COINCIDENCE OF MYERSON ALLOCATION RULE WITH SHAPLEY VALUE

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TÜMER KAPAN

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Economics.

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Supervisor

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(Title and Name)
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Approval of the Institute of Economics and Social Sciences

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Director

ABSTRACT

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Kapan, Tümer
M.A., Department of Economics
Supervisor: Prof. Semih Koray

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This thesis studies the coincidence of the Myerson allocation rule in the context of networks with the Shapley value in the context of transferable utility games. We start with a value function defined on networks and derive a transferable utility game from that. We show that without any restrictions on the value function, Myerson allocation rule may not lead to the same payoff vector as the Shapley value of the derived TU game for any network. Under the assumption of monotonicity of the value function, we show the existence of such coincidence and examine the relation of the set of networks satisfying this coincidence to the set of pairwise stable and strongly stable networks. Next, we propose a new stability notion and examine the coincidence of the two vectors under this stability notion. Finally an alternative allocation rule is introduced whose payoff vector coincide with the Shapley value of the derived transferable utility game on the set of efficient networks which coincides with the set of strongly stable networks under this allocation rule.

Key Words: Networks, Myerson allocation rule, Shapley Value, Stability, Coincidence.

ÖZET

MYERSON DAĞITIM KURALI'NIN SHAPLEY DEĞERİ İLE ÖRTÜŞMESİ

Kapan, Tümer
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Tez Yöneticisi: Prof. Dr. Semih Koray

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Bu çalışmamızda ağlar bağlamındaki Myerson Dağıtım Kuralı ile aktarılabilir yarar oyunları bağlamındaki Shapley Değeri'nin örtüşmesini inceledik. İlk olarak ağlar üzerinde tanımlanmış bir değer fonksiyonu alıp ondan bir aktarılabilir yarar oyunu türettik. Bu değer fonksiyonu üzerine herhangi bir kısıtlama konulmazsa hiç bir ağda, o ağ üzerinde Myerson Dağıtım Kuralı'nın belirlediği yarar vektorü ile türetilen aktarılabilir yarar oyununun Shapley Değeri'nin örtüşmeyebileceğini gösterdik. Değer fonksiyonunun tekdüze olduğu varsayımı altında bu örtüşmenin sağlandığı en az bir ağın varlığını gösterip bu örtüşmenin sağlandığı ağlar kümesi ile ikişerli kararlı ağlar kümesinin ve ayrıca kuvvetli kararlı ağlar kümesinin ilişkisini inceledik. Daha sonra yeni bir kararlılık tanımı önerip örtüşmeyi sağlayan ağlar kümesinin bu yeni tanıma göre kararlı olan ağlar kümesiyle ilişkisini inceledik. Son olarak Myerson Dağıtım Kuralı'na almaşık bir dağıtım kuralının verimli ağlar üzerindeki yarar vektörünün türetilen aktarılabilir yarar oyununun Shapley Değeri ile örtüştüğünü ve bu dağıtım kuralı altında kuvvetli kararlı olan ağlar kümesinin verimli ağlar kümesine eşit olduğunu gösterdik.

Anahtar Kelimeler: Ağlar, Myerson Dağıtım Kuralı, Shapley Değeri, Kararlılık, Örtüşme

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CHAPTER I

INTRODUCTION

In many economic settings agents establish relationships that can be represented by a network structure, which turns out to have a crucial role in determining the outcome of the interaction of the agents. For example, the buyers and sellers of a good or a service in a decentralized market form a network structure by establishing trade links, and the outcome depends on which links are established. A person seeking for job opportunities can gather information through personal relations he has formed before or may want to form new relationships for this purpose. Alliances among corporations, trade agreements among nations also can easily be modeled by using networks. It is important to note that in all these situations possibilities for cooperation among agents are reflected by the network structure, i.e. who is "connected" to whom.

Modeling social and economic interaction by network structures has its roots in cooperative game theory. In his seminal work, Myerson (1977) starts with a transferable utility (henceforth, TU) game and a network that represents the communication structure among the players. To distribute the value generated through the given TU game and network pair among the players, he proposes an allocation rule characterized by some "fairness" axioms. This rule –called the Myerson allocation rule– can be extended to the more general network games framework and in this framework it can be described as follows:

$$Y_i^{MV}(g,v) = \sum_{S \subset N \setminus \{i\}} (v(g|S \cup \{i\}) - v(g|S)) \left(\frac{\#S!(n-\#S-1)!}{n!} \right)$$

where *v* stands for a value function for networks and *v(g|T)* represents the value generated by the restriction of the graph g to coalition T. Note that this rule is based on Shapley-like calculations, and the value allocated to player *i* is a weighted sum of the marginal contributions of *i* to all possible coalitions.

In this study, given a value function defined on networks, we derive a TU game by considering the maximal value each coalition can guarantee for itself without involving agents outside the coalition in the network formation. The basic question we ask is: "Does Myerson allocation rule lead to the same payoff vector as the Shapley value of the associated TU game on some set of networks? How is this set of networks on which such a coincidence occurs located relative to networks that are stable in various senses?" These are questions in the spirit of the "Nash Program" in the sense that they deal with the problem of achieving cooperative outcomes through noncooperative means.

The Nash Program, as put by Trockel (2003: 153), "is a research agenda whose goal is to provide a non-cooperative equilibrium foundation for axiomatically defined solutions of cooperative games." First, in 1951 Nash proposes the use of non-cooperative games to study cooperative games in the following way: " One proceeds by constructing a model of the pre-play negotiation so that the steps of negotiation become moves in a larger non-cooperative game [which will have an infinity of pure strategies] describing the total situation." (Nash, 1951: 295). In 1953, Nash attempted to base an axiomatically defined two-person cooperative solution concept on a noncooperative equilibrium by converting the steps of negotiation in the cooperative game into moves in the non-cooperative game.

In the context of non-cooperative games, players cannot cooperate and form coalitions. They cannot vomit to joint actions or make binding threats or promises since no contracts are enforceable: Players can affect others and are affected by others solely through their choices of strategies. The common feature of all equilibrium notions can be stated as everyone doing her/his best given what others are doing under the given circumstances. Different behavioral and informational assumptions lead to different noncooperative equilibrium notions. In the context of cooperative games, however the assumption is that "the players can and will cooperate" (Nash, 1951: 295), and commitment to a joint action on the part of a coalition is enforceable. Axiomatic approaches to cooperative solution concepts typically involve equity and efficiency considerations along with stability.

Regarding Shapley value as a socially desirable cooperative solution concept for TU games, the question we deal with here is whether the payoff distribution prescribed by the Shapley value operator can be achieved under various stability notions in the context of networks if we employ the Myerson allocation rule. We introduce a new stability notion –called componentwise stability– that turns out to lead to a superset of networks for which coincidence obtains. In case we do not impose any restrictions on the value functions for networks, it turns out that there exist value functions for which it is possible that the Myerson allocation rule's payoff vector will not coincide with the Shapley value of the associated TU game on any of the networks. Thus the coincidence cannot be obtained on any set of stable networks whatever stability notion we use. Confining ourselves to monotonic value functions, however, we show that all strongly stable networks satisfy the desired coincidence. Pairwise stability, on the other hand, is shown to be incompatible with this coincidence. That is there exist pairwise stable networks which do not satisfy coincidence, while there are networks that satisfy coincidence but are not pairwise stable. Finally, we show that another allocation rule proposed by Jackson (2003b), which again is based on Shapley-like calculations, assures that the set of networks satisfying coincidence is equal to the set of strongly stable networks even without the monotonicity assumption for value functions.

CHAPTER II

LITERATURE SURVEY

The literature, which uses networks to model social and economic cooperation, starts with the seminal paper by Myerson (1977), which deals with TU games with communication structures. Together with a TU game (N,v) he considers a network g, which describes the possibilities of communication among the players. A network is, in fact, a graph with vertices being the players in N. The graph determines which pairs of individuals are "linked" to each other. If individuals i and j are linked it means that they can communicate with each other. Note that a graph has components, that is connected subgraphs in which every vertex is either directly connected or indirectly connected through a sequence of edges to every other vertex, and these components induce a partition on the set of vertices (players) N. Myerson derives from the given TU game v and network g a *graph-restricted game* v^g in which the value of each coalition S, is defined as the sum of the values of certain subcoalitions of S under the initial TU game v where the subcoalitions considered are the ones that consist of exactly the set of agents who form the set of vertices of a component of g. The interpretation is that a coalition can generate some value only if the players in that coalition can communicate, that is if they are somehow connected to each other in the network.

Myerson uses the term "allocation rule" to define a way of distributing the value generated through the TU game-network pair v and g, among the agents in the society.

Similar to the axiomatic characterization of solution concepts for TU games, he characterizes an allocation rule through two "fairness" axioms he proposes. One is that, two individuals who can add a new link to the existing network should benefit equally from the addition of that link, and the other is that the value generated by a coalition should be distributed among the players in that coalition, that is transfer of value across coalitions should not occur. Myerson shows that, Shapley value of the graph-restricted game v^g is the unique way of distributing the value that satisfies these two axioms. Myerson thereby brought a new perspective to cooperative game theory. Rather than just assuming that members of a coalition can simply "come together" and create a particular value, he allows different possible structures of "coming together" by the members of a coalition, thus a coalition can create possibly different values depending upon its communication structure.

Note that the enrichment brought by Myerson is limited in the sense that it is assumed that coalitions can cooperate if they are connected somehow, and different forms of being connected are not distinguished. Jackson and Wolinsky (1996) introduce a different framework for studying social and economic networks. Rather than starting with TU games with communication structures, they start with a value function v which assigns a real number to every network that can be formed by the agents in the society N. In this framework "the value of a network can depend on exactly how agents are interconnected, not just who they are directly or indirectly connected to." Here an allocation rule associates a payoff vector for every value function and network pair (v,g).

They assume that players have the right to form or break links and they define the notion of pairwise stability in this framework.

First, they propose two specific models to study social and economic interaction, the connections and co-authorship models, and investigate the stability and efficiency properties of networks. In both models they find that stable networks may be inefficient (i.e. not maximizing the value generated). Then, they analyze the general model and find that there exists a value function such that no component additive and anonymous allocation rule can assure that at least one efficient network is pairwise stable. They also show that the two fairness axioms defined by Myerson characterizes an allocation rule that is again based on the Shapley value.[1]

Using the framework introduced by Jackson and Wolinsky many authors have studied different social and economic situations using networks. For example Corominas-Bosch (1999) uses networks to model trade in a decentralized market. The players are divided into two sets as buyers and sellers. A buyer and seller must be connected to each other for a transaction between them to occur. In this model, no links can be formed among buyers or among sellers. Each seller has one unit of an indivisible good which has value for the buyers but not for himself. Corominas-Bosch models a bargaining game between buyers and sellers. In each period those pairs of buyers and sellers that realize a transaction drop from the market and this goes on until there

[1] Note that the allocation rule here is not the same mathematical object as what is called an allocation rule in Myerson (1977).

remains no link between remaining buyers and sellers; that is no more transactions can occur. It is also assumed that the buyers and sellers discount the value of a transaction at each period. They provide some properties of the final payoffs to buyers and sellers according to different kinds of connections between them.

Calvo-Armengol and Jackson (2001) examine a model of the transmission of job information through a network of social contacts. In each period, agents randomly receive information about new jobs and use it to obtain a job if they are unemployed or if the new job is more attractive than their current jobs. If not, they pass it to those whom they are directly connected. Also employed agents randomly lose their jobs in each period. They show that the possibility of receiving information about new jobs increases as one's status in the network improves. They also obtain the result that the possibility of obtaining a job decreases as length of time that an agent has been unemployed increases, which supports the empirical findings in real life job markets. In fact, what matters are the network structure and the initial status of an agent in the network.

Furusawa and Konishi (2002) examine the formation of free trade agreements as a network formation game. A free trade agreement is represented by a link in the network of countries; if two countries are not connected the trade between them includes a tariff. The incentives to sign an agreement depend on the characteristics of the countries like market size and the size of the industrial good industry. They show that if all countries are symmetric, a complete free trade network is pairwise stable.

Network formation itself has been of interest to many researchers. First Aumann and Myerson (1988) proposed an extensive form game to model this process in the context of TU games with cooperation structures. They start with a TU game v and an exogenously given ranking of all possible pairs of players. In each stage of the game, a pair of players, observing the actions of the pairs preceding them, decides whether or not to connect to each other. After the final network g forms, the payoffs to the players are determined by the Shapley value of the graph-restricted game v^g that is derived from v and g in the same way as Myerson (1977). They show that subgame perfect equilibria of this game may lead to inefficient networks. Dutta, van den Nouweland and Tijs (1998) study the formation of networks in the framework of TU games with cooperation structures using a normal form game. "In this game each player announces the set of players with whom he or she wants to form a link, and a link is formed if and only if both players want to form that link." They consider a class of solutions for TU games with cooperation structures, which satisfies some fairness axioms. After the network, or the cooperation structure, is formed, the payoffs are determined by a solution in that class. Their main finding is that, in the world of superadditive TU games, the undominated Nash equlibrium, coalition-proof Nash equilibrium and the strong Nash equlibrium of this game lead to the complete network or a network that is payoff equivalent to the complete network.

Currarini and Morelli (2000) propose a network formation model where the payoff division is endogenous, that is there is no fixed allocation rule in their model. Given an exogenous ranking of players, players move sequentially, and each announces with whom he or she wants to form a link and demands a payoff as a part of his or her action. A link is formed if both the players want to connect to each other. Also the sum of the demands of the players in a component of the final network should not exceed the value generated by that component, otherwise that component does not form and players in that component receive nothing. They show that this game always has a subgame perfect equilibrium and for the class of size monotonic value functions (defined on networks), all the subgame perfect Nash equilibria lead to efficient networks. Thus they provide a framework where the tension between stability and efficiency does not exist.

Dutta and Mutuswami (1997) also model the formation of networks as a normal form game where a strategy of a player is to announce the set of players with whom he or she wants to form a link. An allocation rule and the resulting network determine the payoffs. But they use an implementation approach to resolve the tension between efficiency and stability in the sense that they design an allocation rule. Since one expects only the stable graphs to form, they argue that expecting the allocation rule to satisfy anonymity, again a fairness axiom, on all the graphs is "unnecessarily stringent" (1997: 343). They show that with a mild assumption on the value function one can design an allocation rule, which will assure that the strong Nash equilibria of this game will lead to efficient graphs and which is anonymous on this set of graphs.

Jackson (2003a) examines the stability, efficiency and the compatibility of these two in a more general setting. He defines three different notions of "efficiency" and examines the relations between these notions. He shows that there exists a value function such that under any component balanced allocation rule satisfying equal treatment of equals we have that none of the constrained efficient (a weaker condition than being value maximizing) networks is pairwise stable. He also points to an important aspect of the tension between efficiency and stability. A network is said to have no loose ends if every player in the network is connected to at least two players. Under the assumption that the value function is anonymous, he shows that if there exists an efficient network with no loose ends then there is no tension, i.e. one can construct an anonymous and component balanced allocation rule such that some of the efficient networks will be pairwise stable.

The studies dealing with stability and efficiency generally assume that agents are myopic, in the sense that when deciding on whether to add or break a link they do not consider how the other agents will react to their actions. Recently some authors started to develop models with farsighted agents. Watts (2002) models the formation of networks as an extensive form game. Here the agents are farsighted in the sense that when deciding to form or break a link at some stage, they consider possible networks that might form in the following stages and discount future benefits from those networks. The cost of forming a link is more than its benefits, but agents also benefit from indirect links. So when nobody is connected to each other, none of the agents would want to bear the cost of forming a link if he/she could not discount future benefits. Watts shows that when agents are non-myopic, it is possible that a network shaped like a circle, in which every agent gets a strictly positive payoff, can form as a subgame perfect equilibrium of this game.

CHAPTER III

COINCIDENCE OF MYERSON ALLOCATION RULE WITH SHAPLEY VALUE

3.1 Definitions And Notation

Let $N = \{1,2,....., n\}$ be the set of individuals in the society. We assume that individuals establish bilateral relations among themselves and form a network structure. We will use a non-directed graph to model these relations whose vertex set will be the set of individuals in the society.

Let g^N denote the set of all subsets of N of cardinality 2. Any subset g of g^N will be called a network, and g^N itself will be called the complete network. Note that a network g is a set of pairs of individuals of the form $\{i,j\}$. If $\{i,j\} \in g$ then we say that individuals i and j are linked under the network g.

Edges of a graph g will be called links hereafter and for ease of notation we will write ij to represent the link $\{i, j\}$. Note that $\{i, j\}$ is not an ordered pair, so ij and ji

represent the same link. The network consisting of only the link ij will be denoted by $\{ij\}$.

Let $g^S = \{ij \in g^N \mid i, j \in S\}$. g^S denotes the complete network among the players in S. $G = \{g| \, g \subset g^N\}$ is the set of all possible networks on N. Given a network $g \in G$, let $N(g) = \{i \in N \mid \exists\, j \text{ s.t. } ij \in g\}$, that is the set of individuals who have at least one link in the network g.

Definition: Let $N = \{1,......, n\}$ be given, a function $v : G \rightarrow IR$ is called a *value function*.

The value function represents the "value" created by the individuals in the society under different network structures. Note that it is different from a TU game since the same set of individuals may create different values depending on how they are connected. This formulation allows the value created to depend on exactly how the individuals in the society are connected.

We assume that $v(\emptyset) = 0$, that is without any connections at all a society cannot create any value.

We will denote the set of all value functions, that is all functions of the form

$v : G \rightarrow IR$, by V.

Definition: A network $g \in G$ is said to be efficient with respect to a value function v if $v(g) \geq v(g')$ for every $g' \in G$.

<u>Definition</u>: A function $Y : G \times V \rightarrow IR^N$ such that $\sum_i Y_i(g,v) = v(g)$ for all $v \in V$ and $g \in G$, is called an *allocation rule*.

An allocation rule determines the payoffs of the individuals forming a network. It is worth to note that an allocation rule depends both on g and v, thus takes into account how the individuals are connected and what the roles of individuals in the network are.

In some contexts an allocation rule may represent the payoffs to individuals that are directly determined by their positions in the network. For example in the Connections Model by Jackson (1996), an individual i benefits directly from his links and indirectly from the links that can be reached by a sequence of links which starts from i; but bears only the cost of his direct links. In this setting the payoff of an individual is simply the sum of his benefits minus the sum of his costs. In some other contexts the allocation rules are given exogenously and some axioms are imposed on allocation rules for equity and efficiency considerations. These studies are similar to the axiomatic study of solution concepts for TU games.

<u>Definition</u>: Given a network $g \in G$, a sequence of distinct individuals $i_1,\ldots, i_K$ such that $i_K i_{K+1} \in g$ for each $k \in \{1,\ldots, \text{K-1}\}$, with $i_1 = i$ and $i_K = j$, is called a *path* in g between individuals i and j.

<u>Definition</u>: Given a network $g \in G$, any nonempty subnetwork $g' \subset g$ satisfying the following conditions is called a *component* of g:

1) if $i \in N(g')$ and $j \in N(g')$ where $i \neq j$, then there exists a path in g' between i and j,

2) if $i \in N(g')$ and $j \notin N(g')$ then there does not exist a path in g between i and j.

The components of a network are its maximal connected subgraphs. We will denote the set of components of a network g by $C(g)$.

Example 1:

Let $N = \{1, 2, 3, 4, 5, 6, 7\}$ and $g = \{12, 23, 34, 56, 67, 75\}$

G has two components which are $\{12, 23, 34\}$ and $\{56, 67, 75\}$.

Definition: Let $N = \{1, ..., n\}$ be the set of individuals in the society. Let 2^N denote the set of all subsets of N, i.e. the set of all possible coalitions in the society. A function $f : 2^N \setminus \{\varnothing\} \to IR$ is called a *transferable utility* (TU) game.

Definition: $N = \{1, ..., n\}$ be given. Denote these of all TU games with

player set N by G^N. A function $\psi : \bigcup_{n \in IN} G^N \to \bigcup_{n \in IN} IR^n$ which satisfies

$\psi(f) \in IR^n$ and $\sum_{i \in N} \psi_i(f) = f(N)$ for $\forall$ n $\in IN$ and $\forall f \in G^N$ is called a *value* for TU games.

Definition: Given a TU game f, the Shapley value $\varphi(f)$ of f is defined by

$$\varphi_i(f) = \sum_{S \subset N \setminus \{i\}} (f(S \cup \{i\}) - f(S)) \left(\frac{\#S!(n - \#S - 1)!}{n!} \right) \text{ for each } i \in N.$$

Before defining some notions of stability of a network, it must be stated that the basic assumption is that players can form new links or break links at the existing network. According to a particular, but commonly used rights structure for a new link to form, both of the players involved in that link should give consent; but a player can break an existing link he is involved in without the consent of the other party involved. The following two notions of "pairwise stability" and "strong stability" aim to describe stable networks at which players or groups of players would not benefit from deviating from the existing network. Both notions have their own assumptions on how players can possibly deviate from an existing network structure.

Definition: A network g is said to be *pairwise stable* with respect to an allocation rule Y and a value function v if

1) for all $ij \in g$, $Y_i(g,v) \geq Y_i(g \setminus \{ij\},v)$ and $Y_j(g,v) \geq Y_j(g \setminus \{ij\},v)$ and

2) for all $ij \notin g$, if $Y_i(g \cup \{ij\},v) > Y_i(g,v)$ then $Y_j(g \cup \{ij\},v) < Y_j(g,v)$.

Note that pairwise stability assumes that players consider only deviations that include only one link. Coalitions including at most two agents can form and add a single link to the existing network to increase their payoffs, or a single player can break a link to increase his payoff. It is assumed that, if addition of a link ij makes one of i and j strictly better off and the other not worse off, i and j will want to add that link.

Denote the set of pairwise stable networks with respect to some allocation rule Y and value function v by $PS(Y,v)$.

Definition: A network g' is said to be *obtainable from g via deviations* by S if

1) $ij \in g'$ and $ij \notin g$ imply $\{i,j\} \subset S$ and

2) $ij \in g$ and $ij \notin g'$ imply $\{i,j\} \cap S \neq \emptyset$.

A network g is said to be *strongly stable* with respect to an allocation rule Y and a value function v if for any $S \subset N$, for any g' that is obtainable from g by deviations by

S, and for any $i \in S$ with $Y_i(g',v) > Y_i(g,v)$, there exists $j \in S$ such that $Y_i(g',v) < Y_i(g,v)$.

Strong stability takes into account deviations by coalitions including possibly more than two players. Any subset of players can act together to change an existing network in order to increase their payoffs. Of course they can only achieve deviations that require no help from the players outside the coalition, i.e., they can form links among themselves and can break those links that involve at least one player from their coalition. Obviously a network that is strongly stable, with respect an allocation rule Y and a value function v, is pairwise stable with respect to that allocation rule and value function.

Denote the set of strongly stable networks with respect to some allocation rule Y and value function v by $SS(Y,v)$.

Finally we will define some properties of value functions and allocation rules which are used in the characterization of Myerson allocation rule.

Definition: A value function v is said to be *component additive* if

$$v(g) = \sum_{h \in C(g)} v(h) \text{ for all } g \in G.$$

Note that component additivity requires that value generated by a component should not depend on the structure of the rest of the network.

Definition: An allocation rule Y is said to be *component balanced* if for any component additive v, any $g \in G$, and any $h \in C(g)$

$$\sum_{i \in N(h)} Y_i(h,v) = v(h) \qquad \text{holds.}$$

Component balancedness requires that when the value generated by a component does not depend on the structure of the rest of the network, the value generated by a component should be distributed among the players in that component. Transfer of value among components is not allowed while distributing payoffs.

Definition: An allocation rule Y is said to satisfy *equal bargaining power* if, for any component additive v and any $g \in G$,

$Y_i(g,v) - Y_i(g \setminus \{ij\},v) = Y_j(g,v) - Y_j(g \setminus \{ij\},v)$ holds.

3.2 The Problem

When Myerson (1977) dealt with TU games with communication structures, his basic assumption was that a coalition could generate value only if the players in that coalition could communicate, that is, if they were somehow connected to each other in the communication network. So, in his framework, together with a TU game, a network g representing the communication structure is needed to be able to know the possibilities of value generation by each coalition. He derives a new TU game v^g where the value of a coalition S is the sum of the values of those subcoalitions of S which make up components of $g \mid S$. Note that if the members in a coalition make up a component, that is if they are connected they can generate a certain amount of value independent of the particular way of connection; so there are no optimal and suboptimal ways of connection so long as there is some connection. In this setting Myerson shows that the only way to distribute the value so as to satisfy equal bargaining power and component balancedness is using the Shapley value of the graph-restricted game v^g, when the communication structure is given by g.

In this study, we start with a set of players $N = \{1,..., n\}$ and a real-valued function v which is defined on the set of all networks that can be formed by the agents in the society N, i.e., a value function for networks. The value generated by the players depends directly on the particular network structure they form, so there may exist optimal and suboptimal ways of connection by players. Jackson and Wolinsky (1996) extend Myerson's result to this setting and show that whenever v is component additive, an allocation rule satisfies equal bargaining power and component balancedness if and only if it is of the following form:

$$Y_i^{MV}(g,v) = \sum_{S \subset N \setminus \{i\}} (v(g|S \cup \{i\}) - v(g|S)) \left(\frac{\#S!(n-\#S-1)!}{n!} \right) \text{ for each } i \in N.$$

where $g|S$ is called the *restriction* of g to the coalition S and is found by deleting all the links in g except the links which connect a player in S to another player in S, that is

$g|S = \{ij \mid ij \in g \text{ and } i \in S, j \in S\}$.

We will call this rule the Myerson allocation rule.

Note that this rule is based on Shapley-like calculations and the payoff to a player is determined by his marginal contribution to all possible coalitions. But the value of a coalition is the value of the network found by restricting the original network to that coalition. So when evaluating the value of a coalition, Myerson allocation rule takes into account only the network that is the restriction of the original network to that coalition and assumes that the players outside that coalition become totally isolated. Remember that restricting a network to a coalition means deleting all the links of that network except the links that connect a player in that coalition to another player in the same.

Given any value function v we derive a TU game v^* associated with v in the following way:

For any $S \subset N$, $v^*(S) = \max_{g \subset g^S} v(g)$

Given any value function v we will look for the coincidence of the payoff vector under Myerson allocation rule at some network g with the payoff vector of Shapley value of the TU game v^*.

Note that we find the value of S under v^*, by assuming that the players outside S are totally isolated and the players in S are connected optimally among themselves. This definition is based on the assumption that each coalition has the right and possibility to separate itself from the rest of the society and act on its own in achieving the maximal total value for itself. There are some further reasons for the assumptions underlying the definition of v^*. Firstly, assuming that players outside S are isolated while finding $v^*(S)$ is compatible with the definition of Myerson allocation rule according to which the value of a coalition at g is found by restricting g to that coalition and thus leaving the players outside that coalition isolated. Secondly, the Shapley value of v^* is based on calculating marginal contributions as if every coalition were connected optimally in itself, since we are not only interested in the marginal contributions at the particular network but we also take into consideration what will be the marginal contributions if every coalition were connected optimally in itself. In the case of coincidence at some network *g*, these two calculations will yield the same outcome even if every coalition is not connected optimally at *g*.

Proposition 1: *There exists a society N and a value function v such that for any g* $\in$ *G we have* $Y^{MV}(g,v) \neq \varphi\,(v^*)$.

Proof: Let $N = \{1, 2, 3\}$ and let $v(12) = 3$, $v(13) = v(23) = 2$,

and for all other networks $g \in G$, $v(g) = 0$.

The TU game associated with v is as follows: $v^*(1) = v^*(2) = v^*(3) = 0$,

$v^*(1, 2) = 3$, $v^*(1, 3) = v^*(2, 3) = 2$ and $v^*(1, 2, 3) = 3$.

The Shapley value of v^* will be $\varphi\,(v^*) = (\frac{7}{6}, \frac{7}{6}, \frac{4}{6})$

But the Myerson allocation rule's distribution of payoffs will be as follows:

For $g^1 = \{12\}$, $Y^{MV}(g^1,v) = (\frac{3}{2}, \frac{3}{2}, 0)$

for $g^2 = \{13\}$, $Y^{MV}(g^2,v) = (1, 0, 1)$

for $g^3 = \{23\}$, $Y^{MV}(g^3,v) = (0, 1, 1)$

for $g^4 = \{12, 23\}$, $Y^{MV}(g^4,v) = (-\frac{1}{6}, \frac{5}{6}, -\frac{4}{6})$

for $g^5 = \{12, 13\}$, $Y^{MV}(g^5,v) = (\frac{5}{6}, -\frac{1}{6}, -\frac{4}{6})$

for $g^6 = \{13, 23\}$, $Y^{MV}(g^6,v) = (-\frac{2}{6}, -\frac{2}{6}, \frac{4}{6})$

for $g^7 = \{12, 23, 13\}$, $Y^{MV}(g^7,v) = (\frac{1}{6}, \frac{1}{6}, -\frac{2}{6})$

So for this particular example, obtaining the payoff vector of the Shapley value of the TU game v^*, under Myerson allocation rule at some network g is not possible at all.

Note that in this case the network $g = \{12\}$ is strongly stable. To see this consider any coalition S which may improve upon g by deviations in S. Note that at g $Y_i^{MV}(g,v) \geq 0$ for every $i \in N$ that is $\sum_{i \in S} Y_i^{MV}(g',v) \geq 0$. Since improvement by S requires that at the new graph g' all players in S should be at least as well off as at g, and at least one player in S should be strictly better off. So at the new graph g', $\sum_{i \in S} Y_i^{MV}(g',v) > 0$ should hold. But only the networks {13} and {23} have value greater than zero except for the initial network {12}, so the above condition could possibly be satisfied only at these networks. Assume there exists an $S \subset N$ which can alter the network $g = \{12\}$ to $g' = \{13\}$ by deviations in itself and improve. Note that when passing from g to g' the link 13 is added. This can only happen with the consent of player 1, that is S must include 1. But $Y_1^{MV}(g,v) = \frac{3}{2}$ and $Y_1^{MV}(g',v) = 1$, so player 1 will not add that link. So no coalition can alter the network g to g' and improve. Assume there exists a coalition $S \subset N$ which can alter the network $g = \{12\}$ to $g'' = \{23\}$ by deviations in itself and improve. Note that when passing from g to g'' the link 23 is added. This can only happen with the consent of player 2, that is S must include 2. But $Y_2^{MV}(g,v) = \frac{3}{2}$ and $Y_2^{MV}(g'',v) = 1$, so player 1 will not add that link. So no coalition can alter the network g to g'' and improve. So no coalition $S \subset N$ can improve upon g. Thus g is strongly stable with respect to Y^{MV} and v. So $SS(Y^{MV}, v\ ) \neq$ &.

Note that in the above example only those networks with one link could generate a positive value. Adding a link to those networks leads to a decrease in the value.

<u>Definition</u>: A value function v is said to be *monotonic* if $g \subset g' \Rightarrow v(g) \leq v(g')$ for any $g, g' \in G$.

Assuming that the value function is monotonic rules out the case where the set of networks satisfying coincidence is empty. We will show this result in a few steps.

Proposition 2: *Assume that v is monotonic, then we have that a network g is pairwise stable, with respect to the Myerson allocation rule and value function v, if and only if* $\forall\ S \subset N, \forall\ ij \in \{kl \in g^N \setminus g \mid k, l \in S\}$ *we have* $v((\{ij\} \cup g) \mid S) \leq v(g|S)$ *(1).*

Condition (1) says that there should not exist a link ij which is not in g such that when added to g the value of this new graph's restriction to some $S \subset N$, which contains both players i and j, is greater than the value of g's restriction to S. *Proof*: First note that when v is monotonic, at any network g, a player i cannot improve alone, that is without cooperating with other players. To see this, note a player i can unilaterally deviate from an existing network g by only breaking existing links he is involved in at g. Remember that for a new link to form, both of the parties involved in that link should give consent. Take any $g \in G$ and any $i \in N$, consider any set of links $\{ ij_1,..., ij_n \}$. Note that for any $S \subset N \setminus \{i\}$, $g|S = (g \setminus \{ ij_1,..., ij_n \}) \mid S$ since restricting a network to a coalition S means deleting all the existing links at g except the ones that are between the players in S. So

$v(g|S) = v((g \setminus \{ ij_1,..., ij_n \}) \,|S)$. Also for any $S \subset N \setminus \{i\}$, $v(g|S \cup\{i\}) \geq v((g \setminus \{ ij_1,..., ij_n \}) \,|S \cup\{i\})$ since $(g \setminus \{ ij_1,..., ij_n \}) \,|S \cup\{i\} \subset g|S \cup\{i\}$ and v is monotonic. Subtracting the first equation from the second one we obtain $v(g|S \cup\{i\})$ - $v(g|S)) \geq (v((\text{g} \setminus \{ ij_1,..., ij_n \})|S \cup\{i\}) - v((g \setminus \{ ij_1,..., ij_n \})|S))$ for any $S \subset N \setminus \{\text{i}\}$. Multiplying both sides by $\left(\frac{\# S!(n-\# S-1)!}{n!}\right)$ and summing these inequalities over all $S \subset N \setminus \{i\}$, we obtain $\sum_{\mathrm{S}\subset\mathrm{N}\setminus\{\mathrm{i}\}} (v(g|S \cup\{i\})$- $v(g|S)) \left(\frac{\# S!(n-\# S-1)!}{n!}\right) \geq \sum_{\mathrm{S}\subset\mathrm{N}\setminus\{\mathrm{i}\}} (v((g \setminus \{ ij_1,..., ij_n \})|S \cup\{i\})$ - $v((g \setminus \{ ij_1,..., ij_n \})|S)) \left(\frac{\# S!(n-\# S-1)!}{n!}\right)$. That is, $Y_i^{MV}(g,v) \geq Y_i^{MV}(g \setminus \{ ij_1,..., ij_n \},v)$, so i cannot improve by only breaking links he is involved .

Turning back to our claim, assume that condition (1) holds but g is not pairwise stable. Then there must exist a player who can improve upon g by adding a new link to g or by breaking an existing link in g. We have seen that a player cannot improve by only breaking a link, so player i should be improving by adding a new link to g. So there exists $j \in N \setminus \{i\}$ such that $Y_i^{MV}(\{\text{ij}\} \cup g,v) > Y_i^{MV}(g ,v)$ holds. But by condition (1) we have $v((\{ij\} \cup g) \,|S) \leq v(g|S)$ for $\forall\ S \subset N$, $\forall\ ij \in \{kl \in g^N \setminus g \mid k, l \in S \}$ and we know that, for any coalition T which does not include i, $v((\{ij\} \cup g)|T) = \text{v}(g|T)$ holds. So we have $v((\{ij\} \cup g) \,|T \cup\{i\})$ - $v((ij \cup g)|T) \leq v(g|T \cup\{i\})$- $v(g|T)$ for any coalition T which does not include i, implying that $\left(\frac{\# T!(n-\# T-1)!}{n!}\right) \big(v((\{ij\} \cup g)|T \cup\{i\})$ - $v((ij$

$\cup g)|T)\Big) \leq \Big(\frac{\#T!(n-\#T-1)!}{n!}\Big)\ \Big(v(g|T\cup\{i\})\text{- } v(g|T)\Big)$ for any coalition T which does

not include i since $\Big(\frac{\#T!(n-\#T-1)!}{n!}\Big) > 0$ for every $T \subset N \setminus \{i\}$. Summing these

inequalities over all such coalitions we obtain $\sum_{T\subset N\setminus\{i\}} (v((\{ij\} \cup g)|T\cup\{i\})\text{- } v(g|T))$

$\Big(\frac{\#T!(n-\#T-1)!}{n!}\Big) \leq \sum_{T\subset N\setminus\{i\}} (v(g|T \cup\{i\})\text{- } v(g|T))\ \Big(\frac{\#T!(n-\#T-1)!}{n!}\Big)$. That is,

$Y_i^{MV}(\{ij\} \cup g,v) \leq Y_i^{MV}(g\ ,v)$ in contradiction with $Y_i^{MV}(\{ij\} \cup g,v) > Y_i^{MV}(g\ ,v)$.

So there cannot exist a player i who can improve upon g by adding a new link to g, so g must be pairwise stable.

Conversely assume that, g is pairwise stable and assume that Condition (1) does not hold. That is $\exists\ S \subset N$ and $\exists\ ij \in \{kl \in g^N \mid kl \notin g$ and $k, l \in S\}$ such that $v((\{ij\} \cup g)|S) > v(g|S)$. Note that $(\{ij\} \cup g)|S \setminus \{i\}= g|S \setminus \{i\}$, so $v((\{ij\} \cup g)|S \setminus \{i\}) = v(g|S \setminus \{i\})$. Subtracting second equation from the first one we have $v((\{ij\}\cup g)|S)$ - $v((\{ij\}\cup g)|S \setminus \{i\}) > v(g|S)$ - $v(g|S \setminus \{i\})$, which in turn implies $\Big(\frac{(\#S-1)!(n-\#S-2)!}{n!}\Big)\ \Big(v((\{ij\}$

$\cup g)\ |S) - v((\{ij\} \cup g)\ |S \setminus \{i\})\Big) > \Big(\frac{(\#S-1)!(n-\#S-2)!}{n!}\Big)\ \Big(v(g|S)\ \text{-}\ v(g|S \setminus \{i\})\Big)$

since $\Big(\frac{(\#S-1)!(n-\#S-2)!}{n!}\Big) > 0$ for every $S \subset N$. Also for every $T \subset N$ with $T \neq S$

we have $v((\{ij\} \cup g) \mid T) \geq v(g \mid T)$ since v is monotonic, and $v((\{ij\} \cup g) \mid T \setminus \{i\}) = v(g \mid T \setminus \{i\})$ since $(\{ij\} \cup g) \mid T \setminus \{i\} = g \mid T \setminus \{i\}$. That is $v((\{ij\} \cup g) \mid T) - v((\{ij\} \cup g) \mid T \setminus \{i\}) \geq v(g \mid T) - v(g \mid T \setminus \{i\})$ for every $T \subset N$ with $T \neq S$, which implies that $\left(\frac{(\#T-1)!(n-\#T-2)!}{n!}\right) \left(v((\{ij\} \cup g) \mid T) - v((\{ij\} \cup g) \mid T \setminus \{i\})\right) \geq$

$\left(\frac{(\#T-1)!(n-\#T-2)!}{n!}\right) \left(v(g \mid T) - v(g \mid T \setminus \{i\})\right)$ for every $T \subset N$ with $T \neq S$, since

$\left(\frac{(\#T-1)!(n-\#T-2)!}{n!}\right) > 0$ for every such T. Summing these inequalities, we obtain

$\left(\frac{(\#S-1)!(n-\#S-2)!}{n!}\right) \left(v((\{ij\} \cup g) \mid S) - v((\{ij\} \cup g) \mid S \setminus \{i\})\right) + \sum_{T \subset N \,\&\, T \neq S} (v((\{ij\} \cup$

$g) \mid T) - v((\{ij\} \cup g) \mid T \setminus \{i\})) > \left(\frac{(\#S-1)!(n-\#S-2)!}{n!}\right) \left(v(g \mid S) - v(g \mid S \setminus \{i))\right) +$

$\sum_{T \subset N \,\&\, T \neq S} (v(g \mid T) - v(g \mid T \setminus \{i\}))$. Rewriting this inequality, $\sum_{M \subset N \setminus \{i\}} (v((\{ij\} \cup g) \mid M$

$\cup \{i\}) - v(g \mid M)) \left(\frac{\#M!(n-\#M-1)!}{n!}\right)$

$> \sum_{M \subset N \setminus \{i\}} (v(g \mid M \cup \{i\}) - v(g \mid M)) \left(\frac{\#M!(n-\#M-1)!}{n!}\right)$, that is $Y_i^{MV}(\{ij\} \cup g, v) >$

$Y_i^{MV}(g, v)$. Now writing j instead of i and following the same arguments above we will have $Y_j^{MV}(\{ij\} \cup g, v) > Y_j^{MV}(g, v)$. That is by adding the link ij to the graph g both players i and j become strictly better off, so g is not pairwise stable, yielding the desired contradiction.

Proposition 3: *Assume that v is monotonic, then we have that a network g is strongly stable with respect to the Myerson allocation rule and value function v only if* $\forall\, n \in \mathbb{N},\ \forall\, S \subset N,\ \forall\, \{i_r j_r \mid r = 1,\ldots, n\} \subset \{kl \in g^N \setminus g \mid k, l \in S\}$, *we have* $v((\{i_1 j_1,\ldots, i_n j_n\} \cup g) \mid S) \leq v(g \mid S)$ *(2).*

Similar to Condition (1), Condition (2) says that there should not exist a sequence of links $i_1 j_1,\ldots, i_n j_n$ which are not in g such that when added to g, the value of this new graph's restriction to some $S \subset N$, which contains all the players $i_1, j_1, \ldots, i_n, j_n$ (some of which may of course coincide with each other) is greater than the value of g's restriction to S.

Proof: Assume g is strongly stable but Condition (2) does not hold. Then $\exists\, n \in \mathbb{N}$ and $\exists\, S \subset N$ and $\exists\, \{i_r j_r \mid r = 1,\ldots, n\} \subset \{kl \in g^N \setminus g \mid k, l \in S\}$ such that we have $v((\{i_1 j_1,\ldots, i_n j_n\} \cup g) \mid S) > v(g \mid S)$. Of course there may exist more than one coalition $T \subset N$ such that $\exists\, \{i_r j_r \mid r = 1,\ldots, n\} \subset \{kl \in g^N \setminus g \mid k, l \in T\}$ such that we have $v((\{i_1 j_1,\ldots, i_n j_n\} \cup g) \mid T) > v(g \mid T)$. And for each such coalition they may exist more than one set of links $\{i_r j_r \mid r = 1,\ldots, n\} \subset \{kl \in g^N \setminus g \mid k, l \in T\}$ such that $v((\{i_1 j_1,\ldots, i_n j_n\} \cup g) \mid T) > \mathrm{v}(g \mid T)$. For each such coalition find a minimal set of links (that is of minimum cardinality) such $\{i_r j_r \mid r = 1,\ldots, n^T\} \subset \{kl \in g^N \setminus g \mid k, l \in T\}$ such that $\mathrm{v}((\{i_1 j_1,\ldots, i_{n^T} j_{n^T}\} \cup g) \mid T) > v(g?T)$. Of course there may exist more than one such minimal set of links of every such coalition T, choose and fix one of those minimal set of links for every such set T. Let us denote those sets of links by $\{i_r^T j_r^T \mid r = 1,\ldots, n^T\}$

for each such coalition T. Note that for any such T, $\{i_r^T j_r^T \mid r = 1,..., n^T\}$ is the minimal set which satisfies $v((\{i_1^T j_1^T,..., i_{n^T}^T j_{n^T}^T\} \cup g) \mid T) > v(g \mid T)$ where $\{i_r^T j_r^T \mid r = 1,...,n^T\} \subset \{kl \in g^N \setminus g \mid k, l \in T\}$. That is, adding any proper subset $\{i_1^T j_1^T,..., i_p^T j_p^T\}$ of $\{i_r^T j_r^T,..., i_{n^T}^T j_{n^T}^T\}$ to g will result in $v((\{i_1^T j_1^T,..., i_p^T j_p^T\} \cup g) \mid T) = v(g \mid T)$. Now among all those minimal sets of links (each corresponding to a different such T) which increase value as described above, choose a minimal one. Let us call the coalition that this set of links corresponds to M. Now we have a coalition $M \subset N$, and a set of links $\{i_r^M j_r^M \mid r = 1,...,n^M\}$ such that $v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) \mid M) > v(g \mid M)$. Note that since $\{i_r^M j_r^M \mid r = 1,...,n^M\}$ is a minimal set among the sets each of which is a minimal set that has the effect $v((\{i_1^T j_1^T,..., i_{n^T}^T j_{n^T}^T\} \cup g) \mid T) > v(g \mid T)$, adding a proper subset of $\{i_1^M j_1^M,..., i_p^M j_p^M\}$ of $\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\}$ will result in $v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) \mid S) = v(g \mid S)$ for any coalition $S \subset N$.

We claim that the players i_1^M, j_1^M ,..., i_n^M , j_n^M (again some of which may coincide with each other) could improve upon g. Take any $k \in \{i_r^M j_r^M \mid r = 1,...,n^M\}$, consider $Y_k^{MV}((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) \mid M, v)$. $Y_k^{MV}((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) \mid M, v)$

$$= \sum_{S \subset N \setminus \{k\}} (v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) \mid S \cup \{k\}) - v(g \mid S)) \left(\frac{\#S!(n-\#S-1)!}{n!}\right).$$

Now we know that for that particular $S \subset N \setminus \{k\}$ which satisfies $S \cup \{k\} = \{i_r^M j_r^M \mid r = 1,...,n^M\}$, we have $v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) \mid S \cup \{k\}) > v(g \mid S \cup \{k\})$. Since $k \notin S$, $(\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) \mid S$ does not contain $\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\}$, but contains only a proper subset of it. But we know from the very choice of $\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\}$ that adding a proper subset of this set to g will not increase value at any restriction. That is $v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g)| S) = v(g? S)$. So we have $v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g)| S \cup\{k\}) - v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) | S) > v(g| S \cup\{k\}) - v(g| S)$ which in turn implies $\left(\frac{\#S!(n-\#S-1)!}{n!}\right) \left(v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) | S \cup\{k\}) - v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) | S)\right) > \left(\frac{\#S!(n-\#S-1)!}{n!}\right) \left(v(g| S \cup\{k\}) - v(g| S)\right)$. Now consider the remaining coalitions, that is any $T \subset N \setminus \{k\}$ such that $T \neq S$. Since v

is monotonic $v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) | T \cup\{k\}) \geq v(g| T \cup\{k\})$. Again since $k \notin T$, $(\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) | T$ does not contain $\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\}$, but contains only a proper subset of it. Again we know that adding a proper subset of this set to g will not increase value at any restriction. That is $v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) | T) = v(g? T)$ for every $T \subset N \setminus \{k\}$ such that $T \neq S$. So we have $v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) | T \cup\{k\}) - v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) |T) \geq v(g| T \cup\{k\}) - v(g|T)$ for every $T \subset N \setminus \{k\}$ such that $T \neq S$, which implies $\left(\frac{\#T!(n-\#T-1)!}{n!}\right) \left(v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) | T \cup\{k\}) - v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) | T)\right) \geq \left(\frac{\#T!(n-\#T-1)!}{n!}\right) \left(v(g| T \cup\{k\}) - v(g| T)\right)$ for every $T \subset N \setminus \{k\}$ such that $T \neq S$. Summing these inequalities, we obtain $\left(\frac{\#S!(n-\#S-1)!}{n!}\right) \left(v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup g) | S \cup\{k\}) - v((\{i_1^M j_1^M,..., i_{n^M}^M j_{n^M}^M\} \cup\right.$

$g) \mid S)\Big) + \sum_{T\subset N\backslash\{k\}\,\&\,T\neq S} \Big(\frac{\#T!(n-\#T-1)!}{n!}\Big)\ \Big(v((\{i_1^M j_1^M,\ldots,\ i_{n^M}^M j_{n^M}^M\} \cup g) \mid T\cup\{k\})$ -

$v((\{i_1^M j_1^M,\ldots,\ i_{n^M}^M j_{n^M}^M\} \cup g) \mid T)\Big) > \Big(\frac{\#S!(n-\#S-1)!}{n!}\Big)\ \Big(v(g\mid S\cup\{k\}) - v(g\mid S)\Big) +$

$$\sum_{T\subset N\backslash\{k\}\,\&\,T\neq S} \Big(\frac{\#T!(n-\#T-1)!}{n!}\Big)\ \Big(v(g\mid T\cup\{k\}) - v(g\mid T)\Big).$$

Rewriting this inequality, $\sum_{K\subset N\backslash\{i\}} (v((\{i_1^M j_1^M,\ldots,\ i_{n^M}^M j_{n^M}^M\} \cup g) \mid K\cup\{k\})\text{- } v(g\mid K))$ $\Big(\frac{\#K!(n-\#K-1)!}{n!}\Big) > \sum_{K\subset N\backslash\{i\}} (v(g\mid K\cup\{k\})\text{- } v(g\mid K))\ \Big(\frac{\#K!(n-\#K-1)!}{n!}\Big)$, that is $Y_k^{MV}(\{i_1^M j_1^M,\ldots,\ i_{n^M}^M j_{n^M}^M\} \cup g,v) > Y_k^{MV}(g\ ,v)$. Now instead of k, we can write any $l \in\{i_r^M j_r^M \mid r = 1,\ldots,n^M\}$ and follow the same argument and obtain $Y_l^{MV}(\{i_1^M j_1^M,\ldots, i_{n^M}^M j_{n^M}^M\} \cup g,v) > Y_l^{MV}(g\ ,v)$. Every player in the coalition $\{i_r^M j_r^M \mid r = 1,\ldots,n^M\}$ will be strictly better off by adding the links $\{i_1^M j_1^M,\ldots,\ i_{n^M}^M j_{n^M}^M\}$ to g, so there exists a coalition which can improve upon g , that is g is not strongly stable. But this contradicts with our initial assumption that g was strongly stable so Condition (2) must hold.

Proposition 4: *Assume that v is monotonic, then we have*

For any $g \in g^N$, if Condition (2) holds for g then Myerson allocation rule's payoff vector at g will coincide with the Shapley value of the associated TU game v, that is* $Y^{MV}(g,v) = \varphi\ (v^*)$.

Proof: Assume that Condition (2) holds for some g, take any $S \subset N$, consider $v(g\mid S)$. Since v is monotonic $v(g^S) \geq v(g')$ for any $g' \subset g^S$, so in particular $v(g^S) \geq v(g\mid S)$.

Assume that $v(g^S) > v(g|S)$, but then we will have $v((g^S \setminus (g|S)) \cup (g|S)) = v(g^S) > v(g|S)$. Note that $ij \in g^S \setminus (g|S)$ implies $ij \notin g$ but $i, j \in S$. So $v((g^S \setminus (g|S)) \cup (g|S)) > v(g|S)$ implies that $\exists\, n \in IN$, $\exists\, S \subset N$ and $\exists\, \{i_r j_r \mid r = 1,..., n\} \subset \{kl \in g^N \setminus g \mid k, l \in S\}$ such that $v((\{i_1 j_1,..., i_n j_n\} \cup g) \,|S) > v(g|S)$. But this contradicts with Condition (2), so our assumption was wrong, that is $v(g^S) \leq v(g|S)$ must hold. Together with $v(g^S) \geq v(g|S)$ this will imply $v(g^S) = v(g|S)$.

So for any $S \subset N$ we have $v(g|S) = v(g^S) = \max_{g \subset g^S} v(g) = v^*(S)$.

Now take any $i \in N$, since $v(g|S) = v^*(S)$ for every $S \subset N$, we have $\left(\frac{\#S!(n-\#S-1)!}{n!}\right)$

$$\left(v(g|S \cup \{i\}) - v(g|S)\right) = \left(\frac{\#S!(n-\#S-1)!}{n!}\right) \left(v^*(S \cup \{i\}) - v^*(S)\right) \text{ for every } S \subset N.$$

That is $\sum_{S \subset N \setminus \{i\}} (v(g|S \cup i) - v(g|S)) \left(\frac{\#S!(n-\#S-1)!}{n!}\right) = \sum_{S \subset N \setminus \{i\}} (v^*(S \cup \{i\}) - v^*(S))$

$\left(\frac{\#S!(n-\#S-1)!}{n!}\right)$, that is $Y_i^{MV}(g,v) = \varphi\,(v^*)$.

Corollary 1: *Assume that v is monotonic, then we have that if g is strongly stable with respect to Myerson allocation rule and v then g satisfies* $Y^{MV}(g,v) = \varphi\,(v^*)$.

Corollary 1 is directly implied by propositions 2 and 3.

So under the assumption of monotonicity of the value function, strong stability of a network will assure the coincidence of Myerson allocation rule's payoff vector with Shapley value of the associated TU game. Note that Myerson allocation rule's payoff vector is the same on the set of strongly stable networks.

Corollary 2: *Assume that v is monotonic, then there always exists a network g which satisfies* $Y^{MV}(g,v) = \varphi(v^*)$.

Proof: Consider the complete network g^N. Take any $S \subset N$, note that $g^N|S = g^S$. We know that $v(g^S) = v^*(S)$ whenever v is monotonic, so we have $v(g^N|S) = v^*(S)$ for every $S \subset N$. Take any $i \in N$, for any $T \subset N \setminus \{i\}$, $v(g^N|T \cup \{i\}) - v(g^N|T) = v^*(T \cup \{i\}) - v^*(T)$. Multiplying with the corresponding coefficients and summing over all $T \subset N \setminus \{i\}$ we obtain $\sum_{T \subset N \setminus \{i\}} (v(g|T \cup \{i\}) - v(g|T)) \left(\frac{\#T!(n-\#T-1)!}{n!}\right) = \sum_{T \subset N \setminus \{i\}} (v^*(T \cup \{i\}) - v^*(T)) \left(\frac{\#T!(n-\#T-1)!}{n!}\right)$, that is $Y_i^{MV}(g^N,v) = \varphi_i(v^*)$. Since this is true for every $i \in N$ we have $Y^{MV}(g^N,v) = \varphi(v^*)$.

So whenever v is monotonic, the coincidence of Myerson allocation rule's payoff vector with the Shapley value of v^* is no longer impossible.

It is worth noting that when v is monotonic the complete network g^N is also pairwise stable. Since there exists no missing links, a pair of players cannot add a new link to g^N. So the only strategic action a player can take to improve, is to break one link. But we have seen that when v is monotonic a player cannot improve by breaking a link. So the complete network is pairwise stable. Thus under the monotonicity of v, Shapley value of the associated TU game v^* can be supported by at least one pairwise stable network under Myerson allocation rule.

We will give an example to further clarify the relationship of the set of pairwise stable networks and the set of networks satisfying coincidence under the monotonicity of v and Myerson allocation rule. We have seen above that the intersection of these two sets, both of which are nonemtpy, is nonempty.

Example 2: Let $N = \{1, 2, 3, 4\}$, Let $v : G \to IR$ be defined through:

$v(ij, jk, ki) = 4$ for any $i, j, k \in \{1, 2, 3, 4\}$

$v(g) = 4$ for any network which contains a network of the form $\{ ij, jk, ki \}$ for any $i, j, k \in \{1, 2, 3, 4\}$.

$v(12, 23, 34, 41) = 4$

And $v(g) = 0$ for all other networks.

Note that v is a monotonic value function. And the associated TU game v^* is as follows:

$v^*(1) = v^*(2) = v^*(3) = 0$, $v^*(12) = v^*(13) = v^*(14) = v^*(23) = v^*(24) = v^*(34) =0$

$v^*(1, 2, 3) = v^*(1, 2, 4) = v^*(2, 3, 4) = v^*(1, 3, 4) = 4$, and $v^*(1, 2 , 3, 4) = 4$.

The Shapley value of this game is $\varphi (v^*) = (1, 1, 1, 1)$.

Consider the network $g = \{12, 23, 34, 41\}$, $Y^{MV}(g,v) = (1, 1, 1, 1) = \varphi (v^*)$, that is coincidence is satisfied on g.

Let $g' = \{12, 23, 34, 41, 24\} = g \cup \{24\}$, $Y^{MV}(g',v) = (\frac{1}{3}, \frac{5}{3}, \frac{1}{3}, \frac{5}{3})$. Note that $Y_2^{MV} (g \cup \{24\},v) = \frac{5}{3} > 1 = Y_2^{MV} (g ,v)$, and $Y_4^{MV} (g \cup \{24\} ,v) = \frac{5}{3} > 1 = Y_4^{MV} (g ,v)$. So players

2 and 4 can improve upon g by adding the link 24 to *g*, thus *g* is not a pairwise stable network.

Now consider the network $g'' = \{12, 23, 31\}$. Note that $Y^{MV}(g'',v) = (\frac{4}{3},\frac{4}{3},\frac{4}{3},0)$ so $Y^{MV}(g'',v) \neq \varphi\ (v^*)$. Let us check that g'' is pairwise stable. Consider players 1 and 4, they can add the link 14 to g'' trying to improve. But note that $Y^{MV}(g'' \cup \{14\}\ ,v) = (\frac{4}{3},\frac{4}{3},\frac{4}{3},0) = Y^{MV}(g'',v)$, so players 1 and 4 cannot improve upon g''.

Consider players 2 and 4, they can add the link 24 to g'' trying to improve. But note that $Y^{MV}(g'' \cup \{24\}\ ,v) = (\frac{4}{3},\frac{4}{3},\frac{4}{3}, 0) = Y^{MV}(g'',v)$, so players 2 and 4 cannot improve upon g''. Consider players 3 and 4, they can add the link 34 to g'' trying to improve. But note that $Y^{MV}(g'' \cup \{34\}\ ,v) = (\frac{4}{3},\frac{4}{3},\frac{4}{3}, 0) = Y^{MV}(g'',v)$, so players 3 and 4 cannot improve upon g''. Consider players 1 and 2. Since the link $12 \in g''$ and we know that a player *i* cannot improve by breaking an existing link *ij*, players 1 and 2 cannot improve upon g''. For the same reason players 1 and 3, and players 2 and 3 cannot improve upon g''. So there exists no pair of players *i* and *j* who can improve upon g'', that is g'' is pairwise stable.

This example shows that there exists a monotonic value function *v* such that there exists a network *g* which is pairwise stable with respect to *v* and Myerson allocation rule but does not satisfy coincidence; and there exists a network g' such that g' satisfies coincidence but is not pairwise stable with respect to *v* and Myerson allocation rule.

While strongly stable networks, with respect to Y^{MV} and some monotonic v, satisfy coincidence of Myerson allocation rule's payoff vector with the Shapley value of v^*, the above example shows that there may exist networks satisfying this coincidence which are not even pairwise stable.

As a final note on this particular example, note that $Y^{MV}(g^N,v) = (1, 1, 1, 1)$ and $Y^{MV}(\{12, 23,31\},v) = (\frac{4}{3},\frac{4}{3},\frac{4}{3}, 0)$ and the coalition $\{1, 2, 3\}$ can deviate from g^N to form the graph $\{12, 23,31\}$ by just deleting all links with player 4. With this deviation all players in the coalition $\{1, 2, 3\}$ become strictly better off at the new network. So g^N is not a strongly stable network, but we know that it is pairwise stable and it satisfies coincidence.

3.3 A New Code of Rights

In this study our aim was to investigate the possible coincidence of payoff vector under Myerson allocation rule, at a given value function v and some network g, with the payoff vector of Shapley value of the associated TU game v^*. We tried to relate the set of networks satisfying this coincidence to the stability notions at hand (namely pairwise stability and strong stability. These notions had their own assumptions about the possibilities of forming coalitions with the aim of deviating from the existing network. As for pairwise stability, recall that, for a new link to form both of the players involved in that should give consent and that a player can break an existing link he is involved without the consent of the other party involved.

Note that, together with the particular value we use to find the solution of the TU game and the particular allocation rule to determine payoffs at the network setting the stability notions at hand played an important role for our purposes. The two stability notions differed on their assumptions of which coalitions can form. But in both cases, the rights structure to form and break a link was based on the ability and willingness of these coalitions. In a different context Sertel (2002) proposed that alteration of states, alteration of networks in our setting, should be examined together with a *code of rights* structure not only on the basis of ability and willingness of a coalition to alter a state. He proposed a list of coalitions, corresponding to every possible alteration by any coalition, whose approval is needed to alter that state. In what follows we propose a new code of rights for determining the "allowed" alterations of existing networks.

Assume that the value function v is component additive. We know that Myerson value is a component balanced allocation rule. So when the value generated by a coalition does not depend on the structure of the rest of the network Myerson allocation rule distributes to each coalition exactly the value generated by that coalition. We assume that any coalition can form to deviate from an existing network to increase their payoffs. But now a coalition needs the consent of some other members of the society to alter the existing network even if they are going to form new links among themselves.

Given a network $g \in G$, any $S \subset N$ should need approval of "others" while making the usual actions of deviations, that is forming new links among the members of S and breaking those links that involve at least one player from S. Since under a component additive value function the value generated by a component depends only on the structure of that component and since Myerson allocation rule distributes value without making any transfers among coalitions, when a player i wants to alter the existing situation, alone or in cooperation with others, it would be somewhat natural to require the approval of the other players in i's component of the desired alteration by i. Think of an autarchic economy which makes absolutely no trade with the rest of the world, so they can consume only the goods and services that is produced in that country. When an individual from this country wants to import a product and sell it in this country to improve his situation, this trade can harm the local producers of that product. As long as the local producers do not give consent, the code of rights we propose prohibits that trade, even if the total societal welfare of that country would increase with that trade. The same holds for deviations in a component of course, if a member of a component becomes worse off due to alterations within that component, by some other members in that component, that member can "block" those alterations.

Formally given any $g \in G$, any $S \subset N$ can form links among the members of S and can break those links that involve at least one player from S only if each agent i in S can get consent from all the players in his own component at g. That is, each i should get consent from $A^i = \{j \in N \mid \{i, j\} \subset N(h)$, where $h \in C(g)\}$, which means that all the agents in the set $A^S = \{i \in N \mid \exists\, j \in S$ such that $\{i, j\} \subset N(h)$, where $h \in C(g)\} \setminus S\}$should approve the alteration intended by S. Under this new code of rights, if a player i belongs to the same component with some member j of the deviating coalition S (at g), that player i has the right to block that deviation. Note that for a deviation from g to g' by S, not to be blocked, the payoffs at the new network g' to all the players whose consent is needed should be at least as high as the payoffs they receive at *g'*. That is $Y_i^{MV}(g',v) \geq Y_i^{MV}(g,v)$ should hold for all $i \in A^S$. This has the effect that when a coalition *S* wants to deviate, it has to consider payoffs not only to themselves but also to the players in $A^S \setminus S$.

Note that possibilities of deviation are now restricted when compared to strong stability. If a coalition *S* can deviate under this code of rights, it can deviate under strong stability also. After all, without hurting "others" *S* could guarantee that " $Y_i^{MV}(g',v) \geq Y_i^{MV}(g,v)$ for every $i \in S$ and there exists some $j \in S$ such that $Y_j^{MV}(g',v) > Y_j^{MV}(g,v)$", and this is enough under strong stability for *S* to deviate. Let us call the set of stable networks under this code of rights as componentwise stable networks and denote it by $CS\ (Y^{MV},v)$. Then we have $SS\ (Y^{MV},v) \subset CS\ (Y^{MV},v)$.

Proposition 5: *Given any component additive value function v, if g ∈ G satisfies coincidence of Myerson allocation rule's payoff vector with the Shapley value of v*, then g is componentwise stable.*

Proof: Assume *g* satisfies coincidence, then *g* must be an efficient graph since $Y_i^{MV}(g,v) = \varphi_i(v^*)$ for every $i \in N$, thus $\sum_{i \in N} Y_i^{MV}(g,v) = \sum_{i \in N} \varphi_i(v^*)$. But we know by definition of a value that $\sum_{i \in N} \varphi_i(v^*) = v^*(N) = \max_{g' \subset g^S} v(g')$.

Since we also know $v(g) = \sum_{i \in N} Y_i^{MV}(g,v)$ we have $v(g) = \max_{g' \subset g^S} v(g')$.

Assume $\exists\ S \subset N$ which can improve upon *g*, by altering *g* to *g'*. Now $A^S \cup S$ is the set of all agents who should be at least as well off at *g'* as *g*. But we know that there exists $i \in S$ such that $Y_i^{MV}(g',v) > Y_i^{MV}(g,v)$, and $Y_j^{MV}(g',v) \geq Y_j^{MV}(g,v)$ for every $j \in S$. So we have $\sum_{i \in S} Y_i^{MV}(g',v) > \sum_{i \in S} Y_i^{MV}(g,v)$. Note that the players in $B = N \setminus (A^S \cup S)$ are neither in S nor have the right to block the deviation. So there does not exist any link between any player k in B and a player l in $A^S \cup S$ at the network g since otherwise k would be either in S or would be connected to a player in S in which case k should be in A^S. So none of the components of g includes any two players m, n such that m $\in$ B and n $\in A^S \cup S$, thus we have $g = (g|B) \cup (g|A^S \cup S)$. With the same argument as above, there does not exist any link between any player k in B and a player l in $A^S \cup S$ at the network g' and we have $g = (g'|B) \cup (g'|\ A^S \cup S)$. Since players in B are not in $A^S \cup S$ they have not been effected by the deviation of S from g to g' (that is no links with players in B are established or broken during these alterations) we have $g'|B = g|B$. Now $v(g) = v(g|B) + v(g|A^S \cup S)$, and $v(g') = v(g'|B) + v(g'|A^S \cup S)$ since v is component additive. But we also know that g is an efficient network so we have $v(g) \geq v(g')$ that is $v(g|\ B) + v(g|\ A^S \cup S) \geq v(g'|B) + v(g'|A^S \cup S)$. Since $g'|B = g|B$ we have $v(g|B) = v(g'|B)$, so we have $v(g|A^S \cup S) \geq v(g'|A^S \cup S)$. That is, the total value generated by the players in $A^S \cup S$ does not increase when passing from g to g'; since Myerson allocation rule is component balanced. This, in turn, implies that the total value to be distributed among the players in $A^S \cup S$ does not increase when passing from g to g'. But we have seen that $\sum_{i \in S} Y_i^{MV}(g',v) > \sum_{i \in S} Y_i^{MV}(g,v)$, so it must be the case that $\sum_{i \in A^S} Y_i^{MV}(g',v) < \sum_{i \in A^S} Y_i^{MV}(g,v)$. This implies that there $Y_j^{MV}(g',v) < Y_j^{MV}(g,v)$. But that agent can block the alteration by S. Since our choice of improving S was arbitrary any such $S \subset N$ which can improve upon g, by altering g to some g', will be blocked by a player $j \in A^S$. So g is componentwise stable.

Unfortunately the converse of the above statement is not true. To see this consider the following example:

Example 3: Let $N = \{1, 2, 3, 4\}$ and let $v : G \rightarrow IR$ be defined through:

$v(12) = v(23) = v(34) = v(41) = 1$,

$v(g) = 1$ for any $g \in G$ containing at least one of the links 12, 23, 34, 41

and $v(g) = 0$ for all other networks.

The associated TU game v^* is as follows:

$v^*(1) = v^*(2) = v^*(3) = v^*(4) = 0$, $v^*(1) = v^*(1, 2) = v^*(2, 3) = v^*(3, 4) = v^*(1, 4) = 1$, $v^*(1, 3) = v^*(2, 4) = 0$, $v^*(1, 2, 3) = v^*(1, 2, 4) = v^*(1, 3, 4) = v^*(2, 3, 4) = 1$ and $v^*(1, 2, 3, 4) = 1$, and the Shapley value of this TU game is $\varphi(v^*) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

Consider the following network $g = \{12, 34, 41\}$. The Myerson allocation for this network is $(\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3})$, so $Y^{MV}(g,v) \neq \varphi(v^*)$, but g is componentwise stable. Note that at g, there is a single coalition that includes all the players in the society. So whenever a coalition $S \subset N$ wants to alter the existing network g it has to get approval of all of the society N. Assume that there is a coalition $S \subset N$ which can improve by altering g to g'. Then, we have $\sum_{i \in S} Y_i^{MV}(g',v) > \sum_{i \in S} Y_i^{MV}(g,v)$ by definition of an improvement by S. But note that g is an efficient network which implies $v(g) \geq v(g')$. That is $\sum_{i \in S} Y_i^{MV}(g,v)$

$+ \sum_{i \in N \setminus S} Y_i^{MV}(g,v) \geq \sum_{i \in S} Y_i^{MV}(g',v) + \sum_{i \in N \setminus S} Y_i^{MV}(g',v)$, together with the above inequality this implies $\sum_{i \in N \setminus S} Y_i^{MV}(g',v) < \sum_{i \in N \setminus S} Y_i^{MV}(g,v)$. So there exists $j \in N \setminus S$ such that $Y_j^{MV}(g',v) < Y_j^{MV}(g,v)$. But that player has the right to block that alteration. So no $S \subset N$ can alter the network g, g is componentwise stable.

In fact, players 2 and 3 can alter the network to $g' = \{12, 23, 34, 41\}$ by adding the link 23, at this new graph g' we have $Y^{MV}(g,v) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = \varphi\,(v^*)$. Players 1 and 4 receive more than what they should under the Shapley value of v^*, and players 2 and 3 receive less than what they should under the Shapley value of v^* but players 1 and 4 have the right to block the alteration of the network g to g' under which the payoff vector of Myerson allocation rule coincides with the Shapley value of v^*.

3.4 An Alternative Allocation Rule

When Myerson (1977) dealt with TU games with communication structures he proposed two fairness axioms that should be satisfied when distributing payoffs to players. What these axioms characterized was an extension of Shapley value to TU games with a communication structures setting. Again the payoff to a player is determined by his marginal contribution to coalitions but the value of a coalition is the value of the network found by restricting the original network to that coalition. So Myerson allocation rule takes into account only the restrictions of the given network to coalitions but does not take into account alternative networks. Jackson (2003b) proposes an alternative approach which does not take the network structure as fixed but assumes that it can be changed. “Here, I take the view that the network is not a permanent fixture, but is something that is either being formed or might change in the future.” (Jackson 2003c: 4). Thus he concludes that the allocation of value at a given network can and should depend on the value that might accrue to alternative potential networks. He criticizes equal bargaining power and component balancedness axioms and proposes an alternative allocation rule that is again based on Shapley-like calculations.

Given a value function v, he defines its *monotonic cover* $v\hat{}$ by

$$v\hat{}(g) = \max_{g' \subset g} v(g') \qquad \text{at any } g \in G.$$

and the allocation rule he proposes is:

$$Y_i^{PBFN}(g,v) = \frac{\mathrm{v(g)}}{\mathrm{v\hat{}\ (gN)}} \sum_{S \subset N \setminus \{i\}} \left(v\hat{}(g^{S \cup \{i\}}) - v\hat{}(g^S) \right) \left(\frac{\#S!(n-\#S-1)!}{n!} \right)$$

It is assumed here that there exists at least one network that generates a positive value so that the value function is not completely trivial. Note that under this assumption we have $v\hat{}(g^N) > 0$.

This rule is called the *Player-Based Flexible Network allocation rule*.

Note that for each S ⊂ N, the monotonic cover $v\hat{}$ associates with the network g^S the maximum value the players in S can generate assuming that the players in N \ S are totally isolated. Here, Jackson’s argument is similar to the argument we proposed for the coincidence of payoff vector of Myerson allocation rule at some *g* with the Shapley value of v^*, that is the allocation of value should depend on the value of efficient networks coalitions can form.

Returning back to the problem of coincidence of the payoff vector of Myerson allocation rule at some *g*, with the Shapley value of the associated TU game we see that one can obtain another coincidence result if the Player-Based Flexible Network allocation rule is chosen instead of the Myerson allocation rule.

Proposition 6: *Given any* $v \in V$ *such that there exists* $g' \in G$ *such that* $v(g') > 0$, and any $g \in G$, *the following are equivalent:*

1) g is efficient relative to v,

2) $Y^{PBFN}(g,v) = \varphi(v^*)$,

3) *g is strongly stable with respect to the Player-Based Flexible Network allocation rule and value function v.*

Proof: 1) $\Rightarrow$ 2)

Assume that g is efficient relative to *v*. Then $\frac{v(g)}{v\hat{}(gN)} = 1$ since $v\hat{}(g^N) = \max_{g' \subset g^N} v(g')$,

thus $Y^{PBFN}(g,v) = \sum_{S \subset N \setminus \{i\}} \left(v\hat{}(g^{S \cup \{i\}}) - v\hat{}(g^S) \right) \left(\frac{\#S!(n-\#S-1)!}{n!} \right)$ for every $i \in N$.

Since $v\hat{}(g^S) = v^*(S)$ for every $S \subset N$ we have $Y_i^{PBFN}(g,v) = \varphi_i(v^*)$ for every $i \in N$.

2) $\Rightarrow$ 3)

Assume $Y^{PBFN}(g,v) = \varphi\,(v^*)$. We know that Shapley value distributes the value

$v^*(N) = \max_{g' \subset g^N} v(g')$ so we have $\sum_{S \subset N \setminus \{i\}} Y_i^{PBFN}(g,v) = \max_{g' \subset g^N} v(g')$ that is g is efficient.

Take any $S \subset N$ and assume that S can alter the network g to another network g'. Note that $\sum_{S \subset N \setminus \{i\}} (\, v\hat{}(g^{S \cup \{i\}}) - v\hat{}(g^S)) \left(\frac{\# S!(n - \# S - 1)!}{n!}\right)$ part of the formula does not depend on the particular network g, these parts are equal for g and g'. But since g is efficient $v(g) \geq v(g')$ and $\frac{v(g)}{v\hat{}(gN)} \geq \frac{v(g')}{v\hat{}(gN)}$. Thus we have $\frac{v(g)}{v\hat{}(gN)} \sum_{S \subset N \setminus \{i\}} (\, v\hat{}\,(g^{S \cup \{i\}}) - v\hat{}\,(g^S\,)) \left(\frac{\# S!(n - \# S - 1)!}{n!}\right) \geq \frac{v(g')}{v\hat{}(gN)} \sum_{S \subset N \setminus \{i\}} (\, v\hat{}\,(g^{S \cup \{i\}}) - v\hat{}\,(g^S\,)) \left(\frac{\# S!(n - \# S - 1)!}{n!}\right)$ for every $i \in S$. That is S cannot improve by altering the network g.

Since our choice of S was arbitrary, no coalition can improve by altering the network g, that is g is strongly stable with respect to $Y^{PBFN}(g,v)$ and v.

3)⇒ 1)

Assume that g is strongly stable with respect to the Player-Based Network allocation rule and value function v, but suppose that g is not efficient with respect to v. Then there exists a $g' \in G$ such that $v(g') > v(g)$, which implies that $\frac{v(g')}{v\hat{}(gN)} > \frac{v(g)}{v\hat{}(gN)}$. Now take any $i \in N$ again since $\sum_{S \subset N \setminus \{i\}} (\, v\hat{}(g^{S \cup \{i\}}) - v\hat{}(g^S)) \left(\frac{\# S!(n - \# S - 1)!}{n!}\right)$ part of the calculation does not depend on the particular network *g*, these parts are equal for *g* and *g'* for every $i \in N$. Then we have $\frac{v(g')}{v\hat{}(gN)} \sum_{S \subset N \setminus \{i\}} (v\hat{}(g^{S \cup \{i\}}) - v\hat{}(g^{S}))$ $\left(\frac{\# S!(n-\# S-1)!}{n!}\right)$ > $\frac{v(g)}{v\hat{}(gN)} \sum_{S \subset N \setminus \{i\}} (v\hat{}(g^{S \cup \{i\}}) - v\hat{}(g^{S}))$ $\left(\frac{\# S!(n-\# S-1)!}{n!}\right)$ for every $i \in N$. Since the grand coalition *N* can alter the network *g* to *g'* and since the above inequality holds for

every $i \in N$, we have that *N* can improve upon *g* by deviations, which contradicts with the fact that *g* is strongly stable. So our assumption was wrong, that is *g* is efficient with respect to *v*.

CHAPTER IV

CONCLUSION

In his study on TU games with communication structures, Myerson (1977) proposed an allocation rule that was an extension of Shapley value to the TU games with a communication structures setting. Later it was shown that this rule has an extension to the more general networks setting we used in this study. Myerson allocation rule is based on Shapley-like calculations and the payoff to a player is determined by his marginal contributions to all possible coalitions in the networks setting. So given a value function defined on networks we derived a TU game v^* from v and we started with the question of whether Myerson allocation rule's payoff vector coincide with the Shapley value of the associated TU game v^* on some set of networks. Due to the particular way we derived v^* from v, the coincidence of Myerson allocation rule's payoff vector with the Shapley value of the associated TU game v^* on some network g would imply the efficiency of that network.

We have found that without any assumptions on the value function it is possible that the coincidence is satisfied on none of the networks a society can form. Under the assumption of monotonicity of the value function v, we have shown that the set of strongly stable networks lies within the set of networks satisfying the coincidence and there always exists a network satisfying the coincidence. The relation of pairwise stability to the networks satisfying coincidence is also examined under the assumption of monotonicity of v. But still there are networks satisfying coincidence and which are not stable under the stability definitions at hand.

Applying the “code of rights” idea of Sertel (2002) to the netwoks setting we proposed a new notion of stability. Under this new code of rights, a coalition cannot deviate without the consent of some other members of the society even if that coalition has the ability to deviate and will benefit from that deviation. We have shown that any network satisfying coincidence will be stable under componentwise stability. But the converse is not true. A line of research following this study would naturally be to find some “finer” notion of stability under which the set of coincidence will exactly coincide with the stable set of networks.

Finally keeping the Shapley value and strong stability notions fixed, we used another allocation rule, which is again based on Shapley-like calculations, proposed by Jackson (2003c). We have shown that the set of networks satisfying the coincidence of the payoff vector of this new allocation rule with Shapley value of the associated TU game v^* exactly coincides with the set of strongly stable networks with respect to this allocation rule and value function v, which is also the set of efficient networks.

Using other values for TU games and searching for the proper stability concept, allocation rule pair that will satisfy the coincidence property, one can carry out another line of research following this study.

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