GRADUATE ADMISSION PROBLEM WITH QUOTA AND BUDGET CONSTRAINTS

The Institute of Economics and Social Sciences
of
Bilkent University

by

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In Partial Fulfilment of the Requirements for the Degree of
MASTER OF ARTS IN ECONOMICS

in

THE DEPARTMENT OF
ECONOMICS
BİLKENT UNIVERSITY
ANKARA

September, 2003

I certify that I have read this thesis and have found that it is fully adequate,in scope and in quality, as a thesis for the degree of Master of Economics.

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ABSTRACT

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In this thesis, we have studied the graduate admission problem with quota and budget constraints as a two sided matching market. We constructed algorithms which are extensions of the Gale - Shapley algorithm and showed that if the algorithms stop then the resulting matchings are core stable (and thus Pareto optimal). However the algorithms may not stop for some problems. Also it is possible that the algorithms do not stop and there is a core stable matching. Also there is no department optimal matching and no student optimal matching under budget constraints. Hence straightforward extensions of the Gale - Shapley algorithm do not work for the graduate admission problem with quota and budget constraints. The presence of budget constraints play an important role in these results.

Keywords: pairwise stable matching, core stable matching, Pareto optimal matching, the Gale - Shapley algorithm, quota and budget constraints.

ÖZET

KOTA VE BÜTÇE KISITLARI ALTINDA DOKTORA KABUL PROBLEMİ

Mehmet Karakaya
Ekonomi, Yüksek Lisans
Tez Yöneticsi: Prof. Dr. Semih Koray
Eylül, 2003

Bu tez çalışmasında kota ve bütçe kısıtları altında doktora kabul problemi iki taraflı eşleşme olarak incelenmiştir. Gale - Shapley algoritmasının uzantıları olan çeşitli algoritmalar yazılmış ve bu algoritmalardan biri için algoritma durursa oluşan eşleşmenin çekirdek kararlı (ve böylece Pareto en iyi) olduğu gösterilmiştir. Fakat bu algoritmalar bazı problemler için durmadığı gibi, algoritmaların durmadığı ve çekirdek kararlı bir eşleşmenin bulunduğu durumlar da mevcuttur. Ayrıca bütçe kısıtı altında bölüm optimal eşleşme ve öğrenci optimal eşleşme yoktur. Bu yüzden Gale - Shapley algoritmasının uzantıları olan algoritmalar kota ve bütçe kısıtları altında doktora kabul problemi için kendilerinden beklenen işlevi yerine getirmemektedir. Bütçe kısıtının varlığı bu sonuçlarda önemli bir rol oynamaktadır.

Anahtar sözcükler: ikili kararlı eşleşme, çekirdek kararlı eşleşme, Pareto en iyi eşleşme, Gale - Shapley algoritması, kota ve bütçe kısıtları.

Acknowledgement

I am deeply indebted to Prof. Semih Koray and I would like to express my special thanks for his supervision, encouragement and never - ending morale support throughout all stages of my study. He devoted so much time and effort to the completion of my study and without him I would never have finished this thesis. I am proud that I have had the privilege of being among his students.

I am also indebted to Dr. Tarık Kara who helped me throughout all stages of my study. I would like to express my special thanks for his helps, encouragement and contribution to my thesis. I have learned a lot from him about writing a thesis.

My thanks also go to Dr. Süheyla Özyıldırım for the insightful comments she made during my defense of the thesis.

I am grateful to Prof. Hervé Moulin and Dr. İpek Ö. Sanver for very helpful discussions and suggestions.

I thank all participants of the BWED XXV (The Twenty - Fifth Bosphorus Workshop on Economic Desing) and the Study Group on Economic Theory (at Economics Deparment of Bilkent University) for their valuable comments.

I am grateful to my friends Barış Çiftçi, Pelin Pasin, Engin Emlek and the graduate students at Economics Department of Bilkent University for their friendship and excellent morale support and encouragement all the time. Special thanks go to my friend Cem Sevik for his helps on TEX to complete the final draft of this thesis.

Finally, my special thanks and gratitude are for my family for their endless support and encouragement throughout all my years of study.

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Chapter 1

Introduction

A typical two-sided matching market consists of two disjoint finite sets, for example a set of men and a set of women; colleges and students; firms and workers. A matching is called a one-to-one matching if a member of one set is allowed to match with at most one member of other set, for example a man (woman) can match with only one woman (man). However, a firm hires many workers, but a worker works for one firm only. This type of matching is called a many-to-one matching.

There is a rich literature on matching theory (see Roth and Sotomayor (1990) for an excellent survey for a period covering all classical results in the field) including both theoretical and empirical studies. Even though there is an extensive literature on matching theory, there is no study considering both quota and budget constraints simultaneously. There are studies where colleges (or firms) have either quota constraint or budget constraint but not both. In this thesis, we study the graduate admission problem under quota and budget constraints. There is a set of departments belonging to one university and a set of students (applicants) who wish to enter these departments. Each department faces both quota and budget constraints which are determined by the university.

Hall (1935) considered a marriage problem involving a group of men and women in a society where a man may marry a woman only if they have previously been introduced. Men have no preferences for women and women have no preferences for men. The aim is to maximize the number of people that can be matched. Hall showed that a complete set of marriages is possible if and only if every subset of men has collectively been introduced to at least as many women as the number of men in that subset, and vice versa.

Gale and Shapley (1962) described a model for *college admissions problem*. A college admission problem consists of a finite set of students and a finite set of colleges where each college faces a quota constraint. Each student has a linear preference relation over colleges and each college has a linear preference relation over sets of students. A student matches with a college or with herself (i.e., stays unmatched) and a college matches with a group of students whose size does not exceed its quota. A matching is blocked by a student iff she prefers to match with herself to getting matched with the college that she is assigned under that matching. A matching is blocked by a college iff it prefers a strict subset of the group of students that it matched under the given matching. A matching is blocked by a student - college pair iff the student prefers that college to her match and the college prefers the union of a proper subset of its match with the student to its present match. A matching is stable iff it is not blocked by a student, by a college and by a student - college pair. From each given set of students a college selects its most prefferred such set of students obeying the quota constraint. This most preffered set of students is referred as the choice of that college from among the group of students it faces. A stable matching is student optimal iff each student likes this matching at least as well as any other stable matching. A stable matching is college optimal iff each college likes this matching at least as well as any other stable matching.

The following algorithm is referred as *the Gale - Shapley student optimal algorithm*:

Step 1: Each student proposes to her most preferred college. Each college rejects all but those who comprise its choice among its proposers.

In general, at step k,

Step k: Each student who was rejected in the previous step proposes to her

next prefferred college. Each college rejects all but those who comprise its choice within the students it has been holding together with its new proposers.
The algorithm stops if there is no student such that her proposal is rejected. Then each student is matched with a college that she proposed at the last step and was not rejected by that college. The Gale - Shapley college optimal algorithm is similarly defined with colleges proposing to group of students by obeying their quota constraints.

A college has substitutable preferences if it regards students as substitutes rather than as complements, i.e., the college prefers to enroll a student who is in its choice set even if some of the other students in its choice set become unavailable. When colleges have substitutable preferences the set of stable matchings is non-empty. That is the Gale - Shapley student optimal algorithm produces a stable student optimal matching (similarly the Gale - Shapley college optimal algorithm produces a stable college optimal matching).

Note that the Gale-Shapley algorithm has been used since 1951 (before Gale and Shapley's paper) in the United States to match medical residents to hospitals (for the analysis of the matching program see Roth (1984)).

Kelso and Crawford (1982) considered a model for labor markets as a many to one matching market. There are a finite set of workers and a finite set of firms. Firms do not face quota or budget constraints. Each worker has a utility of working for a firm with a salary that is paid by that firm. It is assumed that all workers are gross substitutes from the viewpoint of each firm. This assumption is referred as *gross substitutes condition*. In order to define this condition formally, we need some notation which is introduced below following Kelso and Crawford: Let w denote a generic element of the set of workers and f a generic element of the set of firms. Firm f's gross product (measured in terms of salaries) is denoted by $y^f(C^f)$, where C^f is the set of workers hired by firm f. The net profit of firm f is defined by $\pi^f(C^f, s^f) = y^f(C^f) - \sum_{w \in C^f} s_{wf}$, where $s^f = (s_{1f}, \ldots, s_{mf})$ is the vector of salaries faced by firm f. When firm f faces a vector of salaries $s^f = (s_{1f}, \ldots, s_{mf})$, firm f chooses a set C^f of workers which maximizes its net profit. Let $M^f(s^f)$ denote the sets of workers that maximize net profit of firm f.

Consider two vectors of salaries s^f and $\tilde{s}^f$ faced by firm f. Let $T^f(C^f) = \{w \mid w \in C^f \text{ and } \tilde{s}_{wf} = s_{wf}\}$. The *gross substitutes condition* is that
for all firms, if $C^f \in M^f(s^f)$ and $\tilde{s}^f \geqq s^f$, then there exists $\tilde{C}^f \in M^f(\tilde{s}^f)$ such that $T^f(C^f) \subseteq \tilde{C}^f$.
That is firms regard workers as substitutes rather than as complements. "The gross substitutes assumption states that all workers be (weak) gross substitutes to each firm, in the sense that increases in other workers' salaries can never cause a firm to withdraw an offer from a worker whose salary has not risen." Thus the production technology is such that workers are not complements.

Kelso and Crawford (1982) showed the existence of a core allocation by an extension of the Gale - Shapley algorithm. That is there is a matching such that there is no subgroup consisting of firms and workers which blocks that matching. They also showed that there is a firm optimal core allocation, i.e., there is a core matching that each firm likes at least as well as any other core matching.

Mongell and Roth (1986) considered the model of Kelso and Crawford together with budget constraints for firms. They showed by an example that the core of the market may be empty. They also gave an example to show that if the set of core allocations is non-empty, it is possible that there be no firm optimal core matching.

In this thesis, we consider *graduate admission problem* as a two-sided matching market. There are a set of students and a set of departments which belong to one university. Each department faces quota and budget constraints which are determined centrally by the university. Students apply to these departments for their graduate studies and each student has a value added to each department. If a student matches with a department she may be paid by the department or she may pay to the department. If a student pays for her graduate study, that payment is not added to the department's budget for graduate admissions. That payment goes to the university which gives some percentage of that payment to the department for its office expenditures. Departments use their budgets for the payments to graduate students, and if a department has some of its budget left after these payments, the remaining part is used for office expenditures by the department. Each department gets a benefit from its accepted students and its office expenditures. The total benefit of a department from its accepted students is the sum of each accepted student's value added to the department. Each department wants to maximize its gross benefit which is sum of the benefits from accepted students and from office expenditures. We assume that, for any department, the largest benefit from office expenditures is less than any qualified student's benefit to the department no matter how large the office expenditures are. Therefore, each department wants to maximize its gross benefit by accepting more qualified students at a minimum cost. Each student wants to make graduate study at her most preferred department.

Our model differs from the previous models in the sense that departments face both quota and budget constraints. Here we construct some algorithms which are extensions of the Gale - Shapley algorithm and show that, if the algorithms stop, the resulting matchings are core stable (and thus Pareto optimal). However the algorithms do not always stop and it turns out to be possible that the algorithms do not stop while the set of core stable matchings is non-empty. Hence we can say that for the model considered in this paper (two sided matching market with quota and budget constraints) straightforward extensions of the Gale - Shapley algorithm do not work in contrast to college admissions and labor market models without budget constraints. Moreover, the existence of either a department optimal or a student optimal matching is not guaranteed in our setting. In summary, the presence of budget constraints seems to change the picture in a radical fashion.

The rest of the thesis is organized as follows: We present the model and definitions in chapter 2. Chapter 3 examines the relationships between different notions introduced in chapter 2. Chapter 4 defines the algorithm and presents the result that the final matching is core stable if the algorithm stops. Chapter 4 also presents two examples in the first of which the algorithm does not stop, and there is no core stable matching. In the second one the algorithm again does not stop but there is now a core stable matching. Chapter 5 considers a modified model where students wish only to have their reservation prices and defines another algorithm for this model. Chapter 6 starts with observations regarding our algorithms and then concludes the thesis.

Chapter 2

Basic Notions

We denote the finite nonempty set of departments of our university by $D = \{d_1, d_2, \ldots, d_m\}$. A finite nonempty set of students denoted by $S = \{s_1, s_2, \ldots, s_n\}$, is regarded as comprising the applicants to this university for graduate programs offered by its departments.

Each department $d \in D$ has a quota q_d and a budget b_d for its graduate program; both of which are determined centrally by the university. A student can enroll to at most one department, and each department accepts a group of students obeying its quota and budget constraints.

We assume that each student $s \in S$ has a qualification level for each department $d \in D$. The qualification level of student s for department d is an integer and denoted by a^s_d. The qualification levels of student s for the departments are denoted by a vector $a^s_D = (a^s_{d_1}, a^s_{d_2}, \ldots, a^s_{d_m})$. Also we assume that each department has a minimal qualification level as a threshold for accepting students. The minimal qualification level of department d is a positive integer and denoted by a^d.

Each student yields a benefit (or adds a value) to each department if accepted to that department. These values are independent of who the other accepted students are, i.e., there are no externalities in this regard. The benefit of department d obtained from accepting a group of students $S^d \subseteq S$ is denoted by $y^d(S^d)$. We assume that department d's benefit $y^d(S^d)$ is additive, i.e., it is the sum of the accepted students' benefits to the department. We assume that the benefit student s provides to department d is equal to her qualification level for department d, i.e., $y^d(\{s\}) = a^s_d$. Therefore the total benefit of department d from accepting a group of students $S^d \subseteq S$ is $y^d(S^d) = \sum_{s \in S^d} a^s_d$.

If a student gets enrolled to a department for graduate study, she may be paid by the department or she may pay to the department. The amount of payment made by department d to student s is an integer m_{sd}. In other words, student s is paid by department d the amount m_{sd} if $m_{sd} > 0$; there is no payment if $m_{sd} = 0$; student s pays to department d the amount m_{sd} if $m_{sd} < 0$. If an accepted student pays for her graduate study at department d, this payment is not added to department d's budget. That payment is taken by the university and the university gives some fixed percentage of this payment to department d, solely to be used, say, for its office expenditures.

We assume that each student s has a reservation price for each department d (the lowest amount of money that student s will accept from department d) which will be denoted by an integer σ_{sd}. We assume that for all $s \in S$ and for all $d \in D$, $\sigma_{sd} \leq b_d$. Student s's reservation prices for departments will be denoted by a vector $\sigma^s = (\sigma_{sd_1}, \ldots, \sigma_{sd_m})$. Note that a reservation price may also be negative, representing the level of willingness on the part of the student to pay to the department in question to get accepted.

If department d has some remaining budget after payments, the remaining money is only used for office expenditures by the department. Let B be the total budget of the university, and let student s be the least qualified student for department d among all students who are qualified for department d, i.e., $a^s_d \geq a^d$ and for all $h \in S \setminus \{s\}$ with $a^h_d \geq a^d$, we have $a^s_d \leq a^h_d$. Let ϵ^d_B be the benefit of department d if it uses the university's entire budget B for its office expenditures. We assume that $y^d(\{s\}) > \epsilon^d_B$. Therefore, the benefit which is gained by spending B for the office expenditures is less than any qualified student's benefit to department d. This means that one can take $a^d = 1$ and $0 < \epsilon_B^d < 1$ for each $d \in D$.

The total benefit of department d is denoted by Y^d and it is the sum of benefits from accepted students and office expenditures. Therefore when $S^d \subseteq S$ is the accepted group of students by department d and ϵ^d is the benefit that department d gets from office expenditures, we have that $Y^d(S^d, \epsilon^d) = y^d(S^d) + \epsilon^d$.

Definition 1 **A graduate admission problem** is a list $(D, S, q, b, a^S, \sigma)$ where

1. D is a finite nonempty set of departments,
2. S is a finite nonempty set of students,
3. $q = (q_d)_{d \in D}$ is the departments' quotas with $q_d \in \mathbb{N}$ for each $d \in D$,
4. $b = (b_d)_{d \in D}$ is the departments' budgets with $b_d \in \mathbb{N}_0$ [1] for each $d \in D$,
5. $a^S = (a_D^s)_{s \in S}$ is the students' qualification levels for departments with $a_d^s \in \mathbb{Z}$ for each $s \in S$, $d \in D$,
6. $\sigma = (\sigma^s)_{s \in S}$ is the students' reservation prices for departments with $\sigma_{sd} \in \mathbb{Z}$ and $\sigma_{sd} \leq b_d$ for each $s \in S$, $d \in D$.

The preferences of departments and students are implicitly contained in the definition of a graduate admission problem and can be made explicit as follows:

The strict preference relation of department d is denoted by P_d. For all $d \in D$, P_d is a linear order [2] on $2^S \times \mathbb{R}$.

Consider two group of students S^d and $\acute{S}^d$. Let c^d denote the cost of group of students S^d to department d, i.e., $c^d = \sum_{s \in \bar{S}^d} m_{sd}$ with $\bar{S}^d = \{s \in S^d \mid m_{sd} > 0\}$, and $\acute{c}^d$ the cost of group of students $\acute{S}^d$ to department d, i.e., $\acute{c}^d = \sum_{s \in \bar{\acute{S}}^d} \acute{m}_{sd}$ with $\bar{\acute{S}}^d = \{s \in \acute{S}^d \mid \acute{m}_{sd} > 0\}$. Let ϵ^d denote the benefit of office expenditures that department d obtains by accepting the group of students S^d at cost c^d, and

[1] $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$

[2] A linear order on a set X is a complete, transitive and antisymmetric (binary) relation.

$\acute{\epsilon}^d$ the benefit of office expenditures that department d obtains by accepting the group of students $\acute{S}^d$ at cost $\acute{c}^d$.

Note that $(S^d, c^d) = (\acute{S}^d, \acute{c}^d)$ does not imply that $Y^d(S^d, \epsilon^d) = Y^d(\acute{S}^d, \acute{\epsilon}^d)$. To clarify this point consider the following example:
Let $S^d = \acute{S}^d = \{s_1, s_2\}$, and $m_{s_1d} = 50 = m_{s_2d}$; $\acute{m}_{s_1d} = 100$, $\acute{m}_{s_2d} = -100$. Note that $c^d = m_{s_1d} + m_{s_2d} = 50 + 50 = 100$ and $\acute{c}^d = \acute{m}_{s_1d} = 100$. We have that $(S^d, c^d) = (\acute{S}^d, \acute{c}^d)$. However $Y^d(\acute{S}^d, \acute{\epsilon}^d) > Y^d(S^d, \epsilon^d)$, since $\acute{\epsilon}^d > \epsilon^d$.

Now department d strictly prefers S^d to $\acute{S}^d$ if $Y^d(S^d, \epsilon^d) > Y^d(\acute{S}^d, \acute{\epsilon}^d)$.

If $Y^d(S^d, \epsilon^d) = Y^d(\acute{S}^d, \acute{\epsilon}^d)$ then department d considers the associated costs of S^d and $\acute{S}^d$. That is whenever $Y^d(S^d, \epsilon^d) = Y^d(\acute{S}^d, \acute{\epsilon}^d)$ department d strictly prefers S^d to $\acute{S}^d$ if $c^d < \acute{c}^d$.

If $Y^d(S^d, \epsilon^d) = Y^d(\acute{S}^d, \acute{\epsilon}^d)$ and $c^d = \acute{c}^d$, then department d makes a lexicographic comparison among S^d and $\acute{S}^d$ in the following way:
Let $\mid S^d \mid = n_1$ and $\mid \acute{S}^d \mid = n_2$. Let $f : \{1, \ldots, n_1\} \rightarrow \{i \mid s_i \in S^d\}$ be a function such that $f(1) < f(2) < \ldots < f(n_1)$. Let $g : \{1, \ldots, n_2\} \rightarrow \{j \mid s_j \in \acute{S}^d\}$ be a function such that $g(1) < g(2) < \ldots < g(n_2)$.
We say that department d **leximin prefers** S^d to $\acute{S}^d$ if and only if $f(1) < g(1)$ or $\exists k \in \{1, \ldots, n\}$ where $n < \min\{n_1, n_2\}$ such that $\forall t \in \{1, \ldots, k\}$ $f(t) = g(t)$ but $f(t+1) < g(t+1)$.

Now we can define P_d formally as follows:
$\forall (S^d, c^d), (\acute{S}^d, \acute{c}^d) \in (2^S \times \mathbb{R})$ with $(S^d, c^d) \neq (\acute{S}^d, \acute{c}^d)$,
$[(S^d, c^d) P_d (\acute{S}^d, \acute{c}^d)]$ if and only if $[Y^d(S^d, \epsilon^d) > Y^d(\acute{S}^d, \acute{\epsilon}^d)]$ or
$[Y^d(S^d, \epsilon^d) = Y^d(\acute{S}^d, \acute{\epsilon}^d)$ and $c^d < \acute{c}^d]$ or
$[Y^d(S^d, \epsilon^d) = Y^d(\acute{S}^d, \acute{\epsilon}^d)$ and $c^d = \acute{c}^d$ and S^d leximin preferred to $\acute{S}^d]$.

R_d is a preference relation of department d induced from P_d and defined as follows:
$\forall (S^d, c^d), (\acute{S}^d, \acute{c}^d) \in (2^S \times \mathbb{R})$,
$[(S^d, c^d) R_d (\acute{S}^d, \acute{c}^d)]$ if and only if $\neg[(\acute{S}^d, \acute{c}^d) P_d (S^d, c^d)]$.

Hence, for any $(S^d, c^d), (\acute{S}^d, \acute{c}^d) \in (2^S \times \mathbb{R})$ with $(S^d, c^d) \neq (\acute{S}^d, \acute{c}^d)$, we have either $[(S^d, c^d) P_d (\acute{S}^d, \acute{c}^d)]$ or $[(\acute{S}^d, \acute{c}^d) P_d (S^d, c^d)]$.

The strict preference relation of student s is denoted by P_s. For all $s \in S$, P_s is a linear order on $(D \times \mathbb{R}) \bigcup \{(\emptyset, 0)\}$.

We assume that, given any $s \in S$, $\sigma_{sd} = \sigma_{s\tilde{d}}$ if and only if $d = \tilde{d}$. We also assume that $(d, \sigma_{sd}) P_s (\emptyset, 0)$ for all $s \in S$ and all $d \in D$, where $(\emptyset, 0)$ stands for the situation that student s is unmatched (or she is matched with herself).[3]

For all $s \in S$, P_s is defined as follows:
$\forall (d, m_{sd}), (\tilde{d}, m_{s\tilde{d}}) \in (D \times \mathbb{R}) \bigcup \{(\emptyset, 0)\}$,
$[(d, m_{sd}) P_s (\tilde{d}, m_{s\tilde{d}})]$ if and only if $[m_{sd} - \sigma_{sd} > m_{s\tilde{d}} - \sigma_{s\tilde{d}}]$ or
$[m_{sd} - \sigma_{sd} = m_{s\tilde{d}} - \sigma_{s\tilde{d}}$ and $\sigma_{sd} < \sigma_{s\tilde{d}}]$.

R_s is a preference relation of student s induced from P_s and defined as follows:
$\forall (d, m_{sd}), (\tilde{d}, m_{s\tilde{d}}) \in (D \times \mathbb{R}) \bigcup \{(\emptyset, 0)\}$,
$[(d, m_{sd}) R_s (\tilde{d}, m_{s\tilde{d}})]$ if and only if $\neg[(\tilde{d}, m_{s\tilde{d}} P_d (d, m_{sd})]$.

Note that whenever $(d, m_{sd}) \neq (\tilde{d}, m_{s\tilde{d}})$, we have either $[(d, m_{sd}) P_s (\tilde{d}, m_{s\tilde{d}})]$ or $[(\tilde{d}, m_{s\tilde{d}} P_d (d, m_{sd})]$.

Note that being unmatched is not the worst situation for a student $s \in S$, because for all $s \in S$, $[(\emptyset, 0) P_s (d, m_{sd})]$ if $[m_{sd} < \sigma_{sd}]$ for any $d \in D$.

Now we will define what we mean by a matching.

Definition 2 By a **matching** we mean a function $\mu : S \longrightarrow (D \times \mathbb{R}) \bigcup \{(\emptyset, 0)\}$ which matches each student s with a member $\mu_1(s)$ of $D \bigcup \{\emptyset\}$ and also specifies the amount of transfer $\mu_2(s)$ made from $\mu_1(s)$ to s such that the following are satisfied:

1. For all $d \in D$, $\mid S^d_\mu \mid \leq q_d$ (quota constraint),
   where $S^d_\mu = \{s \in S \mid \mu_1(s) = d\}$,

2. For all $d \in D$, $c^d_\mu \leq b_d$ (budget constraint),

[3]We assume that for all $s \in S$, $\sigma_{s\emptyset} = 0$.

where $c^d_\mu = \sum_{s \in \bar{S}^d_\mu} m^\mu_{sd}$ with $m^\mu_{sd} = \mu_2(s)$ for $\mu_1(s) = d$ and $\bar{S}^d_\mu$=$\{s \in S^d_\mu \mid m^\mu_{sd} > 0\}$.

3. For all $d \in D$, for all $s \in S^d_\mu$, $a^s_d \geq a^d$ (qualification level constraint).

Student s is matched with a department if $\mu_1(s) \in D$, she is unmatched if $\mu_1(s) = \emptyset$ under μ.

Let Y^d_μ denote the total benefit of department d under μ. Let y^d_μ denote the benefit of department d that it obtains by accepting the group of students S^d_μ and ϵ^d_μ the benefit of department d that it gets from office expenditures under μ.

Department d's preference relation R_d induces a preference relation R^μ_d over matchings in a natural fashion as follows:
For any matchings $\bar{\mu}$ and $\tilde{\mu}$,
$[\bar{\mu} R^\mu_d \tilde{\mu}]$ if and only if $[(S^d_{\bar{\mu}}, c^d_{\bar{\mu}}) R_d (S^d_{\tilde{\mu}}, c^d_{\tilde{\mu}})]$. We abuse notation and we use R_d for R^μ_d.

Students s's preference relation R_s similarly induces a preference relation R^μ_s over matchings as follows:
For any matchings $\bar{\mu}$ and $\tilde{\mu}$,
$[\bar{\mu} R^\mu_s \tilde{\mu}]$ if and only if $[(\bar{\mu}_1(s), m^{\bar{\mu}}_{s\bar{\mu}_1(s)}) R_s (\tilde{\mu}_1(s), m^{\tilde{\mu}}_{s\tilde{\mu}_1(s)})]$. We abuse notation and we use R_s for R^μ_s.

To present a matching μ, we will use a matrix consisting of three rows and n columns, where $n =\mid S \mid$. The first row lists the set of students respecting their original labeling; the second row specifies the departments the students are assigned to and the third row consists of the associated money transfers. That is

$$\mu = \begin{pmatrix} s_1 & s_2 & \dots & s_n \\ \mu_1(s_1) & \mu_1(s_2) & \dots & \mu_1(s_n) \\ m^\mu_{s_1\mu_1(s_1)} & m^\mu_{s_2\mu_1(s_2)} & \dots & m^\mu_{s_n\mu_1(s_n)} \end{pmatrix}$$

Definition 3 A matching μ is **individually rational** if and only if it satisfies the following properties

1. For all $s \in S$, $m^{\mu}_{s\mu_1(s)} \geq \sigma_{s\mu_1(s)}$, and
2. For all $d \in D$, $Y^d_{\mu} \geq 0$.

Definition 4 We say that a matching μ is **blocked** by a student - department pair $(s, d) \in S \times D$ with $\mu_1(s) \neq d$ if and only if there exists a payment $\tilde{m}_{sd}$ such that

1. $(d, \tilde{m}_{sd}) P_s (\mu_1(s), m^{\mu}_{s\mu_1(s)})$, and
2. $[(S^d_{\mu} \setminus B) \bigcup \{s\}, \hat{c}^d] P_d [S^d_{\mu}, c^d_{\mu}]$,
   for some $B \subseteq S^d_{\mu}$, with

$$\hat{c}^d = \begin{cases} \sum_{h \in (\bar{S}^d_{\mu} \setminus B)} m^{\mu}_{hd} + \tilde{m}_{sd} & \text{if } \tilde{m}_{sd} > 0 \\ \sum_{h \in (\bar{S}^d_{\mu} \setminus B)} m^{\mu}_{hd} & \text{otherwise} \end{cases}$$

   such that $[((S^d_{\mu} \setminus B) \bigcup \{s\}), \hat{c}^d]$ satisfies the quota and budget constraints of department d, i.e., $\mid (S^d_{\mu} \setminus B) \bigcup \{s\} \mid \leq q_d$ and $\hat{c}^d \leq b_d$.

A pair (s, d) that satisfies above two conditions is called **a blocking pair** for matching μ.

Definition 5 A matching μ is **pairwise stable** if and only if it is individually rational and there is no pair (s, d) which blocks it.

Now we will define group blocking of a matching μ.

Definition 6 We say that a matching μ is blocked by a group $(\tilde{D}, \tilde{S})$ with $\tilde{D} \subseteq D$ and $\tilde{S} \subseteq S$ if and only if the following two conditions are satisfied:

1. For all $s \in \hat{S}^d$, $[(d, \tilde{m}_{sd}) P_s (\mu_1(s), m^{\mu}_{s\mu_1(s)})]$,
   where $d \in \tilde{D}$, and $\hat{S}^d \subseteq \tilde{S}$ with for all $s \in \hat{S}^d$, $\mu_1(s) \neq d$,
2. For all $d \in \tilde{D}$, $[(S^d_{\mu} \setminus B) \bigcup \hat{S}^d, \hat{c}^d] P_d [S^d_{\mu}, c^d_{\mu}]$,
   for some $B \subseteq S^d_{\mu}$ with $\hat{c}^d = \sum_{h \in (\bar{S}^d_{\mu} \setminus B)} m^{\mu}_{hd} + \sum_{s \in \bar{\hat{S}}^d} \tilde{m}_{sd}$ such that $\bar{\hat{S}}^d = \{s \in \hat{S}^d \mid \tilde{m}_{sd} > 0\}$ and $[((S^d_{\mu} \setminus B) \bigcup \hat{S}^d), \hat{c}^d]$ satisfies the quota and budget constraints of department d, i.e., $\mid [(S^d_{\mu} \setminus B) \bigcup \hat{S}^d] \mid \leq q_d$ and $\hat{c}^d \leq b_d$.

Note that $\hat{S}^d \subseteq \tilde{S}$ denote the group of students who matched with department $d \in \tilde{D}$ by group blocking of μ, so for all $s \in \hat{S}^d$, $\mu_1(s) \neq d$, and $\bigcup_{d \in \tilde{D}} \hat{S}^d = \tilde{S}$. The amount of money $\tilde{m}_{sd}$ denote the transfer between department $d \in \tilde{D}$ and a student $s \in \hat{S}^d$.

Definition 7 We say that a matching μ is **core stable** if and only if μ is individually rational and there exists no group $(\tilde{D}, \tilde{S})$ which blocks μ.

Proposition 1 *A matching μ is core stable if and only if μ is individually rational and there exists no pair (consisting of a department d and a group of students $\tilde{S} \subseteq S$) $(d, \tilde{S})$ which blocks μ.*[4]

Proof Take any core stable matching μ. By definition, μ is individually rational and there exists no pair $(d, \tilde{S})$ which blocks μ.

For the other part of the proof, take any individually rational matching μ such that there exists no pair $(d, \tilde{S})$ which blocks μ. Suppose that μ is not core stable. Then there exists a group (consisting of a group of departments $\tilde{D} \subseteq D$ and a group of students $\tilde{S}$) $(\tilde{D}, \tilde{S})$ which blocks μ. The cardinality of the group of departments must be equal or greater than two, i.e., $\mid \tilde{D} \mid \geq 2$. Otherwise we have a contradiction with the absence of a pair $(d, \tilde{S})$ which blocks μ.

W.l.o.g. assume that $\tilde{D} = \{d, \acute{d}\}$. A student can match with at most one department, so a student $s \in \tilde{S}$ matches with either department d or department $\acute{d}$. Let $\hat{S}^d \subset \tilde{S}$ denote the group of students who matched with department d and $\hat{S}^{\acute{d}} \subset \tilde{S}$ denote the group of students who matched with department $\acute{d}$ in the blocking matching. Now, the following two conditions are satisfied.

1. For all $s \in \hat{S}^d$,$(d, \tilde{m}_{sd}) P_s (\mu_1(s), m_{s\mu_1(s)})$, and
   for $d \in \tilde{D}$, $[(S^d_\mu \setminus B) \bigcup \hat{S}^d, \hat{c}^d] P_d [S^d_\mu, c^d_\mu]$, where $B \subseteq S^d_\mu$,

2. For all $s \in \hat{S}^{\acute{d}}$, $(\acute{d}, \tilde{m}_{s\acute{d}}) P_s (\mu_1(s), m_{s\mu_1(s)})$, and
   for $\acute{d} \in \tilde{D}$, $[(S^{\acute{d}}_\mu \setminus C) \bigcup \hat{S}^{\acute{d}}, \hat{c}^{\acute{d}}] P_{\acute{d}} [S^{\acute{d}}_\mu, c^{\acute{d}}_\mu]$, where $C \subseteq S^{\acute{d}}_\mu$.

[4] *In other words, an essential coalition for group blocking consists of a department and a group of students.*

However above two conditions mean that both $(d, \hat{S}^d)$ and $(\acute{d}, \hat{S}^{\acute{d}})$ block μ, yielding the desired contradiction. Hence μ is core stable. $\square$

Definition 8 We say that a matching μ is Pareto dominated by another matching $\tilde{\mu}$ if and only if

1. for all $i \in (S \bigcup D)$, $\tilde{\mu} R_i \mu$, and
2. for some $i \in (S \bigcup D)$, $\tilde{\mu} P_i \mu$.

Definition 9 A matching μ is **Pareto optimal** if and only if there exists no matching $\tilde{\mu}$ which Pareto dominates μ.

Chapter 3

Relationships Between Pairwise Stability, Core Stability and Pareto Optimality

In this chapter we examine the relationships between the notions of pairwise stability, core stability and Pareto optimality.

Proposition 2 *If a matching μ is core stable, then μ is pairwise stable.*

Proof Obvious. $\square$

However the converse of the above proposition is not true, i.e., a pairwise stable matching may not be core stable.

Example 1 A pairwise stable but not core stable matching

Let $D = \{d_1, d_2\}$ be the set of departments and $S = \{s_1, s_2, s_3\}$ the set of students. The budgets and quotas of the departments are as follows: $b_{d_1} = 30$, $b_{d_2} = 50$; $q_{d_1} = 2$, $q_{d_2} = 2$. The qualification levels and reservation prices of the students are as given in table 3.1.

Consider the following matching μ:

$a_{d_1}^{s_1}$=15	$a_{d_2}^{s_1}$= 8
$a_{d_1}^{s_2}$=12	$a_{d_2}^{s_2}$=30
$a_{d_1}^{s_3}$=20	$a_{d_2}^{s_3}$=25
$\sigma_{s_1d_1}$=12	$\sigma_{s_1d_2}$=10
$\sigma_{s_2d_1}$=25	$\sigma_{s_2d_2}$=40
$\sigma_{s_3d_1}$=11	$\sigma_{s_3d_2}$=20

Table 3.1: Qualification levels and reservation prices of students for example 1

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & d_2 & d_1 \\ 0 & 50 & 30 \end{pmatrix}$$

The matching μ is pairwise stable since there exists no pair $(s, d) \in S \times D$ that blocks μ. Also note that μ is Pareto optimal. However μ is not core stable. Since the group $(d_2, \{s_1, s_3\})$ blocks μ with payments $\tilde{m}_{s_1d_2} = 10$ and $\tilde{m}_{s_3d_2} = 40$. Department d_2 prefers the group of students $\{s_1, s_3\}$ to student $\{s_2\}$, i.e., $[(s_1, s_3), 50]P_{d_2}[s_2, 40]$, since $a_{d_2}^{s_1} + a_{d_2}^{s_3} = 33 > 30 = a_{d_2}^{s_2}$. Student $\{s_1\}$ prefers to be matched with department d_2 at payment $\tilde{m}_{s_1d_2} = 10$ to be unmatched, i.e., $(d_2, 10)P_{s_1}(\emptyset, 0)$. Student $\{s_3\}$ prefers to be matched with department d_2 at payment $\tilde{m}_{s_3d_2} = 40$ to be matched with department d_1 at payment $m_{s_3d_1}^{\mu} = 30$, i.e., $(d_2, 40)P_{s_3}(d_1, 30)$ since $\tilde{m}_{s_3d_2} - \sigma_{s_3d_2} = 40 - 20 = 20 > 19 = 30 - 11 = m_{s_3d_1}^{\mu} - \sigma_{s_3d_1}$.

Hence a pairwise stable matching need not be core stable. This example also shows that a pairwise stable and Pareto optimal matching need not be core stable.

Example 2 A pairwise stable but not Pareto optimal matching

Let $D = \{d_1, d_2\}$ be the set of departments and $S = \{s_1, s_2, s_3, s_4\}$ the set of students. The quotas and budgets of the departments are as follows: $q_{d_1} = 2$, $q_{d_2} = 2$; $b_{d_1} = b_{d_2} = 100$. The qualification levels and reservation prices of the students are as given in table 3.2.

$a^{s_1}_{d_1}$= 0	$a^{s_1}_{d_2}$=10
$a^{s_2}_{d_1}$=20	$a^{s_2}_{d_2}$=15
$a^{s_3}_{d_1}$=10	$a^{s_3}_{d_2}$= 0
$a^{s_4}_{d_1}$=15	$a^{s_4}_{d_2}$=20
$\sigma_{s_1 d_1}$=40	$\sigma_{s_1 d_2}$=25
$\sigma_{s_2 d_1}$=80	$\sigma_{s_2 d_2}$=70
$\sigma_{s_3 d_1}$=25	$\sigma_{s_3 d_2}$=40
$\sigma_{s_4 d_1}$=70	$\sigma_{s_4 d_2}$=80

Table 3.2: Qualification levels and reservation prices of students for example 2

Consider the following matching μ:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ \emptyset & d_1 & \emptyset & d_2 \\ 0 & 80 & 0 & 80 \end{pmatrix}$$

The matching μ is pairwise stable since there is no pair $(s, d) \in S \times D$ which blocks μ.[1] However μ is not Pareto optimal, i.e., there is another matching which Pareto dominates μ.

Now consider the following matching $\tilde{\mu}$:

$$\tilde{\mu} = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ d_2 & d_2 & d_1 & d_1 \\ 25 & 75 & 25 & 75 \end{pmatrix}$$

The matching $\tilde{\mu}$ Pareto dominates the matching μ, see this:
$(d_2, 25)P_{s_1}(\emptyset, 0)$, $(d_2, 75)P_{s_2}(d_1, 80)$,
$(d_1, 25)P_{s_3}(\emptyset, 0)$, $(d_1, 75)P_{s_4}(d_2, 80)$,
$(S^{d_1}_{\tilde{\mu}}, c^{d_1}_{\tilde{\mu}})P_{d_1}(S^{d_1}_{\mu}, c^{d_1}_{\mu})$, $(S^{d_2}_{\tilde{\mu}}, c^{d_2}_{\tilde{\mu}})P_{d_2}(S^{d_2}_{\mu}, c^{d_2}_{\mu})$.

So we have that for all $i \in (S \bigcup D)$, $\tilde{\mu} P_i \mu$, i.e., $\tilde{\mu}$ Pareto dominates μ.[2]

[1] However the matching μ is clearly not core stable as it will turn out not to be Pareto optimal.

[2] The matching $\tilde{\mu}$ is core stable.

Hence a pairwise stable matching need not be Pareto optimal.

Example 3 A Pareto optimal but not pairwise stable matching

Let $D = \{d_1, d_2\}$ be the set of departments, $S = \{s_1, s_2, s_3\}$ the set of students, and the quotas and budgets of the departments are as follows: $q_{d_1} = 1$, $q_{d_2} = 1$; $b_{d_1} = 30$, $b_{d_2} = 50$. The qualification levels and reservation prices of the students are as given in table 3.3.

$a_{d_1}^{s_1} = 4$	$a_{d_2}^{s_1} = 3$
$a_{d_1}^{s_2} = 8$	$a_{d_2}^{s_2} = 10$
$a_{d_1}^{s_3} = 15$	$a_{d_2}^{s_3} = 15$
$\sigma_{s_1 d_1} = 10$	$\sigma_{s_1 d_2} = 15$
$\sigma_{s_2 d_1} = 20$	$\sigma_{s_2 d_2} = 30$
$\sigma_{s_3 d_1} = 30$	$\sigma_{s_3 d_2} = 40$

Table 3.3: Qualification levels and reservation prices of students for example 3

Consider the following matching μ:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & d_2 & d_1 \\ 0 & 30 & 30 \end{pmatrix}$$

The matching μ is Pareto optimal since there is no other matching that Pareto dominates μ. However μ is not pairwise stable because the pair (s_3, d_2) blocks the matching μ with $\tilde{m}_{s_3 d_2} = 41$. To see this, note that $(d_2, 41) P_{s_3} (d_1, 30)$ and $(s_3, 41) P_{d_2} (s_2, 30)$.

Hence a Pareto optimal matching need not be pairwise stable.

It is obvious that if a matching μ is core stable then μ is Pareto optimal.

Chapter 4

Graduate Admission Algorithm

In this section we define an algorithm, to which we will refer to as the graduate admission algorithm (GAA) which is an extension of the Gale - Shapley algorithm for the graduate admission problem. The algorithm GAA is a centralized algorithm, i.e., the departments' and students' preferences are assumed to be known to a planner (or to a computer program) who matches students with departments according to the rule of GAA. Hence, there is no agent who behaves strategically to manipulate the algorithm.

We will show that when the algorithm GAA stops then the resulting matching is core stable (and thus Pareto optimal). However GAA does not always stop. To clarify this situation, we will give two examples at one of which the algorithm GAA does not stop and there is no core stable matching, while in the other example the algorithm GAA does not stop, but there is a core stable matching.

Time is measured discretely in the algorithm. Let $m_{sd}(t)$ denote the offer that department d makes to student s at time t.

According to the scenario behind our algorithm, given b_d, q_d and what offers are permitted, at each time t, department d will maximize its total benefit $Y_t^d = y^d(S_t^d)+\epsilon_t^d$ when it makes a permitted offer to a group of students S_t^d such that the quota and budget constraints are satisfied, i.e., $\mid (S_t^d) \mid \leq q_d$ and $\sum_{s\in \bar{S}_t^d} m_{sd}(t) \leq$

b_d.

Now we can give the details of how the algorithm GAA works.

Graduate Admission Algorithm

$t = 1$: a) Each department d determines the group of students S_1^d that maximizes its total benefit subject to its quota and budget constraints with $m_{sd}(1) = \sigma_{sd}$ for all $s \in S_1^d$. That is, department d offers to students in S_1^d first their reservation prices.

b) Students who have taken one or more offers accept at most one offer and reject the others.

c) Department d tentatively accepts the group of students who accepted its offers. Let T_1^d denote the group of students who accepted department d's offers at time $t = 1$, $T_1^d \subseteq S_1^d$.[1]

Now, at the end of time $t = 1$ we have a matching μ_1 with $S_{\mu_1}^d = T_1^d$.

$t = 2$: a) Again each department d determines the group of students S_2^d that maximizes its total benefit subject to its constraints where the offers now be of the form:

$$m_{sd}(2) = \begin{cases} \sigma_{sd} + 1 & \text{if } s \in S_1^d \setminus T_1^d \\ \sigma_{sd} & \text{otherwise} \end{cases}$$

b) Students who have taken one or more offers accept at most one offer and reject the others.

c) Department d tentatively accepts the group of students $T_2^d \subseteq S_2^d$ who accepted its offers.

In general, at time k,

$t = k$: a) Consider a student s to whom department d made offers before period k the last of which took place in period $\tilde{t}_s < k$. In case this offer was

[1] $S_1^d \setminus T_1^d$ is now the group of students who took an offer from department d and rejected it at time $t = 1$.

rejected by s because she accepted department $\hat{d}$'s offer with which she got again matched at the end of period $k-1$, i.e., $\mu_{k-1}(s) = \hat{d}$, call such a student a rejector of d prior to k. Let F_k^d denote the group of all rejectors of d prior to k.[2]

Each department d determines the group of students S_k^d solving the same kind of optimization problem as before, where the offers are now of the following form:

$$m_{sd}(k) = \begin{cases} \sigma_{sd} & \text{if } s \notin \bigcup_{t=1}^{t=k-1} S_t^d \\ m_{sd}(\tilde{t}_s) + 1 & \text{if } s \in F_k^d \\ m_{sd}(\tilde{t}_s) & \text{otherwise} \end{cases}$$

Note that department d offers $m_{sd}(k)$ to each student $s \in S_k^d$.

b) Students who have taken one or more offers accept at most one offer and reject the others.

c) Department d tentatively accepts the group of students $T_k^d \subseteq S_k^d$ who accepted its offers.

Stopping Rule

$t = t^\star$: The algorithm stops at time $t^\star$ if each department d makes offers to exactly the set of students who accepted its offers in the preceding period, i.e., if we have for all $d \in D$, $S_{t^\star}^d = T_{t^\star-1}^d$.

If the algorithm stops at $t^\star$ the final matching $\mu_{t^\star}$ is regarded as the outcome of the algorithm.

Proposition 3 *If the algorithm GAA stops, then the final matching of the algorithm is core stable (and thus Pareto optimal).*

Proof Assume that the algorithm stops. Let the algorithm stop at time $t^\star$ with $\mu_{t^\star}$ denoting the final matching of the algorithm. So we have that, for all $d \in D$, $S_{t^\star}^d = T_{t^\star-1}^d$. We abuse notation that we use $\mu^\star$ for $\mu_{t^\star}$.

[2] Note that at time $t = 1$, we have that for all $d \in D$, $F_1^d = \emptyset$, and at time $t = 2$, for all $d \in D$, $F_2^d = S_1^d \setminus T_1^d$.

Clearly $\mu^\star$ is individually rational, since $m^{\mu^\star}_{s\mu_1^\star(s)} \geq \sigma_{s\mu_1^\star(s)}$ for all $s \in S$, and $Y^d_{\mu^\star} \geq 0$ for all $d \in D$.

Now suppose that $\mu^\star$ is not core stable. So there is a group $(d, \tilde{S})$ which blocks $\mu^\star$. So we have that

1. for all $s \in \tilde{S}$, $\mu_1^\star(s) \neq d$,
2. for all $s \in \tilde{S}$, $(d, \tilde{m}_{sd}) P_s(\mu_1^\star(s), m^{\mu^\star}_{s\mu_1^\star(s)})$,
3. $[(S^d_{\mu^\star} \setminus B) \bigcup \tilde{S}, \hat{c}^d] P_d [S^d_{\mu^\star}, c^d_{\mu^\star}]$, for some $B \subseteq S^d_{\mu^\star}$.

Note that the algorithm requires department d to make the offers $\tilde{m}_{sd}$ to each student $s \in \tilde{S}$ at time $t^\star$. Now, there are three possible cases.

Case 1. If there is a student $s \in \tilde{S}$ such that $s \notin \bigcup_{t=1}^{t=t^\star-1} S^d_t$, then we have that $\tilde{m}_{sd} = \sigma_{sd}$.

Case 2. Now assume that there is a student $s \in \tilde{S}$ such that $s \in F^d_{t^\star}$, and let $\tilde{t}_s$ denote the time that department d made an offer to student s the last time before time $t^\star$. Now $\tilde{m}_{sd} = m_{sd}(\tilde{t}_s) + 1$.

Case 3. If there is a student $s \in \tilde{S}$ such that $s \notin F^d_{t^\star}$ and department d made an offer to student s at time $\tilde{t}_s$ the last time before time $t^\star$, then $\tilde{m}_{sd} = m_{sd}(\tilde{t}_s)$.

Therefore department d would make the offers $\tilde{m}_{sd}$ to each student $s \in \tilde{S}$ (by 3), and each student in $\tilde{S}$ would accept the offer (by 2). So department d and the group $\tilde{S}$ of students would match at the outcome of the algorithm, in contradiction with (1). Hence $\mu^\star$ is core stable, (and thus also Pareto optimal).

□

However the algorithm does not always stop. The following example demonstrates this situation. Also note that there is no core stable matching for the following example.

Example 4 [3] **The algorithm GAA does not stop and there is no core stable matching**

Let $D = \{d_1, d_2\}$, $S = \{s_1, s_2, s_3\}$, $q_{d_1} = 1$, $q_{d_2} = 2$, $b_{d_1} = 440$, $b_{d_2} = 1075$, and the qualification levels and reservation prices of the students are as given in table 4.1.

$a^{s_1}_{d_1}= 7$	$a^{s_1}_{d_2}= 6$
$a^{s_2}_{d_1}= 0$	$a^{s_2}_{d_2}= 15$
$a^{s_3}_{d_1}= 8$	$a^{s_3}_{d_2}= 11$
$\sigma_{s_1d_1}=400$	$\sigma_{s_1d_2}= 300$
$\sigma_{s_2d_1}=440$	$\sigma_{s_2d_2}=1000$
$\sigma_{s_3d_1}=400$	$\sigma_{s_3d_2}= 700$

Table 4.1: Qualification levels and reservation prices of students for example 4

Now we apply the graduate admission algorithm:

$t = 1$: a) The solution set of department d_1's optimization is $\{s_3\}$ and the optimizing set for department d_2 is $\{s_1, s_3\}$, i.e., $S^{d_1}_1 = \{s_3\}$, $S^{d_2}_1 = \{s_1, s_3\}$.

Department d_1 offers $\sigma_{s_3d_1} = 400$ to student s_3, and
department d_2 offers $\sigma_{s_1d_2} = 300$ to student s_1 and $\sigma_{s_3d_2} = 700$ to student s_3.

b) Student s_1 accepts department d_2's offer $\sigma_{s_1d_2} = 300$,
student s_2 has no offer,
student s_3 accepts department d_1's offer $\sigma_{s_3d_1} = 400$ and rejects department d_2's offer $\sigma_{s_3d_2} = 700$.

c) Department d_1 accepts $\{s_3\}$ and department d_2 accepts $\{s_1\}$, i.e., $T^{d_1}_1 = \{s_3\}$, $T^{d_2}_1 = \{s_1\}$.

We have a matching

[3] This example is a modification of the example of Mongell and Roth (1986).

$$\mu_1 = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_2 & \emptyset & d_1 \\ 300 & 0 & 400 \end{pmatrix}$$

$t = 2$: a) Now $S_2^{d_1} = \{s_3\}$, $S_2^{d_2} = \{s_1, s_3\}$.

Thus department d_1 offers $m_{s_3 d_1}(2) = \sigma_{s_3 d_1} = 400$ to student s_3, department d_2 offers $m_{s_1 d_2}(2) = \sigma_{s_1 d_2} = 300$ to student s_1 and $m_{s_3 d_2}(1) = \sigma_{s_3 d_2} + 1 = 701$ to student s_3.

b) Student s_1 accepts department d_2's offer,
student s_2 has no offer,
student s_3 accepts department d_2's offer and rejects department d_1's offer.

c) Hence $T_2^{d_1} = \emptyset$, $T_2^{d_2} = \{s_1, s_3\}$.

So we have a new matching

$$\mu_2 = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_2 & \emptyset & d_2 \\ 300 & 0 & 701 \end{pmatrix}$$

Note that, in further periods, department d_1 and department d_2 compete for student s_3 by increasing their offers. The maximal offer that department d_1 makes to student s_3 is equal to its budget. Hence, eventually at some time $t = l$ we have following:

$t = l$: a) Now $S_l^{d_1} = \{s_3\}$, $S_l^{d_2} = \{s_1, s_3\}$.

Department d_1 offers $m_{s_3 d_1}(l) = b_{d_1} = 440$ to student s_3,
department d_2 offers $m_{s_1 d_2}(l) = 300$ to student s_1 and $m_{s_3 d_2}(l) = 741$ to student s_3.

b) Student s_1 accepts department d_2's offer,

student s_2 has no offer,
student s_3 accepts department d_2's offer and rejects department d_1's offer.

c) So, $T_l^{d_1} = \emptyset$, $T_l^{d_2} = \{s_1, s_3\}$.

At time l we have a matching

$$\mu_l = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_2 & \emptyset & d_2 \\ 300 & 0 & 741 \end{pmatrix}$$

$t = l + 1$: a) $S_{l+1}^{d_1} = \{s_1\}$, $S_{l+1}^{d_2} = \{s_1, s_3\}$, implying that department d_1 offers $m_{s_1 d_1}(l+1) = \sigma_{s_1 d_1} = 400$ to student s_1,[4]
department d_2 offers $m_{s_1 d_2}(l+1) = 300$ to student s_1 and $m_{s_3 d_2}(l+1) = 741$ to student s_3.

b) Student s_1 accepts department d_2's offer and rejects department d_1's offer,
student s_2 has no offer, and
student s_3 accepts department d_2's offer.

c) So, $T_{l+1}^{d_1} = \emptyset$, $T_{l+1}^{d_2} = \{s_1, s_3\}$, yielding the matching

$$\mu_{l+1} = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_2 & \emptyset & d_2 \\ 300 & 0 & 741 \end{pmatrix}$$

Now department d_1 and department d_2 compete for student s_1. However the maximal offer that department d_2 makes to student s_1 is equal to $334 = 1075 - 741 = b_{d_2} - m_{s_3 d_2}(l+1)$. Hence at some time $\bar{t}$ we have following:

$t = \bar{t}$: a) $S_{\bar{t}}^{d_1} = \{s_1\}$, $S_{\bar{t}}^{d_2} = \{s_1, s_3\}$, so that department d_1 offers $m_{s_1 d_1}(\bar{t}) = 435$ to student $\{s_1\}$,
department d_2 offers $m_{s_1 d_2}(\bar{t}) = 334$ to student s_1 and $m_{s_3 d_2}(\bar{t}) = 741$ to student s_3.

[4]Department d_1 offers to student s_1 as the first time at time $l+1$, so $m_{s_1 d_1}(l+1) = \sigma_{s_1 d_1} = 400$.

b) Student s_1 accepts department d_1's offer and rejects department d_2's offer, student s_2 has no offer, while
student s_3 accepts department d_2's offer.

c) Thus, $T_{\bar{t}}^{d_1} = \{s_1\}$, $T_{\bar{t}}^{d_2} = \{s_3\}$, and we have the matching

$$\mu_{\bar{t}} = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_1 & \emptyset & d_2 \\ 435 & 0 & 741 \end{pmatrix}$$

$t = \bar{t} + 1$: a) Now $S_{\bar{t}+1}^{d_1} = \{s_1\}$, $S_{\bar{t}+1}^{d_2} = \{s_2\}$.

In here, note that department d_2's preference relation violates the gross substitutes condition, since department d_2 broke its tie with student s_3 even though the offer $m_{s_3 d_2}(\bar{t}+1)$ does not increase, i.e., $m_{s_3 d_2}(\bar{t}+1) = m_{s_3 d_2}^{\mu_{\bar{t}}}$ but $s_3 \notin S_{\bar{t}+1}^{d_2}$.

Department d_1 offers $m_{s_1 d_1}(\bar{t}+1) = 435$ to student s_1,
department d_2 offers $m_{s_2 d_2}(\bar{t}+1) = \sigma_{s_2 d_2} = 1000$ to student s_2.

b) Student s_1 accepts department d_1's offer,
student s_2 accepts department d_2's offer, while
student s_3 has no offer.

c) Hence $T_{\bar{t}+1}^{d_1} = \{s_1\}$, $T_{\bar{t}+1}^{d_2} = \{s_2\}$.

At time $\bar{t}+1$ we obtain matching

$$\mu_{\bar{t}+1} = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_1 & d_2 & \emptyset \\ 435 & 1000 & 0 \end{pmatrix}$$

$t = \bar{t} + 2$: a) Here $S_{\bar{t}+2}^{d_1} = \{s_3\}$, $S_{\bar{t}+2}^{d_2} = \{s_2\}$, whence department d_1 offers $m_{s_3 d_1}(\bar{t}+2) = 440 = m_{s_3 d_1}(l)$ to student s_3,[5]
department d_2 offers $m_{s_2 d_2}(\bar{t}+2) = 1000$ to student s_2.

[5]Department d_1 is supposed to make this offer since student s_3 is unmatched under $\mu_{\bar{t}+1}$.

b) Student s_1 has no offer,
student s_2 accepts department d_2's offer,
student s_3 accepts department d_1's offer.

c) So $T^{d_1}_{\bar{t}+2} = \{s_3\}$, $T^{d_2}_{\bar{t}+2} = \{s_2\}$.

At time $t = \bar{t} + 2$, this yields the matching

$$\mu_{\bar{t}+2} = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & d_2 & d_1 \\ 0 & 1000 & 440 \end{pmatrix}$$

$t = \bar{t} + 3$: a) Now $S^{d_1}_{\bar{t}+3} = \{s_3\}$, $S^{d_2}_{\bar{t}+3} = \{s_1, s_3\}$.

Department d_1 offers $m_{s_3 d_1}(\bar{t} + 3) = 440$ to student s_3,
department d_2 offers $m_{s_1 d_2}(\bar{t} + 3) = 334 = m_{s_1 d_2}(\bar{t})$ to student s_1[6] and $m_{s_3 d_2}(\bar{t} + 3) = 741 = m_{s_3 d_2}(\bar{t})$ to student s_3.[7]

b) Student s_1 accepts department d_2's offer,
student s_2 has no offer,
student s_3 accepts department d_2's offer and rejects department d_1's offer.

c) Thus, $T^{d_1}_{\bar{t}+3} = \emptyset$, $T^{d_2}_{\bar{t}+3} = \{s_1, s_3\}$.

At time $\bar{t} + 3$ we have the matching

$$\mu_{\bar{t}+3} = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_2 & \emptyset & d_2 \\ 334 & 0 & 741 \end{pmatrix}$$

$t = \bar{t} + 4$: a) In this period, $S^{d_1}_{\bar{t}+4} = \{s_1\}$, $S^{d_2}_{\bar{t}+4} = \{s_1, s_3\}$.

Department d_1 offers $m_{s_1 d_1}(\bar{t} + 4) = 435$ to student s_1,
department d_2 offers $m_{s_1 d_2}(\bar{t} + 4) = 334$ to student s_1 and $m_{s_3 d_2}(\bar{t} + 4) = 741$ to

[6]Department d_2 makes this offer since student s_1 is unmatched under $\mu_{\bar{t}+2}$.
[7]Department d_2 makes this offer since it broke ties with student s_3 at time $\bar{t}+1$, so $s_3 \notin F^{d_2}_{\bar{t}+3}$.

student s_3.

b) Student s_1 accepts department d_1's offer, and rejects department d_2's offer,
student s_2 has no offer,
student s_3 accepts department d_2's offer.

c) Hence, $T^{d_1}_{\bar{t}+4} = \{s_1\}$, $T^{d_2}_{\bar{t}+4} = \{s_3\}$, yielding the matching

$$\mu_{\bar{t}+4} = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_1 & \emptyset & d_2 \\ 435 & 0 & 741 \end{pmatrix}$$

Note that $\mu_{\bar{t}+4}$=$\mu_{\bar{t}}$. If we continue to apply GAA we get following matchings at further periods:
$\mu_{\bar{t}+5} = \mu_{\bar{t}+1}$, $\mu_{\bar{t}+6} = \mu_{\bar{t}+2}$, $\mu_{\bar{t}+7} = \mu_{\bar{t}+3}$, $\mu_{\bar{t}+8} = \mu_{\bar{t}+4}$=$\mu_{\bar{t}}$.

The finite tuple of matchings $(\mu_{\bar{t}}, \mu_{\bar{t}+1}, \mu_{\bar{t}+2}, \mu_{\bar{t}+3})$ repeats itself infinitely many times in the algorithm. Hence the algorithm does not stop in this example.

Note that there is no core stable matching in this example, since there is neither a core stable matching such that student s_2 is matched with a department, nor a core stable matching under which she is unmatched.

In the previous example, we see that the algorithm GAA does not stop because a finite tuple of matchings repeats itself, that is a cycle occurs in GAA. So we will define formally what we mean by a cycle.

Definition 10 We say that **a cycle occurs in the algorithm** if there is a finite sequence of matchings $(\mu_{t_0}, \mu_{t_0+1}, \ldots, \mu_{\bar{t}-1})$ $(t_0 < \bar{t})$ such that, for every $t > t_0$, $\mu_t = \mu_{t_0+r}$, where $0 \leq r < \bar{t} - t_0$ and $t \equiv r \pmod{\bar{t} - t_0}$.

We have seen it is possible that the algorithm GAA does not stop. But is it also possible that the algorithm GAA does not stop while no cycle occurs in the algorithm? The following proposition answers this question.

Proposition 4 *The algorithm GAA stops if and only if no cycle occurs in the algorithm.*

Proof It is obvious that if the algorithm GAA stops, then no cycle occurs in the algorithm.

For the other part of the proof, assume that the algorithm GAA does not stop. Let M denote the set of all matchings that occur in the algorithm GAA. Note that the set of all possible matchings for a given graduate admission problem is finite, since D and S are finite, and the money transfers between matched agents are integers. Therefore M is finite.

Let O denote the set of all pairs $(s, d) \in S \times D$ such that d makes an offer to s in the algorithm GAA. In the algorithm, there is a time $\bar{t}$ such that for any $(s, d) \in O$, department d proposes its maximal transfer to student s in the algorithm at any $t < \bar{t}$ such that d makes an offer to s in period t.

Letting $\bar{m}_{sd}$ denote the maximal transfer that department d offers to student s in the algorithm GAA, we have, for any $t > \bar{t}$, $m_{sd}(t) = \bar{m}_{sd}$, if d makes an offer to s at t.

Since M is finite, there is a matching $\bar{\mu}$ such that it occurs infinitely many times in the algorithm GAA. Let t_k be a time such that $t_k > \bar{t}$ and $\mu_{t_k} = \bar{\mu}$.

Claim 1: It is impossible that for all times $t > t_k$, $\mu_t = \bar{\mu}$.

Proof of claim 1: Suppose not, i.e., suppose that for all times $t > t_k$, $\mu_t = \bar{\mu}$.

Since the algorithm GAA does not stop, at each time t there is at least one department d such that $S_t^d \neq T_{t-1}^d$.

Moreover, for all times $t > t_k$, we have that, for any $(s, d) \in O$, $m_{sd}(t) = \bar{m}_{sd}$ if s gets an offer from d at t. But this fact together with the finiteness of D and S implies that there is some time $t^\star > t_k$ such that for all $d \in D$, $S_{t^\star}^d = T_{t^\star-1}^d$, in contradiction with that GAA does not stop. Hence it is impossible for all times $t > t_k$ to have $\mu_t = \bar{\mu}$. This completes the proof of claim 1.

Claim 1 implies that there is a matching $\tilde{\mu}$ which is different than $\bar{\mu}$ such that $\mu_{t_k+1} = \tilde{\mu}$.

Applying Claim 1 to $\tilde{\mu}$, we can say that it is impossible for all times $t > t_k + 1$ to have $\mu_t = \tilde{\mu}$. So there is another matching $\hat{\mu}$ which is different than $\tilde{\mu}$ such that $\mu_{t_k+2} = \hat{\mu}$.

As matching $\bar{\mu}$ occurs infinitely many times in the algorithm, at some further time, again we have matching $\bar{\mu}$. That is there is a time $t_l > t_k$ such that $\mu_{t_l} = \bar{\mu}$. Hence we get a finite tuple of matchings $(\bar{\mu}, \tilde{\mu}, \hat{\mu}, \ldots, \mu_{t_l-1})$. Let C denotes this finite tuple of matchings.

Claim 2: $\mu_{t_l+1} = \tilde{\mu}$.

Proof of claim 2: Note that $\mu_{t_k} = \bar{\mu}$ and $\mu_{t_k+1} = \tilde{\mu}$ such that $\tilde{\mu}$ is different than $\bar{\mu}$. So there is at least a department d and a student s such that $\bar{\mu}_1(s) \neq d$ but $\tilde{\mu}_1(s) = d$. That is department d makes an offer to students s at period $t_k + 1$ and s accepts d's offer. Hence $s \notin F^d_{t_k+1}$.

We will show that $s \notin F^d_{t_l+1}$, i.e., the algorithm requires that d makes an offer to s at period $t_l + 1$. We have two cases to consider that either $\bar{\mu}_1(s) = \emptyset$ or $\bar{\mu}_1(s) = \hat{d}$.

If $\bar{\mu}_1(s) = \emptyset$, then we have that she is again unmatched at the end of period t_l, since $\mu_{t_k} = \mu_{t_l} = \bar{\mu}$. So $s \notin F^d_{t_l+1}$.

Now assume that $\bar{\mu}_1(s) = \hat{d}$. Note that d makes an offer to s at period $t_k + 1$ and s accepts d's offer, and we have for all times $t > \bar{t}$, $m_{sd}(t) = \bar{m}_{sd}$ for any $(s, d) \in O$ if s gets an offer from d at period t. Hence we have $(d, \bar{m}_{sd}) P_s (\hat{d}, \bar{m}_{s\hat{d}})$. That is s do not reject d's offer because of $\hat{d}$'s offer. So $s \notin F^d_{t_l+1}$. Hence at time $t_l + 1$, again d makes an offer to s, and she accepts it, i.e., d and s again get matched at the end of period $t_l + 1$. Note that this is true for all pairs (s, d) such that $\bar{\mu}_1(s) \neq d$ but $\tilde{\mu}_1(s) = d$. So we have $\mu_{t_l+1} = \tilde{\mu}$, which completes the proof of claim 2.

Applying claim 2 to $\mu_{t_l+1} = \tilde{\mu}$, we get the matching $\hat{\mu}$ at the end of period

$t_l + 2$.

Hence by applying claim 2 to each matching in C, we see that C repeats itself infinitely many times in the algorithm GAA. This completes the proof of proposition.

□

In the example 4 above the algorithm GAA does not stop and there is no core stable matching. The following example shows that it is also possible that the algorithm GAA does not stop while there is a core stable matching.

Example 5 The algorithm GAA does not stop and there is a core stable matching

Let $D = \{d_1, d_2, d_3, d_4\}$ be the set of departments, $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ the set of students, where the quotas and budgets of the departments are as follows: $q_{d_1} = 1$, $q_{d_2} = 2$, $q_{d_3} = 1$, $q_{d_4} = 2$; $b_{d_1} = 440$, $b_{d_2} = 1075$, $b_{d_3} = 440$, $b_{d_4} = 1075$. The qualification levels and reservation prices of the students are as given in table 4.2.

$a^{s_1}_{d_1} = 7$	$a^{s_1}_{d_2} = 11$	$a^{s_1}_{d_3} = 4$	$a^{s_1}_{d_4} = 0$
$a^{s_2}_{d_1} = 0$	$a^{s_2}_{d_2} = 15$	$a^{s_2}_{d_3} = 0$	$a^{s_2}_{d_4} = 2$
$a^{s_3}_{d_1} = 8$	$a^{s_3}_{d_2} = 12$	$a^{s_3}_{d_3} = 0$	$a^{s_3}_{d_4} = 1$
$a^{s_4}_{d_1} = 4$	$a^{s_4}_{d_2} = 0$	$a^{s_4}_{d_3} = 7$	$a^{s_4}_{d_4} = 11$
$a^{s_5}_{d_1} = 0$	$a^{s_5}_{d_2} = 2$	$a^{s_5}_{d_3} = 0$	$a^{s_5}_{d_4} = 15$
$a^{s_6}_{d_1} = 0$	$a^{s_6}_{d_2} = 1$	$a^{s_6}_{d_3} = 8$	$a^{s_6}_{d_4} = 12$
$\sigma_{s_1d_1} = 400$	$\sigma_{s_1d_2} = 300$	$\sigma_{s_1d_3} = -500$	$\sigma_{s_1d_4} = 440$
$\sigma_{s_2d_1} = 440$	$\sigma_{s_2d_2} = 1075$	$\sigma_{s_2d_3} = 400$	$\sigma_{s_2d_4} = -500$
$\sigma_{s_3d_1} = 400$	$\sigma_{s_3d_2} = 700$	$\sigma_{s_3d_3} = 420$	$\sigma_{s_3d_4} = -500$
$\sigma_{s_4d_1} = -500$	$\sigma_{s_4d_2} = 450$	$\sigma_{s_4d_3} = 400$	$\sigma_{s_4d_4} = 300$
$\sigma_{s_5d_1} = 400$	$\sigma_{s_5d_2} = -500$	$\sigma_{s_5d_3} = 440$	$\sigma_{s_5d_4} = 1075$
$\sigma_{s_6d_1} = 420$	$\sigma_{s_6d_2} = -500$	$\sigma_{s_6d_3} = 400$	$\sigma_{s_6d_4} = 700$

Table 4.2: Qualification levels and reservation prices of students for example 5

If we apply the algorithm GAA, then a cycle occurs consisting of the following three matchings:

$$\mu_{\bar{t}} = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ d_1 & \emptyset & d_2 & d_3 & \emptyset & d_4 \\ 435 & 0 & 741 & 435 & 0 & 741 \end{pmatrix}$$

$$\mu_{\bar{t}+1} = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ d_1 & d_4 & \emptyset & d_3 & d_2 & \emptyset \\ 435 & -500 & 0 & 435 & -500 & 0 \end{pmatrix}$$

$$\mu_{\bar{t}+2} = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \emptyset & d_4 & d_2 & \emptyset & d_2 & d_4 \\ 0 & -500 & 741 & 0 & -500 & 741 \end{pmatrix}$$

However, there is a core stable matching.

Consider the matching

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ d_3 & d_4 & d_4 & d_1 & d_2 & d_2 \\ 440 & 500 & 575 & 440 & 500 & 575 \end{pmatrix}$$

It is easy to see that the matching μ is core stable. Hence, it is possible that the algorithm GAA does not stop, but there is a core stable matching. Therefore, we cannot say that if the algorithm GAA does not stop, then the set of core stable matchings is empty.

We say that a core stable matching is **department optimal** if every department likes it at least as well as any other core stable matching. Similarly, we say that a core stable matching is **student optimal** if every student likes it at least as well as any other core stable matching.

For the college admissions problem with colleges having quota constraints only, we know that there are a college optimal and a student optimal matching.

However, the following examples show that there is neither a department optimal nor a student optimal matching for the graduate admission problem with quota and budget constraints.

Example 6 There is no department optimal matching

Let $D = \{d_1, d_2\}$ be the set of departments, $S = \{s_1, s_2, s_3\}$ the set of students with the quotas and budgets of the departments as follows: $q_{d_1} = q_{d_2} = 2$; $b_{d_1} = b_{d_2} = 100$. The qualification levels and reservation prices of the students are as given in table 4.3.

$a_{d_1}^{s_1} = 1$	$a_{d_2}^{s_1} = 1$
$a_{d_1}^{s_2} = 10$	$a_{d_2}^{s_2} = 0$
$a_{d_1}^{s_3} = 0$	$a_{d_2}^{s_3} = 10$
$\sigma_{s_1 d_1} = 10$	$\sigma_{s_1 d_2} = 20$
$\sigma_{s_2 d_1} = 50$	$\sigma_{s_2 d_2} = 60$
$\sigma_{s_3 d_1} = 50$	$\sigma_{s_3 d_2} = 60$

Table 4.3: Qualification levels and reservation prices of students for example 6

Consider the matching

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_1 & d_1 & d_2 \\ 30 & 50 & 60 \end{pmatrix}$$

which can easily be checked to be the outcome of GAA.

Now considering the matching

$$\tilde{\mu} = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_2 & d_1 & d_2 \\ 20 & 100 & 80 \end{pmatrix}$$

we see that both μ and $\tilde{\mu}$ are core stable, while $\mu P_{d_1} \tilde{\mu}$ and $\tilde{\mu} P_{d_2} \mu$. Moreover, there is no other core stable matching such that both d_1 and d_2 like it at least as well as any other core stable matching. Therefore there is no department optimal matching.

Example 7 There is no student optimal matching

Let $D = \{d_1, d_2\}$ be the set of departments, $S = \{s_1, s_2, s_3\}$ the set of students, where the quotas and budgets of the departments are as follows: $q_{d_1} = 2$, $q_{d_2} = 1$; $b_{d_1} = b_{d_2} = 100$. The qualification levels and reservation prices of the students are as given in table 4.4.

$a^{s_1}_{d_1} = 1$	$a^{s_1}_{d_2} = 0$
$a^{s_2}_{d_1} = 5$	$a^{s_2}_{d_2} = 0$
$a^{s_3}_{d_1} = 0$	$a^{s_3}_{d_2} = 10$
$\sigma_{s_1 d_1} = 50$	$\sigma_{s_1 d_2} = 60$
$\sigma_{s_2 d_1} = 50$	$\sigma_{s_2 d_2} = 60$
$\sigma_{s_3 d_1} = 50$	$\sigma_{s_3 d_2} = 60$

Table 4.4: Qualification levels and reservation prices of students for example 7

Consider the matching

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_1 & d_1 & d_2 \\ 50 & 50 & 60 \end{pmatrix}$$

which turns out to be the outcome of GAA for this problem.

Now consider the matching

$$\tilde{\mu} = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & d_1 & d_2 \\ 0 & 100 & 100 \end{pmatrix}$$

Similarly as above, both μ and $\tilde{\mu}$ are core stable, while $\mu P_{s_1} \tilde{\mu}$ and $\tilde{\mu} P_{s_2} \mu$. Moreover, there is no other core stable matching such that both s_1 and s_2 like it at least as well as any other core stable matching. Therefore there is no student optimal matching.

Note that $\mu_1(s_1) \in D$ but $\tilde{\mu}_1(s_1) = \emptyset$. Therefore it is possible that there be core stable matchings μ and $\tilde{\mu}$ such that there is a student $s \in S$, $\mu_1(s) \in D$ but $\tilde{\mu}_1(s) = \emptyset$.

Chapter 5

Graduate Admission Algorithm with Reservation Prices

In this chapter, we will modify students' preferences in such a way that students now consider only reservation prices and do not derive further utility from money transfer over and above their reservation prices. Then we construct another graduate admission algorithm ($G\tilde{A}A$) by taking the reservation prices of students equal to the money transfers from the department to which they are accepted. The algorithm $G\tilde{A}A$ is another extension of the Gale - Shapley algorithm. However, like GAA, $G\tilde{A}A$ does not always stop, and it is possible that there exists a core stable matching although $G\tilde{A}A$ does not stop.

Students' Preferences

Again we assume that $(d, \sigma_{sd})P_s(\tilde{d}, \sigma_{s\tilde{d}})$ if and only if $\sigma_{sd} < \sigma_{s\tilde{d}}$.

Note that proposition 1 and 2 continue to be true if students consider only reservation prices, and similar examples of chapter 3 can easily be constructed for this model as well.

Now we will define how the algorithm $G\tilde{A}A$ works.

The algorithm $\tilde{GAA}$

The structure of $\tilde{GAA}$ is the same as that of GAA, the only difference being that a department d which makes an offer to a student s is ready to pay σ_{sd} to s no matter at what stage of the algorithm this offer is made. In other words, $m_{sd}(t) = \sigma_{sd}$ for all $s \in S$, $d \in D$ and all times t at which d makes an offer to s.

At each time t in the algorithm $\tilde{GAA}$, each department d chooses a group of admissible students S_t^d satisfying its quota and budget constraints so as to maximize its total benefit Y_t^d.

$t = 1$: a) Each department d determines a group of students $S_1^d \subseteq S$ as denoted above and offers to each student $s \in S_1^d$.

b) Students who have taken one or more offers accept exactly one offer and reject the others.

c) Department d accepts the group of students who accepted its offers. Let T_1^d denote the group of students who accepted department d's offers at time $t = 1$, where clearly $T_1^d \subseteq S_1^d$.

Now, at the end of period $t = 1$ we have a matching μ_1, and so $S_{\mu_1}^d = T_1^d$.

$t = 2$: a) Each department d determines a group of students $S_2^d \subseteq S \setminus (S_1^d \setminus T_1^d)$ and makes an offer to each student $s \in S_2^d$.

b) Students who have taken one or more offers accept exactly one offer and reject the others.

c) Department d accepts the group of students who accepted its offers.

In general, at time k, the algorithm works as follows.

$t = k$: a) Now we will define in general what we mean by an admissible group of students for department d, i.e, we will define the set $F_k^d \subseteq S$ for department d at time k.

Assume that $\tilde{t} < k$ was the last time that d made an offer to s before time k when s rejected d's offer because of another department $\hat{d}$'s offer. Department d cannot make an offer to student s at time k, if $\mu_{k-1}(s) = \hat{d}$. The set F_k^d denotes the group of all such students for department d at time k, i.e., the group of students to whom department d cannot make offers at time k.[1]

Each department d chooses its group of students S_k^d from $S \setminus F_k^d$ and offers to each student $s \in S_k^d$.

b) Students who have taken one or more offers accept exactly one offer and reject the others.

c) Department d accepts the group of students $T_k^d \subseteq S_k^d$ who accepted its offers.

Stopping Rule

$t = t^\star$: The algorithm stops at time $t^\star$ if each department d makes offers exactly to the group of students who accepted its offers at $t^\star - 1$, i.e., if we have $S_{t^\star}^d = T_{t^\star-1}^d$ for all $d \in D$.

If the algorithm stops at time $t^\star$, the matching $\mu_{t^\star}$ is regarded as the outcome of the algorithm.

Proposition 5 *If the algorithm $G\tilde{A}A$ stops, then the final matching of the algorithm is core stable (and thus Pareto optimal).*

Proof Assume that the algorithm stops. Let the algorithm stop at time $t^\star$, and let the matching $\mu_{t^\star}$ denote the outcome of the algorithm. So we have $S_{t^\star}^d = T_{t^\star-1}^d$ for all $d \in D$. We abuse notation that we use $\mu^\star$ for $\mu_{t^\star}$.

Clearly $\mu^\star$ is individually rational, since for all $s \in S$, $m_{s\mu_1^\star(s)}^{\mu^\star} = \sigma_{s\mu_1^\star(s)}$, and for all $d \in D$, $Y_{\mu^\star}^d \geq 0$.

[1]At time $t = 1$, we have that for all $d \in D$, $F_1^d = \emptyset$, so each department d determines its group of students S_1^d over the set of all students S. At time $t = 2$, for all $d \in D$, $F_2^d = S_1^d \setminus T_1^d$, so the admissible group for department d at time 2 is $S \setminus (S_1^d \setminus T_1^d)$.

Now suppose that $\mu^\star$ is not core stable. Then there is a group $(d, \tilde{S})$ which blocks $\mu^\star$. So we have that

1. for all $s \in \tilde{S}$, $\mu_1^\star(s) \neq d$,
2. for all $s \in \tilde{S}$, $(d, \sigma_{sd}) P_s (\mu_1^\star(s), \sigma_{s\mu_1^\star(s)})$,
3. $[(S_{\mu^\star}^d \setminus B) \bigcup \tilde{S}, \hat{c}^d] P_d [S_{\mu^\star}^d, c_{\mu^\star}^d]$, for some $B \subseteq S_{\mu^\star}^d$.

Claim: There is no student $s \in \tilde{S}$ such that $s \in F_{t^\star}^d$.[2]

Proof of claim: Suppose not, i.e., suppose there is a student $s \in \tilde{S}$ such that $s \in F_{t^\star}^d$. That is department d offered to student $s \in \tilde{S}$ at some time $\tilde{t} < t^\star$ as the last time before time $t^\star$ and student s rejected d's offer because of another department's offer, say department $\hat{d}$'s offer, and $\mu_{t^\star-1}(s) = \hat{d}$.[3] So we have that

4. $(\hat{d}, \sigma_{s\hat{d}}) P_s (d, \sigma_{sd})$.

As $\mu_{t^\star-1}(s) = \hat{d}$ and the algorithm stops at time $t^\star$, we have that $\mu_1^\star(s) = \hat{d}$. Now by (2), we have $(d, \sigma_{sd}) P_s (\hat{d}, \sigma_{s\hat{d}})$. This contradicts with (4). Hence there is no student $s \in \tilde{S}$ such that $s \in F_{t^\star}^d$.

The above claim implies that $\tilde{S} \subseteq (S \setminus F_{t^\star}^d)$, i.e., the algorithm allows department d to make offers to each $s \in \tilde{S}$. Therefore department d would offer to each student $s \in \tilde{S}$ (by 3), and each student $s \in \tilde{S}$ would accept it (by 2), in contradiction with that for all $s \in \tilde{S}$, $\mu_1^\star(s) \neq d$. Hence $\mu^\star$ is core stable, and thus Pareto optimal.

□

[2]Note that there is no student $s \in (S_{\mu^\star}^d \setminus B)$ such that $s \in F_{t^\star}^d$, since for all $s \in (S_{\mu^\star}^d \setminus B)$, $\mu_1^\star(s) = d$.

[3]Note that in here we abuse the notation that $\mu_{t^\star-1}(s)$ denotes the department that s is matched under $\mu_{t^\star-1}$.

The following example shows that there can be more than one core stable matching in our model where only reservation prices of students are considered.

Example 8 There is more than one core stable matching

The set of departments is $D = \{d_1, d_2, d_3\}$, the set of students is $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$, and the quotas and budgets of departments are given by $q_{d_1} = 3$, $q_{d_2} = 3$, $q_{d_3} = 2$; $b_{d_1} = 45$, $b_{d_2} = 50$, $b_{d_3} = 25$. The qualification levels and reservation prices of the students are as given in table 5.1.

$a^{s_1}_{d_1}=15$	$a^{s_1}_{d_2}=0$	$a^{s_1}_{d_3}=0$
$a^{s_2}_{d_1}=20$	$a^{s_2}_{d_2}=0$	$a^{s_2}_{d_3}=0$
$a^{s_3}_{d_1}=10$	$a^{s_3}_{d_2}=0$	$a^{s_3}_{d_3}=15$
$a^{s_4}_{d_1}=0$	$a^{s_4}_{d_2}=11$	$a^{s_4}_{d_3}=0$
$a^{s_5}_{d_1}=15$	$a^{s_5}_{d_2}=15$	$a^{s_5}_{d_3}=0$
$a^{s_6}_{d_1}=0$	$a^{s_6}_{d_2}=10$	$a^{s_6}_{d_3}=12$
$a^{s_7}_{d_1}=0$	$a^{s_7}_{d_2}=0$	$a^{s_7}_{d_3}=10$
$a^{s_8}_{d_1}=0$	$a^{s_8}_{d_2}=10$	$a^{s_8}_{d_3}=0$
$\sigma_{s_1d_1}=10$	$\sigma_{s_1d_2}=20$	$\sigma_{s_1d_3}=25$
$\sigma_{s_2d_1}=10$	$\sigma_{s_2d_2}=20$	$\sigma_{s_2d_3}=25$
$\sigma_{s_3d_1}=15$	$\sigma_{s_3d_2}=20$	$\sigma_{s_3d_3}=25$
$\sigma_{s_4d_1}=28$	$\sigma_{s_4d_2}=20$	$\sigma_{s_4d_3}=25$
$\sigma_{s_5d_1}=25$	$\sigma_{s_5d_2}=21$	$\sigma_{s_5d_3}=22$
$\sigma_{s_6d_1}=30$	$\sigma_{s_6d_2}=20$	$\sigma_{s_6d_3}=10$
$\sigma_{s_7d_1}=25$	$\sigma_{s_7d_2}=40$	$\sigma_{s_7d_3}=0$
$\sigma_{s_8d_1}=30$	$\sigma_{s_8d_2}=10$	$\sigma_{s_8d_3}=20$

Table 5.1: Qualification levels and reservation prices of students for example 8

When writing a matching, we will not write the money transfers between matched agents, since all money transfers between matched agents are the reservation prices of the students. Consider the following matching

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ d_1 & d_1 & d_1 & d_2 & d_2 & d_3 & d_3 & \emptyset \end{pmatrix}$$

Note that the matching μ is core stable. The best group for department d_1

is $\{s_1, s_2, s_5\}$, and $S^{d_1}_{\mu} = \{s_1, s_2, s_3\}$, and department d_1 prefers student s_5 to student s_3. However $(d_2, 21)P_{s_5}(d_1, 25)$, i.e., student s_5 does not form a blocking pair with department d_1. Hence there is no group involving department d_1 which blocks μ.

The best group for department d_2 is $\{s_4, s_6, s_8\}$, and $S^{d_2}_{\mu} = \{s_4, s_5\}$. The group $(d_2, \{s_6, s_8\})$ cannot block μ, since $(d_3, 10)P_{s_6}(d_2, 20)$. The group (d_2, s_8) cannot block μ, since $\sigma_{s_4 d_2} + \sigma_{s_5 d_2} + \sigma_{s_8 d_2} = 20 + 21 + 10 = 51 > 50 = b_{d_2}$ and $a^{s_4}_{d_2} = 11 > 10 = a^{s_8}_{d_2}$, $a^{s_5}_{d_2} = 15 > 10 = a^{s_8}_{d_2}$. Hence there is no group involving department d_2 which blocks μ.

The best group for department d_3 is $\{s_3, s_7\}$, and $S^{d_3}_{\mu} = \{s_6, s_7\}$. However $(d_1, 15)P_{s_3}(d_3, 25)$, i.e., student s_3 does not form a blocking pair with department d_3. Hence there is no group involving department d_3 which blocks μ.

If we apply the algorithm $G\tilde{A}A$, we get the matching

$$\tilde{\mu} = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \\ d_1 & d_1 & d_3 & d_2 & d_1 & d_2 & d_3 & d_2 \end{pmatrix}$$

Clearly $\tilde{\mu}$ is core stable, and different than μ.

Hence it is possible that there be more than one core stable matching.

This example also shows that there is no student optimal matching for this model. Since $\mu P_{s_2} \tilde{\mu}$ but $\tilde{\mu} P_{s_8} \mu$.

Also note that $\mu_1(s_8) = \emptyset$, $\tilde{\mu}_1(s_8) = d_2$, so it is possible that there be two core stable matchings μ and $\tilde{\mu}$ such that there is a student $s \in S$ with $\mu_1(s) = \emptyset$, $\tilde{\mu}_1(s) \in D$.

Now we will give an example that the algorithm $G\tilde{A}A$ does not stop and there is no core stable matching.

Example 9 The algorithm $\tilde{GAA}$ does not stop and there is no core stable matching

Let $D = \{d_1, d_2\}$, $S = \{s_1, s_2, s_3\}$, $q_{d_1} = 1$, $q_{d_2} = 2$, $b_{d_1} = 50$, $b_{d_2} = 70$, and the qualification levels and reservation prices of the students are as given in table 5.2.

$a_{d_1}^{s_1}$=10	$a_{d_2}^{s_1}$=10
$a_{d_1}^{s_2}$= 1	$a_{d_2}^{s_2}$=15
$a_{d_1}^{s_3}$= 8	$a_{d_2}^{s_3}$= 9
$\sigma_{s_1d_1}$=50	$\sigma_{s_1d_2}$=30
$\sigma_{s_2d_1}$=50	$\sigma_{s_2d_2}$=45
$\sigma_{s_3d_1}$=30	$\sigma_{s_3d_2}$=40

Table 5.2: Qualification levels and reservation prices of students for example 9

Now we apply the algorithm $\tilde{GAA}$,

$t = 1$: a) $S_1^{d_1} = \{s_1\}$, $S_1^{d_2} = \{s_1, s_3\}$; and department d_1 offers to student s_1, department d_2 offers to students $\{s_1\}$ and s_3.

b) Student s_1 accepts d_2's offer and rejects d_1's offer; student s_2 has no offer; student s_3 accepts d_2's offer.

c) We obtain a matching

$$\mu_1 = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_2 & \emptyset & d_2 \end{pmatrix}$$

$t = 2$: a) $S_2^{d_1} = \{s_3\}$, $S_1^{d_2} = \{s_1, s_3\}$; and department d_1 offers to student s_2, department d_2 offers to students $\{s_1\}$ and s_3.

b) Student s_1 accepts d_2's offer; student s_2 has no offer; student s_3 accepts d_1's offer and rejects d_2's offer.

c) We obtain a matching

$$\mu_2 = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_2 & \emptyset & d_1 \end{pmatrix}$$

$t = 3$: a) $S_3^{d_1} = \{s_3\}$, $S_3^{d_2} = \{s_2\}$;[4] and department d_1 offers to student s_3, department d_2 offers to student s_2.

b) Student s_1 has no offer; student s_2 accepts d_2's offer; student s_3 accepts d_1's offer.

c) We obtain a matching

$$\mu_3 = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & d_2 & d_1 \end{pmatrix}$$

$t = 4$: a) $S_4^{d_1} = \{s_1\}$, $S_4^{d_2} = \{s_2\}$; and department d_1 offers to student s_1, department d_2 offers to student s_2.

b) Student s_1 accepts d_1's offer; student s_2 accepts d_2's offer; student s_3 has no offer.

c) We obtain a matching

$$\mu_4 = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_1 & d_2 & \emptyset \end{pmatrix}$$

$t = 5$: a) $S_5^{d_1} = \{s_1\}$, $S_5^{d_2} = \{s_1, s_3\}$; and department d_1 offers to student s_1, department d_2 offers to students s_1 and s_3.

b) Student s_1 accepts d_2's offer and rejects d_1's offer; student s_2 has no offer; student s_3 accepts d_2's offer.

c) We obtain a matching

$$\mu_5 = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_2 & \emptyset & d_2 \end{pmatrix}$$

Note that $\mu_5 = \mu_1$, and if we continue we get that $\mu_6 = \mu_2$, $\mu_7 = \mu_3$, $\mu_8 = \mu_4$.

[4] Note that department d_2's preference relation violates the gross substitutes condition.

That is $(\mu_1, \mu_2, \mu_3, \mu_4)$ repeats itself infinitely many times in the algorithm $G\tilde{A}A$. Hence the algorithm $G\tilde{A}A$ does not stop.

Note that there is no core stable matching for this example, since there is neither a core stable matching such that student s_2 is matched with a department, nor a core stable matching under which she is unmatched.

The following proposition shows that it is impossible that the algorithm $G\tilde{A}A$ does not stop and no cycle occurs in the algorithm.

Proposition 6 *The algorithm $G\tilde{A}A$ stops if and only if no cycle occurs in the algorithm.*

Proof The proof is similar to that of proposition 4. □

The following example shows that the algorithm $G\tilde{A}A$ may not stop even if there is a core stable matching.

Example 10 The algorithm $G\tilde{A}A$ does not stop and there is a core stable matching

Let $D = \{d_1, d_2, d_3, d_4\}$ be the set of departments, $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ the set of students and the quotas and budgets of the departments are given by $q_{d_1} = 1$, $q_{d_2} = 2$, $q_{d_3} = 1$, $q_{d_4} = 2$; $b_{d_1} = 50$, $b_{d_2} = 70$, $b_{d_3} = 50$, $b_{d_4} = 70$. The qualification levels and reservation prices of the students are as given in table 5.3.

If we apply the algorithm $G\tilde{A}A$, we get the following matchings:

At the end of time $t = 1$, we have

$$\mu_1 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ d_2 & \emptyset & d_2 & d_4 & \emptyset & d_4 \end{pmatrix}$$

At the end of time $t = 2$, we have

$$\mu_2 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ d_2 & \emptyset & d_1 & d_4 & \emptyset & d_3 \end{pmatrix}$$

$a^{s_1}_{d_1}$=10	$a^{s_1}_{d_2}$=10	$a^{s_1}_{d_3}$= 5	$a^{s_1}_{d_4}$= 0
$a^{s_2}_{d_1}$= 1	$a^{s_2}_{d_2}$=15	$a^{s_2}_{d_3}$= 0	$a^{s_2}_{d_4}$= 3
$a^{s_3}_{d_1}$= 8	$a^{s_3}_{d_2}$= 9	$a^{s_3}_{d_3}$= 0	$a^{s_3}_{d_4}$= 3
$a^{s_4}_{d_1}$= 6	$a^{s_4}_{d_2}$= 0	$a^{s_4}_{d_3}$=10	$a^{s_4}_{d_4}$=10
$a^{s_5}_{d_1}$= 0	$a^{s_5}_{d_2}$= 3	$a^{s_5}_{d_3}$= 1	$a^{s_5}_{d_4}$=15
$a^{s_6}_{d_1}$= 0	$a^{s_6}_{d_2}$= 3	$a^{s_6}_{d_3}$= 8	$a^{s_6}_{d_4}$= 9
$\sigma_{s_1d_1}$=50	$\sigma_{s_1d_2}$=30	$\sigma_{s_1d_3}$=20	$\sigma_{s_1d_4}$=40
$\sigma_{s_2d_1}$=50	$\sigma_{s_2d_2}$=45	$\sigma_{s_2d_3}$=40	$\sigma_{s_2d_4}$=30
$\sigma_{s_3d_1}$=30	$\sigma_{s_3d_2}$=40	$\sigma_{s_3d_3}$=45	$\sigma_{s_3d_4}$=28
$\sigma_{s_4d_1}$=20	$\sigma_{s_4d_2}$=40	$\sigma_{s_4d_3}$=50	$\sigma_{s_4d_4}$=30
$\sigma_{s_5d_1}$=40	$\sigma_{s_5d_2}$=30	$\sigma_{s_5d_3}$=50	$\sigma_{s_5d_4}$=45
$\sigma_{s_6d_1}$=45	$\sigma_{s_6d_2}$=28	$\sigma_{s_6d_3}$=30	$\sigma_{s_6d_4}$=40

Table 5.3: Qualification levels and reservation prices of students for example 10

At the end of time $t = 3$, we have

$$\mu_3 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \emptyset & d_2 & d_1 & \emptyset & d_4 & d_3 \end{pmatrix}$$

At the end of time $t = 4$, we have

$$\mu_4 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ d_1 & d_2 & \emptyset & d_3 & d_4 & \emptyset \end{pmatrix}$$

If we continue to apply our algorithm we obtain the result that $\mu_5 = \mu_1$, $\mu_6 = \mu_2$, $\mu_7 = \mu_3$, $\mu_8 = \mu_4$. That is $(\mu_1, \mu_2, \mu_3, \mu_4)$ repeats itself infinitely many times in the algorithm $G\tilde{A}A$. Hence the algorithm $G\tilde{A}A$ does not stop.

However there is a core stable matching for this example. Consider the matching

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ d_3 & d_4 & d_4 & d_1 & d_2 & d_2 \end{pmatrix}$$

The matching μ is core stable, since each student $s \in S$ is matched with her best department under μ. So there is no group of student who forms a blocking coalition with any department. Hence it is possible that the algorithm does not stop and there is a core stable matching.

Example 11 There is no department optimal matching if students consider only reservation prices

Let $D = \{d_1, d_2\}$ be the set of departments, $S = \{s_1, s_2, s_3, s_4\}$ the set of students and the quotas and budgets of the departments are given by $q_{d_1} = 2$, $q_{d_2} = 2$; $b_{d_1} = 102$, $b_{d_2} = 100$. The qualification levels and reservation prices of the students are as given in table 5.4.

$a_{d_1}^{s_1}$=20	$a_{d_2}^{s_1}$=20
$a_{d_1}^{s_2}$=15	$a_{d_2}^{s_2}$=15
$a_{d_1}^{s_3}$=12	$a_{d_2}^{s_3}$=16
$a_{d_1}^{s_4}$=13	$a_{d_2}^{s_4}$=16
$\sigma_{s_1d_1}$=55	$\sigma_{s_1d_2}$=60
$\sigma_{s_2d_1}$=45	$\sigma_{s_2d_2}$=40
$\sigma_{s_3d_1}$=51	$\sigma_{s_3d_2}$=50
$\sigma_{s_4d_1}$=51	$\sigma_{s_4d_2}$=50

Table 5.4: Qualification levels and reservation prices of students for example 11

Consider the matching

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ d_1 & d_1 & d_2 & d_2 \end{pmatrix}$$

Note that μ is core stable. Department d_1 is matched with its best group under μ, so there is no group including department d_1 which forms a blocking coalition. The best group for department d_2 is $\{s_1, s_2\}$, while $S_\mu^{d_2} = \{s_3, s_4\}$. Student s_1 does not form a blocking coalition with department d_2 since $(d_1, 55)P_{s_1}(d_2, 60)$. And department d_2 does not form a blocking coalition only with student s_2, since $a_{d_2}^{s_3} = a_{d_2}^{s_4} = 16 > 15 = a_{d_2}^{s_2}$. So there is no group consisting of department d_2 and some students which forms a blocking coalition. Hence μ is core stable.

Now consider the matching

$$\tilde{\mu} = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ d_2 & d_2 & d_1 & d_1 \end{pmatrix}$$

Note that $\tilde{\mu}$ is core stable. Department d_2 is matched with its best group under $\tilde{\mu}$, so there is no group including department d_2 which forms a blocking coalition. The best group for department d_1 is $\{s_1, s_2\}$ and $S_{\tilde{\mu}}^{d_1} = \{s_3, s_4\}$. Student s_2 does not form a blocking coalition with department d_1 since $(d_2, 40)P_{s_2}(d_1, 45)$. And department d_1 cannot form a blocking coalition with student s_3, since $\sigma_{s_3 d_1} + \sigma_{s_1 d_1} = 106 > 102 = b_{d_1}$. Similarly department d_1 can not form a blocking coalition with student s_4, since $\sigma_{s_4 d_1} + \sigma_{s_1 d_1} = 106 > 102 = b_{d_1}$. So there is no group involving department d_1 which forms a blocking coalition. Hence $\tilde{\mu}$ is core stable.

However we have that $\mu P_{d_1} \tilde{\mu}$ and $\tilde{\mu} P_{d_2} \mu$. Hence there is no department optimal matching for the model where students consider only reservation prices.

Chapter 6

Discussion and Conclusion

6.1 Discussion

We start this section with some observations. If we go back to example 4 where the algorithm GAA does not stop, we see that there is a pairwise stable and Pareto optimal matching in the cycle that occurs there.

In that example, the matching $\mu_{\bar{t}+2}$ is both pairwise stable and Pareto optimal, where

$$\mu_{\bar{t}+2} = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & d_2 & d_1 \\ 0 & 1000 & 440 \end{pmatrix}$$

This observation motivated us to write another algorithm to find pairwise stable matchings. We define a new graduate admission algorithm GAA^{ref} which can be regarded as a refinement of the algorithm GAA.

The algorithm GAA^{ref}

The rules of the algorithm GAA^{ref} are the same as the rules of GAA with the exception that, at each time t, each department d can now make only one new offer. That is, at each time t, each department d can make offers in addition to the students in T_{t-1}^d to only one student $s \in S$ with $s \notin T_{t-1}^d$. Hence we have that

$$\left|S_t^d\right| \leq \left|T_{t-1}^d\right| + 1.$$

Proposition 7 *If the algorithm GAA^{ref} stops, then the final matching of the algorithm is pairwise stable.*

Proof Assume that the algorithm GAA^{ref} stops at time $t^\star$. Let $\mu^\star$ denote the final matching of the algorithm. Clearly $\mu^\star$ is individually rational.

Now suppose that $\mu^\star$ is not pairwise stable. Then there is a pair $(s,d) \in S \times D$ with $\mu_1^\star(s) \neq d$ that blocks $\mu^\star$. Let $\tilde{m}_{sd}$ denote the money transfer from department d to student s .

However, as in the the proof of proposition 3, according to the algorithm department d could have offered $\tilde{m}_{sd}$ to student s, and therefore would have matched with student s, if that were the case.

Hence $\mu^\star$ is pairwise stable.

$\square$

However, if the algorithm GAA^{ref} stops, this does not guarantee the Pareto optimality of the final matching. For example, if we apply the algorithm GAA^{ref} to the graduate admission problem of example 2, the algorithm stops and the final matching is μ given in there. As noted in example 2, μ is pairwise stable but not Pareto optimal.

We record the following statement that has not been either proven or disproven as a conjecture.

Conjecture 1: *The algorithm GAA^{ref} stops in a finite time.*

Now consider examples 9 and 10 in which the algorithm $G\tilde{A}A$ does not stop, but there is a pairwise stable and Pareto optimal matching that is a member of the occurring cycle.

In example 9 there is no core stable matching either. The matching μ_4 is however, both pairwise stable and Pareto optimal, where

$$\mu_4 = \begin{pmatrix} s_1 & s_2 & s_3 \\ d_1 & d_2 & \emptyset \end{pmatrix}$$

In example 10, the algorithm $G\tilde{A}A$ does not stop but there is a core stable matching. The matching μ_4 there,is both pairwise stable and Pareto optimal, where

$$\mu_4 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ d_1 & d_2 & \emptyset & d_3 & d_4 & \emptyset \end{pmatrix}$$

These observations motivated us to introduce yet another algorithm in an attempt to find pairwise stable matchings when students care only about getting their reservation prices. Here we define a new graduate admission algorithm ($G\tilde{A}A^{ref}$) which can be regarded as a refinement of the algorithm $G\tilde{A}A$.

The algorithm $G\tilde{A}A^{ref}$

The rules of the algorithm $G\tilde{A}A^{ref}$ are the same as the rules of $G\tilde{A}A$ with the exception that, at each time t, each department d can now make only one new offer. That is at each time t, each department d can make offers in addition to the students in T^d_{t-1} to only one student $s \in S$ with $s \notin T^d_{t-1}$. Hence we have that

$\left|S^d_t\right| \leq \left|T^d_{t-1}\right| + 1.$

Proposition 8 *If the algorithm* $G\tilde{A}A^{ref}$ *stops, then the final matching of the algorithm is pairwise stable.*

Proof Assume that the algorithm $G\tilde{A}A^{ref}$ stops at time $t^\star$. Let $\mu^\star$ denote the final matching of the algorithm. Clearly $\mu^\star$ is individually rational.

Now suppose that $\mu^\star$ is not pairwise stable. Then there is a pair $(s,d) \in S \times D$ with $\mu^\star_1(s) \neq d$ that blocks $\mu^\star$. Remember that the money transfer from department d to student s is simply σ_{sd} if s accepts d's offer.

However, as in the proof of proposition 5, according to the algorithm, department d could have made the offer σ_{sd} to student s, i.e., $s \notin F^d_{t^\star}$. Therefore department d and student s would have matched in the outcome of the algorithm, a contradiction.

Hence $\mu^\star$ is pairwise stable.

$\square$

However, even if the algorithm $\tilde{GAA}^{ref}$ stops then the final matching need not be Pareto optimal.

The following statement is neither proven nor disproven so far and is thus recorded as a conjecture.

Conjecture 2: *The algorithm $\tilde{GAA}^{ref}$ stops in a finite time.*

There are examples where $\tilde{GAA}$ does not stop, but $\tilde{GAA}^{ref}$ does. For example, consider example 9 in which the algorithm $\tilde{GAA}$ does not stop and there is no core stable matching. The block of matchings (μ_1,μ_2,μ_3,μ_4) repeats itself infinitely under $\tilde{GAA}$. Now starting from time $t = 5$, if we apply the algorithm $\tilde{GAA}^{ref}$, the algorithm $\tilde{GAA}^{ref}$ stops at time $t = 6$, and $\mu_6 = \mu_4$ which is pairwise stable and Pareto optimal.

6.2 Conclusion

In this thesis, we have studied the graduate admission problem with quota and budget constraints as a two sided matching market. We have defined an algorithm (GAA) which is an extension of the Gale - Shapley algorithm. We showed that if the algorithm GAA stops then the final matching is core stable (and thus Pareto optimal). However the algorithm GAA does not always stop, and it is possible that the algorithm GAA does not stop while the set of core stable matchings is nonempty. Also there is neither a department optimal matching nor a student optimal matching under budget constraints.

We have also studied the model that students only consider reservation prices. We have defined another algorithm ($\tilde{GAA}$) that is again an extension of the Gale - Shapley algorithm. We showed that if the algorithm $\tilde{GAA}$ stops then the final matching is core stable (and thus Pareto optimal). However, the algorithm $\tilde{GAA}$ does not always stop, and it is again possible that the algorithm $\tilde{GAA}$ does not stop while the set of core stable matchings is nonempty. Also if students only care about their reservation prices there are no department optimal matching and no student optimal matching.

We obtain similar results under algorithms GAA and $\tilde{GAA}$. Hence we can say that in the model defined in this thesis (two sided matching market with quota and budget constraints), straightforward extensions of the Gale - Shapley algorithm do not function as well as it works for college admissions and labor market models without budget constraints.

We also defined certain refinements of GAA and $\tilde{GAA}$, called GAA^{ref} and $\tilde{GAA}^{ref}$ which yield pairwise stable matchings if they stop. However, whether these must stop or not, or whether there always exist pairwise stable matchings in our framework with budget constraints or not stays as an open problem yet to be explored.

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