt heuristic whose search strategy is different than Teitz and Bart (1968). This heuristic is likely to find different local solutions than the previous heuristic. Whitaker (1983) also provided a greedy exchange heuristic but his method does not allow multiple starts so is not very useful for finding good solutions.

Captivo (1991) provided three different heuristics. The first heuristic is a greedy heuristic based on a very simple idea. It places the first facility on the 1-median of the network and second facility to the node, which decreases the total travel cost at most. This process is repeated until all medians are located. The heuristic is improved using ideas of Maranzana (1964) that every local solution found is improved by finding the 1-median of the nodes assigned to a facility and replacing this facility with the new median. The heuristic iterates until no further improvement is possible. This heuristic has a time complexity of $O(n^2p)$. The second heuristic is a dual based heuristic that provides solutions to the dual of the L*P*-relaxation for which the obtained objective function values are lower bounds for the problem. The heuristic is a dual ascent procedure similar to

Erlenkotter's (1978) procedure. This procedure provides better bounds than Galvão (1980). The third heuristic constructs a primal solution based on the best dual solution using Complementary Slackness conditions. The primal heuristic is very fast and provides good bounds, whereas the primal-dual heuristic provides very good solutions but is very time consuming. Later on, a series of heuristics are proposed and implemented (Rushton and Kohler, 1973; Densham and Rushton, 1992a, 1992b), which makes use of efficient data structures and clever search strategies. These heuristics are very fast but Teitz and Bart (1968) still provide better bounds. These heuristics do not provide 1-opt solutions so an additional pass is required to guarantee optimality as stated by Horn (1996). Horn provided comparisons and comments on the p-median heuristics that may be useful for the interested reader.

Metaheuristics have been successfully applied to p-median problems in recent years and promise even better results in the coming years with new technologies and computational improvements. An efficient implementation of tabu search for the p-median problem is provided by Rolland, Schilling, and Current (1997). Tabu search is very successful in terms of good solutions and computation time compared to many previous heuristics such as Goodchild and Noronha (1983) and Densham and Rushton (1992a, 1992b). Two genetic algorithms are provided by Bozkaya, Zhang, and Erkut (2002) and Alp, Erkut, and Drezner (2003). First paper provides experimental results that suggest that convergence is slow whereas the second one presents results within 0.1% of the optimum in 85% of the test instances in short time. The p-median problem has also been solved using Heuristic Concentration (HC) methods. HC is a two stage heuristic. In the first stage, an initial set of solutions is constructed using a heuristic method (usually an exchange heuristic) by starting the heuristic at systematic or random initial points. The best solutions of this set form the Concentration Set for the problem. Stage two restricts the potential facility sites to the sites observed in the solutions in the Concentration Set and solves the model. Usually exact techniques are used in stage two. For example Rosing and Revelle (1997), Rosing (1998), and Rosing, Revelle, Rolland, Schilling, and Current (1998) use ILP formulations and exact solution techniques for the second stage of HC heuristic. In contrast to these papers, Rosing, Revelle, and Schilling (1999) use another heuristic, which is in fact a two stage heuristic, to generate solutions from the Concentration Set in the second stage of an HC for the p-median problem. This heuristic is referred to as a gamma heuristic because it involves three heuristic stages. In fact this metaheuristic can be very useful to provide good solutions for some large p-median problems because it reduces the search space dramatically at the second stage in addition to the fact that it is very simple and fast.

3.2.3 Conditional *p*-medians of a General Network:

Consider the problem of locating p facilities with minisum objective on a network on which there already exist q facilities. This problem is the conditional p-median problem defined as: Given a set $Y \subseteq N$, $|Y| = q$, find a set $Z^* \subseteq N$, $|Z^*| = p$, and $Z^* \in \arg\min_{Z \subseteq N, |Z|=p} \left\{ \sum_{v_i \in V} w_i D(v_i, Y \cup Z) \right\}$. Minieka (1980) solved the problem when a single facility is to be located on a network with a number of existing facilities. The author redefined the entries of the distance matrix and solved the problem with known techniques for the unconditional problems. For the case when multiple conditional facilities are to be located, Drezner (1995) proposed a heuristic, which requires the solution of multiple unconditional problems. Berman and Simchi-Levi (1990) solved the problem on general networks by solving a (p+1)-median problem with one more auxiliary new facility representing the total effect of all old facilities.

3.3 Absolute *p*-Median Problem on Tree Networks:

Although the absolute p-median problem is *NP*-hard on general networks, it is polynomialy solvable on tree networks. Studying the problem on tree networks is useful for a number of reasons. First of all studying the problem on a simpler structure provides insight on the general structure. Secondly, solving the problem on simple components of the general network, for example the spanning trees of the network, provides upper bounds on the objective function value of the general problem. Last but not the least important, most of the widely studied real-world systems like transportation networks across countries or telecommunication networks have just a few cycles and can be well approximated by tree networks.

The work on simple networks, which in turn gave rise to very simple and elegant algorithms for such networks was probably initiated by the work of Goldman and Witzgall (1970). It is proved in this paper that a "gated" subnetwork of a network containing half or more than half of the total demand must include at least one optimal 1-median location. The subnetwork S is gated if there exists a function g: N-$S \rightarrow S$ such that for each $x \in N$-S and $s \in S$, there exists a point $g(x)$ in S, satisfying $d(x, s) = d[x,$

$g(x)] + d[g(x), s]$. This observation reduces the search for the optimal 1-median of a network to a subnetwork satisfying the conditions above. The result is valuable in tree networks, cacti, and networks with many blocks because these networks involve gated subnetworks. Goldman (1972b) presented a less restrictive version of the observation in Goldman and Witzgall (1970) in that a near optimal solution is localized into a subregion of the network, which includes nearly half of the demand. Goldman (1971) devised a very simple $O(n)$ algorithm for tree networks based on the observations above. His algorithm starts with a tip node of a tree network. If the weight of this node is smaller than half of the total weight, it is deleted from the tree with its associated link and its weight is added to the weight of the node adjacent to it. The process is repeated at most n times until a node with demand weight more than or equal to the half of the total demand is found. This node is a 1-median of the tree network. The algorithm is very simple because it does not require the calculation of the shortest path distances of the network, which itself takes $O(n^2)$ time. Goldman (1971) also proposed an algorithm to locate the 1-median of a network with only one cycle and introduced a decomposition procedure for more general networks, which either locates the 1-median of the network or reduces the search to a cyclic component of the network. Chen, Francis, Lawrence, Lowe, and Tufekci (1985) developed an algorithm based on that of Goldman (1971) that either finds the 1-median of the network or finds a block that contains all the 1-medians of the network. The block graph of the network is obtained and the 1-median problem is solved on this block graph using Goldman's algorithm. Although the complexity of the algorithm is $O(n)$, the construction of the block graph, which is $O(m)$ (as stated by Aho, Hopcroft, and Ullman, 1976), dominates the complexity.

Another important observation about median location on tree networks is stated by Kariv and Hakimi (1979): finding a 1-median of a tree is equivalent to finding a *w*-centroid of the tree. The *w*-centroid of a tree is a vertex v_0 whose removal from the tree, divides the tree into $deg(v_0)$ components such that maximum total weight on its components is minimum among all vertices of the tree.

The 2-median problem on tree networks has been investigated by Mirchandani and Oudjit (1980). The main contribution of the paper is the observation that for deterministic tree networks the path connecting the 2-medians of a network passes through the 1-median of the network. An efficient algorithm which has computational complexity of $O(n^2)$ is presented based on this information. The algorithm is a lin*k*-deletion method, which deletes one link of the tree at each iteration and finds 1-medians of the resulting two components. This lin*k*- deletion method has the same time complexity with the general *p*-median algorithms that will be discussed in the following paragraph for $p=2$ but it is computationally more efficient. The problem is also solved on probabilistic tree networks in which the link lengths change in discrete time intervals. The tree has a finite number of states with associated probabilities of occurrence. It is observed that for the probabilistic single facility case Goldman's algorithm solve the problem because the link lengths do not enter the algorithm. Nevertheless, for the 2-median problem the facility that serves a demand point changes with the state of the network and the algorithms devised for deterministic cases do not work. An algorithm is presented by Mirchandani and Oudjit (1980) for the probabilistic trees. Although the computational complexity of the algorithm is equal to that of complete enumeration, the computation time is reported to be less in practice than that of complete enumeration.

An $O(n^3p^2)$ algorithm for finding the p-median of a tree is proposed by Matula and Kolde (1976). Kariv and Hakimi (1979) presented a dynamic programming based algorithm for this problem. The algorithm calculates distance sums over different subtrees of the tree network with k medians ($1 \leq k \leq p$) at the first stage and calculates the p-median value by backtracking in the second stage. The algorithm has a time complexity of $O(n^2p^2)$ which had been the best complexity bound for years until Tamir (1996) improved the bound to $O(n^2p)$. Tamir's algorithm is again a dynamic programming algorithm and it solves more general problems in which each facility may be assigned a fixed establishment cost. So Tamir's algorithm also solves UFLP on tree networks in the same time bound. We note that both dynamic programming algorithms by Tamir (1996) and Kariv and Hakimi(1979) can be used to solve for the conditional p-medians of a tree network Another algorithm which has a complexity bound of $O(n^3p)$ is also provided by Hsu (1982), but this algorithm is dominated by both Kariv and Hakimi(1979) and Tamir (1996). We also note that the p-median problem on a path network is solved by Hassin and Tamir (1991) in $O(np)$.

3.4 The *p*-Median Problem with Mutual Communication:

The facility location problems that involve interactions between pairs of new facilities and between new facilities and existing one are referred to as problems with mutual communication. The problem can be formally stated as follows: Given nonnegative and nondecreasing functions

$f_{ij}(.)$ and $g_{jk}(.)$, find $X^* = (x_1^*, x_2^*, \ldots, x_p^*) \in N^p$ such that $H(X^*) \leq H(X)$ $\forall X \in N^p$ where $H(X) = \sum_{i,j} f_{ij}(d(v_i, x_j)) + \sum_{j,k:j\pi k} g_{jk}(d(x_j, x_k))$. When the functions $f_{ij}(.)$ and $g_{jk}(.)$ are linear with nonnegative slopes w_{ij} and v_{jk}, then the cost function can be expressed as

$$H(X) = \sum_{i,j} w_{ij}(d(v_i, x_j)) + \sum_{j,k:j\pi k} v_{jk}(d(x_j, x_k))$$

and the problem is referred to as the *p*-median problem with mutual communication. The *p*-median problem with mutual communication is *NP*-hard on general networks (Kolen, 1982), the result being valid even if the network is a simple triangle (Tamir, 1993). It is already known that there exists an optimal solution to this problem on the vertices of the network. The problem is first defined by Dearing, Francis, and Lowe (1976) with distance constraints. It is shown in the same paper that the objective function is convex for all data choices if and only if the underlying network structure is a tree.

The *p*-median problem on tree networks with mutual communication is studied by Dearing and Langford (1975) by embedding the tree network into a Euclidean space and solving the problem with techniques developed for rectilinear problems in the Euclidean space. Picard and Ratliff (1978) solved the problem by solving a sequence of minimum cut problems. Each cut problem corresponds to an edge of the tree so *n*-1 problems are solved. Removal of each edge *e* identifies two subtrees $T_1(e)$ and $T_2(e)$. For each edge of the tree an auxiliary network is constructed which has *p* nodes corresponding to the new facilities, a source node representing the total demand of subtree $T_2(e)$, and a sink node for total demand in $T_1(e)$. The capacity of the arc between new facility nodes *i* and *j* is v_{ij} and the capacity of the arc between source *s* (sink *t*) and new facility node *j* is $\sum_{v_i \in T_1(e)} w_{ir}$ ($\sum_{v_i \in T_2(e)} w_{ir}$). Kolen (1982) solved the problem using an algorithm based on the fact that in an optimal *p*-median solution no subset of facility locations can be moved to adjacent vertices such that the objective function value improves.

It is shown that the *p*-median problem on tree networks with mutual communication and distance constraints can be expressed as a mathematical program (Erkut, Francis, Lowe, and Tamir, 1989). Based on this result, the problem is solved using duality theory and column generation algorithms. These solution techniques on tree networks provide insight into devising solution algorithms for general networks as well. Erkut, Francis, and Lowe (1988) studied the problem on general networks and computed strong lower and upper bounds on the objective function value. The problem is transformed into a linear program based on *separation conditions* by Francis, Lowe, and Ratliff (1978). Because the separation conditions are only necessary but not sufficient conditions for general networks, the resulting LP is only a relaxation of the original problem. The solution of the LP provides a lower bound on the objective value. The problem is exactly solved for an arbitrary set of spanning trees of the network to yield upper bounds. These bounds may be useful in branch-and-bound algorithms in order to solve the problem to optimality. Tamir (1993) provided an $O((p^3+n)\log n+np)$ algorithm for the problem on tree networks that is based on solving local problems. Each local problem is a classical minimum cut problem and is related to an edge of the tree. A restricted version of the problem in which facility sites are restricted to proper subsets of the node set is also introduced. It is shown that this restricted problem is *NP*-hard even on tree networks and it is only polynomial for path networks.

Xu, Francis, and Lowe (1994) provided an $O(p^3(n+b))$ algorithm for the p-median problem with mutual communication on general networks which either locates each facility to a vertex or restricts it to be in a particular block of the network, where b is the number of blocks of the network. The algorithm transforms the original problem data into the block graph of the network.

The p-median facility location problems with mutual communication are hard problems on general networks, so researchers exploit special structures of the problem. The algorithms discussed above exploit the special structure of the network and make use of convexity results on the trees. There are also algorithms which solve the problem on general networks in reasonable time. These second type of algorithms exploit the special structure of the interaction graph instead of the network itself. The interaction graph is an auxiliary graph composed of p nodes representing the new facilities to be located. There exists an arc between new facility nodes NF_i and NF_j if the interaction cost v_{ij} is positive. Chhajed and Lowe (1992a) solved the p-median problem with mutual communication on general graphs when the interaction graph is a series-parallel graph. Following the definition by Richey (1989), a graph is series-parallel if it can be reduced to an arc by repeated application of the following operations: series reduction (any node u such that $deg(u) = 2$ is deleted with its adjacent arcs and a new arc is placed between its adjacent vertices), cut reduction (if q is a pendant node adjacent to node u and there exists another node v which is also adjacent to u, q is deleted and a new arc is placed between u and v), and parallel reduction (any two arcs adjacent to the same two nodes are replaced by a new arc). Reformulating it as a graph theoretic node selection problem solves the problem. Chhajed and Lowe (1992b) solved the p-median problem with mutual communication on general graphs when the interaction graph is a Halin graph. A graph is called a Halin graph if there exists a tree with vertex degrees other than two and a cycle which connects the pendant vertices of this tree embedded into the network. The problem is solved using a graph reduction technique, which reduces the Halin graph to a series-parallel graph and solves the problem on the corresponding graph using the techniques in Chhajed and Lowe (1992a). When number of candidate locations for each new facility is at most λ, the algorithm has a time complexity of $O(p\lambda^6)$. This bound is equal to $O(pn^6)$ for the regular p-median problem with mutual communication. The results can be extended to networks with interaction graphs, which are more general than Halin graphs. A generic algorithm, which solves many multifacility problems including p-median problem with mutual communication is also provided by Chhajed and Lowe (1994). This study is mainly an extension of the results in the aforementioned papers by the two authors. An $O(np)$ algorithm is presented for the same problem on tree networks when the interaction graph is series-parallel by Chhajed and Lowe (1992c) which is based on the $O(np^3)$ algorithm for tree graphs by Kolen (1986). This algorithm exploits both the special structure of the network and the interaction graph.

3.5 The *p*-Median Problem with Continuous Link Demands:

In the classical median location problems we have considered above, the nodes of the network generate demands. Nevertheless, in many real world problems the demand is generated by the customers who are continuously distributed on the links of the network. The aggregation of the link demands to nodes results in unsatisfactory models.

Minieka (1977) introduced the first general median problem in which links of the network generate demands and each link is served by a facility by serving the most distant point on the link from the facility. The regular point-to-point distances are replaced by the point-to-edge distances, which is equal to the distance between the point and the most distant point on the edge. The nodal optimality results fail for this type of problems. An algorithm is devised for the single facility case by Minieka (1977) but the extension to the p-facility case remained unsolved until Hansen and Labbé (1989) provided an $O(m^2)$ algorithm for the problem where m is the number of edges. In this paper, it is shown that the possible facility locations on a network can be restricted to the union of the vertex set and the set of middle points of edges. A linear algorithm for finding the set of all general continuous medians of a tree network is also provided. The conditional general p-median problem can also be solved using these algorithms.

Chiu (1987) generalized the 1-median problem to a continuous 1-median problem on a network with discrete demands at the nodes of the network and continuous demands on the links. The formal definition of the 1-median problem with continuous link demands is as follows: Given a network $N=(V, L)$ with node set V and link set L, let l' denote the length of link l, h_i denote the fraction of demands originating from node i, and f_l' denote the fraction of demands originating from link l. Obviously, $\sum_{i \in V} h_i + \sum_{l \in L} f_l' = 1$. The demand that arises on link l is assumed to have a general distribution function $f_l(y)$ for demand $y \in (0,\ l')$. The demands generated by links are weighted by probability in this formulation. The objective is to find a point $x^* \in N$ such that $D(x^*) \leq D(x)$ for all $x \in N$, where $D(x) = \sum_{i \in V} h_i d(v_i, x) + \sum_{l \in L} f_l{}' \int_{y \in l} d(x,y) f_l(y) dy$. The nodal optimality results fail for the continuous problems. The function $D(x)$ is analytically investigated in Chiu(1987). It is unfortunately neither convex nor concave even if it is restricted to the points between two breakpoints of an edge (Note that a breakpoint of an edge is a point y for which $d(v_i, y) = d(v_j, y)$ for two nodes v_i and v_j and both of the distance functions are not decreasing in the same direction). Nevertheless, one can find the local optima in every region between two breakpoints using nonlinear programming techniques and find the global solution by investigating all the local optima. Unfortunately there are $O(n^3)$ breakpoints and if the density function is not very simple, such as a uniform distribution, the analysis becomes intractable even for small values of n. A heuristic that locates the facility on one of the breakpoints is suggested by Chiu (1987) which may be useful if the function $D(x)$ can easily be evaluated. The objective function $D(x)$ is convex on any path of a tree network. This convexity property has led to an algorithm for the 1-median problem with continuous link demands on tree networks (Chiu, 1987). The algorithm is very similar to Goldman's 1-median algorithm on tree networks.

Brandeau, Chiu, and Batta (1986) considered the 2-median problem on tree networks with continuous link demands. They have presented an algorithm for general demand distributions which converges to a local minima based on a sequential location and allocation procedure and the fact that the continuous 1-median of the network lies on the path connecting any pair of optimal 2-medians of the network. Mirchandani and

Oudjit (1980) provided this result for discrete problems. Their proof is also valid for continuous 2-medians. The algorithm finds all local minima and chooses the global minimum among them.

Cavalier and Sherali (1986) solved the continuous p-median problem on path networks when the demand is distributed by a uniform probability distribution function on the links of the network. The algorithm relies on solving very easy linear programs to find all local optima and selects the global solution. The problem is also solved for p=2 on tree networks by solving 2-median problems on several paths of the tree network. A reduction on the number of paths to be considered is also presented. The problem with uniform distribution of demand is also considered by Nkansah and David (1986) on general networks. It is shown that the interior points of an edge may be omitted from the search space if the edge belongs to a circuit. Analogous conditions are presented when the distribution is more general than the uniform distribution but additional assumptions are imposed on the model. This result is further clarified by Batta and Palekar (1987) who showed that for general networks whose edges belong to at least one circuit, the search may be restricted to the nodes of the network. We believe that these results may further be extended to graphs with many blocks using block diagrams.

3.6 Capacitated *p*-Median Problem on a Network:

The capacitated p-median problem arises when there exists capacity restrictions on the total demand assigned to each facility. This problem is a restriction of the Capacitated Facility Location Problem in which the number of facilities is a priori set to a fixed number p. The problem is known to be *NP*-hard (Garey and Johnson, 1979). The problem is studied under different names such as Capacitated Warehouse Problem and Sum-of-Stars Clustering Problem as stated by Maniezzo, Mingozzi, and Baldacci (1998). The problem is also being studied as a set-partitioning problem with side constraints. Pirkul (1987) proposed a branch-and-bound algorithm based on Lagrangean relaxation for the problem. Neebe and Rao (1983) proposed another exact algorithm for the problem based on set partitioning with side constraints formulation. Hansen, Jaumard, and Sanlaville (1994) and Baldacci, Maniezzo, Mingozzi, and Ricciardelli (1995) also provided exact algorithms for the problem.

Mulvey and Beck (1984) provided two heuristic algorithms for the problem. The heuristics are location-allocation procedures similar to Maranzana (1964) but the local search criterion is different. Osman and Christofides (1994) presented another heuristic for the problem, which is a hybid of the two metaheuristics Simulated Annealing and Tabu Search. An additional heuristic that is a multistart procedure, based on the solutions of a series of generalized assignment problems and local search, is devised by Maniezzo, Mingozzi, and Baldacci (1998). They also provided a bionomic algorithm which is a metaheuristic similar to Genetic Algorithms but allows diversification of children. The bionomic algorithm outperforms many previous metaheuristics.

The problem is also studied for continuous link demands. A capacitated 2-median problem with continuous link demands is formulated by Sherali and Nordai (1988) on tree networks. Certain optimality conditions are provided for the problem. The search space is reduced to a limited subset of the tree network based on presented optimality conditions and localization results.

3.7 Further Remarks and Conclusions:

The median problem and related minisum facility location problems have attracted the interest of researchers for the last 40 years. Several problems are defined as extensions of these problems. Slater (1981) defined the *S*-median of a network as the set of points providing service to demands generated by subnetworks of the network. A subnetwork is served when the point that is nearest to the facility is served. The *S*-median of a tree network is found in $O(n)$ time by Slater (1981). Minieka (1983b) defined the pendant medians of a network to be a set of p points on the network that minimizes the total distance to the vertices that are actually pendant vertices of the minimum spanning tree of the network. For tree networks, this problem is easy to solve with Goldman's algorithm by assigning zero weight to vertices that are not pendant. Nevertheless, the problem is not very easy on general networks. Nodal optimality results do not hold for the problem and although no *NP*-completeness result is provided, devising polynomial time algorithms for the problem is conjectured to be impossible.

In classical p-median problems, one level of facilities is considered. Nevertheless, many real world distribution systems consist of many facility layers such as factories, warehouses, customers, and suppliers. Facility location problems with many layers of facilities are usually referred to as hierarchical facility location problems. Narula (1986) presents an inclusive survey of hierarchical location problems that deals with different types of hierarchy. Application areas are also discussed to be varying from location of emergency services to waste disposal centers. Serra and ReVelle (1993, 1994) deal with *pq*-median problems in which two levels of facilities each with median objective are being located. The problem is modeled as a mixed integer program by Serra and ReVelle (1993). Heuristics are proposed for the two-level biobjective model by Serra and ReVelle (1994) and Alminyana, Borras, and Pastor (1998).

The fuzzy models have been widely studied in recent years. A fuzzy formulation of the *p*-median problem is proposed by Canos, Ivorra, and Liern (1999) and Canos, Ivorra, and Liern (2001). An algorithm based on Hakimi (1965) is proposed for the fuzzy problem by Canos, Ivorra, and Liern (1999).

Models that deal with improving the network structure are also being developed recently. In these models, facilities have already been located and adding new arcs or decreasing transportation costs improves the total cost of serving customers. These problems are called inverse location problems and promise large application areas because location problems are long run problems and even if they are solved to optimality under certain conditions at a point in time, they become suboptimal over time if the network structure changes. Interested readers are referred to Berman, Ingco, and Odoni (1992) and Zhang, Liu, and Ma (2000) for reverse problems. Wang, Batta, Bhadury, and Rump (in press) also deal with facility location problems which improve the network by simultaneous opening of new facilities and closing of old ones. These problems arise in reformation procedures of global companies.

The literature on minisum facility locaion problems is summarized below:

Table 6: Nodal Optimality Results For Absolute Multifacility Minisum Location Problems

Author	*Year*	*Problem*	*Nodal optimality*
Hakimi	1964	1-median	Yes
Hakimi	1965	*p*-median	Yes
Goldman and Meyers	1965	*p*-facility minisum problems with concave cost functions	Yes
Levy	1967	*p*-facility minisum problems with concave cost functions and capacity restrictions	Yes
Goldman	1969	General model in which cost function is concave and depends on the source and destination pairs	Yes
Hakimi and Maheshwari	1971	General model in which cost function is concave, facilities are capacitated and there exist multiple commodities and multiple processing stages	Yes
Wendell and Hurter	1973	General model in which cost function is concave, facilities are capacitated and there exist multiple commodities, multiple processing stages and directed arcs	Yes
Frank	1966	maximum probability absolute *R*-median	No
Frank	1967	maximum probability absolute *R*-median (demands follow joint normal distribution)	No
Mirchandani and Odoni	1979	1-median with random discrete demands (directed/undirected)	Yes
Mirchandani	1979	Supporting *p*-median and	Yes

and Odoni		Conditional *p*-median	
Berman, Larson, Chiu	1985	1-median which is a M/G/1 queue	No

Table 7: Literature on Exact Solution Techniques for *p*-Facility Minisum Location Problems on General Networks

Author	***Year***	***Problem***	***Summary***
ReVelle and Swain	1970	*p*-median	Branch-and-Bound
Järvinen, Rajala, and Sinervo	1972	*p*-median	Branch-and-Bound and upper bounds
Garfinkel, Neebe, and Rao	1974	*p*-median	L*P*-relaxation, decomosition, group theoretics, and dynamic recursion
Narula, Ogbu, Samuelsson	1977	*p*-median	Lagrangean relaxation and subgradient optimization
Cornuejols, Fisher, Nemhauser	1977	*p*-median	Greedy heuristic, Lagrangean relaxation and B&B
Mavrides	1979	*p*-median with different fixed costs	UFLP techniques and Lagrangean relaxation
Galvão	1980	*p*-median	B&B and a heuristic for Lagrangean relaxation
Boffey and Karkazis	1984	*p*-median (n=206, p=45)	A series of UFLPs with varying fixed facility costs
Christofides and Beasley	1982	*p*-median ($n \le 200$)	Lagrangean relaxation and subgradient optimization
Beasley	1985	*p*-median ($n \le 900$, $p \le 90$)	Supercomputer
Mirchandani, Oudjit, and Wong	1985	*p*-median ($n \le 200$)	Nested Dual Approach
Hooker	1986	1-median with convex cost functions	Treelike segments and convex programming
Hooker	1989	*p*-median with convex cost functions	Treelike segments, convex programming and segment elimination techniques

Table 8: Literature on Approximate Solution Techniques for p-Facility Minisum Location Problems on General Networks

Author	*Year*	*Problem*	*Summary*
Maranzana	1964	*p*-median	Fast heuristic
Teitz and Bart	1968	*p*-median	1-opt heuristic Fast and widely used
Goodchild and Noronha	1983	*p*-median	1-opt heuristic
Whitaker	1983	*p*-median	Greedy exchange heuristic
Captivo	1991	*p*-median	Three heuristics: Primal ($O(n^2p)$) dual, and primal-dual
Rushton and Kohler Densham, Rushton	1973 1992	*p*-median	Efficient heuristics and data structures
Rolland, Schilling, and Current	1997	*p*-median	Tabu search
Rosing and Revelle Rosing Rosing, Revelle, Rolland, Schilling, and Current Rosing, Revelle, and Schilling	1997 1998 1998 1999	*p*-median	Heuristic concentration

Table 9: Literature on p-Facility Minisum Location Problems on Tree Networks

Author	*Year*	*Problem*	*Summary*
Goldman and Witzgall	1970 1972	1-median	Localization theorems
Goldman	1971	1-median	$O(n)$ algorithm
Kariv and Hakimi	1979	1-median	Equal to *w*-centroid
Mirchandani and Oudjit	1980	2-median	$O(n^2)$ algorithm

Matula and Kolde	1976	*p*-median	$O(n^3p^2)$ algorithm
Hsu	1982	*p*-median	$O(n^3p)$ algorithm
Kariv and Hakimi	1979	*p*-median	$O(n^2p^2)$ algorithm
Tamir	1996	*p*-median	$O(n^2p)$ algorithm

Table 10: Literature on *p*-Facility Minisum Location Problems with Mutual Communication

Author	*Year*	*Problem*	*Summary*
Kolen	1982	*p*-median with mutual comm.	*NP*-hard on general networks
Dearing, Francis, and Lowe	1976	*p*-median with mutual comm. and distance constraints on trees	The objective function is convex if and only if the underlying network structure is a tree
Dearing and Langford	1975	*p*-median with mutual comm. on trees	Embedding into Euclidean space
Picard and Ratliff	1978	*p*-median with mutual comm. on trees	Solving a sequence of minimum cut problems
Erkut, Francis, Lowe, and Tamir	1989	*p*-median with mutual comm. and distance constraints on trees	Expressed as an MP, solved using duality and column generation
Erkut, Francis, and Lowe	1988	*p*-median with mutual comm. and distance constraints on general networks	Computed strong lower and upper bounds on the objective function value
Tamir	1993	*p*-median with mutual comm. and distance constraints on trees	$O((p^3+n)logn+np)$ algorithm
Xu, Francis, and Lowe	1994	*p*-median problem with mutual comm. on general network	$O(p^3(n+b))$ algorithm which localizes the solution
Chhajed and Lowe	1992	*p*-median problem with mutual comm.	Reformulating as a graph theoretic node

		on *N* when the interaction graph is series-paralel and Halin	selection problem
Chhajed and Lowe	1992	*p*-median problem with mutual comm. on *T* when the interaction graph is series-paralel	*O(np)* algorithm

Chapter 4

DISTANCE CONSTRAINED FACILITY LOCATION PROBLEM ON NETWORKS:

The distance constraints arise when there exist upper and / or lower bounds on the distances between facilities and customers and pairs of facilities. When facilities are identical, the following distance constraints are used:

$$D(\delta, X) \le u_\delta, \forall \delta \in \Delta \qquad DC1.u$$

$$D(\delta, X) \ge l_\delta, \forall \delta \in \Delta \qquad DC1.l$$

$$\min_{x' \in X, x' \neq x} d(x, x') \le u, \forall x \in X \qquad DC2.u$$

$$\min_{x' \in X, x' \neq x} d(x, x') \ge l, \forall x \in X \qquad DC2.l$$

If facilities are nonhomogeneous:

$$d(\delta, x_i) \le u_{\delta i}, \forall \delta \in \Delta, i \in \{1,.., p\} \qquad nDC1.u$$

$$d(\delta, x_i) \ge l_{\delta i}, \forall \delta \in \Delta, i \in \{1,.., p\} \qquad nDC1.l$$

$$d(x_i, x_j) \le u_{ij}, 1 \le i < j \le p \qquad nDC2.u$$

$$d(x_i, x_j) \ge l_{ij}, 1 \le i < j \le p \qquad nDC2.l$$

The distance constraints may arise in many real world applications. For example, if the facilities are identical and there exists a customer δ that must be served by a facility no more (less) than a distance u_δ (l_δ) away, then constraint *DC1.u (DC1.l)* is included to the model. If facilities are not identical then upper and lower bounds are specified for each customer-facility pair and constraints *nDC1.u* and *nDC1.l* are in effect. Similarly, it may be preferable to have at least one more facility within a distance u from any facility (when facilities are subject to breakdowns) or it may be undesirable to have facilities close to each other within a distance l (when there is high competition between facilities or facilities are mutually obnoxious), then constraints *DC2.u* and *DC2.l* are used for homogeneous facilities. In the nonhomogeneous case, distances between pairs of facilities may be bounded and constraints *nDC2.u* and *nDC2.l* are used. Motivating examples from the real world can be found in Francis, Lowe, and Ratliff (1978), Tansel, Francis, and Lowe (1980), Tansel, Francis, Lowe, and Chen (1982), Erkut, Francis, and Tamir (1992) and Tansel and Yesilkokcen (1993, 1996).

4.1 The Literature:

Although distance constrained network location problems may have lower and upper bounds as we have mentioned before, the related literature focuses on the problem with nonhomogeneous facilities and upper bounds on distance functions. This is the problem that involves locating p facilities on N so as to satisfy upper bounds on distances between pairs of new facilities and pairs of new and existing facilities. Formally, the problem is defined as follows: Given the nonempty sets I_C and I_B and the positive upper bounds c_{ij} and b_{jk}, find a location vector X in

N^m, if it exists, such that $d(x_i, v_j) \leq c_{ij}$, $(i, j) \in I_C$, and $d(x_j, x_k) \leq b_{ij}$, $(j, k) \in I_B$. The distance constrained facility location problem is closely related to the multifacility minimax facility location problem with mutual communication between facilities. In fact the distance-constrained problem is the recognition form of the later problem. If there exists a polynomial algorithm for the distance constrained problem, there also exists a polynomial algorithm for the minimax problem with mutual communication between facilities. Solving a series of the distance-constrained problems usually solves the later problem.

The problems are *NP*-hard on general networks (Kolen, 1986), but there exist polynomial algorithms for tree networks. It is shown that the distance constraints define convex sets for all data choices if and only if the underlying network is a tree (Dearing, Francis, and Lowe, 1976). Based on this result, Francis, Lowe, and Ratliff (1978) provided the necessary and sufficient conditions, called the separation conditions, for the distance constraints to be consistent for tree networks. In order to obtain the separation conditions, an auxiliary network N_{BC} is constructed. N_{BC} consists of n vertices $E_1,..,E_n$ denoting the existing facilities and p vertices $N_1,\ldots,N_p$ denoting p new facilities. There exists an arc (N_i, E_j) of length c_{ij} for every $(i, j) \in I_C$ and an arc (E_j, E_k) of length b_{jk} for every $(j, k) \in I_B$. If network N_{BC} is not connected the problem can be decomposed into a number of independent problems with a smaller number of constraints, so we assume that N_{BC} is connected. The separation conditions state that the problem is consistent if and only if $d(v_j, v_k) \leq L(E_j, E_k)$ where $L(E_j, E_k)$ denote the length of a shortest path between existing facilities E_j and E_k in network N_{BC}. The separation conditions are necessary for general networks for consistency but they are not sufficient while sufficiency also holds for tree networks. Francis, Lowe, and Ratliff (1978) provided an algorithm called the Sequential Location Procedure (SLP) which has a computational complexity of $O(p(p+n))$. This algorithm is extended by Tansel, Francis, and Lowe (1980). The relations between tight separation conditions (Separation conditions that hold as an equality, i.e. $d(v_j, v_k)=L(E_j, E_k)$) and the solution of multifacility minimax problems are highlighted. It is also shown that SLP is a best order algorithm that solves the distance constrained problem. That is, any other algorithm must have a worse case time complexity of $O(p(p+n)$. Averbakh and Berman (1996) solved a more generalized version of this problem on tree networks. The model locates p distinguishable facilities on a tree network subject to upper bounds on interfacility distances. There exist no customers in the model. A feasible region that is not necessarily connected is given for each facility. The feasible regions are assumed to be chosen according to customer locations. A Sequential Location Scheme similar to SLP is presented together with conditions for feasibility. It is shown that when the set of feasible regions for each faciliy is finite the feasible facility locations can be found in $O(np^2)$ time if a feasible solution exists. Erkut, Francis, and, Lowe (1989) showed how to develop a mathematical model of the problem using separation conditions. This program solves the problem optimally when the network is a tree; it provides a lower bound otherwise. When the facilities are to be located at the vertices of the graph and the facility interactions induce a series-parallel graph, polynomial algorithms are devised by Chhajed and Lowe (1992a, 1992b). A generic polynomial algorithm which solves several location problems including multifacility minimax problem with mutual communication between facilities is also developed later by Chhajed and Lowe (1994) when the facility interactions again induce a series-parallel graph.

DISTANCE CONSTRAINED FACILITY LOCATION PROBLEM ON NETWORKS

When the facility interactions induce a tree network Tansel and Yesilkokcen (1993, 1996) provide a polynomial algorithm to find the feasible regions of a distance-constrained facility location problem with upper bounds. The concept of feasible regions may be very useful in taking managerial decisions. The distance-constrained problem in which facility interactions induce a tree network further extended by Tansel (1994) to cases which include both upper and lower bounds on the interfacility distances. A Sequential Capture/Intersection Procedure similar to SLP is provided for the problem that finds the feasible regions for each facility in the first phase and locates facilities in these regions in the second phase.

Erkut, Francis, and Tamir (1992) solved the multifacility minimax problem with facility interactions and distance constraints in polynomial time ($O[mn(m+nlogm)+n^3logn]$) for tree networks. Observe that although the distance-constrained facility location problem is equivalent to the MMLP with facility interactions, the MMLP with facility interactions and distance constraints is a different and harder problem.

Moon and Chaudhry (1984) provide a valuable survey on distance constrained network location problems. They assume that facilities are indistinguishable and provide a classification scheme and integer programming formulations of many problems, which may be useful to stimulate research in this area.

The results presented in this chapter are summarized below:

DISTANCE CONSTRAINED FACILITY LOCATION PROBLEM ON NETWORKS

Table 11: Literature on Distance Constrained Location Problems on Networks

Author	*Year*	*Problem*	*Summary*
Kolen	1986	Distance-constrained facility location	*NP*-hard on general networks
Dearing, Francis, and Lowe	1976	Distance-constrained facility location	Convexity on tree networks
Francis, Lowe, and Ratliff	1978	Distance-constrained facility location	Separation conditions Sequential Location Procedure $O(p(p+n))$
Tansel, Francis, and Lowe	1980	Distance-constrained Facility Location	Extended Sequential Location Procedure
Erkut, Fracis, and, Lowe	1989	Distance-constrained facility location	Mathematical programming models
Chhajed and Lowe	1992 1994	MMLP with mutual comm. when interaction graph is series-paralel	Polynomial algorithms
Tansel and Yesilkokcen	1993 1996	Distance-constrained facility location when interaction graph is tree	Polynomial algorithm to find the feasible regions
Averbakh and Berman	1996	Distance-constrained facility location on trees	Sequential Location Scheme

Chapter 5

MULTIOBJECTIVE NETWORK LOCATION PROBLEMS:

The minimax (center type) and minisum (median type) problems are widely studied in network location theory. Minimax objective is used in locating emergency services in order to minimize the worst case cost whereas minisum objective is used in locating services that provide regular service such as daily delivery services, in order to minimize the average travel cost. We have outlined the numerous applications, models, exact and approximate algorithms related to these problems in the previous chapters. Although we are able to solve very large instances of these single objective models, modeling real world problems using only one objective seems unrealistic for many cases. For example, it might be useful to minimize the average travel cost provided that the worse case cost is not too high. Otherwise the facility may provide different types of services, which have different demand weights and transportation costs, and multiple minisum or minimax objectives may be used at the same time. Furthermore when the network is dynamic in the sense that the parameters such as customer demands and link lengths are subject to change at finite and discrete points in time, multiple objectives each corresponding to a time interval may be minimized simultaneously. Although multiobjective models have wide application areas, there are few studies on these models compared to single objective models. This is mainly because it is hard to combine the objectives in a reasonable way. Two common methods are used to handle multiobjectivity: weighted objective function and constrained single objective function. We will deal with problems with both center and median objectives in the rest of the chapter.

5.1 Center - Median Biobjective Models:

The biobjective network location problems with minimax and minisum objectives are referred to as the "cent-dian" or "medi-center" problems. Halpern (1976), who used a convex combination of the two objective functions, first introduced the cent-dian problem. The medi-center problem was initially studied as a median problem in which maximum travel cost to the customers is constrained. Both problems are biobjective models in which two antagonistic objectives are optimized and closely related to each other as Halpern (1980) states. We will refer to both of the problems as the cent-dian problem. The single facility cent-dian problem may be formally stated as follows: Given a network *N=(V, E)*, let $z_m(x) = \sum_{v_i \in V} w_i d(v_i, x)$ and $z_c(x) = \max_{v_i \in V} u_i d(v_i, x)$ denote median and center functions respectively. The cent-dian problem may be defined as one of the following three problems:

$$P1: \quad \min_{x \in N} \lambda z_m(x) + (1-\lambda) z_c(x) \qquad \lambda \in [0,1]$$

$$P2: \quad \min_{x \in N} z_m(x) \quad s.t. \quad z_c(x) \le \mu \qquad \mu \in \Re^+$$

$$P3: \quad \min_{x \in N} z_c(x) \quad s.t. \quad z_m(x) \le \theta \qquad \theta \in \Re^+$$

Halpern (1980) showed that the problems *P*2 and *P*3 are dual to each other in the sense that given an optimal solution x^* with objective function value $z_m(x)$ to *P*2 with parameter μ, this solution is also an optimal solution to *P*3 with objective function value μ and parameter $\theta = z_m(x)$ and vice versa. Solving problem *P*2 for all values of $\mu \in [z_m^*, z_c^*]$ yields all nondominated solutions of the biobjective problem. It is also shown by Halpern (1980) that *P*1 is a special case of the other two constrained problems in the sense that the solutions generated for *P*1 for all values of $\lambda \in [0,1]$ are included in the solution set to the other two problems.

*P*1 is solved for tree networks by Halpern (1976) for all values of λ. He showed that the 1-cent-dian of a tree network is on the path between the 1-center and a 1-median of the tree. Furthermore it is either a vertex on this path or the absolute center itself. He explicitly characterized the location of the 1-cent-dian for all values of λ using a simple algorithm. He also showed that the 1-cent-dian problem might be transformed to a median problem on a larger network. Handler (1985) solved *P*2 on tree networks for all possible values of μ using a simple algorithm. He also showed that the two objectives could be combined by a median model in which locating a facility far away from individual customers is penalized by the cost function itself. For example, using an exponential cost function, which severely penalizes the placement of a facility far away from the customer, may be useful. Handler solved this model with single median objective with exponential cost functions on tree networks. Halpern (1978) solved *P*1 on general networks for all values of λ. He characterized the cent-dian function on general graphs and showed that it lies on a path between the center and the median of the graph. He also showed that the function attains its minimum on an edge on one of the finite number of points called breakpoints. He enumerated all possible breakpoints using an upper bound for eliminating some entire edges. His work is very insightful for understanding the problem on general graphs. He also showed how to transform the problem to a median problem with increased number of vertices and edges.

The cent-dian problems may also be extended to cases in which multiple facilities are to be located. The multiple facility cent-dian problem referred to as the *p*-cent-dian problem may be formally stated as follows: Given a network *N=(V, E)* where *V* is the vertex set and *E* is the edge set, let X be a set of p points, and $z_m(X) = \sum_{v_i \in V} w_i D(v_i, X)$ and $z_c(X) = \max_{v_i \in V} u_i D(v_i, X)$ denote median and center functions respectively. The cent-dian problem is the problem of finding a solution $X^* \subseteq N$, $|X^*| = p$ to one of the following three problems:

$$mP1: \qquad \min_{X \in N} (\lambda z_m(X) + (1-\lambda) z_c(X)) \qquad \lambda \in [0,1]$$

$$mP2: \qquad \min_{X \in N} z_m(X) \qquad s.t. \qquad z_c(X) \le \mu \qquad \mu \in \Re^+$$

$$mP3: \qquad \min_{X \in N} z_c(X) \qquad s.t. \qquad z_m(X) \le \theta \qquad \theta \in \Re^+$$

Hooker, Garfinkel, and Chen (1991) presented a theoretical result that the finite dominating set for the 1-cent-dian problem which is the union of vertex set and breakpoints is at the same time a dominating set for the *p*-cent-dian problem. Nevertheless, Perez-Brito, Moreno-Perez, and Rodriguez-Martin (1998) presented a counterexample for the 2-cent-dian problem. They defined a new finite dominating set which consists of the vertex set, the breakpoints and the extreme points whose range is equal to the range of a breakpoint or to the distance between two vertices (the range of a breakpoint is the distance between two identifying vertices). This finite dominating set consists of $O(m^2n^3)$ points where m is the number of edges and n is the number of vertices. They also developed algorithms for the 2-cent-dian problem for general networks and tree networks which has time complexities of $O(m^2n^4)$ and $O(n^2)$, respectively. The algorithm on tree networks is a link deletion method whereas the one for general networks is a clever implicit enumeration technique.

The finite dominating set for the p-cent-dian problem is shown to be the same as the one for the 2-cent-dian problem defined above by Perez-Brito, Moreno-Perez, Rodriguez-Martin (1997). Garfinkel and Hooker (1998) also identified the finite dominating set for the p-cent-dian problem, thereby correcting the misunderstanding in Hooker, Garfinkel, and Chen (1991). All the work above considered the cent-dian problems when the imbedded center problems are unweighted, i.e. $u_i = 1$ for all i. Tamir, Perez-Brito, and Moreno-Perez (1998) studied the weighted problem on tree networks. They formulated the p-cent-dian problem as a restricted p-median problem and identified the finite dominating set for the problem. They solved the problem on tree networks in $O(pn^4)$ time and on path networks in $O(pn^3)$ time using the $O(pn^2)$ algorithm for the p-median problem by Tamir (1996).

The cent-dian problem is also extended to the location of structured facilities such as subtrees and paths on a network. Interested reader is referred to Averbakh and Berman (1999) for locating a cent-dian path on a tree network and Tamir, Puerto, and Perez-Brito (2002) for locating a cent-dian subtree on a tree network.

The models for the median and center problems gave rise to many extended problems. Same extensions may be considered for the cent-dian problems in the future. For example, the *p*-cent-dian problem may be investigated on stochastic networks, under capacity restrictions on the facilities or with continuously distributed demand along the links of the network. The results on centdian problems are summarized below:

Table 12: Literature on *p*-Centdian Problems on Networks

Author	***Year***	***Problem***	***Summary***
Halpern	1976 1978	1-centdian on trees	Definition Solved convex combination problem
Halpern	1980	1-centdian on trees	Equivalence of constrained and convex combination problems
Handler	1985	1-centdian on trees	Solved constrained version
Perez-Brito, Moreno-Perez, Rodriguez-Martin	1998	2-centdian	Finite dominating set $O(m^2n^4)$ for N $O(n^2)$ for T
Perez-Brito, Moreno-Perez, Rodriguez-Martin	1997	*p*-centdian	Finite dominating set
Tamir, Perez-Brito, and Moreno-Perez	1998	*p*-centdian	$O(pn^4)$ for T $O(pn^3)$ for P

5.2 Other Multiobjective Models:

Lowe (1978) considered the location of a single facility on a tree network with multiple objectives. Methods to find efficient solutions are provided when the objective functions are convex.

Tansel, Francis, and Lowe (1982) studied a biobjective MMLP on tree networks in which the objectives are minimizing the maximum distance between customers and new facilities and minimizing the maximum distance between pairs of new facilities. The efficient frontier of the problem is constructed and the problem is extended to the case with with more than two objectives.

Chapter 6

UNDESIRABLE FACILITY LOCATION ON NETWORKS:

The literature on location theory is dominated by problems that deal with desirable facilities. These models are used to locate facilities such as schools, police stations, fire stations, hospitals, and supermarkets. The objectives used in these models usually involve minimization of a function of distance or time between the facilities and potential customers. Nevertheless, there exist other types of facilities such as landfills, nuclear power stations, military bases, and chemical factories that are necessary but undesirable for the common householders. Such facilities may produce hazardous wastes, produce high levels of noise, and explode by accident or military attack. The location theory has been studying the location of such undesirable facilities since late 80's and very successful results have been obtained.

The location of undesirable facilities is typically more complicated than their desirable counterparts. Usually a sound measure of undesirability is not available to the analyst. Instead, it is assumed that undesirability can be expressed as a function of the distance between the facilities and customers. It is typical that there are multiple objectives in the location of undesirable facilities such as minimizing the undesirable effects to the customers and minimizing the travel costs from/to the facility. The location decision generates routing problems in the transportation network because the materials transported from/to an undesirable facility are usually hazardous and their transportation must be handled carefully in order to guarantee minimum risk to environment and society. Location-routing problems are studied widely and the interested reader may consult the survey papers by Erdogan, Erdogan, and Tansel (2003), Min, Jayaraman, and Srivastava (1998), and Laporte (1988).

Many undesirable facility location models involve dispersion of facilities from each other and from the customers. Such dispersion strategies are useful when facilities that are mutually undesirable are being located. For example, when military bases are being located they are dispersed as much as possible in order to eliminate the effects to the others of an attack on one of the bases. Similarly, the franchises of a burger chain should be dispersed in order to reduce competitiveness between the franchises of the same chain. Dispersion has also applications in decision analysis using multiple objectives (in that the nondominated set of solutions may be quite large and a representative set of solutions, which are far apart from each other, may be presented to the decision maker); marketing a set of products with diverse set of attributes; and providing multiple diverse set of starting points to a heuristic (Chandra and Halldorsson, 2000).

In this chapter, we survey the location of undesirable facilities on networks. Many problems, especially the ones involving air pollution, noise, and risk of explosion, are more appropriate to study with the planar models because most of the hazardous effects spread through a geographical space without following any network structure. Nevertheless, there may also be cases such as location of prisons, which cause crime to households or location of franchises, which are affected by the network structure. We investigate problems that involve the maximization of a function of distances between new facilities and existing facilities and/or between pairs of new facilities. The resulting models together with the algorithms devised for general networks and special networks are presented in the subsequent sections.

The analytical models devised for both planar and network problems are presented in the survey paper of Erkut and Neuman (1989). This paper presents a classification scheme for undesirable facility location, which we find very useful to fully understand the area. This paper is a milestone in the undesirable facility location, which presents past research, provides insight on the models and opens new research areas.

6.1 Maxisum Facility Location on Networks:

6.1.1 Single Facility:

The single facility maxisum dispersion problem locates a facility on a network such that the total of weighted distances from the existing facilities to the new facility are minimized. The model can be expressed as follows: Given a network *N = (V, E)*, find a point x^* in *N* such that $x^* \in \arg\max_{x \in N} \sum_{v_i \in V} w_i d(x, v_i)$. The single facility maxisum dispersion problem is similar to the absolute median problem on networks and it is sometimes referred to as the maxian or antimedian problem. Unfortunately, the vertex optimality results do not hold for the maxisum dispersion problem.

The problem has been first studied by Church and Garfinkel (1978). They have shown that there exists a finite set of points, which is the union of bottleneck points and vertices of degree one, which contain a solution to the maxisum problem. When the network is a tree, the solution should be on one of the pendant vertices of the tree. The authors present an $O(n^3)$ algorithm to find the 1-maxian point on a general graph based on these observations which finds local maxima on each edge. A very similar approach is presented by Minieka (1983a), which characterizes the solution of the single facility maxisum dispersion problem on undirected and directed networks, for vertex-restricted and absolute cases. It is well known that the objective function is a piecewise linear concave function when restricted to a single edge. Tamir (1991) proposed an $O(n)$ algorithm to solve the local problem on an edge using the algorithm by Zemel (1984) which yields an $O(nm)$ algorithm for the maxisum dispersion problem on general networks.

The methods by Church and Garfinkel (1978) and Minieka (1983) yield $O(n^2)$ algorithms when the network is a tree. Ting (1984) presented an $O(n)$ algorithm for the single facility maxisum problem on tree networks. The algorithm is elegant and makes use of a special representation of the tree network.

6.1.2 Multiple Facilities:

The single facility models for the maxisum dispersion problem may be extended to locate multiple facilities on a network. Nevertheless, when the facilities are located with respect to existing customers without considering facility interactions, the model would locate all facilities on a single point, which solves the single facility model. It is obvious that such a model does not make sense because usually undesirable facilities affect each other and locating them on a single point would harm both the new facilities and the existing facilities severely. Multiple facility maxisum models locate p facilities on a network such that the total weighted distance between the pairs of new facilities plus the total distance between new and existing facilities are maximized. Such problems are also referred to as p-maxian problems (Erkut, Baptie, and Hohenbalken, 1990). The p-maxian problem can be expressed as follows: Given a network $N = (V, E)$ find a set $X^* = \{x_1^*, \ldots, x_p^*\}$ in N such that

$$X^* \in \arg \max_{X \in N, |X| = p} \sum_i \sum_j \alpha_{ij} d(v_i, x_j) + \sum_i \sum_j \beta_{ij} d(x_i, x_j)$$

When $\alpha_{ij} = 0$, i.e. there exists no existing facilities in the system, the problem is referred to as the p-defense-sum problem or the maxisum dispersion problem (Kuby, 1987). We will refer to the problems involving existing facilities as p-maxian problems and problems not involving existing facilities as p-defense-sum problems from this point on.

It is stated by Erkut, Baptie, and Hohenbalken (1990) that the p-defense-sum problem is proven to be *NP*-hard by Hansen and Moon (1988). Tamir (1991) also showed that the p-defense-sum problem is *NP*-hard on general graphs even if the graph is as simple as a single edge, by reduction from the Maximum Cut Problem. He also showed that the unweighted or homogeneous p-defense-sum problem in which $\beta_{ij} = 1$, is *NP*-hard on general graphs via reducing the problem to the Independent Set Problem.

The p-defense-sum problem has been studied by Kuby (1987). The author formulated the problem as a binary IP for discrete cases, which involve the vertex-restricted network problems as well. He used a standard solver to solve the model for instances with 25 nodes and 10 new facilities. Although Kuby (1987) is important because it involves the first mathematical formulation of the model, it is not practically possible to solve large instances of the problem using standard solvers.

Erkut, Baptie, and Hohenbalken (1990) presented a very effective branch-and-bound technique to solve the vertex-restricted p-maxian problem. They provided upper bounds via solving a set of easy knapsack problems and lower bounds using a simple but very effective heuristic. The algorithm can be used to solve the p-defense-sum problem as well.

For the absolute p-maxian problem, Tamir (1991) provided an $O(m^p n)$ algorithm, which solves a local problem on every p subset of edges using the algorithm provided by Zemel (1984).

A special case of p-defense-sum problem with $\beta_{ij} = 1$, referred to as the homogeneous problem, is studied by Ting (1988) and Hansen and Moon (1988). They presented $O(n^2)$ algorithms for the problem on tree networks for absolute and vertex-restricted cases.

Similarly, a special case of the p-maxian problem which is referred to as the homogeneous problem and satisfies $\alpha_{ij} = \alpha_i$ for all j and $\beta_{ij} = 1 \ \forall\ i, j$ is studied by Tamir (1991) on tree networks. An $O(np)$ algorithm is presented for the problem which is also an improvement for the homogeneous *p*-defense-sum algorithms mentioned above. Tamir (1991) stated that the complexity of the algorithm further reduces to *O(n)* when the tree is a star using the result by Ibaraki and Katoh (1988).

The maxisum problems on general networks are very hard problems, so heuristic methods are used for the general graphs. As we mentioned above, Erkut, Baptie, and Hohenbalken (1990) provided a greedy heuristic for the discrete case, which makes use of a neighborhood search at each iteration. This heuristic is very simple and successful which is surprising with respect to the hardness of the problem. Ravi, Rosenkrantz, and Tayi (1994) showed that this heuristic is a 4-approxiamation algorithm and no algorithm which has a performance guarantee less than 2 can be devised for the problem unless $P = NP$. Kincaid (1992) presented two metaheuristics for the problem, namely, Simulated Annealing and Tabu Search. He also presented a computational experiment on the best values of the parameters to be used in the design of the heuristics. It is observed empirically that Tabu Search provides better results than the Simulated Annealing or greedy algorithm of Erkut et. al (1990).

6.2 Maximin Facility Location on Networks:

6.2.1 Single Facility:

The single facility maximin problem is to locate a facility on a network such that the minimum weighted distance from the new facility to the existing facilities are maximized. The problem can be expressed as follows: Given a network $N = (V, E)$ and a set of existing facilities F, find x^* in N such that $x^* \in \arg\max_{x \in N} \min_{i \in F} w_i d(i,x)$.

When the existing facilities are on the vertices of the network and the weights w_i are equal to one, the problem is trivially solved in *O(m)* time by locating the facility at the midpoint of the longest edge in the network. When the existing facilities are on the pendant vertices of a tree and weights are again equal, the problem is solved in *O(n)* time by a simple algorithm by Moon (1989).

The weighted maximin problem on a path network was solved in $O(n^3)$ time by Drezner and Wesolowsky (1985). This bound is improved by Tamir (1988) to *O(nlogn)* using special data structures. Burkard, Dollani, Lin, and Rote (1998) provided a linear time algorithm for this problem. The algorithm is based on the division of the objective function into two parts which are piecewise linear functions along the path. An algorithm to solve the problem on star networks in *O(n)* time is also provided. In this algorithm, a linear program is solved which is developed based on observations obtained for path networks. It is also shown that the problem can be solved in *O(n+blogn)* time in an extended star where *b* is the number of branches (an extended star is a tree which has a single vertex with degree greater than 2 and the paths from this vertex to the pendant vertices are called the branches of the tree).

Tamir (1991) showed that for a tree network the objective function value of a 1-maximin problem is an element of the following finite set: $R = \{ \frac{d(v_i, v_j)}{w_i^{-1} + w_j^{-1}} \big| 1 \le i \ne j \le n \} \}$ The 1-maximin problem may be solved by the related anticover problem, in that the objective function value of the 1-maximin problem is the largest element z^* in this set for which there exists a point x in N which is at least z^*/α_i distance away from each vertex v_i. The problem is solved in $O(nlog^2n)$ by constructing a binary search on the members of this set and an $O(n)$ algorithm for the anticover problem. The relationship between the anticenter and anticover problems is analogous to the relation between center and cover problems. Burkard, Dolloni, Lin, and Rote (1998) provide an $O(nlogn)$ algorithm which is a modification of the algorithm by Tamir (1991). It is shown that, when the weights w_i are equal to each other the problem is solved in linear time, by Burkard, Dolloni, Lin, and Rote (1998).

When the existing facilities are on the vertices of the network and the weights are not all equal to one at the same time, the problem in solved in $O(nm)$ time independently by Melachrinoudis and Zhang (1999) and Berman and Drezner (2000). The algorithm by Berman and Drezner is simple and depends on finding the local maximum on each edge via solving an easy LP.

Welch and Salhi (1997) proposed a different formulation for an undesirable facility spreading air pollution to its surroundings by using a pollution dispersion model where the relationship between pollution levels, distance and wind strength is considered. The usage of pollution dispersion model had first appeared in Karkazis (1991). However, his model minimizes the sum of dispersed air pollution in the plane whereas Welch and Salhi's (1997) model minimizes the maximum amount of air pollution spread on a network. They also placed a minimum distance constraint to prevent locating the facility in the immediate neighborhood of a node with a relatively smaller weight.

6.2.2 Multiple Facilities:

The single facility maximin problem on networks can be generalized to the case of multiple facilities. Due to similar reasons for the maxisum problem, the multiple facilities maximin problem involves not only the interaction between new and existing facilities but also the interaction between pairs of facilities. The multiple facilities maximin problem is the problem of selecting p points on a network such that the minimum weighted distance between pairs of new facilities and between new and old facilities is maximized. Formally, the problem may be stated as follows: Given a network $N = (V, E)$, find a set $X^* = \{x_1^*,...,x_p^*\}$ in N such that

$$X^* \in \arg \max_{X \in N, |X| = p} \min\{\min_{i,j} \alpha_{ij} d(v_i, x_j), \min_{i,j} \beta_{ij} d(x_i, x_j)\}$$

When $\beta_{ij}=\infty$, the interaction between pairs of facilities are not considered and new facilities are located such that the minimum weighted distance from the existing facilities is maximized. This problem is referred to as anti-p-center by Klein and Kincaid (1994). The problem is polynomially solvable and an $O(nm^2)$ algorithm is provided for the discrete case for all values of p.

The multifacility maximin problem is widely investigated when there are no existing facilities and the objective is to disperse the new facilities as much as possible on the network. This problem is called the p-dispersion problem. The p-dispersion problem is related to the well-known *p*-center problem. It is shown that these two problems are strongly dual to each other on tree networks, i.e. the objective function value of a solution to a *p*-dispersion problem is twice the objective function value of a (*p*-1)-center problem on the same tree network (Shier, 1977). The duality is weak for general networks, in that twice the objective function value of a (*p*-1)-center problem provides an upper bound for the objective function value of a solution to a *p*-dispersion problem (Tamir, 1991). The duality results are extended to problems with nonlinear cost functions by Tansel, Francis, Lowe, and Chen (1982). The results are also valid when the existing and new facilities are restricted to discrete subsets of the network as shown by Chandrasekaran and Tamir (1980, 1982). The *p*-dispersion problem is equivalent to the *r*-separation problem. Given a real *r*, the *r*-separation problem is the problem of finding a feasible set of *p* points, which are at least *r* units apart from each other. The *p*-dispersion problem can be solved by solving a series of *r*-separation problems.

Erkut (1990) proved that the discrete (vertex-restricted) *p*-dispersion problem is *NP*-Hard on general networks via reduction from the Clique Problem. The discrete *p*-dispersion problem was first solved by Kuby (1987). Kuby (1987) provided an IP formulation and solved the problem using a standard solver. Erkut (1990) provided a branch-and-bound algorithm for this problem using a heuristic to obtain lower bounds. This is a two-stage heuristic that constructs a greedy solution in the first stage and improves the solution using neighborhood search in the second stage. Ravi, Rosenkrantz, and Tayi (1994) show that this heuristic in fact has a performance guarantee of 2 and devising an algorithm with a better performance guarantee proves $P = NP$. White (1991) provides a "First Point Outside the Neighborhood" heuristic (FPON) for the problem. In this heuristic, a solution is constructed for a real number r such that in each iteration the first point in the list that is at least r units away from the preselected points is added to the solution. The number r is changed until a solution of p units is found. White (1991) shows that this heuristic has a performance guarantee of 2 for certain values of p and 3 for all p. Erkut, Ulkusal, and Yenicerioglu (1994) provides a valuable survey and computational study based on their experiments with 10 different heuristics. It is observed that Simulated Annealing is very effective for the p-dispersion problem and a combination of several heuristics may improve the results severely. Simulated Annealing and Tabu Search heuristics are used by Kincaid (1992) for the discrete p-dispersion problem and yield very good solutions. The heuristics for these problems are very successful because there exists multiple optima for the maximin facility location problems and the chance to stop at an optimal solution is very high compared to other models such as maxisum facility location models.

The continuous (absolute) p-dispersion problem on general networks are *NP*-hard even if the problem is homogeneous, i.e. $\beta_{ij}=1$ (Tamir, 1991). It is also shown that if there exists a polynomial time ε-approximation with $\varepsilon<2/3$, then $P = NP$ (Tamir, 1991). Tamir (1991) provides an approximation algorithm with a performance guarantee of 1/2 for the homogeneous problem. The algorithm is very simple since it begins with an arbitrary point and selects the farthest point from the preselected points at each iteration.

The maximin facility location problems that involve existing facilities are called the p-anti-center-dispersion problems (Erkut, 1990) in order to distinguish them from the p-dispersion problem that do not involve existing facilities. The absolute p-anti-center-dispersion problems are *NP*-hard on general graphs even if the graph consists of a single edge (Tamir, 1991). The unweighted p-anti-center-dispersion problems can be solved by solving a series of r-anticover problems. Given r, r-anticover problem finds the maximum number of points on a network such that the points are at least r units apart from each other and from existing facilities. The r-anticover problem is solved by Moon and Goldman (1989) on tree networks but this algorithm is very complicated to be used in a solution procedure. Chandrasekaran and Daughety (1981) provide an *O(nolgn)* algorithm for the problem and solve the related anticenter problem in polynomial time. Tamir (1991) presents a linear time algorithm for the r-anticover problem that yields even more efficient algorithms for tree networks. The discrete p-anti-center-dispersion problem on a general graph is solved by Erkut (1990) using a branch-and-bound algorithm and efficient bounding procedures.

6.3 Other Single Objective Models:

Although the literature on single objective undesirable facility location on networks is dominated by maxisum and maximin objectives, other models are devised to handle undesirability. The maximin model is a conservative approach that maximizes the worst-case performance and the maxisum model can result in a solution in which some undesirable facilities are located in the neighborhood of a community center or another facilty. Based on similar arguments, new models of dispersion are necessary to handle a broad range of situations.

The p-defense problem, first defined by Moon and Chaudry (1984), can be expressed as follows: Given $N = (V, E)$, find a set $X^* = \{x_1^*,...,x_p^*\}$ in N such that $X^* \in \arg\max_{X \in N, |X|=p} \sum_i \min_{j \neq i} \beta_{ij} d(x_i, x_j)$. This model is not as conservative as the p-dispersion problem and do not allow new facilities to be very close to each other like the p-defense-sum problem. This model is formulated as an IP by Erkut and Neuman (1991) and solved using a branch-and-bound technique. Tamir (1991) solved this problem on tree networks in $O(p^2n^3)$ time. Erkut and Neuman (1991) defined another new problem, which will be referred to as the p-dispersion-sum problem. This problem is defined as follows: Given $N = (V, E)$, find a set $X^* = \{x_1^*,...,x_p^*\}$ in N such that $X^* \in \arg\max_{X \in N, |X|=p} \min_i \sum_j \beta_{ij} d(x_i, x_j)$. This problem is also solved by Erkut and Neuman (1991) by branch-and-bound. A two-stage (greedy-pair wise interchange) heuristic is devised for both models, which is very effective in terms of producing optimal and near optimal solutions in a short time. Although these models are devised for cases that do not involve existing facilities, the models can easily be extended to include existing facilities and the algorithms devised can be used for these extended models.

Another model, which is an extension of the p-defense-sum model, which includes existing facilities, is given by Ting (1988). This model is different from the p-maxian model because it incorporates only the distance between an existing facility and the nearest new facility as opposed to the p-maxian problem which includes distances between every pair of new and existing facilities. The model can be expressed as follows: Given $N = (V, E)$, find a set $X^* = \{x_1^*,...,x_p^*\}$ in N such that $X^* \in$

$\arg \max_{X \in N, |X|=p} \sum_i \alpha_i D(v_i, X) + \sum_i \sum_j \beta_{ij} d(x_i, x_j)$. Tamir (1991) presents an $O(p^2 n^2)$ algorithm for this problem on trees.

Chandra and Halldorsson (2000) suggests a new unified model, which includes the previous models (*p*-dispersion, *p*-defense-sum), the models introduced in this section (*p*-defense, *p*-dispersion-sum) and gives rise to new models. The paper presents new approximation algorithms and performance analysis for a broad range of problems.

Most of the literature on location theory is based on certain assumptions on the network parameters. The demand weights and distances are assumed to be nonnegative constants which is considered to be the normal interpretation. Nevertheless, allowing negative weights on a network gives rise to flexible models in which a facility may be considered as a desirable facility for some existing facilities whereas it may be undesirable for some others. For example, an airport is highly desirable for an industrial organization that imports and exports a high volume of goods, but it is undesirable for a householder. Burkard and Krarup (1998) studied the 1-median problem with positive/negative weights on a cactus and developed an $O(n)$ algorithm for the problem. The algorithm makes use of the block diagram of the graph and finds the local minimum in each block. When the objective is to minimize the sum of minimum weighted distances of existing facilities from the new facilities, the 2-median of a pos/neg weighted tree, star and path is found in $O(n^2)$, $O(nlogn)$, and $O(n)$ times, respectively by Burkard, Cela, and Dollani (2000). Algorithms devised for the *p*-median problem are also provided in this paper, which may be quite intuitive for the interested reader.

6.4 Minimum Covering Problem on Networks:

The minimum covering problem is to find a location for a new facility on a network such that the total weight of existing facilities within a specified distance is minimized. Formally, given a real number *r*, find a point x^* in *N* such that $x^* \in \arg\min_{x \in N} \sum_{i \in N(x,r)} w_i$ where $N(x, r) = \{y: d(x, y) \leq r\}$.

The minimum covering problem is trivial if and only if the longest edge in the network is at least twice the covering distance long (2*r*). In this case the optimum location of the facility will be on the mid-point of such an edge. The first paper to model the location of an obnoxious facility on a network using the minimum-covering criterion is by Sung and Joo (1993). They state that the objective function of the model is continuous piecewise concave and there is at least one optimum point. Using this property an efficient solution algorithm is derived. Then being unaware of Sung and Joo (1994), Berman, Drezner and Wesolowsky (1996) studied the same problem. Their paper includes an analysis of the problem, identification of special cases where the problem is easily solved, an algorithm to solve the problem in general based on identifying the optimal segment on each edge, and a sensitivity analysis with respect to the covering distance *r*.

6.5 Multiobjective Models:

The undesirable facility location is multiobjective in nature. The single facility models we have considered above aim to locate the undesirable facilities as far away from the population centers and from each other as possible. Nevertheless, minimizing the undesirable effects usually results in very high transportation costs and travel times. Thus the solutions to the single objective models are usually impractical for applications. For undesirable facilities location, multiobjective models are used in order to locate the facilities safely and cheaply.

Multiobjective models are also widely used to locate semi-desirable facilities, whose desirable and undesirable effects are perceived to be equal. Semi-desirable facility models are necessarily multiobjective in nature because the facilities must be sufficiently far away to guarantee safety and near enough to guarantee accessibility. Airports are typical examples for semi-desirable facilities.

Ratick and White (1988) was probably the first who developed a multiobjective model for locating undesirable facilities. Their model included facility size and risk factors as well as cost. They developed an IP model of the problem and solved the problem using a standart solver. This model is important because it provides valuable insight on undesirable facility location.

Zhang and Melachrinoudis (2001) studied a biobjective maximin-maxisum objective model. Both objective functions are piecewise linear and concave. The edges are divided into segments, which are analogous to Hooker's treelike segments (1986) and an algorithm based on elimination of inefficient edge segments is proposed. The proposed algorithm runs in $O(n^2 logn)$ time for unweighted trees, and in $max\{O(n^3), O(|R|logn)\}$ for weighted trees, where $|R|$ is the number of intersection points of the line segments. On a general network the algorithm runs in

$max\{O(mn^2),O((mn+|R|)log(mn))\}$ for both the unweighted and weighted cases.

Hamacher, Labbé, Nickel and Skriver (2002) proposed a multiobjective model for the semi-desirable facilities. They considered *q* minisum and / or maxisum objectives. They stated that the objective functions are all piecewise linear, and they partitioned the network into segments where the objective functions are linear. The solution method is based on pairwise comparisons of these segments. They proposed that this algorithm could also be applied to maximin-minimax biobjective problem. They also considered the biobjective version of the problem (q=2) as a special case and provided an efficient algorithm. The problem is also considered on directed networks.

Skriver and Andersen (2000) provided a general biobjective semi-desirable facility location model which minimizes the transportation cost and obnoxiousness at the same time for both the planar and the network problems. For the network case they proposed an algorithm, which is on fact a modification of the BSSS (Big Square Small Square) method. They modified the BSSS method by dividing edges into sub-edges instead of dividing the big squares into small squares.

The literature on undesirable facilty location is presented in the following tables:

Table 13: Literature on Single Facility Maxisum Facility Location on Networks

Author	***Year***	***Summary***
Church and Garfinkel	1978	Finite dominating set $O(n^3)$ algorithm for *N* $O(n^2)$ algorithm for *T*
Minieka	1978	Characterized the solution for directed and undirected *N* $O(n^2)$ algorithm for *T*
Tamir	1991	$O(nm)$ algorithm for *N*
Ting	1984	$O(n)$ algorithm for *T*

Table 14: Literature on Multiple Facility Maxisum Facility Location on Networks

Author	***Year***	***Problem***	***Summary***
Hansen and Moon	1988	*p*-defense-sum	*NP*-hard
Tamir	1991	*p*-defense-sum	*NP*-hard even on a single edge
Kubys	1987	Vertex-restricted *p*-defense-sum	MP Standard solver
Erkut, Baptie, and Hohenbalken	1990	Vertex-restricted *p*-maxian	B&B and heuristics
Tamir	1991	Absolute *p*-maxian	$O(m^p n)$ algorithm
Ting Hansen and Moon	1988 1988	Homogeneous *p*-defense-sum on trees	$O(n^2)$ algorithm
Tamir	1991	Homogeneous *p*-maxian on trees	$O(np)$ algorithm for *T* $O(np)$ algorithm for star networks
Ravi, Rosenkrantz, and Tayi	1994	Maxisum problem on general networks	Best heuristic can be 2-approximation
Kincaid	1992	Maxisum problem on general networks	Simulated Annealing and Tabu Search

Table 15: Literature on Single Facility Maximin Facility Location on Networks

Author	***Year***	***Problem***	***Summary***
Moon	1989	Unweighted maximin problem when customers are on leafs of a tree	$O(n)$algorithm
Drezner and Wesolowsky	1985	Weighted maximin problem on a path	$O(n^3)$ algorithm
Tamir	1988	Weighted maximin problem on a path	$O(nlogn)$ algorithm
Burkard, Dollani, Lin, and Rote	1998	Weighted maximin problem on a star	$O(n)$ algorithm
Tamir	1991	Weighted maximin problem on trees	$O(nlog^2n)$ algorithm
Burkard, Dolloni, Lin, and Rote	1998	Maximin problem on trees	$O(nlogn)$ –weighted $O(n)$ –unweighted
Melachrinoudis and Zhang Berman and Drezner	1999 2000	Weighted maximin problem on general networks	$O(nm)$ algorithm

Table 16: Literature on Multiple Facility Maximin Facility Location on Networks

Author	***Year***	***Problem***	***Summary***
Klein and Kincaid	1994	Vertex-restricted anti-*p*-center	$O(nm^2)$ algorithm for all *p*
Shier	1977	*p*-dispersion on trees	Duality with *p*-center
Tansel, Francis, and Lowe	1982	*p*-dispersion on trees with nonlinear costs	Duality results
Tamir	1991	*p*-dispersion on *N*	Duality with *p*-center
Chandrasekaran and Tamir	1980 1982	Vertex-restricted *p*-dispersion	Duality with *p*-center Solving a series of *r*-separation problems
Erkut	1990	Vertex-restricted *p*-	*NP*-Hard on general

		dispersion	networks
Kuby	1987	Vertex-restricted *p*-dispersion	IP formulation
Erkut	1990	Vertex-restricted *p*-dispersion	B&B and heuristics
Ravi, Rosenkrantz, and Tayi	1994	Vertex-restricted *p*-dispersion	ε-approximation ($\varepsilon<2$) proves $P = NP$
White	1991	Vertex-restricted *p*-dispersion	First Point Outside Neighborhood heuristic
Erkut, Ulkusal, and Yenicerioglu	1994	Vertex-restricted *p*-dispersion	Computational survey on heuristics
Kincaid	1992	Vertex-restricted *p*-dispersion	Simulated Annealing and Tabu Search
Tamir	1991	Absolute *p*-dispersion	*NP*-hard ½-approximation alg.
Tamir	1991	Absolute *p*-anti-center-dispersion	*NP*-hard on single edge Solving a series of *r*-anticover problems
Chandrasekaran and Daughety Tamir	1981 1991	Absolute *p*-anti-center-dispersion on trees	Polynomial algorithm
Erkut	1990	Discrete *p*-anti-center-dispersion	B&B and efficient bounds

Chapter 7

STRUCTURE LOCATION PROBLEMS ON NETWORKS:

We have studied in detail point location problems on networks in the previous chapters. Although many facilities are very small compared to the underlying structure in which they are placed and the point facility assumption is valid for these facilities, there are other situations in which facilities of large sizes are placed. Usually these facilities are network structures such as paths, cycles, trees, subnetworks, etc. and the facilities they represent are some kind of transportation or communication routes. We refer to these problems as "Structure Location Problems". An alternative term is "Extensive Facility Location" used by Mesa and Boffey (1996).

The literature on structure location problems is somewhat out of order because the problems in this area are usually studied as vehicle routing problems and the location aspect of the problems are undiscovered. Beasley and Nascimento (1996) define a Vehicle Routing-Allocation Problem that involves many of the problems we cover in our survey as special cases. This paper constitutes a framework in understanding the vehicle routing and location-allocation aspects of Structure Location Problems. Mesa and Boffey (1996) and Hakimi, Schmeichel, and Labbè (1993) provide surveys together with classification schemes on the structure location problems and we provide an extension on these surveys that include additional problem types as well as more recent papers. We investigate problems that locate paths, trees, and cycles on a network. Structure location problems are often biobjective in nature. Usually, the first objective is the minimization / maximization of the length of the facility to be placed whereas the second objective varies from problem to problem. The second objective can be a covering type objective in that the facility to be located should be within a prespecified distance of customers while the number of customers covered is maximized or minimized. The objective may also be the minimization / maximization of the total distance of customers to the facility, which we will refer to as the distance objective, or it may be the minimization / maximiation of the maximum distance of customers to the facility, which we will refer to as the eccentricity objective. There are also single objective problems in which the length of the structure is fixed and one of three types of objectives mentioned above is used as the single objective.

7.1 Covering Objective:

7.1.1 Covering Path Problems:

The Shortest Covering Path Problem (SCP) is the problem of placing a path-shaped facility between two specified points on the network such that all nodes are within a specified distance form the facility and the length of the facility is minimized. SCP is a single objective problem first defined by Current, Cohon and ReVelle (1984). This problem is observed to be a synthesis of the well known Shortest Path and Set-Cover Location Problems. Current, Cohon and ReVelle (1984) presented an ILP formulation of the problem and a branch-and-cut algorithm that utilizes subtour elimination constraints. Two example problems are also solved. Because of the clarity and conciseness of their exposition, this paper is highly recommended for researchers who have limited familiarity with branch-and-bound and constraint relaxation techniques. Later, algorithms with better computational performance are presented for this problem by Current, Pirkul, and Roland (1994).

Current, ReVelle, and Cohon (1985) introduced the Maximal Covering Shortest Path Problem (MCSP) that is defined to be the biobjective problem of finding a path between a given source and a destination so as to minimize the path length and maximize the total demand that is covered by the path. A demand is covered if it is on the path or within a prespecified distance of the path. A special case is the Maximal Population Shortest Path (MPSP) problem where the covering distance is assumed to be zero. Boffey and Narula (1998) studied the 2-MPSP where, instead of one path, two vertex disjoint paths are located between a source and a destination. They presented two solution procedures based on the ILP formulation of the problem. One of the procedures is the weighting method (Lagrangean relaxation) and the other is the k-shortest path method. They did not implement any of the procedures they have proposed but presented valuable modeling insights on the possible extensions of the problem.

The Minimum-Covering Shortest Path Problem (MinCSP) is introduced by Current, Revelle, and Cohon (1988). MinCSP aims to locate a path between two prespecified nodes in the network. The objectives are simultaneous minimization of the length of the tour and of the total demand covered by the path. A node is covered if it lies within a given distance from a closest node of the path. MinCSP has many applications in hazardous materials transportation. Current, ReVelle, and Cohon (1988) presented an ILP formulation of the problem and solved a test problem using the weighting method traditionally used in multiobjective optimization. In their discussion of the related literature, they provided many useful comments on various solution techniques of the problem.

7.1.2 Covering Tour Problems:

Current and Schilling (1989) formulated the Covering Tour Problem (CTP). CTP is defined as follows: Given a network $G = (V \cup W, E)$ where V is the set of nodes that can be visited and W is the set of nodes that must be covered and a set $T \subseteq V$, where T is the set of nodes that must be visited, CTP determines a minimum length tour or a Hamiltonian cycle over a subset of V such that the tour contains all vertices of T and every vertex of W is covered by the tour, i.e. lies within a prespecified distance from a closest vertex of the tour. The problem has many application areas in distribution and transportation models such as post box placement in a neighbourhood. Gendreau, Laporte, and Semet (1997) formulated the CTP as an ILP, analysed the corresponding polytope, and solved the integer formulation by a branch-and-cut algorithm. Their algorithm works in a reasonable amount of computation time even for very large networks consisting of 600 nodes in total with 100 potential sites. The paper also contains an efficient heuristic that provides results within 3% of optimal objective value. This study is notable because the authors managed to find the exact solution of such a large problem. Moreover, the identified properties of the polytope and cutting planes that are generated may be useful in many related problems. Maniezzo, Baldacci, Boschetti, and Zamboni (1999) provided another ILP formulation and applied three metaheuristics to the problem. Motta, Ochi, and Martinhon (2001) presented a reduction technique, which reduces the size of the problem significantly. This technique can be useful in solving the problem optimally or approximately. The Covering Tour model has application areas in the planning of daily routes of mobile emergency services such as police and ambulance patrols. As an example, this model has been successfully used in a mobile health care system in Ghana by Hodgson, Laporte, and Semet (1996).

Current and Schilling (1994) presented the biobjective Maximal Covering Tour Problem (MCTP). In MCTP, a tour passing through p nodes is found where one objective is to minimize the length of the tour and the other is to maximize the total demand within some prespecified travel distance from a tour node. ILP formulation of the problem is presented in the paper.

Labbè, Laporte, and Soriano (1998) studied the Cycle Cover Problem (CCP), which is defined to be the problem of covering all edges of a graph with simple cycles consisting of at least three edges so as to minimize the total length of cycles. They provided a lower bounding procedure and six heuristics based on a relaxation of CCP that results in well-known Chinese Postman Problem. Their heuristics produce optimal or near-optimal solutions for the 100 test problems in a very short time.

7.1.3 Covering Tree Problems:

STRUCTURE LOCATION PROBLEMS ON NETWORKS

Kim, Lowe, Ward, and Francis (1989) considered the general problem of finding a minimum length covering subgraph of a network. They found that the minimum-covering subgraph of any network is always a subtree. So, they renamed the problem as the Subtree *r*-Cover Problem. They emphasized that the problem is *NP*-Hard for general networks, so they focused on special networks. They provided efficient solution methods via exploiting the special structure of the network whenever possible. They devised a generic algorithm that exploits the structure and specialized this algorithm to a polynomial time algorithm for cactus graphs.

Kim, Lowe, Ward, and Francis (1990) studied the Subtree *r*-Cover Problem on a tree network. They observed that the problem is very close to the Point *r*-Cover Problem of Tansel, Francis, Lowe and Chen (1982). In fact, an optimal minimum cost covering subtree of a tree can be found by modifying the point *r*-cover algorithm of Tansel et al. (1982). The algorithm simply finds all point covers and constructs a subtree whose pendant vertices are the point covers. The modified algorithm runs in $O(m^2)$ time where m is the number of edges. The paper also contains a proof of optimality based on duality theorems.

Kim, Lowe, Tamir, and Ward (1996) studied the problem of locating a tree-shaped central facility on a tree network and defined two covering tree problems: Direct Subtree Covering Problem (DSCP) and Indirect Subtree Covering Problem (ISCP). In DSCP a customer is covered if it is a member of the facility and each uncovered customer pays a penalty. The objective is to minimize the total cost associated with the length of the facility and the sum of the penalties. An $O(n)$ algorithm is given, based on dynamic programming, for the discrete case. In the Indirect Subtree Covering problem, a customer is covered if it is within a prespecified distance from the facility. Again, each uncovered customer pays a penalty. The problem is solved in $O(n\log^2 n)$ time for the discrete case. The continuous cases for these problems in which subedges are allowed can be handled by adding vertices corresponding to critical points.

Another related problem is the Maximal Direct Covering Tree problem (MDCTP) introduced by Hutson and ReVelle (1989). They defined MDCTP to be the problem of identifying a subtree of a given tree network that minimizes the total cost of the subtree and maximizes the total demand located at nodes covered by the subtree. Church and Current (1993) later studied the MDCTP and gave an $O(n^2)$ exact algorithm based on their ILP formulation. They also extended their formulation to various cases with side constraints. They were able to solve test instances with 35 nodes in less than 5 CPU seconds.

7.2 Distance Objective:

7.2.1 Distance Path Problems:

Minieka (1985) studied the problem of finding an optimal location of a path-shaped facility of a specified size in a tree network under minimizing distance sum and maximizing distance sum objectives. The structure to be located may contain partial arcs. The minimum distance sum path (may be referred to as the median path in analogy to the point median problem) and the maximum distance sum paths are located in polynomial time.

The Median Shortest Path Problem (MSPP) is a bi-criteria problem introduced by Current, Revelle, and Cohon (1987). MSPP aims to optimally locate a path between two prespecified nodes. One criterion is to minimize the length of the path and the other one is to minimize the total travel time required for nodes not on the path to reach a closest node on the path. Current et al. (1987) gave an ILP formulation of the problem. They introduced an algorithm called MONET which is basically a complete enumeration algorithm based on the solution of the k-shortest path problem. They identified nondominated solutions of a certain test problem using both MONET and an exact branch-and-bound algorithm. Their results show that, within a fixed amount of computation time, MONET is able to generate many more nondominated solutions than the competing branch-and-bound algorithm. Although the results are promising, more experimentation is needed to reach a firmer conclusion.

When the objective is to minimize or maximize the total distance of customers to the facility, locating multiple path-shaped facilities with a given total length on a tree network is shown to be polynomial for fixed number of facilities; it is *NP*-hard when the number of facilities is variable and partial arcs are not allowed. Moreover, the problem is *NP*-hard on general graphs for any number of facilities and even if partial arcs are allowed. The case with partial arcs on tree graph is an *NP*-open problem (Hakimi, Schemeichel, and Labbé, 1993).

7.2.2 Distance Tour Problems:

The bi-criteria Median Tour Problem (MTP), introduced by Current and Schilling (1994), is the "tour version" of the MSPP. In MTP a tour passing through p nodes must be constructed so as to minimize the total tour length and the total travel distance of the remaining customers not on the tour to their closest nodes in the tour. Current and Schilling (1994) emphasized that the problem has many application areas including, the design of mobile service delivery systems, overnight parcel delivery, and distributed computer networks. They provided an ILP formulation of the problem. They provided an approximation algorithm for finding the efficient frontier of the problem and applied the solution procedure to a 681-node network. The heuristic starts with a feasible solution, usually optimal for one objective, and improves the solution with respect to the remaining objective.

Labbé, Laporte, Rodriquez-Martin, and González (1999) solved two versions of the MTP. The first problem seeks to minimize the total cost of the tour and the total distance of customers to the tour whereas the second problem seeks to minimize the tour length subject to an upper bound on the total distance of customers to the tour. Efficient branch-and-cut algorithms and heuristics are provided for both problems. Foulds, Wilson, and Yamaguchi (2000) provided a branch-and-bound algorithm for the problem depending on subtour elimination constraints and L*P*-relaxation. Moreno Pérez, Moreno-Vega and Rodríguez Martín (2002) provided a Tabu Search algorithm for the problem while Renaud, Boctor, and Laporte (2004) provided two heuristics (one greedy and the other being Genetic Algorithm) for this problem.

7.2.3 Distance Tree Problems:

Minieka (1985) studied the problem of finding an optimal location of a tree-shaped facility of a specified size in a tree network under minimizing distance sum and maximizing distance sum objectives. The structure to be located may contain partial arcs. The tree which minimizes the distance sum is located in polynomial time whereas locating the tree which maximizes the distance sum is *NP*-Hard as proved by Hakimi, Schmeichel, and Labbè (1993). When the length of the tree is not specified but restricted to be smaller than a given number the Tree Median Problem arises. This problem is studied by Shigeno and Shioura (1995) who solved the problem in which partial arcs are allowed in linear time by formulating the problem as a continuous Knapsack Problem. The case in which the partial arcs is not allowed is *NP*-Hard and can be formulated as a 0/1-Knapsack Problem. Approximation algorithms can be devised using this formulation for the problem.

Kim, Lowe, Tamir, and Ward (1996) presented the single objective Median Subtree Location Problem (MSLP) in which the length of the tree-shaped facility plus the total distance form the customers to the facility is minimized. On a tree network, the MSLP is solved in $O(n)$ time when the subtree do not contain partial arcs. George and ReVelle (2003) solved the biobjective MSLP on tree networks, where the first objective is the minimization of the tree length whereas the second is the minimization of the total weighted distance between the customers and tree-shaped facility. Although the single objective case is easy to solve the biobjective problem is harder. The authors present ILP formulations of the problem and solved the problem using branch-and-bound and L*P*-relaxation techniques.

The problem of locating multiple tree-shaped facilities is considered by Hakimi, Schmeichel, and Labbé (1993) and it is shown that the problem is *NP*-hard even on tree networks for both minimizing and maximizing the distance sum objectives.

7.3 Eccentricity Objective:

Although the eccentricity objective is widely used in single point location problems such as center problems, anticenter problems, dispersion problems, etc., few studies exist in structure location theory that deal with this objective.

7.3.1 Eccentricity Path Problems:

Minieka (1985) provided polynomial time algorithms for finding the locations of path-shaped facilities of a specified length on a tree network that minimizes or maximizes the maximum distance of any customer to the facility. The solution is very simple for the minimum ecentricity case and follows from the observation that the minimum eccentricity path (or tree) must include the center of the tree. The problem is extended to the multiple facilities case by Tamir and Lowe (1990) who refer the problem as the Generalized p-Forest Problem. They provided a polynomial algorithm for finding the location of p tree shaped facilities on a tree network where the objective is minimizing or maximizing the maximum distance traveled by the customers to a nearest facility and p is fixed. When the facilities are located on a general network, the problem is *NP*-Hard (Hakimi, Schmeichel, and Labbé, 1993).

7.3.2 Eccentricity Cycle Problems:

The Cycle Center Problem (CCP) is defined by Foulds, Wilson, and Yamaguchi (2000) as follows: Given a network $G = (V,E)$, identify a cycle C in G that minimizes the maximum of the distances between any vertex not in C to a closest vertex in C, such that C is of minimal length among all such cycles. An ILP formulation of the problem is presented by the authors and solved by branch-and-bound and subtour elimination constraints. Although instances up to 25 nodes are solved, an efficient heuristic is needed for larger instances.

7.3.3 Eccentricity Tree Problems:

Shioura and Shigeno (1995, 1997) studied the Tree Center Problem, which is the problem of finding a subtree of a network such that the maximum distance from other vertices of the network to the subtree is minimized provided that the length of the subtree in smaller than a prespecified value. They have formulated the problem as an ILP and showed that it is equal to a Bottleneck Knapsack Problem when the underlying graph is a tree network. The problem is solved in $O(n)$ time when the Subtree may or may not contain subedges. The case in which the subtree cannot contain subedges is also solved by Minieka (1985) based on the observation that the Subtree always contains the center of the problem. The problem is extended to the multiple facilities case by Tamir and Lowe (1990) and a polynomial time algorithm is provided for locating p tree-shaped facilities on a tree network with the objective o minimization of the maximum distance. This problem is *NP*-hard on general networks (Hakimi, Schmeichel, and Labbé, 1993). When the objective is maximizing the maximum distance, the problem is polynomial (Hakimi, Schmeichel, and Labbé, 1993) on tree networks and general networks.

A summary of the results presented in this chapter can be found in the following tables:

Table 17: Literature on Structure Facility Location on Networks with Covering Objective

Author	***Year***	***Problem***	***Summary***
Current, ReVelle, and Cohon	1984	Shortest Covering Path	ILP formulation, B&B
Current, Pirkul, and Roland	1994	Shortest Covering Path	B&B
Current, ReVelle, and Cohon	1985	Maximal Covering Shortest Path	Definition
Boffey and Narula	1998	Maximal Population Shortest Path	ILP formulation 2 algorithms
Current, Revelle, and Cohon	1988	Minimum-Covering Shortest Path	ILP formulation
Current and Schilling	1989	Covering Tour	Definition
Gendreau, Laporte, and Semet	1997	Covering Tour	ILP, B&B Polyhedral analysis
Maniezzo, Baldacci, Boschetti, Zamboni	1999	Covering Tour	Metaheuristics
Motta, Ochi, and Martinhon	2001	Covering Tour	Reduction technique
Hodgson, Laporte, and Semet	1996	Covering Tour	Application in a health care system
Current and Schilling	1994	Maximal Covering Tour	ILP formulation
Labbè, Laporte, and Soriano	1998	Cycle Cover	6 heuristics
Kim, Lowe, Ward, and Francis	1989	Minimum Length Covering Subgraph	*NP*-Hard for general networks

			O(n) for cacti
Kim, Lowe, Ward, and Francis	1990	Subtree *r*-Cover on trees	$O(m^2)$ algorithm
Kim, Lowe, Tamir, and Ward	1996	Direct (Indirect) Subtree Covering	$O(n)$ $[O(n\log^2 n)]$ algorithm
Hutson and ReVelle	1989	Maximal Direct Covering Tree of a tree	Definition
Church and Current	1993	Maximal Direct Covering Tree of a tree	$O(n^2)$ algorithm

Table 18: Literature on Structure Facility Location on Networks with Distance Objective

Author	***Year***	***Problem***	***Summary***
Minieka	1985	Minimize (Maximize) Distance Sum of a Path-Shaped Facility	Polynomial algorithms for tree networks
Current, Revelle, and Cohon	1987	Median Shortest Path	ILP formulation Complete enum.
Hakimi, Schemeichel, Labbé	1993	Minimize (Maximize) Distance Sum of Path- and Tree-Shaped Facilities	*NP*-hardness results
Current and Schilling	1994	Median Tour	ILP formulation Heuristic
Labbé, Laporte, Rodriquez-Martin, and González	1999	Median Tour	B&B Heuristics
Foulds, Wilson, and Yamaguchi	2000	Median Tour	B&B
Moreno Pérez, Moreno-Vega and Rodríguez Martín	2002	Median Tour	Tabu Search
Renaud, Boctor, and Laporte	2004	Median Tour	Genetic Algorithm
Shigeno and Shioura	1995	Tree Median on trees	*O(n)* -continuous *NP*-Hard –discrete
Kim, Lowe, Tamir,	1996	Median Subtree on	*O(n)* algorithm

and Ward		trees (single objective)	
George and ReVelle	2003	Median Subtree on trees (biobjective)	ILP formulations B&B

Table 19: Literature on Structure Facility Location on Networks with Eccentricity Objective

Author	***Year***	***Problem***	***Summary***
Minieka	1985	Minimize (Maximize) Maximum Distance of a Path- or Tree-Shaped Facility	Polynomial algorithms for tree networks
Tamir and Lowe	1990	Generalized *p*-Forest	Polynomial algorithms for tree networks
Foulds, Wilson, and Yamaguchi	2000	Cycle Center	ILP formulation B&B
Shioura and Shigeno	1995 1997	Tree Center	ILP formulation *O(n)* –trees
Hakimi, Schemeichel, Labbé	1993	Minimize (Maximize) Maximum Distance of Path- and Tree-Shaped Facilities	*NP*-hardness results

Chapter 8

COMPETITIVE FACILITY LOCATION ON NETWORKS:

Facility location decisions are quite complicated in the real world in contrast to the simplicity of the proposed models in the literature. The choice of a location depends on many factors such as customer demand patterns, transportation costs, infrastructure, labor costs, environmental issues, politics, accessibility to important facilities such as airports, hospitals, etc. Basic models of location theory deal with problems in which only demand configurations and transportation costs are involved. These models generally assume that the organization that makes decisions to locate its facilities is either a non-profit organization or it is monopolistic in nature and no competitors who wish to provide the same products or some substitute goods are available in the market. This assumption may hold for some public services such as fire fighting or police coverage, but it is virtually meaningless in modeling private sector where all companies compete. In fact, competition is such an important factor in many industries that it affects prices, quality, volume, trends, and even life styles. Millions of dollars are spent for advertisements and promotions to create and capture customer demand. It is obvious that facility location decisions are highly affected by the competitive environment because the organizations struggle to be close to the customers in order to attract them to their retailers or to be able to provide goods to demand points at lower prices. Prices may in turn affect the demand weights and customers' choices between facilities. Consequently, competitive location is a complicated area of research that attracts the attention of economists, geographers, and operations researchers.

In modeling competitive location problems, many assumptions are made, so that the resulting models can be handled with available mathematical and computational resources. These assumptions are various and lead to many different models in the area. Unfortunately, every model is unique and a slight change in a single assumption creates a new model with completely unpredictable characteristics. We believe that the assumptions proposed in these models induce a natural taxonomy for the competitive location models and we follow the taxonomy presented below which is similar to the taxonomy presented in the bibliographic study of Eiselt, Laporte, and Thisse (1993). We extended this bibliography in light of the survey papers by Eiselt and Laporte (1989a), Hakimi (1990), Drezner (1995) and Plastria (2001). Based on this bibliography, we will survey the papers in competitive location literature by focusing mainly on the studies that involve network distances.

8.1 Taxonomy and Problem Features:

8.1.1 The space:

The first competitive location problem has been proposed for locating two competing firms on a line segment by Hotelling (1929). Although a long time passed since the first identification of the problem, many economists continue to study the problem in linear markets due to the obvious simplicity of the problem. The papers on linear markets are numerous (more than 60) and most of them appear in the economics journals: Anderson (1987, 1988), Anderson and de Palma (1988), Anderson, de Palma, and Thisse (1992), Anderson and Neven (1991), Artle and Carruthers (1988), Asami, Fujita, and Thisse (1993), Beckman (1972), Ben-Akiva, de Palma, and Thisse (1989), Bester (1989), Bonanno (1987), Boyer, Lafont, Mahenc, and Moreau (1990), Boyer, and Moreau (1990), Braid (1988), Capozza and Van Order (1980, 1989), Cremer, Marshand, and Thisse (1991), D'Aspremont, Gabszewicz, and Thisse (1979), Dasci and Laporte (forthcoming), de Palma, Ginsburgh, Labbé, and Thisse (1989), de Palma, Ginsburgh, Papageorgiou, and Thisse (1985), de Palma, Pontes and Thisse (1987), Eaton (1972, 1976), Eaton and Lipsey (1975, 1976, 1982), Economides (1986, 1989), Eiselt (1991), Fujita, Ogawa, and Thisse (1988), Fujita and Thisse (1986), Ghosh (1996), Ghosh and Buckanan (1988), Hamilton, Thisse, and Weskamp (1989), Kats (1987), Lerner and Singer (1937), Osborne and Pitchik (1986, 1987), Shilonyi (1981), Smithies (1941), Teitz (1968), and Weber (1990).

The problem also has been extended to employ other continuous spaces such as circular markets, the plane and the *m*-dimensional real space. The circular markets are appropriate to eliminate any boundary effects induced by the linear bounded markets. Lerner and Singer (1937), Eaton and Lipsey (1975), Salop (1979), Novshek (1980), Kats (1987, 1990), Economides (1989), and Kats and Thisse (1990) studied many variants of competitive location problem on circular markets. The planar problems are mostly studied by geographers to locate physical facilities. The papers that study the problem in the plane are those of Beaumont

(1980), Beckman (1972), Capozza and Van Order (1978), Carpenter (1989), Drezner (1981), Drezner (1994), Drezner and Drezner (1997, 1998), Drezner, Drezner, and Eiselt (1996), Drezner, Drezner, and Shiode (2002), Eaton and Lipsey (1975, 1976), Hamilton and Thisse (1992), Hanjoul and Thill (1987), Hurter and Lederer (1985), Lederer and Hurter (1986), Mills and Law (1964), Okabe and Aoyagy (1991), Okabe and Suzuki (1987), Shaked (1982), and Wendell and McKelvey (1981). Solving the problem in the plane is sometimes very tedious and aggregation techniques are used for planar problems. Aggregation of continuous demand into a finite number of points and how to reduce aggregation error is discussed in Drezner and Drezner (1997).

The *m*-dimensional real space is used by decision analysts to model abstract entities, such as candidates in a political arena and products in the attribute space. Few papers study these problems: Bester (1989), Choi, DeSarbo, and Harker (1990), and MacLeod, Norman, and Thisse (1987, 1988).

The problem has also been widely studied by operations researchers in discrete space, networks, and special networks such as trees. We discuss these in some detail in the following sections.

8.1.2 Number of Competitors:

Although in most of the papers in the literature there exist only two competing organizations, there are few studies that deal with more than two competitors (players). Each competitor can locate any number of facilities. Usually the number of facilities each player will locate is assumed to be fixed and known. In a few cases the number of players and facilities are not known a priori but determined by the model itself, these models are called “free-entrance” models in economics.

8.1.3 Pricing and other policies:

In many real world situations, companies compete with each other not only via facility locations but also by determining their prices, quantity of the product offered to the market, quality of the service offered and size/type of the facilities to be located. The prices can be used as decision variables, may be fixed or nonexistant (no price), may be fixed for every customer at the facility with a transportation cost that is paid by each customer to access the facility (mill price), may be fixed for all customers with the transportation costs being paid by the facility (uniform delivered price), or may be differently priced for different customers (spatial discriminatory price). In addition to models that involve prices there exist few models that include other variables such as volume, quantity, and facility size. The interested reader may refer to Karkazis (1989) for a multicriteria model that involves distance and quality as objective criteria and location and facility levels as decision variables.

8.1.4 Rules of the competition:

Many researchers model the competitive location problems as multiplayer games and find equilibrium solutions based on the assumptions and rules of the game. The first rule that comes into one’s mind is about the timing of the decisions. The models are divided into two main categories: Simultaneous location and sequential location of facilities.

When facilities are simultaneously located and prices are fixed or nonexistant, a *Cournot-Nash* solution, which is widely used in game theoretical models, is searched for. This is a solution for the game in which no competitor has any incentive to relocate his facilities. If the competitors play a dynamic game in which they can relocate their facilities in turn by starting at any solution, the game may eventually reach an equilibrium state, known as a *Cournot-Nash* solution, if such an equilibrium exists.

When facilities are simultaneously located and prices are variable, there are two conventions to model the problems. The first variation is a two-stage game in which the players simultaneously determine the locations of the facilities at the first stage and they simultaneously determine their prices in the second stage. The solution to this two-stage game is referred to as the *Subgame Perfect Nash Equilibrium*. The term "subgame" hints the fact that each stage is solved optimally using a Cournot-Nash Equilibrium. The second method is to simultaneously determine the locations and prices. Nevertheless, these models are very complicated and there exist only a few studies concerning them.

When facilities are located sequentially, completely different games arise. The entrance of firms is assumed to follow an order and the first entering firm is referred to as the leader whereas the second one is called the follower. Two different optimization problems arise in this situation: the leader's problem in which a firm enters a virgin market having the knowledge that a second firm will enter the market soon but perhaps not having perfect information about the configuration of future demands and the follower's problem in which a firm enters a market where there are already existing facilities. Many problematic issues arise in sequential problems and assumptions must be clear and reasonable in these models. For example, if the follower is allowed to locate facilities at the same points where the leader has located its facilities and a customer equidistant to two facilities splits its demand between old and new facilities then the follower always guarantee to capture as many customers as the leader by locating its facilities on top of the old facilities. In such a situation each firm will prefer to enter a market as a follower and the business will be over before it has been started. There must be some benefits for the leader firms such as a time restriction, which prevents followers entering a market until the leader harvests the initial fruits. The sequential location problems and their solutions are referred to as *Stackelberg Games* and *Stackelberg Equilibriums*, respectively. Hakimi (1983) used the terms centroid and medianoid for the leader's and follower's problems under fixed or non-existing prices, which is somehow confusing because the objectives are quite different from the well-known median and center problems. Eiselt and Laporte (1996) presented a very valuable survey on sequential location problems including complexity results, discussions on modeling and remarkable insights.

The objectives used in competitive location problems can be very different. Usually firms aim to maximize their market capture (number of customers patronizing their facilities). Other objectives such as minimizing the follower's market share for the leader firm, maximizing profits, maximizing the probability that a given profit is attained and guaranteeing half of the demand (voting games) may also be used in competitive location problems.

8.1.5 Customer Behavior:

The customers choose facilities according to some preference rules. An attraction function, whose inputs are distance of the customer to the facilities, quality of the facility expressed in some measures such as facility size, parking lot availability, product variability, etc. is devised for each customer and the choices are made based on this function. The customer behavior may be binary (deterministic) in which each customer patronizes the facility to which she is attracted most, or it may be based on customer preferences (probabilistic) in which each facility can be patronized with some probability which is inversely proportional with the distance and directly proportional with the quality of the facility. These models are sometimes referred to as Huff models. There are also models, which are between the two models, referred to as the partially binary models where each customer patronizes the nearest facility of each organization with some probability function.

The tie breaking rules are especially important in binary models. Ties may be broken in favor of existing facilities or new facilities (in sequential models) or the demand may be divided between tied facilities according to some function such as total market share of each organization, etc.

The customer demand weights may also depend on the location of facilities. When the demand is essential such as health services, bread, education, etc., the demand weights are independent of the distance of the facility to the customer. Nevertheless, for non-essential demand such as entertainment, restaurants, parks, etc., the demand weight is a nonincreasing function of the distance to the facility and the models for non-essential goods must involve the variable characteristics of the demand weights. In fact, locating new facilities in some areas may also generate new demand points, which is an untouched issue in the literature.

8.1.6 Information:

The information available for each player is also an important factor for game theoretical models of competitive facility location. The players may have complete information about the market or may have perceptions of the market. If there is no full information, then competitors must use some estimate of each other's market perception in order to develop concrete models. The value of information may be so high in some games that firms pay money to discover each other. The value of the information as well as how much information is needed to understand the game are discussed in Eiselt (1998).

8.2 Simultaneous Entry Models:

8.2.1 Deterministic (Binary) Customer Preferences:

As we have mentioned before, the Cournot-Nash equilibrium concept is used for the solution of competitive location problems with simultaneous entry of the firms into the market. Tobin and Friezs (1986) presented two models in which a firm enters a competitive market and the location of the firm's production site and the production amount are decision variables. The firm entering the market is producing large amounts of the product and the price is a function of the total quantity of the goods supplied to the market. A mathematical model, based on the price equilibrium models, is presented in Harker (1986) with a nonlinear objective function and linear constraints. It was computationally infeasible to solve such a large nonlinear problem so it has been solved for many different prices using a heuristic approach. The model is further developed by Labbé and Hakimi (1991) in which two firms simultaneously enter the market by opening one production site each. The firms first choose their facilities' locations then set production quantities. The price of the good is a linear decreasing function of the total amount produced by both of the firms. It is proved that a Subgame Perfect Nash Equilibrium always exists and an $O(n^3)$ algorithm is provided to find the solution when the facility sites are restricted to the vertices of the network.

Lederer (1986) presented a different model for two competing firms entering a market simultaneously. The firms first design their networks, then determine prices knowing each other's network structure. This study stands out because it is the unique study in competitive location literature that involves the design of a network. The problem has application areas in transportation and distribution sectors. It is shown that under certain conditions Nash Equilibrium exists which is socially beneficial for customers. This analysis is related to the studies of Lederer (1981) and Lederer and Hurter (1986), which analyse discriminatory pricing and location for problems in the plane.

Lederer and Thisse (1990) developed a model for two competitors on a network when the locations are restricted to the vertices of the network. First the firms determine their locations and the production technology they will use on these facilities (there exists a finite number of technologies available), and then they set prices knowing each other's decisions. This is a two-stage game and the equilibrium is attained using a Subgame Perfect Nash Equilibrium solution. The solution is again optimal from the customers' point of view and is somehow a generalized 2-median of the network.

Fischer (2002) developed a model in which two competitors enter a market by determining their facilities' locations and their prices. The demand weight for each customer is dependent on the price of the good and every customer is charged a different price from other customers by each facility. When the firms decide on the location and price at the same time, the resulting model is nonlinear and hard to solve. On the other hand, when the prices are adjusted after the location decisions, Nash Equilibrium is reached. It is observed that firms try to avoid sharing markets.

8.2.2 Probabilistic Customer Preferences:

De Palma, Ginsburgh, Labbé, and Thisse (1989) studied the problem when m firms enter the market. Each firm i opens m_i facilities and the customer demand is divided among the nearest facilities of each firm according to some probability function that assigns higher probabilities to nearer facilities. A dispersion factor is also included in the attraction function that determines the level of different tastes in customer preferences. When the customers' preferences are diverse enough, it is shown that Nash Equilibrium is reached when each firm locates its facilities at the m_i-medians of the network.

8.3 Sequential Entry Models:

8.3.1 Deterministic (Binary) Customer Preferences:

Wendell and McKelvey (1981) studied the two-facility competitive location problem from the leader's point of view. They aimed to find a point in the location space such that the leader guarantees at least as many customers as the follower regardless of the follower's location. Customers' choice is solely dependent on the distance to the facility. This is also referred as the *Voting Game* in which a candidate tries to guarantee half of the votes in an election. Wendell and McKelvey (1981) studied the problem on the line, in the plane, and on a network. Local and global solutions are characterized and it is shown that a symmetry property holds when a global optimum exists. It is shown that when the number of vertices is odd, then the optimum solution occurs on a vertex of the graph. The solution to this problem is also referred to as the Condorcet Solution in some references (Hansen and Thisse, 1981; Hansen, Thisse, and Wendell, 1986). It is shown that the set of solutions to this problem (Condorcet Solution), the 1-median problem and the two-facility competitive location problem in which customer demands are divided between equidistant facilities (Nash Solution or Plurality Solution) are equivalent on tree networks because of the convexity of the distance function in tree networks (Hansen and Thisse, 1981, Wendell and McKelvey, 1981). For general networks Hansen, Thisse, and Wendell (1986) prove that the local solutions to the three problems is equivalent where a local solution is a solution which is optimal with respect to the points in a small neighborhood around. Furthermore, the Condorcet solution is a 3-approximatoin to the 1-median solution on a general network in the worse case (Hansen and Thisse, 1981). Bandelt (1985) characterized the networks for which Condorcet Solutions and 1-medians coincide. The Condorcet Solutions may not exist for some networks; instead a related solution concept introduced by Simpson (1969) can be used. A Simpson Solution is a point on the network which minimizes the largest total weight of demand points closer to any other point. The Simpson Solution concept may be useful when a leader aims to minimize the market capture of its follower. Hansen and Labbé (1988) provided polynomial time algorithms to find Condorcet and Simpson Solutions of a general network.

Megiddo, Zemel, and Hakimi (1983) introduced the Maximum Coverage Location Problem. The problem aims to locate r facilities on a network in which customers are being served by old facilities in order to maximize the number of customers patronizing new facilities. The locations of the old facilities are not considered but the critical distance at which each customer wishes to switch from an old facility to a new facility is known. The attraction function is dependent only on the distance and ties are broken in favor of old facilities. The problem is *NP*-hard on general graphs as shown by the authors by reducing from the Minimum Dominating Set Problem. A finite dominating set is identified which is order of $O(n)$ for trees and $O(mn)$ for general networks, where m is the number of edges. An $O(n^2r)$ dynamic programming based algorithm is provided for tree networks based on this observation. It is also noted that when the objective is to capture the entire market and the number of facilities to open is not known, the algorithms devised for the Covering Problem can be used.

The Maximum Coverage Problem becomes simpler when the locations of old facilities are known. This problem is called the ($r \mid X_p$)-medianoid problem and can be stated formally as follows: Given a network $N=(V, E)$ and a set X_p of p facilities already established on the network, find a set of r facilities Y_r^* on N such that $Y_r^* = \arg \max_{Y_r \in N, |Y_r| = r} W(Y_r \mid X_p)$ where $W(Y_r \mid X_r) = \sum \{w(v) \mid D(v, Y_r) < D(v, X_p)\}$. Finding the absolute and vertex-restricted ($r \mid X_1$)-medianoid is *NP*-hard on general graphs as shown by Hakimi (1983) and Hakimi (1990), respectively. Although the problem is *NP*-hard for variable number of facilities, polynomial algorithms may be devised when r is fixed. Megiddo, Zemel, and Hakimi (1983) provided an $O(n^r m^r / r!)$ algorithm for finding the ($r \mid X_p$)-medianoid of a general graph. Hakimi (1990) proved that the nodal optimality theorems do not hold for the medianoid problems even the problem is as simple as a ($1 \mid X_1$)-medianoid. Megiddo, Zemel and Hakimi (1983) provided an $O(mn)$ algorithm for the ($1 \mid X_1$)-medianoid problem.

Medianoid problems are the follower's problems, but what about the leaders? The problems of locating p facilities knowing that a follower will locate r facilities in competition is called the ($r \mid p$)-centroid problem and formally defined as follows: Given a network $N=(V, E)$ and a find the set of p facilities X_p^* on N such that $X_p^* = \arg \max_{X_p \in N, |X_p| = p} W(Y_r^*(X_p) \mid X_p)$ where Y_r^* is a ($r \mid X_p$)-medianoid. The ($1 \mid 1$)-centroid of a general graph may not be on a vertex of the graph (Wendell and McKelvey, 1981; Hakimi, 1983), but there always exist a ($r \mid 1$)-centroid that is a vertex for r>1 (Hakimi, 1990). The ($1 \mid 1$)-centroid of a tree network is always on a node and it is the 1-median of the tree (Slater, 1975; Wendell and

McKelvey, 1981). The $(1|1)$-centroid of a general network was solved by a $O(m^4m^2 logmnlogD)$ algorithm by Megiddo, Zemel, and Hakimi (1983), where D is the total demand weight. It is shown that finding absolute or vertex-restricted $(1|p)$-centroid of a general network and finding an approximation algorithm with a performance guarantee to the problem is *NP*-hard (Hakimi, 1990).

The $(r|p)$-centroid and $(r|X_p)$-medianoid problems are extended to the cases when the demands are nonessential and depend on the distance to the facilities and demands are distributed according to customer preferences. Several nodal optimality results together with important insights of the problems is provided in Hakimi (1990). The paper also provides insight on the multi-period games in which facilities may be relocated and new rules of the game are introduced. The $(r|p)$-centroid problem is also analyzed for networks with stochastic demand weights. In stochastic problems, it is assumed that the leader does not have complete information on the future demand weights when the follower enters the market. The stochastic $(1|1)$-centroid problem is solved by Shiode and Drezner (2003) on tree networks based a nodal optimality result and bisection search.

The vertex-restricted $(r|p)$-centroid problem is referred to as the Maximum Capture Problem by ReVelle (1986). The problem is formulated as an IP, based on the classical Maximal Covering Problem by Church and ReVelle (1974). Eiselt and Laporte (1989b) modified the model to include attraction parameters. Their model assigns an attraction value to each facility-customer pair based on the inverse square distance between them and referred to as the "gravity" model. ReVelle and Serra (1991) modified the model to a dynamic one, which includes relocation of old facilities and opening new facilities in each period. The model is further extended to involve hierarchical facilities and competition at each level of hierarchy by Serra, Marianov, and ReVelle (1992). It is also extended to involve uncertain demand weights and two models are proposed for the problem. The first model maximizes the minimum possible market capture, whereas the second minimizes the maximum regret. A branch-and-bound algorithm and a 1-opt heuristic are provided for the problem. The maximum capture objective of the MaxCap Problem was modified to a preemptive one in which the leader firm locates *p* facilities in order to minimize the follower's market capture where the follower also locates *p* facilities in Serra and ReVelle (1994). When the two facilities are equidistant to a customer, the market is shared so that the follower always guarantees capturing half of the demand. An IP formulation is presented and two heuristics are proposed. The first heuristic locates the leader's *p* facilities then solves the MaxCap Problem of follower's to optimality using branch-and-bound, then iterates by changing one of the leader's facilities' locations. This heuristic is in fact a 1-opt procedure. The second heuristic uses another heuristic for the MaxCap Problem. The heuristics are compared in terms computation time and solution quality. The algorithms may also be used when the numbers of facilities each firm locates are different from each other (Serra, Ratick, and ReVelle, 1996).

Dobson and Karmarkar (1987) studied a very different version of the problem in which a leader firm chooses a set of points such that no other firm can open a facility which is profitable. The number of facilities is not known a priori but is a model parameter. Several versions of stability are discussed and IP formulations are provided to identify stable sets. The problem is proven to be *NP*-hard and an enumeration algorithm is presented.

Brandeau and Chiu (1994) solved a two-facility competitive location problem on tree networks using a different attraction function, which includes market externalities such as congestion at the facility, delay time in the cashier queue, and etc. A facility's attractiveness is inversely proportional with the distance and market externality factor associated with it. Ties are broken in the favor of the firm whose market share is greater. The optimal solutions are characterized and an $O(n^2)$ algorithm is provided to solve the problem on tree networks. When the firms are equally attractive for each customer, the solution is the 1-median of the tree.

The MaxCap or $(r \mid X_p)$-medianoid problem is solved by Dasci, Eiselt, and Laporte (2002) on networks in which demand is distributed along the edges. It is shown that $(r \mid X_p)$-medianoid problem is *NP*-hard on general graphs with edge demands only. It is shown that an optimal solution may not exist for the single facility case. Neverthless, a finite set of $O(nm)$ points is identified which includes all optimal or ε-optimal solutions. An $O(nm^2)$ algorithm is presented to find optimal or suboptimal solutions for the $(1 \mid X_p)$-medianoid problem with edge demands which is similar to that of Megiddo, Zemel, and Hakimi (1983).

8.3.2 Probabilistic Customer Preferences:

The Condorcet Solutions are extended to involve probabilistic customer preferences by Bauer, Domschke, and Pesch (1990, 1993). Two competitive facilities are to be open on the network and the leader wants to locate its facility on a point that guarantees as many customers as the other facility regardless of the follower's location. It is shown that if there exists an optimal solution, then at least one optimal location is on one of the nodes of the network. An algorithm for providing optimal (if they exist) and sub-optimal solutions is presented.

The MaxCap Problem is extended to include the probabilistic customer preferences. The attraction function for each customer is not known in advance but is a continuous random variable of distance (Benati, 1999). Under certain assumptions the problem is modeled as an IP. Two brach-and-bound algorithms are provided for the problem based on Lagrangean Relaxation and submodularity of the objective function, respectively. The algorithms are very effective in that large instances of the problem (100 nodes) are solved in a few seconds. Benati and Hansen (2002) also studied this problem with a more general attraction function. The resulting model is a special IP whose terms in the objective function are ratios. The problem is new in the literature and proven to be *NP*-hard. A branch-and-bound algorithm together with an efficient heuristic is provided for the problem.

Colome and Serra (2001) compared 3 different probabilistic MaxCap models with the deterministic MaxCap formulation. Based on results obtained from test instances, it is discussed that when the appropriate model to use is not be known in advance; the deterministic model provides the minimum error.

As opposed to the Max Cap problem, consider a case when a firm wants to enter a competitive market and the number of facilities to be opened in this market is not predetermined. In this case, a competitive version of the Uncapacitated Facilitiy Location Problem arises. Benati (2003) studied such a problem in which customer preferences are heterogeneous and a probability function is used to represent the customer behavior. The problem is modeled as a nonlinear integer program while the objective function is concave and submodular. A branch-and-bound algorithm is developed for instances smaller than 50 and a metaheuristic similar to Heuristic Concentration of Rosing and Revelle (1997) is used for larger instances.

The single facility MaxCap or $(1|X_p)$-medianoid problem was solved for networks in which demand is not only generated by the nodes of the network but it is uniformly generated on the links of the network. Okunuki and Okabe (2002) solved this problem when the customer preferences are probabilistic and devised an $O(n^2 logn)$ algorithm for general networks.

Berman and Krass (2002) considered a competitive location model with probabilistic customer preferences. Their model is different than previous models because the customer demands change as new facilities enter the market. The demand is affected in two ways: first it increases because new firms create new demands called "market expansion" and second and more familiar, the demand decreases because new facilities share customers of old facilities owned by the same organization called "cannibalization". Berman and Krass (2002) characterize optimal and suboptimal solutions to the problem considering variable expenditure functions and market expansion and cannibalization effects.

8.4 Flow-Intercepting Competitive Location Models:

In the real world, many customers go to retailers such as supermarkets, gas stations on their way to home or office. They can either make special purpose trips to facilities or they are intercepted by the facility on their route to other destinations. This type of models is referred to as *Flow Intercepting Spatial Interaction* models (Berman, Hogdson, and Krass 1995). Berman and Krass (1998) studied the competitive version of the Flow Intercepting model. The problem is solved via branch-and-bound and a very efficient heuristic is provided for the model with worst-case performance analysis. When customers make no special trips but only intercepted by the facilities on their route, the problem is referred to as the *Flow-Capturing Problem* (Hodgson, 1990). The competitive version is solved by Wu and Lin (2003) who developed a mathematical model and a greedy heuristic for the problem.

The literature on competitive facility location problems is summarized below:

Table 20: Literature on Competitive Facility Location when Competitors Simultaneously Enter the Market

Author	***Year***	***Problem***	***Summary***
Harker	1986	Nonlinear, price equilibrium model	Heuristic
Labbé and Hakimi	1991	2 firms, 2 facilities	Subgame Perfect Nash Equilibrium $O(n^3)$ algorithm
Lederer	1986	2 firms, network design and price setting	Nash Equilibrium

Lederer and Thisse	1990	Vertex-restricted 2 firms including production tech.	Subgame Perfect Nash Equilibrium
Fischer	2002	2 firms when demand weights are dependent and price	Nash Equilibrium
De Palma, Ginsburgh, Labbé, and Thisse	1989	Multiple firms, probabilistic customer preferences	Nash Equilibrium

Table 21: Literature on Competitive Facility Location when Competitors Sequentially Enter the Market

Author	***Year***	***Problem***	***Summary***
Wendell and McKelvey	1981	*Voting Game*	Local and global solutions are characterized
Hansen and Thisse	1981	*Voting Game*	Equal to 1-median on trees
Simpson	1969	*Voting Game*	Simpson Solution
Hansen and Labbé	1988	*Voting Game*	Polynomial algorithms for Condorcet and Simpson Solutions
Megiddo, Zemel, and Hakimi	1983	Maximum Coverage	*NP*-hard on general graphs Finite dominating set $O(n^2r)$ alg. for trees
Hakimi	1983 1990	$(r \mid X_p)$-medianoid	*NP*-hardness results $O(n^r m^r / r!)$ algorithm
Hakimi	1983 1990	$(r \mid p)$-centroid	*NP*-hardness results
Megiddo, Zemel and Hakimi	1983	$(1 \mid X_1)$-medianoid $(1 \mid 1)$-centroid	$O(mn)$ algorithm $O(m^4 m^2 logmnlogD)$
Shiode and Drezner	2003	Stochastic $(1 \mid 1)$-centroid	Nodal optimality for trees
ReVelle	1986	Maximum Capture	IP formulation
Eiselt and Laporte	1989	Maximum Capture	IP formulation

		with attraction parameters	
ReVelle and Serra	1991	Dynamic Maximal Covering	IP formulation
Serra, Marianov, and ReVelle	1992	Hierarchical Maximal Covering	IP formulation Heuristics
Serra and ReVelle	1994	Preemptive Maximal Covering	IP formulation Heuristics
Dobson and Karmarkar	1987	Unknown number of facilities	IP formulations Enumeration alg.
Brandeau and Chiu	1994	2 facility problem with market externalities	$O(n^2)$ algorithm for trees
Dasci, Eiselt, and Laporte	2002	$(r \mid X_p)$-medianoid with cont. link demands	$O(nm^2)$ algorithm when r=1
Bauer, Domschke and Pesch	1990 1993	Condorcet Solutions with probabilistic custumer preferences	Nodal optimality
Benati Benati and Hansen	1999 2002	MaxCap with probabilistic custumer preferences	IP formulation B&B
Colome and Serra	2001	MaxCap	Compared binary and probabilistic models
Benati	2003	Unknown number of facilities with probabilistic customer preferences	Nonlinear Integer Program B&B Metaheuristics
Okunuki and Okabe	2002	$(1 \mid X_p)$-medianoid with continous link demands and probabilistic cust. Preferences	$O(n^2 logn)$ algorithm
Berman and Krass	2002	Market expansion and demand canibalization	Characterize optimal and suboptimal solutions

Chapter 9

ROBUST FACILITY LOCATION ON NETWORKS:

The reliability and validity of the location models used in decision making depend on the data used. Although the models become much simpler when the data is deterministic and accurate, it is nearly impossible to expect to have deterministic data for many real world problems. One primary reason for this is that most location decisions affect a long time horizon so that the data used at the time of decision-making is just an estimate of what is expected to occur in the future. The second reason is that most of the data used in models such as demand volumes or travel times are not deterministically known by the analyst but obtained by statistical methods such as data sampling. In either case, the data at hand is uncertain and may or may not obey an a priori available probability density function. When the data follows a probability distribution, stochastic programming may be used for modeling and solving these problems. Nevertheless, when the data is totally random with no specific pattern then a set of scenarios is used. The set of scenarios may be finite so that the problem is solved via solving a number of problems on each scenario or may be infinite in which case a lower and an upper bound is assumed to be available for each parameter so that the realizations of each parameter is assumed to be confined to an interval. In both situations, robust approaches may be appropriate in which the performance in a worst case scenario according to some objective criterion is optimized.

9.1 Robustness:

When the data is uncertain, the decision maker may be pessimistic and be concerned with the worst solutions in order to avoid serious failure of the business. Furthermore, he may not only be concerned with the cost function and how it varies with the actual realizations of model parameters but also with the difference between the costs associated with the location decision and the optimal decision that would have been made if the parameters were perfectly known a priori to the decision. These concerns give rise to *robust approaches* that aim to produce solutions that are not very far away from the optimal decisions for every possible realization of model parameters (scenario). A robust solution to a problem may be interpreted as an ε-optimal solution for any realization of the parameters (Averbakh and Berman, 2000b) and robust models try to minimize ε. Two robust approaches are used in the literature:

Absolute robust criterion: The maximum objective function value among all possible scenarios is minimized. More formally, given a network $N=(V, E)$, a set S of scenarios consisting of all possible values of node weights and edge lengths, a set of functions $f_s(.)$ which is the objective function to be minimized under scenario s, and a set F which is the space of all feasible solutions; the absolute robust problem is to find a set of points $X \subseteq N$ such that $X \in \arg\min_{X \in F} \max_{s \in S} f_s(X)$.

Robust deviation criterion (Minimax regret): The regret of a location decision with respect to a scenario is defined to be the difference between the objective function of the location decision and the optimal objective function value associated with the scenario. The minimax regret criterion looks for a location decision whose maximum regret among all possible scenarios is minimum. More formally, given a network *N=(V, E)*, a set S of scenarios consisting of all possible values of node weights and edge lengths, a set of functions $f_s(.)$ which is the objective function to be minimized under scenario s, and the set F which is the space of all feasible solutions; the minimax regret problem is to find a set of points $X \subseteq N$ such that $X \in \arg\min_{X \in F} \max_{s \in S}[f_s(X) - f_s(X_s^*)]$ where X_s^* is an optimal solution for the problem under scenario s.

Robust problems became popular in recent years and many problems in the field of optimization are solved based on the above robustness criteria (Ben-Tal and Nemirovsky, 2002). Furthermore, new robustness measures are introduced and used by researchers for many problems. The state of the art on handling robust discrete problems is presented by Kouvelis and Yu (1997) and the interested reader is referred to this extensive book for further discussion on advantages of minimax regret approach to problems with uncertain data. However, in location problems on network the two criteria presented above are widely used and few other measures are introduced. In fact, the literature on robust network location problems is devoted to median and center type of problems with absolute and deviation robust measures and we will focus only on these problems in the following sections.

9.2 *p*-Median Problem with Uncertain Data:

Tansel and Scheuenstuhl (1988) studied the 1-median problem when the node weights are not known in advance but restricted to be in specific intervals on tree networks. Three solution concepts are defined for this problem; weak, strong and permanent solutions. A point is a weak solution if it minimizes the total weighted distances from every vertex for at least one scenario. A point is a permanent solution if it minimizes the sum weighted distances from every vertex under all scenarios. Lastly, a point is a strong solution if it minimizes the total weighted distances with some positive probability. It is shown that the set of weak solutions constitute a subtree of the tree and a linear time algorithm, which trims the tree until the set of weak solutions remain, is presented. Furthermore, it is shown that the permanent solution is either a vertex of the tree or it simply does not exist. It is also shown that if a probability distribution function is assumed for each point of the tree to be an optimal solution then the strong solutions can be found by evaluating the vertices of the tree that belong to the set of weak solutions. The concept of permanent solutions are further extended by Demir, Tansel, and Scheuenstuhl (forthcoming) to unionwise permanent solutions. A set of solutions is unionwise permanent if they collectively behave like a permanent solution. Methods for finding unionwise permanent solutions for the 1-median problem on tree networks are presented. The unionwise solutions may be further examined by the decision maker and a single point solution may be chosen among them. These two papers are different than the other papers in the literature according to the solution concepts used and a simple discussion about the comparison of permanent solution and minimax regret solution concepts is presented in this paper.

Kouvelis, Vairaktakis, and Yu (1994) studied the robust 1-median problem on a tree network when the data is imprecise. When the data is discrete and is available in the form of a number of scenarios, the absolute robust and minimax regret solutions can be found in $O(sn)$ time, where s is the number of possible scenarios and n is the number of nodes, based on the facts that the median function on each edge for each scenario is linear, and the objective function localized on an edge is the maximum of a set of linear functions and is a convex piecewise linear function. When the data is given as intervals for node and link lengths, a worst case scenario is constructed for each point of the network which gives the worst objective function value if a facility is placed on point x. It is shown that in every worst case scenario the link lengths are equal to their upper bounds when the network is a tree so the link lengths are considered to be deterministic and set to their upper bounds. Furthermore, a finite set of scenarios is identified which contains all worst case scenarios. It is shown by a simple example that the nodal optimality results do not hold for robust problems. When the location of the facility is restricted to the set of nodes, an $O(n^3)$ algorithm is presented for the problem. When the location of the facility is unrestricted an $O(n^4)$ algorithm is given. Chen and Lin (1998) studied the same problem and shown that the vertex-restricted robust 1-median of a tree belongs to the set of scenario medians and is a subset of V. Furthermore the absolute (unrestricted) robust 1-median of a tree is on one of the edges adjacent to the vertex-restricted robust 1-median so the search for the restricted and unrestricted robust 1-median is reduced significantly. An $O(n^3)$ algorithm is presented for the unrestricted case. The objective function is convex on any path of the tree network as observed by Averbakh and Berman (2000a) who proposed an improved $O(n^2)$ algorithm for the node-restricted problem. It is also possible to solve the restricted and unrestricted problems by a complicated algorithm in $O(nlog^2n)$ time (Averbakh and Berman, 1996) as stated by the authors themselves.

Averbakh and Berman (2000a) studied the robust 1-median problem on general networks for the first time and presented the first polynomial time algorithm for the problem when the demand weights are uncertain and restricted to specified intervals but link lengths are deterministic. The algorithm divides each edge into treelike segments on which the objective function is convex and solves the problem in $O(mn^2logn)$ time for the unrestricted case. It is shown that when the link lengths are also uncertain and belong to specified intervals the problem is strongly *NP*-hard on general networks (Averbakh, 2003).

Burkard and Dollani (1999) studied the robust 1-median problem on tree networks when the vertex weights are given in intervals and may assume negative values as well as positive ones. This problem may be useful in modeling the location of obnoxious facilities. It is shown that there exists at least one vertex that is optimal when the absolute robustness criterion is used. This problem is handled by solving the problem on each treelike segment on every edge. The time complexity of the algorithm is linear. When the minimax regret approach is used instead of absolute robust criterion, the problem is solved in $O(n^2)$ time. The number of scenarios is reduced to a finite number and worst case scenarios are identified for certain pairs of points on the network under each criterion. Burkard and Dollani (1999) also introduced and solved the *Dynamic Robust 1-Median* problem on tree networks in which the vertex and edge weights are dynamically changing according to some variable, which represents the time, and node weights may assume negative and positive values. It is shown that nodal optimality is not present in such problems. The absolute robust problem is solved in linear time whereas the robust deviation problem is solved in $O(n^2\alpha(n)logn)$ where $\alpha(n)$ is the inverse Ackermann function.

9.3 *p*-Center Problem with Uncertain Data:

The robust *p*-center problem is easily handled on networks when only the node weights are uncertain and assume interval data. Averbakh and Berman (1997) showed that the absolute robust *p*-center problem could be solved by setting all node weights to their maximum values and solving the resulting single *p*-center problem. Likewise the deviation robust (minimax regret) *p*-center problem can be solved by solving n+1 *p*-center problems on the network with each center problem corresponding to a specific scenario. Thus the robust *p*-center problem with only uncertain node weights is polynomialy solvable for the cases in which the *p*-center problem is polynomialy solvable. This is also true for some other problems with minimax objective (Averbakh, 2000).

The robust 1-center problem like the median version is shown to be strongly *NP*-hard on general graphs when the vertex and link weights are uncertain and only interval estimates of these parameters are available at hand (Averbakh, 2003). The problem is studied on tree networks extensively by Averbakh and Berman (2000b). For each point on the tree, a worst case scenario is characterized in which the objective is the worst when the facility is on this point and it is shown that the robust 1-center of a tree is a unique point. An algorithm which finds the edge that contains the optimal solution in $O(n^2 logn)$ time is presented, this algorithm of course solves the node-restricted version in the same time whereas requires more effort to find the unrestricted solution. The absolute problem is solved in $O(n^6)$ time whose high computational complexity proves that the problem is not such an easy one even on tree networks. The problem is also solved in $O(n^2 logn)$ time for the unweighted case in which the node weights are deterministic and equal to 1 and link lengths are uncertain. The results of Averbakh and Berman (2000b) are improved by Burkard and Dollani (2002). Burkard and Dollani (2002) showed that the edge that contains the optimal solution can be found in $O(nlogn)$ time and when the problem is unweighted the optimal solution on this edge can be placed in linear time which results in an algorithm of $O(nlogn)$ for the unweighted case. Furthermore the authors showed that when the solution is restricted to a single edge a finite number of worst case scenarios may be identified which is in the order of $O(n^3)$ and the problem can be solved in $O(n^3 logn)$ time. This paper contains valuable discussion on the behavior of the objective function and recommended for those who seek to grasp the technical details of the problem.

9.4 Further Remarks:

There are few other models in the network location literature, which involve uncertainty but are different than the models introduced above. We would like to mention the ones that attracted our attention.

The models we have covered assume that the number of facilities to be placed is known in advance. Nevertheless, there may be cases in which the number of facilities to be placed is uncertain and a number of scenarios are given each corresponding to a different realization of the parameters: the number of facilities to be placed and the node weights. Current, Ratick, and ReVelle (1997) studied such a problem, which is referred to as the NOFUN problem. Two versions of the problem are considered: the stochastic one in which each scenario is assigned a probability of occurrence and the expected opportunity loss is minimized and the robust one in which the occurrence of the scenarios is totally random and the maximum regret over all scenarios is minimized. ILP formulations are presented for each model and complete enumeration is used to solve the models.

Another model is presented by Daskin and Hesse (1997). This model is called the α-reliable p-minimax regret model, which is developed in order to avoid some disadvantages of minimax regret models. It is discussed that the worst case or average case models are not realistic to be used in the real world because the worst case model is too costly and the average model is too risky for the real world. The α-reliable p-minimax regret model is a hybrid approach that assigns probabilities to scenarios and minimizes the maximum regret over a subset of scenarios whose total probability of occurrence is greater than a threshold value α. The model is formulated as an ILP and an 88-node instance of the model is solved using branch-and-bound. We believe that this approach may be useful in many applications and interested readers are referred to the paper in order to have an idea of possible extensions and future research areas related to this approach.

The last model we find important is by Killmer, Anandalingam, and Malcolm (2001) who studied the location of a noxious facility on a network in which the demand weights, link lengths, and production costs are uncertain. This problem is modeled as a nonlinear program and solved using GAMS. The model is a multiobjective one, which minimizes maximum regret and the expected cost simultaneously and provides intuition for developing new hybrid models.

The literature presented in this chapter is summarized in the following tables:

Table 22: Literature on Robust Minisum Facility Location on Networks

Author	***Year***	***Problem***	***Summary***
Tansel and Scheuenstuhl	1988	1-Median with interval data	Weak, strong and permanent solutions
Demir, Tansel, and Scheuenstuhl	Coming	1-Median with interval data	Unionwise permanent solutions
Kouvelis, Vairaktakis, and Yu	1994	Robust 1-median of a tree	$O(sn)$ – discrete $O(n^3)$ – interval data, vertex-restricted $O(n^4)$ – interval data, absolute
Chen and Lin	1998	Robust 1-median of a tree	$O(n^3)$ algorithm for the absolute problem
Averbakh and Berman	2000a	Robust 1-median of a tree	$O(n^2)$ – vertex-restricted
Averbakh and Berman	1996	Robust 1-median of a tree	$O(n\log^2 n))$ algorithm
Averbakh and Berman	2000a	Robust 1-median of a network	$O(mn^2\log n)$ algorithm
Averbakh	2003	Robust 1-median of a network	*NP*-hardness results when link lenghts are also uncertain
Burkard and Dollani	1999	Robust 1-median of a tree with pos/neg weights	$O(n)$ –absolute robust $O(n)$ –minmax

			regret
Burkard and Dollani	1999	Dynamic robust 1-median of a tree with pos/neg weights	$O(n)$ –absolute robust $O(n^2 \alpha(n) logn)$ – minmax regret

Table 23: Literature on Robust Minimax Facility Location on Networks

Author	***Year***	***Problem***	***Summary***
Averbakh and Berman	1997	Robust *p*-center	Solving *n*+1 *p*-center problems
Averbakh	2003	Robust *p*-center with uncertain link length	*NP*-hardness results
Averbakh	2000b	Robust *p*-center on tree networks with uncertain link length	$O(n^2 logn)$ – unweighted
Burkard and Dollani	2002	Robust *p*-center on tree networks with uncertain link length	$O(nlogn)$ – unweighted

Chapter 10

AGGREGATION METHODS FOR LOCATION PROBLEMS:

We have reviewed the existing literature on location problems on networks in the previous chapters. These problems are preferred by most of the location analysts because their data requirements are not very high compared to other location models. Usually, demand and distance data is sufficient to describe a problem. Although considerable effort has been devoted to the development of exact solution methods for these models, relatively less attention has been given to the gathering and analysis of the data used in these models. Most of the data sets are constructed without having the specific model at hand so they are far away from being error free and detailed enough for the specific problem. Moreover, most of them are aggregated data and contain errors that are unknown to the analysts. It is obvious that when the data sets include considerable deviations from the actual data, the efforts paid to solve the model to optimality become meaningless. On the other hand, most of the real world problems involve millions of demand points so the data used must be aggregated into a smaller data set. For example, when we need to locate emergency services in a city, every household is a demand point and it is computationally infeasible to solve a location problem with millions of demands. Consequently, the demands are usually aggregated according to the postal codes. Of course, every aggregation scheme will introduce some error into the model and there exists a trade off between data tractability and accuracy. Aggregation is crucial to handle such problems but the error induced must be controlled cleverly. That is the main reason why aggregation models of location problems have received serious attention in the last decade. The development of information technologies such as Geographical Information Systems (GIS) that handle large amounts of data with a user-friendly graphical interface has also dramatically increased the importance of aggregation because unaggregated data is available for many geographical areas and researchers have the opportunity to aggregate the data themselves via having a high level of control on the error introduced into the model. The interested reader is referred to Church (2002) for a detailed discussion on the integration of location science and GIS. We feel that concluding a survey on location problems without dealing with aggregation issues will make it incomplete and this chapter is devoted to fill this gap.

10.1 Aggregation Models:

Assume that we are solving a location problem on a very large demand set P with m demand points and aim to locate n facilities on the candidate facility sites (which can be same as or different than P). We would like to decrease the number of demand points from m to q where q is much smaller than m but greater than p so that the resulting problem is nontrivial. We replace each demand point P_i in P with an aggregate point P_i' such that P_i's are not necessarily different from each other and the demand set $P = \{P_1, \ldots P_m\}$ is aggregated into a smaller demand set P' consisting of the aggregation points $P'_1, \ldots P'_q$. If the original objective function is $f(X, P)$, the objective function of the aggregated problem can be denoted by $f(X, P')$ where $f(.,.)$ can be any function defined on any metric space. The aggregation model is defined solely by the assignment of P_i to P_i'.

Two types of errors are induced by the aggregation: the cost error that results from the incorrectness of the objective function value and the optimality error that results from the incorrectness of the facility locations (Casillas, 1987). When the unaggregated data is available, the cost error can be removed by using the original data but optimality error is serious in that facilities may be far away from their true optimal locations and high costs may be incurred due to this error. Most of the literature focuses on reducing the optimality error which can be expressed as the problem of bounding the cost error $|f(X, P) - f(X, P')|$ from above. It is shown by Francis and Lowe (1992) that the error incurred is bounded above as follows:

p-median problem: $|f(X, P) - f(X, P')| \leq \Sigma \; \{w_i D(P_i, P_i'): 1 \leq i \leq m\}$

p-center problem: : $|f(X, P) - f(X, P')| \leq \max \; \{w_i D(P_i, P_i'): 1 \leq i \leq m\}$

Then, an ideal aggregation model should find the locations of aggregated points, i.e P_i's, so that the error bounds are minimized. Observe that the error bounds are again *p*-median and *p*-center functions and minimizing the error bounds require the solutions of larger location problems that have the same structure as the original problem but with an increased number of facilities to be located. This is referred to as the *paradox of aggregation* after Francis and Lowe (1992). Although the aggregation model may not be optimally solved, it can be approximately solved using the a priori designed approximation algorithms for the original *p*-median and *p*-center problems. Goodchild (1979) claimed that there exists no general rule for aggregation and a specific aggregation procedure must be devised for each specific problem. Francis and Lowe (1992) also stress on this fact and suggest exploiting problem structures to derive good approximation schemes. Error bounds are derived for Conditional p-Median, Conditional p-Center, Multifacility Minisum, Multifacility Minimax, Quadratic Assignment, Supporting Median, Round-Trip, Cent-Dian, Obnoxious Facility Location Problems by Francis, Lowe, and Tamir (1997, 2000). These bound are valuable in that they can be used to develop good aggregation methods for various location problems via exploiting the problem structures. The paper includes a general method to derive aggregate location models and their associated upper bounds for some other problems not included above and constitutes a milestone in the literature.

The optimality error decomposes into three types of error, called source A, source B, and source C errors (Hillsman and Rhoda, 1978). Source A errors are defined to be the sum of the differences between the distances of the actual demand points to their nearest facilities and those of the aggregated demand points to their nearest facilities. Source B errors are special types of Source A errors. When a facility is placed on a point, the distance between this facility and the demands aggregated into this point are considered to be zero whereas it is strictly positive when unaggregated data is used. Source B errors are the total of such errors for all facility locations. Source C errors arise when demand points are not assigned to the nearest facility because of aggregation. For the p-median problem, methods to eliminate Source A and B errors are presented by Current and Schilling (1987). They introduced a method based on replacing the demand-weighted distance between an aggregated point and a facility with the demand-weighted distance between original demand points associated with the aggregated point and the facility. Mirchandani and Reilly (1986) proposed a method to eliminate Source A and B errors for zonal or polygon-based problems. Hodgson and Neuman (1993) proposed a method to reduce Source C errors in zonal problems. Based on methods by Current and Schilling (1987) and Hodgson and Neuman (1993), a method to eliminate all three types of errors is presented by Bowerman, Calamai, and Hall (1999). Their method is an iterative method which applies Current and Schilling's method at the first stage to eliminate the Source A and B errors, then the aggregated demand points are partitioned according to the selected facility sites to form new set of aggregated points which does not contain Source C errors. The process is repeated until all errors are eliminated.

Most of the studies related to aggregation are experimental in nature in that an aggregation scheme is used to aggregate large problems into smaller ones and errors are calculated using techniques like simulation. The most common aggregation technique used is the centroid aggregation in which the plane is divided into a number of zones and each demand point in a zone is aggregated to the centroid of that zone. In fact this aggregation scheme that comes into mind at first is a very effective aggregation scheme for continuous p-median problems as discussed by Plastria (1996). He showed that the weighted distance function to the centroid is asymptotically equal to the continuous 1-median function in most of the problems and centroid aggregation seems to be a reasonable choice for demand aggregation. For the planar Euclidean p-median problems, Zhao and Batta (1999) analytically studied the Source A, B, and C errors using centroid aggregation scheme. They developed upper bounds for each type of error for this aggregation scheme. This study stands out because it presents analytical results compared to widely known empirical studies.

Francis, Lowe, and Rayco (1993, 1996) are probably the first papers that devise aggregation schemes with known error bounds. They proposed a Row-Column Aggregation for the p-Median problem called MRC for problems in the plane with rectilinear distances. The algorithm imposes a grid structure over the demand locations and adjusts the grid spacing via solving simpler median problems on each coordinate. An attainable error bound is derived for the method. The algorithm is of polylogorithmic complexity in the number of demand points and the error bound is not attained in most of the test problems used. Similar to this algorithm, a Transformed Row-Column Aggregation called TRC is devised for the p-center problem in the plane with rectilinear distances by Rayco, Francis, and Lowe (1995). The algorithm involves a 45° rotation of the axes and imposes a grid structure on the demands. The grids are not identical and the dimension of grids are found by solving simpler center problems on the axes. An error bound is derived for the aggregation procedure and conditions under which the bound is attainable are presented in the paper. Computational experiments are conducted using this method and it is observed that the error bound is attained in most of the test instances, which is in direct contrast with the experimental results for the p-median problem. The authors suggest that there exists self-cancellation of errors in the p-median problem and it is more robust to aggregation than the center problem. Francis and Rayco (1995) proposed an aggregation scheme for the unweighted p-center problem in the plane with rectilinear distances. The aggregation scheme is asymptotically optimal with respect to the number of aggregate points in that the error bound converges to the bound in Rayco, Francis, and Lowe (1995) as the number of aggregate points increases. Another aggregation scheme for *p*-center problems in the plane is presented by Rayco, Francis, and Tamir (1999) that imposes a grid structure onto the plane consisting of identical diamonds of specified dimensions. For the 1-median problem in the plane with rectilinear distances, Francis, Lowe, Rayco, and Tamir (2000) proposed a new representation of aggregation error called the maximum error. This is the maximum of the errors associated with each possible location of new facility. The authors use this new type of error because it allows the analysis of self-cancellation effects involved in median type problems. A new Row-Column Aggregation method is proposed which minimizes this error for the single facility case. The aggregation algorithm proves to be useful for the multifacility problems as well, as stated by the authors. Interested readers are also referred to Erkut and Bozkaya (1999) for a more detailed discussion on the aggregation issues on the *p*-median problem including other new error functions.

Andersson, Francis, Normark, and Rayco (1995, 1998) presented aggregation methods for the *p*-center and *p*-median problems on networks. A row-column aggregation method is used at the first step of the algorithm that is very similar to MRC and TRC. Then network problems are solved on the largest component in each grid to find the aggregate points. Computational experiments based on real world networks are presented in the paper. Another aggregation approach for networks is presented by Zhao and Batta (2000) for networks with continuous link demands. The nodes of the network only represent the road intersections and discrete demands are allowed on the links. First of all, it is shown that nodal solutions can be used for this problem with an associated error in the objective function value. This error is dependent on the demands associated with links. For such networks, they show that demands on some intervals of a link may be aggregated into a single point on these intervals. It is shown that if the intervals are taken to be between breakpoints on the links, the aggregation procedure does not introduce any error to the original problem but reduces the number of demand points from infinity to a finite number.

10.2 Conclusion:

As we have mentioned above aggregation problems arise frequently in the location literature. Many experimental studies have shown that the methods used in the aggregation seriously affect the final solution. In recent years, errors incurred in the aggregation are studied analytically and error bounds are derived for specific problems (Francis, Lowe, and Tamir, 1997). It is important to have an idea about how much we pay in terms of objective function while reducing the size of the problem when we are comparing different aggregation techniques. We believe that these studies will be useful for researchers in developing better aggregation schemes for location problems. A review of the results can be found below:

Table 24: Literature on Aggregation Methods for Location Problems

Author	***Year***	***Problem***	***Summary***
Casillas	1987	Aggregation errors	Definition
Francis and Lowe	1992	Aggregation errors in p-median and p-center	Upper bounds Paradox of aggregation
Francis, Lowe, and Tamir	1997 2000	Broad range of problems	Upper bounds
Hillsman and Rhoda	1978	Aggregation errors	Source A, B, and C

			errors
Current and Schilling	1987	Aggregation errors	Elimination of Source A and B errors
Mirchandani and Reilly	1986	Aggregation errors for zonal regions	Elimination of Source A and B errors
Hodgson and Neuman	1993	Aggregation errors for zonal regions	Elimination of Source A and B errors
Bowerman, Calamai, and Hall	1999	Aggregation errors	Elimination of Source A, B and C errors
Plastria	1996	Aggregation for continuous *p*-median	Centroid aggregation
Zhao and Batta	1999	Aggregation for continuous *p*-median	Upper bounds for source A, B, and C errors using centroid aggregation scheme
Francis, Lowe and Rayco	1993 1996	*p*-Median	Row-Column Aggregation
Rayco, Francis, and Lowe	1995	*p*-Center	Transformed Row-Column Aggregation
Francis and Rayco	1995	Unweighted *p*-center	Error bounds
Rayco, Francis, and Tamir	1999	*p*-center	Error bounds
Francis, Lowe, Rayco, and Tamir	2000	1-median	New error: Maximum error
Erkut and Bozkaya	1999	Aggragation	Survey
Andersson, Francis, Normark, and Rayco	1995 1998	*p*-center and *p*-median on networks	A row-column aggregation method
Zhao and Batta	2000	*p*-center and *p*-median on networks with continuous link demands	Aggragation of demand to nodes

Chapter 11

SUMMARY AND CONCLUSION

In this thesis, we have reviewed facility location problems on networks. Although there exist some problems we have not mentioned due to time limitations, we believe that we provide a broad perspective on problem types, solution techniques, and computational results. There exist many journals, conferences and technical reports all over the world and providing a complete survey of the literature seems out of reach. Nevertheless, we hope that we have covered most of the related work. We apologize to those authors who have published in the area but have not been mentioned here. We conclude the thesis by summarizing the literature we have reviewed in the previous chapters and presenting some concluding remarks.

The facility location problems with the objective of minimizing the maximum distance from the customers to the facilities are widely studied. There exist mant variants of the problem including linear and nonlinear versions, discrete demands and continuous demands, capacitated and uncapacitated facilities, deterministic and stochastic data, etc. The problem is well solved on tree networks. Most of the problem variants are solved via identification of a finite set of points that include optimal facility locations and solving a series of covering problems using the distances between identified facility locations and demand points. The single facility case is relatively easy and solved in general networks, tree networks, and special networks such as cacti. The literature on single facility location problems with minimax objective on general networks, trees, and special networks is presented in Table 1, Table 2, and Table 3, respectively.

The problem is proven to be *NP*-hard on general networks for general p but polynomial algorithms are provided for given p when the cost functions are linear. The literature for multiple facility location problems with minimax objective on general networks and tree networks is summarized in Table 4 and Table 5, respectively.

The continuous version of the minimax facility locations are investigated widely when the demands are distributed uniformly on links of the networks. These problems may be further extended to include more general distribution functions instead of uniform distribution. Moreover, the version of the problem with capacitated facilities is solved on tree networks and this problem may be studied in more general networks.

The facility location problems with minisum objective are also widely studied and well solved. The problems are *NP*-hard on general networks. Nodal optimality results are provided for many variants of the problem. We summarize the nodal optimality results in table 6.

Nodal optimality results give rise to integer programming formulations of the problems and IP techniques are widely used to solve multiple facility problems to optimality. The literature on exact methods to solve minisum multifacility location problems on general networks are presented in Table 7.

Although there exist algorithms that solve large instances of the problem, these are usually very time consuming. Thus, approximation algorithms are important to solve problems and there exist many approximation algorithms and metaheuristics for the problem some of which are presented in Table 8.

The problems are relatively easy on tree networks and there exist many efficient polynomial algorithms for the problem on tree networks that are summarized in Table 9.

When there exist mutual interaction between the facilities, the distances between the new facilities are also included in the objective functions. These problems are hard on general networks but algorithms exploiting either the network structure or the structure of the interaction are provided. The literature on facility location problems with minisum objective and mutual communication are provided in Table 10.

The minimax facility location problems with mutual communication are closely related to the distance-constrained facility location problems that are again solved via exploiting the structure of the problem. Polynomial algorithms are provided for both problems on tree networks. You may refer to Table 11 for the results on distance-constrained facility location problems.

The minisum and minimax objectives may not be appropriate for every situation in the real world, so multiobjective models are used in which minisum and minimax objectives are simultaneously used. These models are solved via identification of a finite dominating set and multiobjective optimization techniques. Although the problem is well solved on trees, the problem on general networks may further be studied in the following years.

SUMMARY AND CONCLUSION

The literature on biobjective minisum/minimax facility location models are presented in Table 12.

The obnoxious facility location problems are also studied both on general and tree networks. IP formulations and heuristics are widely used for problems on general networks whereas polynomial algorithms are devised for most of the problems on tree networks. The literature on obnoxious facility location is summarized in Tables 13-16. We believe that the objectives used in these models are not sophisticated enough to model the obnoxiousness of the hazardous facilities, so models that can handle more general situations can be developed in the future.

The location of structures on a network is closely related to vehicle routing problems but the facility location perspective helps developing efficient algorithms and proving *NP*-hardness results. A part of the literature on Structure Location Problems with covering, distance, and eccentricity objectives are presented in Tables 17-19. Most of the problems are solved on tree networks but these problems can be extended to more general cases and new problems can be defined in this area.

Competitive location models are very complicated. There exist many problems in the literatue and many other problems can be defined by slightly changing the assumptions. Although there exist some well solved problems in the literature, most of the problems are untouched. We believe that this area deserves more interest. New models may be developed and existing models may be extended to more realistic cases. The literature in competitive facilty location is summarized in Table 20 and Table 21.

SUMMARY AND CONCLUSION

The robust approaches are popular in the recent years and many models most of which use the minmax regret concept are developed for facility location problems. The literature on robust median and center problems are presented in Table 22 and Table 23. Robust solutions for other facility location problems such as structure location problems and centdian problems may also be developed in addition to the basic models presented in the literature.

Table 24 presents results on aggregation techniques for location problems. It is known that the aggragation technique to be used depends on the problem at hand. Aggregation models for a few well-known location problems are developed. New models for other location problems can be a further research area.

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