ON MONOMIAL BURNSIDE RINGS

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By
Ergün Yaraneri
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Laurance J. Barker (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Asst. Prof. Dr. Ergün Yalçın

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Asst. Prof. Dr. Semra Kaptanoğlu

Approved for the Institute of Engineering and Science:

Prof. Dr. Mehmet B. Baray
Director of the Institute Engineering and Science

ABSTRACT

ON MONOMIAL BURNSIDE RINGS

Ergün Yaraneri
M.S. in Mathematics
Supervisor: Assoc. Prof. Dr. Laurance J. Barker
September, 2003

This thesis is concerned with some different aspects of the monomial Burnside rings, including an extensive, self contained introduction of the $A-$fibred $G-$sets, and the monomial Burnside rings. However, this work has two main subjects that are studied in chapters 6 and 7.

There are certain important maps studied by Yoshida in [16] which are very helpful in understanding the structure of the Burnside rings and their unit groups. In chapter 6, we extend these maps to the monomial Burnside rings and find the images of the primitive idempotents of the monomial Burnside $\mathbb{C}-$algebras. For two of these maps, the images of the primitive idempotents appear for the first time in this work.

In chapter 7, developing a line of research persued by Dress [9], Boltje [6], Barker [1], we study the prime ideals of monomial Burnside rings, and the primitive idempotents of monomial Burnside algebras. The new results include;
(a): If A is a $\pi-$group, then the primitive idempotents of $\mathbb{Z}_{(\pi)}B(A,G)$ and $\mathbb{Z}_{(\pi)}B(G)$ are the same
(b): If G is a $\pi'-$group, then the primitive idempotents of $\mathbb{Z}_{(\pi)}B(A,G)$ and $\mathbb{Q}B(A,G)$ are the same
(c): If G is a nilpotent group, then there is a bijection between the primitive idempotents of $\mathbb{Z}_{(\pi)}B(A,G)$ and the primitive idempotents of $\mathbb{Q}B(A,K)$ where K is the unique Hall $\pi'-$subgroup of G.
($\mathbb{Z}_{(\pi)} = \{a/b \in \mathbb{Q} : b \notin \cup_{p\in\pi} p\mathbb{Z}\}$, π =a set of prime numbers).

Keywords: Monomial Burnside rings, ghost ring, primitive idempotents, inflation map, invariance map, orbit map, conjugation map, restriction map, induction map, prime ideals, prime spectrum .

ÖZET

TEK TERİMLİ BURNSIDE HALKALARI

Ergün Yaraneri
Matematik, Yüksek Lisans
Tez Yöneticisi: Doç. Dr. Laurance J. Barker
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Bu tezde tek terimli Burnside halkalarının değişik yönlerini inceledik. Fakat bu çalışma iki önemli konu içermektedir, ve bunlar 6. ve 7. kısımlarda ele alınmıştır.

Burnside halkaları üzerinde önemli fonksiyonlar tanımlanmıştır. 6. kısımda bu fonksiyonları Burnside halkalarını alt halka olarak içeren tek terimli Burnside halkalarına genişlettik. Ayrıca yine 6. kısımda tek terimli Burnside $\mathbb{C}-$ cebirlerinin ilkel idempotentlerinin genişlettiğimiz fonksiyonlar altındaki görüntülerini bulduk. Söz konusu fonksiyonlardan ikisi için ilkel idempotentlerin görüntüleri ilk olarak bu çalışmada yer almaktadır.

Kısım 7 de ise tek terimli Burnside halkalarının asal ideallerini inceledik ve bazı tek terimli Burnside cebirlerinin ilkel idempotentleri hakkında bilgiler edindik. Elde ettiğimiz sonuçlar daha önceden başka çalışmalarda yer almayan yeni sonuçlar da içermektedir. 7. kısımdaki bu yeni sonuçlar arasında aşağıdaki üç sonuç en önemlileridir. ($\mathbb{Z}_{(\pi)} = \{a/b \in \mathbb{Q} : b \notin \cup_{p\in\pi} p\mathbb{Z}\}$, π =asal sayılardan oluşan bir küme).
(a): Eğer A bir $\pi-$grup ise $\mathbb{Z}_{(\pi)}B(A, G)$ ve $\mathbb{Z}_{(\pi)}B(G)$ aynı ilkel idempotentlere sahiptir.
(b): Eğer G bir $\pi'-$grup ise $\mathbb{Z}_{(\pi)}B(A, G)$ ve $\mathbb{Q}B(A, G)$ aynı ilkel idempotentlere sahiptir
(c): Eger G bir nilpotent grup ise $\mathbb{Z}_{(\pi)}B(A, G)$ ve $\mathbb{Q}B(A, K)$ 'nin ilkel idempotentleri arasinda bire-bir eşleme yapabiliriz. Burada K G'nin biricik Hall $\pi'-$alt grubudur.

Anahtar sözcükler: Tek terimli Burnside Halkaları, hayalet halka, ilkel idempotentler, infilasyon foksiyonu, stabil elemanlar foksiyonu, yörünge fonksiyonu, eşlenik foksiyonu, daraltma fonksiyonu, genişletme fonksiyonu, asal idealler, asal idealler spekturumu.

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Contents

Chapter 1

Introduction

The concept of fibred permutation sets arises naturally in topics closely connected with many aspects of representation theory: character theory, induction theorems etc.

The theory of fibred permutation sets for a finite group G is a quite easy extension of the theory of permutation sets where the role played by points of the permutation sets is now played by fibres, which are copies of a fixed finite abelian group A called the fibre group. We define an $A-$fibred $G-$set to be a finite $A-$free $A \times G-$set. We concentrate exclusively on the study of isomorphism classes of $A-$fibred $G-$sets. These classes may be added and multiplied in a natural fashion and in this manner they generate a commutative ring known as the monomial Burnside ring of A and G, and denoted by $B(A, G)$. In the special case where A is trivial, we recover the ordinary Burnside ring $B(1, G) = B(G)$.

The ordinary Burnside rings have many uses in representation theory, the theory of $G-$spheres, and group theory. They are Mackey functors. The importance of the ordinary Burnside rings in such areas of algebra leads to an extension of the ordinary Burnside ring. The first extension was given by Dress in [9].

Following Dress [9], the monomial Burnside rings, explicitly or implicitly, have been studied in contexts related to induction theorems. See, for instance, Boltje

[2], [3], [4], [5], [6].

The main contributor to the subject is undoubtedly Dress who introduced the monomial Burnside rings, and discovered a number of deep and striking results in [9]. One of his celebrated results in [9] asserts that a finite group G is solvable if and only if the monomial Burnside ring $B(A, G)$ has no nontrivial idempotents.

Because of the importance of idempotents, the idempotents of the ordinary Burnside rings have received much attentions, first by Dress in [8], a formula for the primitive idempotents of $\mathbb{Q}B(G)$ in terms of the transitive basis of $B(G)$ appeared for the first time in [10] by Gluck. After that, in [15], Yoshida found a formula for the primitive idempotents of $RB(G)$ where R is any integral domain of characteristic 0 from which the idempotent formula of Gluck follows as an easy corollary. For the monomial Burnside rings there is a similar history. In [6], Boltje gave an idempotent formula for the primitive idempotents of $KB(A, G)$ in terms of the transitive basis of $KB(A, G)$ where K is a field of characteristic 0 containing enough roots of unity, and A is the unit group of an algebraically closed field. Later, in [1], Barker gave idempotent formulas for the primitive idempotents of $\mathbb{C}B(A, G)$, $KB(A, G)$ and $B(A, G)$ in terms of the transitive basis of $B(A, G)$ where K is any field of characteristic 0 from which the idempotent formula of Boltje follows as an easy consequence. Since $B(1, G) = B(G)$, the idempotent formula of $\mathbb{Q}B(G)$ obtained by Gluck in [10] follows from the idempotent formula of Barker in [1].

The monomial Burnside rings introduced by Dress in [9] are more general than the monomial Burnside rings considered by Boltje, Barker, and us. We consider the same monomial Burnside rings as Barker [1].

We study some different aspects of the monomial Burnside rings and try to extend some theory from the Burnside rings. We made much use of the paper [9] by Dress, which is a fundamental paper on this subject especially in chapters 2, 3, 4 and 7. However because of the full generalities of Drees' paper [9], these chapters, while influenced by [9], have different flavors.

Chapter 2 contains an extensive account of the theory of fibred permutation sets.

In chapter 3 and 4, the monomial Burnside ring is defined and its basic properties are studied.

There are certain maps defined on the Burnside rings which appear in [16]. With these maps the Burnside rings become Mackey functors. In chapter 6, we extend these maps to the monomial Burnside rings and find the images of the primitive idempotents of $\mathbb{C}B(A,G)$ under these maps.

In the paper [9] by Dress, there is a section dealing with prime ideals of his ring. Because of the full generalities of his ring and his ghost ring, in chapter 7 our approach is indeed different from his approach, although inspired from his paper. Although the name of chapter 7 is prime ideals, our main object there is finding the primitive idempotents of the monomial Burnside rings tensored over $\mathbb{Z}$ with an integral domain of characteristic 0. We give some partial answers when the integral domain satisfies some restrictive conditions.

In chapter 8, we give some maps whose domains or codomains are the monomial Burnside rings.

In chapter 9, we give some ring theoretic propertis of the monomial Burnside rings.

Finally, let us summarize some of the new results in this thesis. In chapter 1, we prove all the results about $A-$fibred $G-$sets whose proofs are left to the reader in Dress [9] and Barker [1]. In chapter 2, and 3, we try to exhibit which results in [1] can be deduced from [9]. In chapter 6, we extend the maps studied by Yoshida in [16] to the monomial Burnside rings and find the images of the primitive idempotents of the monomial Burnside $\mathbb{C}-$algebras. For two of these maps, the images of the primitive idempotents appear for the first time in this work. The results obtained in chapter 7 include new facts which did not appear in the papers [9] by Dress, and [1] by Barker. The new results obtained in chapter 7 includes, for instance, some facts about the primitive idempotents of $\mathbb{Z}_{(\pi)}B(A,G)$ where A is a $\pi-$group, or G is a $\pi'-$group, or G is a nilpotent group

(Here, $\mathbb{Z}_{(\pi)} = \{a/b \in \mathbb{Q} : b \notin \cup_{p\in\pi} p\mathbb{Z}\}$, $\pi =$a set of prime numbers).

To facilitate the reading, important definitions and results have been repeated where necessary.

Chapter 2

$A-$Fibred $G-$Sets

The monomial Burnside rings were introduced by Dress in [9]. We will see in the next chapter that the elements of the monomial Burnside rings are closely related to the $A-$fibred $G-$sets. So we first need to introduce an account of the theory of $A-$fibred $G-$sets. In [9], Dress gave a very short introduction to $A-$fibred $G-$sets leaving details to the reader. Also he considered more general $A-$fibred $G-$sets than we want to consider here. However, we mainly follow [9] but using the notations of Barker in [1].

In this chapter we introduce $A-$fibred $G-$sets and give some properties. We need some facts about $G-$sets. The following facts about $G-$sets are well-known and can be found in [14]. Let G be a finite group.

(1) Let G be a group. A finite set S is called a $G-$set if there is a map $G \times S \to S$, $(g, s) \mapsto gs$, satisfying; $1s = s$ and $(gh)s = g(hs)$ for all $g, h \in G, s \in S$.

(2) Let S and T be $G-$sets. A map $f : S \to T$ is called a $G-$map if $f(gs) = gf(s)$ for all $g \in G, s \in S$.

(3) Let S be $G-$set. For any $s \in S$, we write $orb_G(s) = \{gs : g \in G\}$ and $stab_G(s) = \{g \in G : gs = s\}$. They are called $G-$orbit of s and $G-$stabilizer of s, respectively. Moreover, $orb_G(s)$ is a $G-$set and $stab_G(s)$ is a subgroup of G.

(4) A $G-$set S is said to be transitive if for any $s_1, s_2 \in S$ there is a $g \in G$ such that $gs_1 = s_2$.

(5) A $G-$set S is transitive if and only if any $G-$map from a $G-$set T into S is

surjective.

(6) For any subgroup H of G, the set of left cosets of H in G becomes a $G-$set by left multiplication.

(7) Let S be a $G-$set. For any $s \in S$, the map $G/stab_G(s) \to orb_G(s)$ given by $gstab_G(s) \mapsto gs$ is a bijective $G-$map ($G-$isomorphism) and so the $G-$sets $G/stab_G(s)$ and $orb_G(s)$ are isomorphic. Hence in particular, $|G : stab_G(s)| = |orb_G(s)|$. We write $T_1 \simeq_G T_2$ for isomorphic $G-$sets T_1 and T_2.

(8) Let S and T be $G-$sets. Take any $s \in S$ and $t \in T$. Then $orb_G(s) \simeq_G orb_G(t)$ if and only if $stab_G(s) =_G stab_G(t)$.

(9) For a $G-$set S and a subgroup H of G we write S^H to denote the set of $H-$fixed points of S.

(10) For $G-$sets S and T, $Hom_G(S,T)$ denotes the set of all $G-$maps from S to T.

(11) Let S be a $G-$set and H be a subgroup of G. Then we have a bijection between the sets $Hom_G(G/H, S)$ and S^H given by $(f : G/H \to S) \mapsto f(H)$.

(12) For $H, K \leq G$; $(G/K)^H = \{gK : H \leq {}^gK\} = \{g_it_jK : 1 = 1, ..., r; j = 1, ..., s\}$ where $g_1Kg_1^{-1}, ..., g_rKg_r^{-1}$ are all distinct $G-$conjugates of K containing H and $t_1, ..., t_s$ are the left coset representatives of K in $N_G(K)$.

(13) Let H, K be subgroups of G. Then $(G/K)^H = \biguplus_{H \leq W, W =_G K} (G/K)^W$.

(14) Let $H_1, ..., H_n$ be all distinct nonconjugate subgroups of G, and S be a $G-$set. Then $S \simeq \biguplus_i \lambda_i(G/H_i)$ where $\lambda_i = \frac{|S_i|}{|G:H_i|}$ and $S_i = \{s \in S : stab_G(s) =_G H_i\}$.

(15) (Burnside)For any $G-$set S, the number of $G-$orbits of S is $\frac{1}{|G|}\sum_{g \in G} |S^{<g>}|$.

(16) A $G-$set S is called $G-$free if $stab_G(s) = 1$ for all $s \in S$. For such $G-$sets each $G-$orbits have the same number of elements.

(17) Let S be a $G-$set. Writing S as a disjoint union of its $G-$orbits we can express S in the form $GX = \{gx : g \in G, x \in X\}$ where X is a set of orbit representatives. Once an X is chosen, any element of S can be written uniquely in the form gx where $g \in G, x \in X$.

Now we can begin to study $A-$fibred $G-$sets. Let A be a finite abelian group and G be a finite group. We write AG for $A \times G$ by identifying $a \in A$ with $(a, 1) \in A \times G$ and $g \in G$ by $(1, g) \in G$. Note that by our notational convention

$ag = ga$ for any $a \in A$ and $g \in G$. A finite $A-$free $AG-$set is called an $A-$fibred $G-$set, and its $A-$orbits $orb_A(s), s \in S$, are called fibres. We sometimes use the notation As instead of $orb_A(s)$ for fibres. Let S be an $A-$fibred $G-$set. Writing S as a disjoint union of its fibres we can express S as $AX = \{ax : a \in A, x \in X\}$ where X is a set of representatives of fibres. Any element of $S = AX$ can be written uniquely in the form ax where $a \in A, x \in X$. Hence, given any $A-$fibred $G-$set S we can see S as a set of formal products $AX = \{ax : a \in A, x \in X\}$, and $a_1x_1 = a_2x_2$ if and only if $a_1 = a_2, x_1 = x_2$. Consequently, by an $A-$fibred $G-$set (equivalently by an $A-$free $AG-$set) we mean a set of formal products $AX = \{ax : a \in A, x \in X\}$ such that AX is a $G-$set and X is a finite set.

Let AX be an $A-$fibred $G-$set. Since it is an $AG-$set, it must be isomorphic to a disjoint union of the sets of left cosets of some subgroups of AG. However, it is not true for all subgroups of AG that the set of left cosets forms an $A-$ fibred $G-$set because it may not be $A-$free.

Remark 2.1 *Let $V \leq G$ and $\nu \in Hom(V, A)$. Then $\{\nu(v^{-1})v : v \in V\}$ is a subgroup of AG, and the set of its left cosets in AG forms an $A-$fibred $G-$set.*

Proof: Put $\triangle_{(V,\nu)} = \{\nu(v^{-1})v : v \in V\}$. It is clear that $\triangle_{(V,\nu)}$ is a subgroup of AG and $AG/\triangle_{(V,\nu)}$ is an $AG-$ set (by left multiplication). Hence we only need to check its $A-$stabilizers. Take any $\nu(v^{-1})v \in \triangle_{(V,\nu)}$. Then $a \in A$ is in $stab_A(\nu(v^{-1})v)$ if and only if $a\nu(v^{-1})v = \nu(v^{-1})v$. Thus, $stab_A(\nu(v^{-1})v) = 1$ and so $AG/\triangle_{(V,\nu)}$ is an $A-$fibred $G-$set.

□

We use the notation $\triangle_{(V,\nu)}$ for the subgroup $\{\nu(v^{-1})v : v \in V\}$ of AG for any $V \leq G$ and $\nu \in Hom(V, A)$. Later we will show that any transitive $A-$fibred $G-$set is $AG-$isomorphic to $AG/\triangle_{(V,\nu)}$ for some $V \leq G$ and $\nu \in Hom(V, A)$.

For an $A-$fibred $G-$set AX, the set of its $A-$orbits (fibres) $\{Ax : x \in X\}$ is denoted by $A \setminus AX$.

Remark 2.2 *Let AX be an $A-$fibred $G-$set. Then $A \setminus AX$ is a $G-$set with the action;*

$G \times (A \setminus AX) \to (A \setminus AX)$, $(g, Ax) \mapsto gAx = Agx$.

Proof: Obvious.

$\square$

Note that for an $A-$fibred $G-$set AX we have the following immediate properties;
$orb_{AG}(ax) = orb_{AG}(x)$, $stab_{AG}(ax) = stab_{AG}(x)$, $orb_G(Aax) = orb_G(Ax)$, and $stab_G(Aax) = stab_G(Ax)$ for all $a \in A, x \in X$.

Remark 2.3 *Let AX be an $A-$fibred $G-$set. Then AX is a transitive $AG-$set if and only if $A \setminus AX$ is a transitive $G-$set.*

Proof: ($\Rightarrow$) Take any two fibres Ax_1 and Ax_2. Since AX is a transitive $AG-$set and $x_1, x_2 \in AX$ there is an $ag \in AG$ such that $agx_1 = x_2$. But then $gAx_1 = Ax_2$ and so $A \setminus AX$ is a transitive $G-$set.
($\Leftarrow$) Take any $a_1x_1, a_2x_2 \in AX$. Since $A \setminus AX$ is a transitive $G-$set, there is a $g \in G$ such that $gAx_1 = Ax_2$. But then $a_2x_2 = ga_3x_1$ for some $a_3 \in A$ and so $(a_1^{-1}a_3g)(a_1x_1) = a_2x_2$ implying that AX is a transitive $AG-$set.

$\square$

We call AX a transitive $A-$fibred $G-$set if AX is a transitive $AG-$set, or equivalently if $A \setminus AX$ is a transitive $G-$set.

Remark 2.4 *Let $S = AX$ be an $A-$fibred $G-$set, and $s \in S$. Then;*
(i) The map $\pi_s : stab_{AG}(s) \to stab_G(As)$ given by $\pi_s(ag) = g$ is a group isomorphism,
(ii) For any $g \in stab_G(As)$, there is a unique $a_g \in A$ such that $a_ggs = s$,
(iii) The map $\nu_s : stab_G(As) \to A$ given by $\nu_s(g) = a_g^{-1}$ is a group homomorphism,
(iv) $stab_{AG}(s) = \triangle_{(stab_G(As),\nu_s)}$,
(v) $AG/\triangle_{(stab_G(As),\nu_s)}$ is a transitive $A-$fibred $G-$set,
(vi) $AG/\triangle_{(stab_G(As),\nu_s)} \simeq_{AG} orb_{AG}(s)$.

Proof: (i) It is a straightforward checking.
(ii) Since π_s is bijective, for any $g \in stab_G(As)$ there is a unique $a_g \in A$ such that

$\pi_s(a_g g) = g$. Note that $a_g gs = s$.

(iii) Given $g, h \in stab_G(As)$, we find unique elements $a_g, a_h, a_{gh} \in A$ such that $a_g gs = a_h hs = a_{gh}(gh)s = s$ by part (ii). Then $s = a_{gh}(gh)s = a_{gh}g(hs) = a_{gh}a_h^{-1}(gs) = a_{gh}a_h^{-1}a_g^{-1}s$ and so $a_{gh} = a_g a_h$ because S is $A-$free. Hence $\nu_s \in Hom(stab_G(As), A)$.

(iv) If $ag \in stab_{AG}(s)$, then $g \in stab_G(As)$ and $s = ags = a_g gs = \nu_s(g^{-1})gs$. Thus $as = \nu_s(g^{-1})s$, and since S is $A-$free $\nu_s(g^{-1}) = a$. So $ag = \nu_s(g^{-1})g \in \{\nu_s(g^{-1})g : g \in stab_G(As)\} = \triangle_{(stab_G(As),\nu_s)}$. Therefore, $stab_{AG}(s)$ is contained in the set $\triangle_{(stab_G(As),\nu_s)}$. Converse direction is clear because an element $\nu_s(g^{-1})g$ of $\triangle_{(stab_G(As),\nu_s)}$ is equal to $a_g g$ where $a_g gs = s$.

(v) Clearly it is a transitive $AG-$set. Moreover it is $A-$free from 2.1.

(vi) Obvious.

□

For any $A-$fibred $G-$set S we know from 2.4 that the $A-$fibred $G-$sets $orb_{AG}(s)$ and $AG/\triangle_{(stab_G(As),\nu_s)}$ are isomorphic (as $AG-$sets) where ν_s is the uniquely determined element of $Hom(stab_G(As), \nu_s)$ by the condition: $gs = \nu_s(g)s$ for all $g \in stab_G(As)$. Hence, in particular any transitive $A-$fibred $G-$set S is isomorphic to $AG/\triangle_{(stab_G(As),\nu_s)}$ where s is any element of S. And conversely for any $V \leq G$ and $\nu \in Hom(V, A)$ the set $AG/\triangle_{(V,\nu)}$ is a transitive $A-$fibred $G-$set. We use the notation $A_\nu G/V$ to denote the transitive $A-$fibred $G-$set $AG/\triangle_{(V,\nu)}$, and use $[A_\nu G/V]$ to denote its isomorphism class. Also for any $A-$fibred $G-$set AX, we write $[AX]$ for the isomorphism class of AX.

Remark 2.5 *Let $S = AX$ be an $A-$fibred $G-$set, and for $s \in S$ let $\nu_s : stab_G(As) \to A$ be the uniquely determined element of $Hom(stab_G(As), A)$ by the condition: $gs = \nu_s(g)s$ for all $g \in stab_G(As)$. Then*
$\nu_{gs} = {}^g\nu_s$ and $\nu_{as} = \nu_s$ for any $g \in G$ and $a \in A$.

Proof: $h \in stab_G(As)$ if and only if $Ahs = As$, or equivalently $A(ghg^{-1})gs = Ags$. Hence $h \in stab_G(As)$ if and only if $ghg^{-1} \in stab_G(Ags)$. Moreover, $hs = \nu_s(h)s$ and $(ghg^{-1})gs = \nu_{gs}(ghg^{-1})gs$ imply that $\nu_s(h)s = \nu_{gs}(ghg^{-1})s$. Since S is $A-$free, $\nu_{gs}(ghg^{-1}) = \nu_s(h)$ implying that $\nu_{gs} = {}^g\nu_s$.

Note that $A(as) = As$. So the functions ν_{as} and ν_s have the same domain. Also

$g(as) = \nu_{as}(g)(as)$ implies that $gs = \nu_{as}(g)s$. On the other hand, $gs = \nu_s(g)s$ and so $\nu_{as}(g)s = \nu_s(g)s$. Because S is $A-$free, $\nu_{as} = \nu_s$.

$\square$

Let $ch(A,G) = \{(V,\nu) : V \leq G, \nu \in Hom(V,A)\}$. Then G acts on $ch(A,G)$ by conjugation; $(g,(V,\nu)) \mapsto {}^g(V,\nu) = ({}^gV, {}^g\nu)$ where ${}^g\nu : {}^gV \to A$ is given by ${}^g\nu(gvg^{-1}) = \nu(v)$ for all $v \in V$. We write $(V,\nu) =_G (W,\omega)$ if the elements $(V,\nu),(W,\omega) \in ch(A,G)$ are in the same $G-$orbit of $ch(A,G)$.

Remark 2.6 *Let* $(V,\nu),(W,\omega) \in ch(A,G)$. *Then,*
$A_\nu G/V \simeq_{AG} A_\omega G/W$ *(equivalently,* $[A_\nu G/V] = [A_\omega G/W]$*) if and only if* $(V,\nu) =_G (W,\omega)$.

Proof: $A_\nu G/V \simeq_{AG} A_\omega G/W$ if and only if the subgroups $\triangle_{(V,\nu)}, \triangle_{(W,\omega)}$ of AG are $AG-$conjugates. Now if ${}^{ah}\triangle_{(V,\nu)} = \triangle_{(W,\omega)}$ for some $ah \in AG$, then
$\{\nu(g^{-1})hgh^{-1} : g \in V\} = \{\omega(g^{-1})g : g \in W\}$.
But $\{\nu(g^{-1})hgh^{-1} : g \in V\} = \{{}^h\nu((hgh^{-1})^{-1})(hgh^{-1}) : g \in V\} = \{{}^h\nu(u^{-1})u : u \in {}^hV\}$. So, ${}^hV = W$ and ${}^h\nu = \omega$ implying that ${}^h(V,\nu) = (W,\omega)$.

$\square$

Consider $A_\nu G/V$ which denotes the $A-$fibred $G-$set $AG/\triangle_{(V,\nu)}$ where $\triangle_{(V,\nu)}$ $= \{\nu(v^{-1})v : v \in V\}$. Let us denote $ag\triangle_{(V,\nu)}$ by $ag\triangle$. Two fibres $A(ag\triangle)$ and $A(bh\triangle)$ are equal if and only if there is a $c \in A$ such that $cg\triangle = h\triangle$ if and only if $h^{-1}g \in V$ and $c = \nu(g^{-1}h) \in A$. Thus, the fibres $A(ag\triangle)$ and $A(bh\triangle)$ are equal if and only if $gV = hV$. Hence, we have a bijective map $A \setminus (A_\nu G/V) \to G/V$ given by $A(ag\triangle) \mapsto gV$. It is clear that this map is a $G-$map. Consequently, $A \setminus (A_\nu G/V) \simeq_G G/V$.
Suppose $A_\nu G/V$ is given. It can be written in the form AX where X is a set of $A-$orbits representatives. Since it is $A-$free, each $A-$orbit has the same number of elements which is $|A|$, and we showed above that the number of $A-$orbits is equal to $|G/V|$. Therefore, to represent the $A-$fibred $G-$set $A_\nu G/V = AG/\triangle_{(V,\nu)}$ in the form AX we can take for example X as the set $\{g\triangle_{(V,\nu)} : gV \subseteq G\}$ (AG acts on AX by left multiplication). Note that $stab_G(A(g\triangle_{(V,\nu)})) = {}^gV$, $\nu_{g\triangle_{(V,\nu)}} = {}^g\nu$ and $|A_\nu G/V| = |A||G : V|$.

Chapter 3

Monomial Burnside Rings

We are still assuming that A is a finite abelian group and G is a finite group. We introduce two binary operations on $A-$fibred $G-$sets. The most obvious one is the disjoint union. The other one is slightly more complicated than the disjoint union, and we introduce it now as exactly Dress did in [9] but we are using the notations of [1].

Suppose $S = AX$ and $T = AY$ are both $A-$fibred $G-$sets. Then their cartesian product $S \times T$ is an $A-$set with the action: $a(s,t) = (as, a^{-1}t)$.
Let $S \otimes T$ denote the set of $A-$orbits of the cartesian product $S \times T$ with respect to the above $A-$action. We write $s \otimes t$ for the $A-$orbit containing $(s,t) \in S \times T$. Thus,

$$S \otimes T = \{s \otimes t : s \in S, t \in T\}, \ s \otimes t = \{(as, a^{-1}t) : a \in A\}.$$

Note that $(as) \otimes t = s \otimes (at)$ for any $a \in A$ and $(s,t) \in S \times T$.
We let AG act on $S \otimes T$ as: $ag(s \otimes t) = (ags) \otimes (gt)$.

Remark 3.1 *$S \otimes T$ constructed above is an $A-$fibred $G-$set.*

Proof: The action is well-defined: Let $s \otimes t = s' \otimes t'$ for some $s, s' \in S$ and $t, t' \in T$. We want to show that $ag(s \otimes t) = ag(s' \otimes t')$ for any $ag \in AG$. Since

$s \otimes t = s' \otimes t'$, (s,t) and (s',t') are in the same $A-$orbit of $S \times T$. Thus there is an $a \in A$ such that $(s',t') = (a^{-1}s, at)$. But then $(ags', gt') = (gs, agt)$ and so $ags' \otimes gt' = gs \otimes agt = ags \otimes gt$. Hence $ag(s \otimes t) = ag(s' \otimes t')$.

$S \otimes T$ is $A-$free: Take an element $s \otimes t \in S \otimes T$ and compute its $A-$stabilizer. $a \in A$ is in the $A-$stabilizer of $s \otimes t$ if and only if $as \otimes t = s \otimes t$ which is to say that $(as,t),(s,t) \in S \times T$ are in the same $A-$orbit of $S \times T$. Then, $(as,t),(s,t) \in S \times T$ are in the same $A-$orbit of $S \times T$ if and only if $(bas, b^{-1}t) = (s,t)$ for some $b \in A$, or equivalently $a = b = 1$. Hence, $S \otimes T$ is $A-$free.

The action properties are satisfied: It is obvious that $1(s \otimes t) = s \otimes t$ and $((ag)(bh))s \otimes t = (ag)((bh)s \otimes t)$.

□

Theorem 3.2 *For any* $(V,\nu),(W,\omega) \in ch(A,G)$ *we have;*

$$A_\nu G/V \otimes A_\omega G/W \simeq_{AG} \biguplus_{VgW \subseteq G} A_{\nu.^g\omega} G/V \cap {}^gW.$$

Proof: Remember that $A_\nu G/V = AG/\triangle_{(V,\nu)}$ and $A_\omega G/W = AG/\triangle_{(W,\omega)}$. Put $\triangle_{(V,\nu)} = \triangle$, $\triangle_{(V,\nu)} = \triangle'$, and $S = A_\nu G/V \otimes A_\omega G/W$. Since we can express any $A-$fibred $G-$set S as a disjoint union of its $AG-$orbits, and since $orb_{AG}(s) \simeq_{AG} AG/stab_{AG}(s) = AG/\triangle_{(stab_G(As),\nu_s)}$; we can proceed as follows.

Take any element $ag\triangle \otimes bh\triangle'$ of S. Then $orb_{AG}(ag\triangle \otimes bh\triangle') = orb_{AG}(\triangle \otimes abg^{-1}h\triangle')$. Hence we calculate $stab_{AG}(s)$ for elements $s \in S$ of the form $\triangle \otimes g\triangle'$. $ah \in AG$ is in $stab_{AG}(\triangle \otimes g\triangle')$ if and only if $ah\triangle \otimes hg\triangle' = \triangle \otimes g\triangle'$. But $ah\triangle \otimes hg\triangle' = \triangle \otimes g\triangle'$ if and only if $(ah\triangle, hg\triangle')$ and $(\triangle, g\triangle')$ are in the same $A-$orbit of $A_\nu G/V \times A_\omega G/W$ which is equivalent to, $(ah\triangle, hg\triangle') = (b\triangle, b^{-1}g\triangle')$ for some $b \in A$. Now $(ah\triangle, hg\triangle') = (b\triangle, b^{-1}g\triangle')$ is the same as with $ab^{-1}h \in \triangle$ and $bg^{-1}hg \in \triangle'$, or equivalently $h \in V$, $\nu(h^{-1}) = ab^{-1}$, $g^{-1}hg \in W$ and $\omega((g^{-1}hg)^{-1}) = b$. Hence, $ah \in AG$ is in $stab_{AG}(\triangle \otimes g\triangle')$ if and only if $h \in V \cap {}^gW$ and $a = \nu(h^{-1})\omega((g^{-1}hg)^{-1}) = (\nu.{}^g\omega)(h^{-1})$. Therefore,
$stab_{AG}(\triangle \otimes g\triangle') = \triangle_{(V \cap {}^gW, \nu.{}^g\omega)}$ and so $orb_{AG}(\triangle \otimes g\triangle') \simeq_{AG} A_{\nu.^g\omega} G/V \cap {}^gW$.

Now, $orb_{AG}(\triangle \otimes g\triangle') = orb_{AG}(\triangle \otimes h\triangle')$ if and only if there is an $ak \in AG$ such that $ak\triangle \otimes kg\triangle' = \triangle \otimes h\triangle'$, which is to say that $(ak\triangle, kg\triangle')$ and $(\triangle, h\triangle')$ are in the same $A-$orbit of $A_\nu G/V \times A_\omega G/W$. But this holds if and only if there is a $b \in A$ such that $(ak\triangle, kg\triangle') = (b\triangle, b^{-1}h\triangle')$, which is equivalent to $ab^{-1}k \in \triangle$ and $bh^{-1}kg \in \triangle'$. Using the definitions of $\triangle$ and $\triangle'$, we see that $ab^{-1}k \in \triangle$ and $bh^{-1}kg \in \triangle'$ if and only if $k \in V$, $\nu(k^{-1}) = ab^{-1}$, $h^{-1}kg \in W$ and $\omega((h^{-1}kg)^{-1}) = b$, or equivalently $h^{-1}kg \in W$ and $k \in V$. But then by $k \in V$, $orb_{AG}(\triangle \otimes g\triangle') = orb_{AG}(\triangle \otimes h\triangle')$ if and only if $VhW = VgW$. Hence,

$$A_\nu G/V \otimes A_\omega G/W \simeq_{AG} \biguplus_{VgW \subseteq G} A_{\nu.{}^g\omega} G/V \cap {}^gW.$$

$\square$

The formula in 3.2 is known as the Mackey product formula.

Define an addition and multiplication on the isomorphism classes of $A-$fibred $G-$sets as follows:
$[S] + [T] = [S \uplus T]$ and $[S][T] = [S \otimes T]$.
It is clear that the above operations are well-defined, commutative, associative, and moreover the multiplication is distributive over the addition. Thus, the set of isomorphism classes of $A-$fibred $G-$sets forms a commutative semiring. We write $B(A,G)$ for the associated Grothendieck ring and call it monomial Burnside ring. Therefore $B(A,G)$ is a set of formal differences of isomorphism classes of $A-$fibred $G-$sets, and it is a commutative ring with 1 with respect to the following operations;
$[AX] + [AY] = [A(X \uplus Y)]$ and $[AX][AY] = [AX \otimes AY] = [A(X \otimes Y)]$
where the action of AG on $A(X \otimes Y)$ is given by $ag(x \otimes y) = agx \otimes y$.
Note that the multiplicative identity of the ring is $[A_\tau G/G]$ where τ is the trivial group homomorphism from G to A.
Remember that $ch(A,G) = \{(V,\nu) : V \leq G, \nu \in Hom(V,A)\}$ is a $G-$set by conjugation.

Remark 3.3 *(i) $B(A,G)$ is a commutative ring with 1.*
(ii) $B(A,G)$ is a free $\mathbb{Z}-$module with so called transitive basis $\{[A_\nu G/V] : (V,\nu) \in ch(A,G)\}$.
(iii) $B(A,G) = \bigoplus_{(V,\nu) \in_G ch(A,G)} \mathbb{Z}[A_\nu G/V]$ where the notation under the direct sum means that (V,ν) runs over a set of representatives of nonconjugate elements of $ch(A,G)$.

(iv) The multiplication of $B(A,G)$ on its transitive basis given as;

$$[A_\nu G/V][A_\omega G/W] = \sum_{VgW\subseteq G} [A_{\nu.{}^g\omega} G/V \cap {}^gW].$$

Proof: In chapter 2 we proved the following three facts.
(1) Any transitive $A-$fibred $G-$set is isomorphic to $A_\nu G/V$ for some (V,ν).
(2) Any set of the form $A_\nu G/V$ is a transitive $A-$fibred $G-$set.
(3) $A_\nu G/V \simeq A_\omega G/W$ if and only if $(V,\nu) =_G (W,\omega)$.
Hence; (i), (ii), and (iii) follows from the above three and from the definition of $B(A,G)$. We proved (iv) in 3.2.

□

For a $G-$set S, let $AS = \{as : a \in A, s \in S\}$ be the set of formal products. Thus, $a_1s_1 = a_2s_2$ if and only if $a_1 = a_2$ and $s_1 = s_2$. We let AG act on AS as: $(bg)(as) = (ab)(gs)$ for all $bg \in AG$ and $as \in AS$. Then, AS becomes an $A-$fibred $G-$set. If $[S]$ denotes the isomorphism class of the $G-$set S, then it is clear that $[S] = [T]$ implies $[AS] = [AT]$. So we have a well-defined map
$\psi_1 : B(G) \to B(A,G)$ given by $\psi_1([S]) = [AS]$ for any $G-$set S.

Remark 3.4 *(i) $\psi_1([G/V]) = [A_\tau G/V]$ for any $V \leq G$, where τ is the trivial element of the group $Hom(V,A)$.*
(ii) ψ_1 is a unital ring monomorphism.
(iii) $B(G)$ can be regarded as a subring of $B(A,G)$.

Proof: (i) $\psi_1([G/V]) = [A(G/V)]$, $A(G/V) = \{a(gV) : a \in A, gV \in G/V\}$, and $A_\tau G/V = AG/\triangle_{(V,\tau)}$ where $\triangle_{(V,\tau)} = \{1v : v \in V\} \leq AG$.
Define a map $f : A(G/V) \to AG/\triangle_{(V,\tau)}$ where $f(a(gV)) = ag\triangle_{(V,\tau)}$. The elements $a(gV)$ and $b(hV)$ of $A(G/V)$ are equal if and only if $a = b$ and $gV = hV$ which is to say that $(bh)^{-1}(ag) = 1(h^{-1}g) \in \triangle_{(V,\tau)}$. Thus, the elements $a(gV)$ and $b(hV)$ of $A(G/V)$ are equal if and only if $ag\triangle_{(V,\tau)} = bh\triangle_{(V,\tau)}$. Hence, f is well-defined, and injective.
Obviously from its definition, f is surjective and an $AG-$map.

Now since we proved that $A(G/V)$ and $AG/\triangle_{(V,\tau)}$ are isomorphic (by the map f) we have $\psi_1([G/V]) = [A_\tau G/V]$.
(ii) It follows easily from the multiplication formula given in 3.3 (iv).
(iii) Since ψ_1 is a unital ring monomorphism, by identifying $B(G)$ with $\psi_1(B(G))$ we can regard $B(G)$ as a subring of $B(A,G)$.

□

Remember that for any $A-$fibred $G-$set AX, the set $A \setminus AX$ of its $A-$orbits (fibres) is a $G-$set with respect to the $G-$action: $gAx = Agx$. Also in chapter 2 we showed that $A \setminus (A_\nu G/V) \simeq_G G/V$. Hence we have a well-defined map
$\psi_2 : B(A,G) \to B(G)$ given by $\psi_2([AX]) = [A \setminus AX]$.

Remark 3.5 *(i)* $\psi_2([A_\nu G/V]) = [G/V]$ *for any* $(V,\nu) \in ch(A,G)$.
(ii) ψ_2 *is a unital ring epimorphism.*

Proof: Follows immediately from the above explanation and from the multiplication formula given in 3.3 (iv).

□

Let A' and A be two finite abelian groups such that $A' \leq A$. Hence for any $V \leq G$, a group homomorphism $\nu : V \to A'$ can be seen as a group homomorphism $V \to A$. By this way the $A'-$fibred $G-$set $A'_\nu G/V$ can be seen as the $A-$fibred $G-$set $A_\nu G/V$. Moreover, if $A'_\nu G/V \simeq_{A'G} A'_\omega G/W$ then $(V,\nu) =_G (W,\omega)$ and so $A_\nu G/V \simeq_{AG} A_\omega G/W$. Thus we have a well-defined map $\psi_3 : B(A',G) \to B(A,G)$ given by $\psi_3([A'_\nu G/V]) = [A_\nu G/V]$. It is clear that ψ_3 is a unital ring monomorphism.

We constructed the ring homomorphisms ψ_1 and ψ_2 in 3.4 and 3.5. Now we consider the composition map $\phi = \psi_1 \circ \psi_2 : B(A,G) \to B(A,G)$ where $\phi([A_\nu G/V]) = [A_\tau G/V]$ for any $(V,\nu) \in ch(A,G)$ where τ denotes the trivial homomorphism.

Remark 3.6 *(i)* ϕ *is a unital ring homomorphism, and it can be also seen as* $B(G)-$*module endomorphism of the* $B(G)-$*module* $B(A,G)$.
(ii) ϕ *is a projection onto* $B(G) \leq B(A,G)$.
(iii) $B(A,G) = B(G) \oplus Ker(\phi)$ *as* $B(G)-$*modules or* $\mathbb{Z}-$*modules.*

Proof: All parts are obvious. (Note that $B(G)$ is a unital subring of $B(A,G)$ and $Ker(\phi)$ is an ideal of $B(A,G)$. So, the decomposition in (iii) is not merely a submodule decomposition.)

□

We close this chapter after giving an elementary consequence of the multiplication formula given in 3.3 (iv). For any $\nu \in Hom(G,A)$, we have $[A_\nu G/G]^n = [A_{\nu^n} G/G]$. Since $\nu(g) \in A$ for any $g \in G$, $[A_\nu G/G]^{|A|} = 1$. Therefore, $[A_\nu G/G]$ is a unit in $B(A,G)$ for any $\nu \in Hom(G,A)$. Moreover $\mathfrak{K} = \{[A_\nu G/G] : \nu \in Hom(G,A)\}$ is a multiplicatively closed subset of $B(A,G)$ containing 1. Hence $\mathfrak{K}$ is a subgroup of the unit group $B(A,G)^*$ of $B(A,G)$. In fact, this shows that $Hom(G,A)$ embeds in $B(A,G)^*$ by $\nu \mapsto [A_\nu G/G]$ for any $\nu \in ch(A,G)$.

Chapter 4

Possible Ghost Rings

The Burnside ring $B(G)$ can be embedded in the ring $\mathbb{Z}^n$ where n is the number of noncojugate subgroups of G. That is, there is a ring monomorphism from $B(G)$ to $\mathbb{Z}^n$ and so the image of $B(G)$ is a subring of $\mathbb{Z}^n$ and it is called the ghost ring of $B(G)$. That is why the name of this chapter is possible ghost rings. In this chapter we embed $B(A, G)$ to two rings which are easier to work with than $B(A, G)$. The first one is a direct product of some group rings, and the second one which is easier is a direct product of $\mathbb{C}$. The first one was studied in [9] and the second one in [1]. We will mainly follow [9] for the first ghost ring using the notations in [1] and supply details skipped in [9]. We are still assuming that A is a finite abelian group. However, for the second ghost ring we have to assume that A is cyclic. Since the monomial Burnside rings introduced by Dress in [9] are more general than the monomial Burnside rings we are considering, the second ghost ring introduced by Barker in [1] will be used more than the first ghost ring in the next chapters.

For any $H \leq G$, let $\mathbb{Z}Hom(H, A)$ be the group ring. Conjugation by an element $g \in G$ induces a group isomorphism $Hom(H, A) \to Hom({}^gH, A)$ which can be extended to a group ring isomorphism from $\mathbb{Z}Hom(H, A)$ to $\mathbb{Z}Hom({}^gH, A)$. For an $A-$fibred $G-$set $S = AX$ and $s \in S$, remember that ν_s is the uniquely determined element of $Hom(stab_G(As), A)$ by the condition: $gs = \nu_s(g)s$ for all $g \in stab_G(As)$.

Lemma 4.1 *Let AX and AY be $A-$fibred $G-$sets. For any $x \in X$ and $y \in Y$ we have;*
(i) $stab_G(Ax \otimes y) = stab_G(Ax) \cap stab_G(Ay)$,
(ii) $\nu_{x\otimes y} = \nu_x \nu_y$.

Proof: (i) Let $g \in G$. If $gx = a_1x$ and $gy = a_2y$ for some $a_1, a_2 \in A$, then $g(x\otimes y) = a_1a_2(x\otimes y)$ and, more generally, $ga(x\otimes y) = aa_1a_2(x\otimes y)$ for all $a \in A$. On the other hand, if $g(x \otimes y) = b(x \otimes y)$ for some $b \in A$, then $gx = cx$ for some $c \in A$, and we must have $gy = c^{-1}by$. So we have shown that $g(Ax \otimes y) = Ax \otimes y$ if and only if $gAx = Ax$ and $gAy = Ay$. So part (i) follows.
(ii) For any $g \in stab_G(Ax \otimes y) = stab_G(Ax) \cap stab_G(Ay)$, we have $gx = \nu_x(g)x$, $gy = \nu_y(g)y$ and $g(x \otimes y) = \nu_{x\otimes y}(g)(x \otimes y)$. Then $\nu_{x\otimes y}(g)(x \otimes y) = g(x \otimes y) = (gx) \otimes (gy) = (\nu_x(g)x) \otimes (\nu_y(g)y) = \nu_x(g)\nu_y(g)(x \otimes y)$. Since $AX \otimes AY$ is $A-$free, it follows that $\nu_{x\otimes y}(g) = \nu_x(g)\nu_y(g)$.

$\square$

For any $H \leq G$, we define a map from $B(A, G)$ to $\mathbb{Z}Hom(H, A)$ as:

$$\psi_H : B(A, G) \to \mathbb{Z}Hom(H, A), \qquad [AX] \mapsto \sum_{x\in X, H\leq stab_G(Ax)} \nu_x|_H$$

where $\nu_x|_H$ denotes the restriction of ν_x to H. We usually omit $|_H$ and use ν_x for $\nu_x|_H$.

Lemma 4.2 *ψ_H is well-defined.*

Proof: Suppose $[AX] = [AY]$. We want to show that $\psi([AX]) = \psi([AY])$. Now, AX and AY are two isomorphic $AG-$sets. Thus, there is a bijective $AG-$map $f : AX \to AY$. For any $x \in X$, there are unique elements $a_x \in A$ and $y_x \in Y$ such that $f(x) = a_xy_x$.
Note that for $x, x' \in X$ if $y_x = y_{x'}$ then $f(a_{x'}x) = a_{x'}a_xy_x$ and $f(a_xx') = a_xa_{x'}y_{x'}$. So $f(a_{x'}x) = f(a_xx')$ implying that $x = x'$ and $a_x = a_{x'}$. That is, $Y = \{y_x : x \in X\}$ and y_x are all distinct where x range in X.
Take an $x \in X$. Then $g \in stab_G(Ax)$ if and only if $Agx = Ax$ which is to say

that $Agf(x) = Af(x)$ (equivalently $g \in stab_G(Af(x))$), because f respects the $AG-$action. Thus the maps ν_x and $\nu_{f(x)}$ are defined in the same domain, and $stab_G(Ax) = stab_G(Af(x)) = stab_G(Ay_x)$. Moreover, for any $g \in stab_G(Ax)$ we have $gx = \nu_x(g)x$ and $gf(x) = \nu_{f(x)}(g)f(x)$. On the other hand from $gx = \nu_x(g)x$ we get $f(gx) = f(\nu_x(g)x)$ implying that $gf(x) = \nu_x(g)f(x)$. Hence, $\nu_{f(x)}(g)f(x) = \nu_x(g)f(x)$. Since AY is $A-$free, $\nu_{f(x)}(g) = \nu_x(g)$. So we proved that $\nu_x = \nu_{f(x)} = \nu_{a_x y_x} = \nu_{y_x}$ (the last equality follows from 2.5).
Finally, using $Y = \{y_x : x \in X\}, stab_G(Ax) = stab_G(Ay_x)$, and $\nu_x = \nu_{y_x}$ we compute

$$\psi([AX]) = \sum_{x \in X, H \leq stab_G(Ax)} \nu_x = \sum_{x \in X, H \leq stab_G(Ay_x)} \nu_{y_x}$$
$$= \sum_{y \in Y, H \leq stab_G(Ay)} \nu_y = \psi([AY]).$$

□

Theorem 4.3 *(i) ψ_H is a unital ring homomorphism.*
(ii) For any $(V, \nu) \in ch(A, G)$;

$$\psi_H([A_\nu G/V]) = \sum_{gV \subseteq G, H \leq {}^gV} {}^g\nu.$$

(iii) If $H \not\leq_G V$, then $\psi_H([A_\nu G/V]) = 0$.

Proof: (i) Additivity is clear because $[AX] + [AY] = [A(X \uplus Y)]$.
Let AX and AY be $A-$fibred $G-$sets. Then using 4.1;

$$\psi_H([AX][AY]) = \psi_H([AX \otimes AY]) = \psi_H([A(X \otimes Y)])$$
$$= \sum_{x \otimes y \in X \otimes Y, H \leq stab_G(Ax \otimes y)} \nu_{x \otimes y}$$
$$= \sum_{x \in X, y \in Y, H \leq stab_G(Ax), H \leq stab_G(Ay)} \nu_x \nu_y$$
$$= (\sum_{x \in X, H \leq stab_G(Ax)} \nu_x)(\sum_{y \in Y, H \leq stab_G(Ay)} \nu_y)$$
$$= \psi_H([AX])\psi_H([AY]).$$

(ii) Remember that $A_\nu G/V = AG/\triangle_{(V,\nu)}$ where $\triangle_{(V,\nu)} = \{\nu(v^{-1})v : v \in V\}$. In chapter 2 we showed that to write $A_\nu G/V$ in the form AX we can take $X = \{g\triangle_{(V,\nu)} : gV \subseteq G\}$. Now we easily find that $stab_G(A(g\triangle_{(V,\nu)})) = {}^gV$ and $\nu_{g\triangle_{(V,\nu)}} = {}^g\nu$. So, $\psi_H([A_\nu G/V]) = \sum_{gV\subseteq G, H\leq {}^gV} {}^g\nu$.
(iii) It is obvious because $\{gV \subseteq G : H \leq {}^gV\}$ is empty set if $H \not\leq_G V$.

$\square$

Since conjugation by an element $g \in G$ induce a group ring isomorphism $\mathbb{Z}Hom(H,A) \to \mathbb{Z}Hom({}^gH,A)$, conjugation by g changes the map $\psi_H : B(A,G) \to \mathbb{Z}Hom(H,A)$ to a map, ${}^g\psi_H : B(A,G) \to \mathbb{Z}Hom({}^gH,A)$, by taking $g-$conjugates of the image of ψ_H. Note that ${}^g\psi_H = \psi_{{}^gH}$ because:

$$\psi_{{}^gH}([A_\nu G/V]) = \sum_{kV\subseteq G, {}^gH\leq {}^kV} {}^k\nu = \sum_{g^{-1}kV\subseteq G, H\leq {}^{g^{-1}k}V} {}^g({}^{g^{-1}k}\nu)$$

$$= {}^g(\sum_{g^{-1}kV\subseteq G, H\leq {}^{g^{-1}k}V} {}^{g^{-1}k}\nu) = {}^g\psi_H([A_\nu G/V]).$$

Note that for any $H \leq G$, the group $N_G(H)/H = N(H)$ acts on $Hom(H,A)$ by conjugation: $(gH,\nu) \mapsto {}^g\nu : {}^gH \to A$ where ${}^g\nu({}^gh) = \nu(h)$ for all $h \in H$. Thus by $\mathbb{Z}-$linear extension, $N(H)$ acts on $\mathbb{Z}Hom(H,A)$.

Remark 4.4 *The action of $N(H)$ on $\mathbb{Z}Hom(H,A)$ fixes $\psi_H(B(A,G))$ setwise.*

Proof: From the explanation above we have ${}^g\psi_H = \psi_{{}^gH}$ for any $g \in G$. Hence the result follows because ${}^gH = H$ for $gH \in N(H)$.

$\square$

Remark 4.5 *(i) $\psi_H([A_\nu G/H]) = \sum_{gH\subseteq N_G(H)} {}^g\nu$ for any $\nu \in Hom(H,A)$.*
(ii) $\psi_H([A_\nu G/H]) = |stab_{N(H)}(\nu)| \sum_\omega \omega$ where ω ranges over all distinct $N(H)-$conjugates of ν.
(iii) For any $(V,\nu) \in ch(A,G)$,

$$\psi_H([A_\nu G/V]) = \sum_{H\leq W\leq G, W=_G V} \psi_W([A_\nu G/V]).$$

Proof: (i) Obvious.
(ii) It is clear because $Hom(H, A)$ is an $N(H)-$set by conjugation.
(iii) $\psi_H([A_\nu G/V]) = \sum_{gV\subseteq G, H\leq {}^gV} {}^g\nu$. Note that the indices of the sum range in the set $\{gV \subseteq G : H \leq {}^gV\} = (G/V)^H$. We can write $(G/V)^H$ as $\uplus_{H\leq W, W=_G V}(G/V)^W$ (this property is stated in the beginning of chapter 2, (13)). So,
$\{gV \subseteq G : H \leq {}^gV\} = \uplus_{H\leq W, W=_G V}\{gV \subseteq G : W \leq {}^gV\}$. Then

$$\psi_H([A_\nu G/V]) = \sum_{gV\subseteq G, H\leq {}^gV} {}^g\nu$$

$$= \sum_{H\leq W, W=_G V} (\sum_{gV\subseteq G, W\leq {}^gV} {}^g\nu)$$

$$= \sum_{H\leq W\leq G, W=_G V} \psi_W([A_\nu G/V]).$$

□

Now we show that the product map $\prod_{H\leq G}\psi_H$ is an injective ring homomorphism from $B(A, G)$ to $\prod_{H\leq G}\mathbb{Z}Hom(H, A)$. We need the following lemma.
Let $\mathcal{F}$ be a subset of the subgroups of G such that if $V \in \mathcal{F}$, then ${}^gH \in \mathcal{F}$ for any $H \leq V$, and $g \in G$. We put

$$B(A, G, \mathcal{F}) = \bigoplus_{(V,\nu)\in_G ch(A,G), V\in\mathcal{F}} \mathbb{Z}[A_\nu G/V].$$

Lemma 4.6

$$B(A, G, \mathcal{F}) = \bigcap_{H\notin\mathcal{F}} Ker(\psi_H).$$

Proof: If $H \not\leq_G V$ then $\psi_H([A_\nu G/V]) = 0$. Hence,

$$B(A, G, \mathcal{F}) \subseteq \bigcap_{H\notin\mathcal{F}} Ker(\psi_H).$$

Take any $z = \sum_{(W,\omega)\in_G ch(A,G)} \lambda_{W,\omega}[A_\omega G/W] \in B(A, G)$ such that z is in $\bigcap_{H\notin\mathcal{F}} Ker(\psi_H)$ where $\lambda_{W,\omega}$ is an integer for $(W, \omega) \in ch(A, G)$. We want to show that $z \in B(A, G, \mathcal{F})$. So it suffices to show that if $\lambda_{W,\omega} \neq 0$, then $W \in \mathcal{F}$. Assume the contrary. Then there is a $W \notin \mathcal{F}$ with $\lambda_{W,\omega} \neq 0$ for some $\omega \in Hom(W, A)$.

Let V_0 be such an element which is maximal with respect to $\leq_G$. Thus for some $\nu_0 \in Hom(V_0, A)$ $\lambda_{V_0,\nu_0} \neq 0$, $V_0 \notin \mathcal{F}$, and if $\lambda_{V,\nu} \neq 0$ and $V \notin \mathcal{F}$ then $V_0 \nleq_G V$. Now,

$$0 = \psi_{V_0}(z) = \sum_{(W,\omega)\in_G ch(A,G)} \lambda_{W,\omega}\psi_{V_0}([A_\omega G/W]).$$

Because $H \nleq_G V$ implies that $\psi_H([A_\nu G/V]) = 0$,

$$0 = \psi_{V_0}(z) = \sum_{(W,\omega), V_0 \leq_G W} \lambda_{W,\omega}\psi_{V_0}([A_\omega G/W]).$$

For any W appearing in the last sum, if $W \in \mathcal{F}$ then $V_0 \in \mathcal{F}$ which is not the case. So,

$$0 = \psi_{V_0}(z) = \sum_{(W,\omega), V_0 \leq_G W, W \notin \mathcal{F}} \lambda_{W,\omega}\psi_{V_0}([A_\omega G/W]).$$

By the maximality of V_0,

$$0 = \psi_{V_0}(z) = \sum_{\omega \in_{N(V_0)} Hom(V_0,A)} \lambda_{V_0,\omega}\psi_{V_0}([A_\omega G/V_0]).$$

Then using 4.5(ii);

$$0 = \psi_{V_0}(z) = \sum_{\omega \in_{N(V_0)} Hom(V_0,A)} \lambda_{V_0,\omega}|stab_{N(V_0)}(\omega)| \sum_{\mu} \mu$$

where for a fixed $\omega \in Hom(V_0, A)$ the index μ ranges over all distinct $N(V_0)-$conjugates of ω. So all μ appearing in the last sum are distinct. Since the elements of $Hom(V_0, A)$ are linearly independent over $\mathbb{Z}$, we must have $\lambda_{V_0,\nu_0} = 0$ which is a contradiction. Hence we proved that

$$B(A, G, \mathcal{F}) \supseteq \bigcap_{H \notin \mathcal{F}} Ker(\psi_H).$$

□

Theorem 4.7 *The map* $\prod_{H \leq G} \psi_H : B(A, G) \to \prod_{H \leq G} \mathbb{Z}Hom(H, A)$ *is an injective ring homomorphism.*

Proof: Let $\mathcal{F}$ be the empty set. Then 4.6 implies that

$$0 = B(A, G, \mathcal{F}) = \bigcap_{H \leq G} Ker(\psi_H) = Ker(\prod_{H \leq G} \psi_H).$$

So the product map is injective.

□

Let R be a commutative ring with 1, we write $RB(A,G)$ for $R \otimes_{\mathbb{Z}} B(A,G)$ and $RHom(H,A)$ for $R \otimes_{\mathbb{Z}} \mathbb{Z}Hom(H,A)$. We denote the $R-$linear extension of the map ψ_H by again ψ_H. The results 4.6 and 4.7 are still true when $\mathbb{Z}$ is replaced by any commutative ring R with identity such that $|G|$ is not a zero divisor of R (proofs are exactly the same).
For any $(V,\nu) \in ch(A,G)$ put $n_{V,\nu} = |stab_{N(V)}(\nu)|$. Note that for $(V,\nu) =_G (W,\omega)$ $n_{V,\nu} = n_{W,\omega}$. Now for any $H \leq G$ and $(V,\nu) \in ch(A,G)$, by 4.5(iii);

$$\psi_H([A_\nu G/V]) = \sum_{H \leq W \leq G, W =_G V} \psi_W([A_\nu G/V]).$$

Suppose $W_1, ..., W_n$ are all distinct conjugates of V containing H. Let $W_i = {}^{g_i}V$. Then since ${}^g\psi_H = \psi_{{}^gH}$,

$$\psi_H([A_\nu G/V]) = \sum_i {}^{g_i}\psi_V([A_\nu G/V]).$$

Using 4.5(i) and (ii);

$$\psi_H([A_\nu G/V]) = \sum_i {}^{g_i}\Big(\sum_{gV \subseteq N_G(V)} {}^g\nu\Big) = \sum_i {}^{g_i}\Big(|stab_{N(V)}(\nu)| \sum_\omega \omega\Big).$$

Hence for any $(V,\nu) \in ch(A,G)$; $\psi_H(\frac{1}{n_{V,\nu}}[A_\nu G/V]) \in \mathbb{Z}Hom(H,A)$ and $\psi_V(\frac{1}{n_{V,\nu}}[A_\nu G/V]) = [\nu]^+$ where $[\nu]^+$ denotes the sum of all distinct conjugates of ν. Let $\mathbb{Z}Hom(V,A)^{N(V)}$ be the set of $N(V)-$fixed points of $\mathbb{Z}Hom(V,A)$. Then obviously it is a subring of $\mathbb{Z}Hom(V,A)$ and it is a free $\mathbb{Z}-$module with basis $[\nu]^+$. Hence, images of the different elements $\frac{1}{n_{V,\nu}}[A_\nu G/V]$ under the map ψ_V form a $\mathbb{Z}-$basis of $\mathbb{Z}Hom(V,A)^{N(V)}$.

Remark 4.8 *The $\mathbb{Z}-$linear span of the set $\{\frac{1}{n_{V,\nu}}[A_\nu G/V] : (V,\nu) \in ch(A,G)\}$ is a subring of $\mathbb{Q}B(A,G)$ and isomorphic to $\prod_{V \leq_G G} \mathbb{Z}Hom(V,A)^{N(V)}$.*

Proof: It follows from the explanation above. Note that the product is taken over all nonconjugate subgroups of G. It is because: G acts on $\prod_{V \leq G} \mathbb{Z}Hom(V,A)$ and g sends the term $\mathbb{Z}Hom(V,A)$ onto $\mathbb{Z}Hom({}^gV,A)$ isomorphicly. Thus since ${}^g\psi_V = \psi_{{}^gV}$, the map $\prod_{V\leq G}\psi_V : \mathbb{Q}B(A,G) \to \prod_{V\leq G}\mathbb{Q}Hom(V,A)$ maps the $\mathbb{Z}-$linear span of $\{\frac{1}{n_{V,\nu}}[A_\nu G/V] : (V,\nu) \in ch(A,G)\}$ isomorphicly onto $(\prod_{V\leq G}\mathbb{Z}Hom(V,A))^G$. Also note that we have

$$(\prod_{V\leq G}\mathbb{Z}Hom(V,A))^G \simeq \prod_{V\leq_G G}\mathbb{Z}Hom(V,A)^{N(V)}.$$

□

Now we study the ghost ring introduced in [1]. For this purpose we have to assume that A is cyclic. Hence, from now on in this chapter A is a finite cyclic group and we assume that $A \leq \mathbb{C}^*$.

We have already some algebra maps $\psi_H : \mathbb{C}B(A,G) \to \mathbb{C}Hom(H,A)$ and we want to construct algebra maps from $\mathbb{C}B(A,G)$ to $\mathbb{C}$. For any $h \in H$, define a map $ev(h) : \mathbb{C}Hom(H,A) \to \mathbb{C}$ given by $ev(h)(\sum_{\nu\in Hom(H,A)}\lambda_\nu\nu) = \sum_{\nu\in Hom(H,A)}\lambda_\nu\nu(h)$. It is clear that $ev(h)$ is a $\mathbb{C}-$algebra epimorphism. Note that $ev(h_1) = ev(h_2)$ if and only if $\nu(h_1) = \nu(h_2)$ for all ν in $Hom(H,A)$ if and only if $h_2^{-1}h_1 \in Ker\nu$ for all $\nu \in Hom(H,A)$. Let for any $V \leq G$, $O(V) = \cap_{\nu\in Hom(V,A)}Ker\nu$. We saw above that $ev(v_1) = ev(v_2)$ if and only if $v_1O(V) = v_2O(V)$.

For any $H \leq G$ and $h \in H$ define a map $S^G_{H,h} : \mathbb{C}B(A,G) \to \mathbb{C}$ as $S^G_{H,h} = ev(h) \circ \psi_H$. It is clear that $S^G_{H,h}$ is a $\mathbb{C}-$algebra homomorphism (because it is a composition of two such maps), and

$$S^G_{H,h}([A_\nu G/V]) = ev(h)(\sum_{gV\subseteq G, H\leq {}^gV} {}^g\nu) = \sum_{gV\subseteq G, H\leq {}^gV} {}^g\nu(h).$$

Remark 4.9 *For any $H \leq G$ and $h \in H$ we have;*
(i) For any $[A_\nu G/V] \in B(A,G)$

$$S^G_{H,h}([A_\nu G/V]) = \sum_{gV\subseteq G, H\leq {}^gV} {}^g\nu(h),$$

(ii) For any $A-$fibred $G-$set AX

$$S^G_{H,h}([AX]) = \sum_{x\in X, H\leq stab_G(Ax)} \nu_x(h),$$

(iii) $S^G_{H,h} : \mathbb{C}B(A,G) \to \mathbb{C}$ *is a* $\mathbb{C}-$*algebra epimorphism,*
(iv) $S^G_{H,h} = S^G_{{}^gH,{}^gh}$ *for any* $g \in G$.

Proof: (i) It is proved above.
(ii) Using the definition $S^G_{H,h} = ev(h) \circ \psi_H$ we can easily find the desired result because we know the rules of the maps $ev(h)$ and ψ_H.
(iii) Because it is a composition of two $\mathbb{C}-$algebra maps, the result follows.
(iv) $S^G_{{}^gH,{}^gh} = ev({}^gh) \circ {}^g\psi_H = ev({}^gh) \circ \psi_{{}^gH} = S^G_{H,h}$.

□

We define the following two sets (first one is already defined before);

$$ch(A,G) = \{(V,\nu) : V \leq G, \nu \in Hom(V,A)\},$$

$$el(A,G) = \{(H,h) : H \leq G, hO(H) \in H/O(H)\}$$

where $O(H) = \cap_{\nu \in Hom(H,A)} Ker\nu$, or equivalently $O(H)$ is the minimal normal subgroup of H such that $\bar{H} = H/O(H)$ is an abelian group of exponent dividing $|A|$. The sets $ch(A,G)$ and $el(A,G)$ are called the set of $A-$subcharacters of G and the set of $A-$subelements of G, respectively. See [1] for a more detailed explanation of these two sets. We just state the following.

Remark 4.10 *(i) The sets* $ch(A,G)$ *and* $el(A,G)$ *are* $G-$*sets by the conjugation action of* G.
(ii) $|ch(A,G)| = |el(A,G)|$, $|G \setminus ch(A,G)| = |G \setminus el(A,G)|$ *and* $|Hom(H,A)| = |\bar{H}|$, *where* $G \setminus el(A,G)$ *and* $G \setminus el(A,G)$ *denote* $G-$*orbit representatives.*

Proof: See [1].

□

We write $(H,h) =_G (K,k)$ if the elements $(H,h),(K,k) \in el(A,G)$ are in the same $G-$orbit of $el(A,G)$.

Theorem 4.11 *For any (H, h), consider the map $S^G_{H,h} : \mathbb{K}B(A, G) \to \mathbb{K}$ given by $S^G_{H,h}([AX]) = \sum_{x \in X, H \leq stab_G(Ax)} \nu_x(h)$ where $\mathbb{K}$ is a field of characteristic 0 containing enough roots of unity to ensure that $A \leq \mathbb{K}^*$. Then;*
(i) For any $(V, \nu) \in ch(A, G)$,

$$S^G_{H,h}([A_\nu G/V]) = \sum_{gV \subseteq G, H \leq {}^gV} {}^g\nu(h).$$

(ii) $S^G_{H,h}$ is a $\mathbb{K}-$algebra epimorphism.
(iii) Any $\mathbb{K}-$algebra homomorphism from $\mathbb{K}B(A, G)$ to $\mathbb{K}$ is of the form $S^G_{H,h}$ for some $(H, h) \in el(A, G)$.
(iv) $S^G_{H,h} = S^G_{K,k}$ if and only if $(H, h) =_G (K, k)$.
(v) $\mathbb{K}B(A, G)$ is a semisimple algebra.
(vi) The following map is a $\mathbb{C}-$algebra isomorphism,

$$\prod_{(H,h) \in_G el(A,G)} S^G_{H,h} : \mathbb{C}B(A, G) \to \prod_{(H,h) \in_G el(A,G)} \mathbb{C}.$$

Proof: Indeed we know (i), (ii) and half of (iv) from 4.9. For the rest, or for all of them see [1].

□

Theorem 4.11 which was obtained by Barker in [1] is very important, and it will be used in the next chapters.

Chapter 5

Primitive Idempotents of $\mathbb{C}B(A,G)$

We are still assuming that A is a finite cyclic group and $A \leq \mathbb{C}^*$. An explicit formula for the primitive idempotents of $\mathbb{C}B(A,G)$ in terms of the transitive basis can be found in [1]. We just state in this chapter some results that we need in later chapters, for details see [1].

From 4.11 (vi) we know that $\varphi = \prod_{(H,h)\in_G el(A,G)} S^G_{H,h}$ is a $\mathbb{C}-$algebra isomorphism from $\mathbb{C}B(A,G)$ to $\prod_{(H,h)\in_G el(A,G)} \mathbb{C}$. We know that $\mathcal{B} = \{[A_\nu G/V] : (V,\nu) \in ch(A,G)\}$ is a $\mathbb{C}-$basis of the $\mathbb{C}-$algebra $\mathbb{C}B(A,G)$. Let $\mathcal{B}'$ be the standard basis of $\prod_{(H,h)\in_G el(A,G)} \mathbb{C}$. That is, it consists of all vectors $(0,..,1,..,0)$ with only one nonzero entry which is 1 in the $(H,h)^{th}$ place. Suppose we order $\mathcal{B}$ and $\mathcal{B}'$. Then φ has an $n \times n$ matrix say ${}_{\mathcal{B}}[\varphi]_{\mathcal{B}'}$ with respect to the ordered basis $\mathcal{B}$ and $\mathcal{B}'$ where $n = |G \setminus ch(A,G)| = |G \setminus el(A,G)|$. Thus we have for any $z \in \mathbb{C}B(A,G)$ that $[\varphi(z)]_{\mathcal{B}'} = {}_{\mathcal{B}}[\varphi]_{\mathcal{B}'}[z]_{\mathcal{B}}$ where $[\varphi(z)]_{\mathcal{B}'}$ denotes the coordinate matrix of $\varphi(z)$ with respect to $\mathcal{B}'$, and $[z]_{\mathcal{B}}$ denotes the coordinate matrix of z with respect to $\mathcal{B}$. Since φ is an isomorphism, primitive idempotents maps onto primitive idempotents. Also the primitive idempotents of $\prod_{(H,h)\in_G el(A,G)} \mathbb{C}$ are just the elements of $\mathcal{B}'$. Hence $\varphi^{-1}(\mathcal{B}')$ must be the set of primitive idempotents of $\mathbb{C}B(A,G)$. Hence, in concrete examples we can find the primitive idempotents of $\mathbb{C}B(A,G)$ by evaluating the inverse of the matrix ${}_{\mathcal{B}}[\varphi]_{\mathcal{B}'}$.

Remark 5.1 *(i) The primitive idempotents of* $\mathbb{C}B(A,G)$ *are of the form* $e^G_{H,h}$ *where* (H,h) *runs over all nonconjugate* $A-$*subelements of* G.
(ii) $e^G_{H,h}e^G_{K,k} = \begin{cases} 1 & \textit{if } (H,h) =_G (K,k) \\ 0 & \textit{otherwise.} \end{cases}$
(iii) $\sum_{(H,h)\in_G el(A,G)} e^G_{H,h} = 1.$
(iv) $e^G_{H,h}$ *is the unique element of* $\mathbb{C}B(A,G)$ *satisfying the following condition for all* $(K,k) \in el(A,G)$*;*
$S^G_{K,k}(e^G_{H,h}) = \begin{cases} 1 & \textit{if } (H,h) =_G (K,k) \\ 0 & \textit{otherwise.} \end{cases}$
(v)

$$\mathbb{C}B(A,G) = \bigoplus_{(V,\nu)\in_G ch(A,G)} \mathbb{C}[A_\nu G/V] = \bigoplus_{(H,h)\in_G el(A,G)} \mathbb{C}e^G_{H,h}.$$

(vi) For any $z \in \mathbb{C}B(A,G)$,

$$z = \sum_{(H,h)\in_G el(A,G)} S^G_{H,h}(z)e^G_{H,h}.$$

(vii) For any $z \in \mathbb{C}B(A,G)$ *and* $(H,h) \in el(A,G)$, $ze^G_{H,h} = S^G_{H,h}(z)e^G_{H,h}$.

Proof: See [1].

$\square$

Remark 5.1 was obtained by Barker in [1]. It is very important and used throughout in the next chapters without referring sometimes.

Chapter 6

Some Maps

There are certain important maps defined in [16] for the Burnside rings. To realize $B(A,G)$ as a Mackey functor, in this chapter we extend these maps to $B(A,G)$ which contains $B(G)$ as a unital subring, and we find the images of the primitive idempotents of $\mathbb{C}B(A,G)$ under all these maps, except one, namely, the orbit map.

If A is taken to be the trivial group, then our results recover the corresponding results about these maps defined on $B(G)$.

We are still assuming that A is a finite abelian group. However, for the places in which the algebra maps $S^G_{H,h}$ or the primitive idempotents $e^G_{H,h}$ appear we have to assume that A is a finite cyclic group regarded as a subgroup of $\mathbb{C}^*$. Moreover, wherever $S^G_{H,h}$ or $e^G_{H,h}$ appear, it must be understood that we extended these maps by $\mathbb{C}-$linear extension from $B(A,G)$ to $\mathbb{C}B(A,G)$.

In fact, there are six maps that we want to consider. One of them , namely, conjugation map, is very trivial. Two of them were studied, and the images of the primitive idempotents under these two maps were found by Barker in [1]. Hence, for these three maps we just state the results without proofs.

6.1 The Inflation Map

Let $N \trianglelefteq G$ and $S = AX$ be an $A-$fibred $G/N-$set. Define the inflated set $inf_N^G(S) = S$ and let AG act on $inf_N^G(S)$ as;
$(ag, s) \mapsto ags = a(gN)s$ for all $ag \in AG$ and $s \in inf_N^G(S)$.

Remark 6.1 *Let $S = AX$ and $T = AY$ be $A-$fibred $G/N-$sets where $N \trianglelefteq G$.*
(i) $inf_N^G(S)$ is an $A-$fibred $G-$set.
(ii) $S \simeq_{A(G/N)} T$ if and only if $inf_N^G(S) \simeq_{AG} inf_N^G(T)$.
(iii) $inf_N^G(S \uplus T) = inf_N^G(S) \uplus inf_N^G(T)$.
(iv) $inf_N^G(S \otimes T) = inf_N^G(S) \otimes inf_N^G(T)$.
(v) S is a transitive $A-$fibred $G/N-$set if and only if $inf_N^G(S)$ is a transitive $A-$fibred $G-$set.

Proof: (i) The action is well-defined: Suppose $a_1g_1 = a_2g_2 \in AG$ and $s_1 = s_2 \in inf_N^G(S) = S$. We want to show that $a_1g_1s_1 = a_2g_2s_2$, equivalently $(a_1(g_1N))s_1 = (a_2(g_2N))s_2$. Since $a_1(g_1N) = a_2(g_2N)$ and $s_1 = s_2$ we have already $(a_1(g_1N))s_1 = (a_2(g_2N))s_2$ because the action of $A(G/N)$ on S is well defined.
The action properties are satisfied: Obvious.
$inf_N^G(S)$ is $A-$free: It is clear because the actions of A on S and $inf_N^G(S)$ are the same.
(ii) Since $inf_N^G(S) = S$ and $inf_N^G(T) = T$, any bijective map from S to T is a bijective map from $inf_N^G(S)$ to $inf_N^G(T)$ and conversely. It is clear from the definition of the $AG-$action on inflated sets that a bijective map from S to T respects the $A(G/N)-$action if and only if it respects the $AG-$action.
(iii) and (iv) It is immediate because $inf_N^G(S) = S$ and $inf_N^G(T) = T$.
(v) It is clear by the action of AG on $inf_N^G(S)$.

□

Hence, by 6.1 we have a well-defined map, called the inflation map,
$Inf_N^G : B(A, G/N) \to B(A, G)$ given by $Inf_N^G([S]) \mapsto [inf_N^G(S)]$ for any $A-$fibred $G/N-$set S.

Remark 6.2 *$Inf_N^G : B(A, G/N) \to B(A, G)$ is a ring monomorphism.*

Proof: It is a well-defined map from 6.1(i) and (ii). It is a ring homomorphism from (6.1)(iii) and (iv). Finally, injectivity follows from 6.1(ii).

$\square$

Let $N \trianglelefteq G$ and $N \leq V \leq G$. For any $\nu \in Hom(V/N, A)$, we write $\hat{\nu}$ for the group homomorphism $V \to A$ given by $\hat{\nu}(v) = \nu(vN)$ for all $v \in V$.

Remark 6.3 *Let $N \trianglelefteq G$ and $(V/N, \nu) \in ch(A, G/N)$. Then*

$$Inf_N^G([A_\nu \frac{(G/N)}{(V/N)}]) = [A_{\hat{\nu}} G/V].$$

Proof: $Inf_N^G([A_\nu \frac{(G/N)}{(V/N)}]) = [inf_N^G(A_\nu \frac{(G/N)}{(V/N)})]$, and by 6.1($v$)
$inf_N^G(A_\nu \frac{(G/N)}{(V/N)}) = A_\nu \frac{(G/N)}{(V/N)}$ is a transitive $A-$fibred $G-$set. Hence the $AG-$orbits of its elements are all equal to $A_\nu \frac{(G/N)}{(V/N)}$.
Remember that $A_\nu \frac{(G/N)}{(V/N)} = \frac{A(G/N)}{\triangle_{(V/N,\nu)}}$ where $\triangle_{(V/N,\nu)} = \{\nu((vN)^{-1})(vN) : vN \in V/N\}$ which is a subgroup of $A(G/N)$. Put $\triangle = \triangle_{(V/N,\nu)}$.
We find the $AG-$stabilizer of $\triangle = 1(1N)\triangle \in \frac{A(G/N)}{\triangle}$:
Let $ag \in AG$. Then ag is in the stabilizer if and only if $a(gN)\triangle = \triangle$, which is equivalent to $g \in V$ and $\nu((gN)^{-1}) = \hat{\nu}(g^{-1}) = a$. So, $ag \in AG$ is in the stabilizer if and only if $ag \in \{\hat{\nu}(g^{-1})g : g \in V\} = \triangle_{(V,\hat{\nu})}$. Therefore;
$inf_N^G(A_\nu \frac{A(G/N)}{V/N}) = orb_{AG}(\triangle) \simeq_{AG} (AG)/\triangle_{(V,\hat{\nu})} = A_{\hat{\nu}} G/V$, as desired.

$\square$

For the rest of this section, we consider the $\mathbb{C}-$linear extension of the inflation map; $Inf_N^G : \mathbb{C}B(A, G/N) \to \mathbb{C}B(A, G)$.

Lemma 6.4 *Let $N \trianglelefteq G$ and $(H, h) \in el(A, G)$. Then for any $A-$fibred $G/N-$set $S = AX$ we have;*

$$S_{H,h}^G(Inf_N^G([S])) = S_{(NH)/N,hN}^{G/N}([S]).$$

Proof: It suffices to prove this lemma for transitive $A-$fibred $G/N-$sets. Hence take any $(V/N, \nu) \in ch(A, G/N)$. Remember that $\hat{\nu}$ is the group homomorphism from V to A given by $\hat{\nu}(v) = \nu(vN)$. Using 6.3;

$$S^G_{H,h}(Inf^G_N([A_\nu \frac{(G/N)}{(V/N)}])) = S^G_{H,h}([A_{\hat{\nu}} G/V]) = \sum_{gV \subseteq G, H \leq {}^gV} {}^g\hat{\nu}(h).$$

Note that; since $N \leq V$, $V/N = (NV)/N$. So, $H \leq {}^gV$ if and only if $((NH)/N) \leq {}^{gN}((NV)/N) = {}^{gN}(V)$. Hence $\{gV \subseteq G : H \leq {}^gV\} = \{(gN)(V/N) \subseteq (G/N) : ((NH)/N) \leq {}^{gN}(V/N)\}$. Also ${}^g\hat{\nu}(h) = {}^{gN}\nu(hN)$. Therefore the last sum can be written as;

$$\sum_{gV \subseteq G, H \leq {}^gV} {}^g\hat{\nu}(h) = \sum_{(gN)(V/N) \subseteq (G/N), ((NH)/N) \leq {}^{gN}(V/N)} {}^{gN}\nu(hN)$$

$$= S^{G/N}_{(NH)/N, hN}([A_\nu \frac{(G/N)}{(V/N)}]).$$

□

Theorem 6.5 *Let $N \trianglelefteq G$ and $(K/N, kN) \in el(A, G/N)$. Then*

$$Inf^G_N(e^{G/N}_{K/N,kN}) = \sum_{(H,h) \in_G el(A,G), ((NH)/N, hN) =_{G/N} (K/N, kN)} e^G_{H,h}.$$

Proof: For some complex numbers $\lambda_{H,h}$;

$$Inf^G_N(e^{G/N}_{K/N,kN}) = \sum_{(H,h) \in_G el(A,G)} \lambda_{H,h} e^G_{H,h}.$$

Then by 6.4;

$$\lambda_{H,h} = S^G_{H,h}(Inf^G_N(e^{G/N}_{K/N,kN})) = S^{G/N}_{(NH)/N,hN}(e^{G/N}_{K/N,kN}).$$

Therefore

$$\lambda_{H,h} = \begin{cases} 1 \ , & ((NH)/N, hN) =_{G/N} (K/N, kN) \\ 0 \ , & \text{otherwise.} \end{cases}$$

Thus, the result follows.

□

6.2 The Invariance Map

Let $N \trianglelefteq G$ and $S = AX$ be an $A-$fibred $G-$set . Let $inv_N^G(S) = \{s \in S : Ns = s\}$. We let $A(G/N)$ act on $inv_N^G(S)$ as: $(a(gN), s) \mapsto (a(gN))s = ags$ for all $a(gN) \in A(G/N)$ and $s \in inv_N^G(S)$.

Remark 6.6 *Let $N \trianglelefteq G$ and $S = AX$ be an $A-$fibred $G-$set. Then $inv_N^G(S)$ is an $A-$fibred $G/N-$set.*

Proof: $inv_N^G(S)$ is closed under the $A(G/N)-$action: Let $a(gN) \in A(G/N)$ and $s \in inv_N^G(S)$. We want to show that $a(gN)s = ags \in in_N^G(S)$. For any $n \in N$, $n(ags) = ag((g^{-1}ng)s)$. By the normality of N, $g^{-1}ng \in N$, and since $s \in inv_N^G(S)$ is fixed by N we have $n(ags) = ags$ implying that $inv_N^G(S)$ is closed under the action of $A(G/N)$.
$inv_N^G(S)$ is an $A(G/N)-$set: It is a straightforward checking of action properties.
$inv_N^G(S)$ is $A-$free: It is clear because the actions of A on $inv_N^G(S)$ and S are the same.

$\square$

Remark 6.7 *Let $S = AX$ and $T = AY$ be $A-$fibred $G-$sets. Then*
(i) If $S \simeq_{AG} T$, then $inv_N^G(S) \simeq_{A(G/N)} inv_N^G(T)$.
(ii) $inv_N^G(S \uplus T) = inv_N^G(S) \uplus inv_N^G(T)$.
(iii) If S is a transitive $A-$fibred $G-$set, then $inv_N^G(S)$ is a transitive $A-$fibred $G/N-$set.

Proof: (i) Suppose $S \simeq_{AG} T$. Then there is a bijective $AG-$map from S to T. It is clear that the restriction of this map to $inv_N^G(S)$ yields a bijective $A(G/N)-$map from $inv_N^G(S)$ to $inv_N^G(T)$ which shows that $inv_N^G(S) \simeq_{A(G/N)} inv_N^G(T)$.
(ii) and (iii) are obvious.

$\square$

Hence by 6.6 and 6.7, we have a well-defined map, called the invariance map, $Inv_N^G : B(A, G) \to B(A, G/N)$ given by $Inv_N^G([S]) = [inv_N^G(S)]$ for any $A-$fibred $G-$set S.

Remark 6.8 *$Inv_N^G : B(A,G) \to B(A,G/N)$ is a $\mathbb{Z}-$module homomorphism.*

Proof: Follows from 6.6 and 6.7.

$\square$

Let $N \trianglelefteq G$ and $V \leq G$. For any $\nu \in Hom(V,A)$ such that $N \leq Ker\nu$, we write $\hat{\nu}$ for the group homomorphism $V/N \to A$ given by $\hat{\nu}(vN) = \nu(v)$ for all $vN \in V/N$. Note that $N \leq Ker\nu$ implies that $N \leq V$, and also that $\hat{\nu}$ is well-defined.

Remark 6.9 *Let $N \trianglelefteq G$ and $(V,\nu) \in ch(A,G)$. Then we have*
$$Inv_N^G([A_\nu G/V]) = \begin{cases} [A_{\hat{\nu}}\frac{(G/N)}{(V/N)}], & N \leq Ker\nu \\ 0, & N \nleq Ker\nu. \end{cases}$$

Proof: $Inv_N^G([A_\nu G/V]) = [inv_N^G(A_\nu G/V)]$. Remember $A_\nu G/V = AG/\triangle_{(V,\nu)}$ where $\triangle_{(V,\nu)} = \{\nu(v^{-1})v : v \in V\}$ which is a subgroup of AG. Put $\triangle_{(V,\nu)} = \triangle$. Now, $ag\triangle \in inv_N^G(A_\nu G/V)$ if and only if $ang\triangle = ag\triangle$ for all $n \in N$ which is equivalent to $g^{-1}ng \in \triangle$ for all $n \in N$. Then by the definition of $\triangle$, $g^{-1}ng \in \triangle$ for all $n \in N$ if and only if $g^{-1}ng \in V$ and $\nu((g^{-1}ng)^{-1}) = 1$ for all $n \in N$ that is to say, $n \in {}^gV$ and ${}^g\nu(n^{-1}) = 1$ for all $n \in N$. Since N is normal, $n \in {}^gV$ and ${}^g\nu(n^{-1}) = 1$ for all $n \in N$ if and only if $N \leq Ker{}^g\nu = {}^g(Ker\nu)$, equivalently $N \leq Ker\nu$. Hence,
$$inv_N^G(A_\nu G/V) = \begin{cases} \text{empty} & if \quad N \nleq Ker\nu \\ A_\nu G/V & if \quad N \leq Ker\nu. \end{cases}$$
So, if $N \nleq Ker\nu$, then $Inv_N^G([A_\nu G/V]) = 0$.
Suppose now $N \leq Ker\nu$. Then $inv_N^G(A_\nu G/V) = A_\nu G/V$, and it is a transitive $A-$fibred $G/N-$set by 6.7. Thus $A(G/N)-$orbits of its elements are all equal to $A_\nu G/V$.
We find the $A(G/N)-$stabilizer of $\triangle = 1.1\triangle \in AG/\triangle$:
$a(gN)$ is in the stabilizer if and only if $ag\triangle = \triangle$, equivalently $ag \in \triangle$. Then by the definition of $\triangle$, $ag \in \triangle$ if and only if $g \in V$ and $\nu(g^{-1}) = a$. Using $N \leq Ker\nu \leq V$, we see that $a(gN)$ is in the stabilizer if and only if $gN \in V/N$ and $\hat{\nu}((gN)^{-1}) = a$. Hence, $stab_{A(G/N)}(\triangle) = \{\hat{\nu}((gN)^{-1})(gN) : gN \in V/N\} = \triangle_{(V/N,\hat{\nu})}$. Consequently for $N \leq Ker\nu$;
$inv_N^G(A_\nu G/V) = A_\nu G/V = orb_{A(G/N)}(\triangle) \simeq_{A(G/N)} \frac{A(G/N)}{\triangle_{(V/N,\hat{\nu})}} = A_{\hat{\nu}}\frac{(G/N)}{(V/N)}$. So we

proved the desired result.

□

Remark 6.10 *Suppose $N \trianglelefteq G$ with $O(N) = N$ and $N \leq V \leq G$. Then for any $\nu \in Hom(V, A)$, $N \leq Ker\nu$.*

Proof: The restriction of ν to N is a group homomorphism from N to A. So $O(N) \leq Ker\nu$.

□

For the rest of this section we consider $\mathbb{C}-$linear extension of the invariance map; $Inv_N^G : \mathbb{C}B(A, G) \to \mathbb{C}B(A, G/N)$.

Lemma 6.11 *Let $N \trianglelefteq G$ with $O(N) = N$ and $(H/N, hN) \in el(A, G/N)$. Then for any $A-$fibred $G-$set $S = AX$ we have;*

$$S_{H/N,hN}^{G/N}(Inv_N^G([S])) = S_{H,h}^G([S]).$$

Proof: It suffices to show this lemma for transitive $A-$fibred $G-$sets. Hence, take any $(V, \nu) \in ch(A, G)$. From 6.10, $N \leq Ker\nu$ if and only if $N \leq V$. Also remember that $\hat{\nu}$ denotes the group homomorphism $V/N \to A$ given by $\hat{\nu}(vN) = \nu(v)$ if $N \leq Ker\nu$.

Case (1): $N \leq V$.

Using 6.9 we have

$$S_{H/N,hN}^{G/N}(Inv_N^G([A_\nu G/V])) = S_{H/N,hN}^{G/N}([A_{\hat{\nu}}\frac{(G/N)}{(V/N)}])$$

$$= \sum_{(gN)(V/N)\subseteq(G/N),(H/N)\leq^{gN}(V/N)} {}^{gN}\hat{\nu}(hN)$$

$$= \sum_{gV\subseteq G,H\leq^{g}V} {}^{g}\nu(h) = S_{H,h}^G([A_\nu G/V]).$$

Case (2): $N \nleq V$.

By 6.9, $Inv_N^G([A_\nu G/V]) = 0$ and so $S_{H/N,hN}^{G/N}(Inv_N^G([A_\nu G/V])) = 0$.

On the other hand $S_{H,h}^G([A_\nu G/V]) = \sum_{gV\subseteq G,H\leq^{g}V} {}^{g}\nu(h) = 0$ because the indices of the last sum range in the set $\{gV \subseteq G : H \leq {}^gV\}$ which is empty. Otherwise; if $H \leq {}^gV$ for some $g \in G$ then from $N \leq H$ we get $N \leq {}^gV$, and so by the normality of N we have $N \leq V$ which is not the case. So for $N \nleq V$, $S^{G/N}_{H/N,hN}(Inv^G_N([A_\nu G/V]))$ and $S^G_{H,h}([A_\nu G/V])$ are both equal to 0.

□

Theorem 6.12 *Let $N \trianglelefteq G$, $O(N) = N$ and $(K,k) \in el(A,G)$. Then*
$Inv^G_N(e^G_{K,k}) = \begin{cases} 0 & \text{if } N \nleq K \\ e^{G/N}_{K/N,kN} & \text{if } N \leq K. \end{cases}$

Proof: For some complex numbers $\lambda_{H/N,hN}$;

$$Inv^G_N(e^G_{K,k}) = \sum_{(H/N,hN)\in_{G/N} el(A,G/N)} \lambda_{H/N,hN} e^{G/N}_{H/N,hN}.$$

Using 6.11;

$$\lambda_{H/N,hN} = S^{G/N}_{H/N,hN}(Inv^G_N(e^G_{K,k})) = S^G_{H,h}(e^G_{K,k}).$$

Hence,

$\lambda_{H/N,hN} = S^G_{H,h}(e^G_{K,k}) = \begin{cases} 1, & (H,h) =_G (K,k) \\ 0, & \text{otherwise.} \end{cases}$

Therefore;

$$Inv^G_N(e^G_{K,k}) = \sum_{(H/N,hN)\in_{G/N} el(A,G/N),(H,h)=_G(K,k)} e^{G/N}_{H/N,hN}.$$

The condition $(H,h) =_G (K,k)$ implies that $N \leq H = {}^gK$ for some $g \in G$, and by the normality of N we get $N \leq K$. Hence, $Inv^G_N(e^G_{K,k}) = 0$ if $N \nleq K$.
Suppose now $N \leq K$. If $(H_1,h_1) =_G (K,k) =_G (H_2,h_2)$ where $H_1 \geq N \leq H_2$, then $(H_1/N, h_1N) =_{G/N} (K/N,kN) =_{G/N} (H_2/N,h_2N)$ ($O(N) = N$ is used here! Note that in this case $N \leq T$ implies that $O(T/N) = O(T)/N$). So the last sum can not contain more than one summand, and clearly $(K/N,kN)$ is a summand. Hence, $Inv^G_N(e^G_{K,k}) = e^{G/N}_{K/N,kN}$ if $N \leq K$.

□

Note that Inv^G_N is not a multiplicative map in general (in contrast to the invariance map defined on $B(G)$). Because, for example, it may happen that there

are $\nu \in Hom(G, A)$ and $\omega \in Hom(G, A)$ such that $N \not\leq Ker\nu$, $N \not\leq Ker\omega$ but $N \leq Ker(\nu.\omega)$. In this case; $Inv_N^G([A_{\nu.\omega}G/G]) \neq 0$ but
$[A_\nu G/G][A_\omega G/G] = [A_{\nu.\omega}G/G]$, $Inv_N^G([A_\nu G/G])Inv_N^G([A_\omega G/G]) = 0$.

Remark 6.13 *If $O(N) = N$, then Inv_N^G is multiplicative and so a ring homomorphism.*

Proof: Take any two elements $x, y \in \mathbb{C}B(A, G)$ where, say,

$$x = \sum_{(H,h) \in_G el(A,G)} \lambda_{H,h} e^G_{H,h}, \ y = \sum_{(K,k) \in_G el(A,G)} \mu_{K,k} e^G_{K,k}.$$

Then using 6.12;

$$Inv_N^G(x) = \sum_{(H,h) \in_G el(A,G), N \leq H} \lambda_{H,h} e^{G/N}_{H/N,hN},$$

$$Inv_N^G(y) = \sum_{(K,k) \in_G el(A,G), N \leq K} \mu_{K,k} e^{G/N}_{K/N,kN}.$$

Thus,

$$Inv_N^G(x)Inv_N^G(y) = \sum_{(T,t) \in_G el(A,G), N \leq T} \lambda_{T,t}\mu_{T,t} e^{G/N}_{T/N,tN}.$$

On the other hand;

$$xy = \sum_{(T,t) \in_G el(A,G)} \lambda_{T,t}\mu_{T,t} e^G_{T,t},$$

implying from 6.12 that

$$Inv_N^G(xy) = \sum_{(T,t) \in_G el(A,G), N \leq T} \lambda_{T,t}\mu_{T,t} e^{G/N}_{T/N,tN}.$$

Consequently, $Inv_N^G(xy) = Inv_N^G(x)Inv_N^G(y)$.

□

Lastly, if A is taken to be the trivial group then $B(A, G) = B(G)$ and $O(N) = N$ for any subgroup N of G and so our invariance map extends the invariance map defined for $B(G)$ in [16].

6.3 The Orbit Map

First we must recall some facts from chapter 2. For any $A-$fibred $G-$set $S = AX$ and $s \in S$, ν_s is the uniquely determined element of $Hom(stab_G(As), A)$ by the condition: $gs = \nu_s(g)s$ for all $g \in stab_G(As)$. Moreover, $orb_{AG}(s) \simeq_{AG} AG/\triangle_{(stab_G(As),\nu_s)} = A_{\nu_s}G/stab_G(As)$ where $\triangle_{(stab_G(As),\nu_s)} = \{\nu_s(g^{-1})g : g \in stab_G(As)\} \leq AG$.

Let $N \trianglelefteq G$ and S be a $G-$set. Let $S \setminus N$ be the set of $N-$orbits of S. So, $S \setminus N = \{orb_N(s) : s \in S\}$. $S \setminus N$ becomes a $G/N-$set with the action: $(gN, orb_N(s)) \mapsto (gN)(orb_N(s)) = orb_N(gs)$.

To extend this definition to $A-$fibred $G-$sets, one can attempt to take all $N-$orbits of an $A-$fibred $G-$set $S = AX$. Then it becomes an $A(G/N)-$set, however it may not be $A-$free.

Let $N \trianglelefteq G$ and $S = AX$ be an $A-$fibred $G-$set. Define

$$S \setminus \setminus N = \{orb_N(s) : s \in S, stab_G(As) \cap N \leq Ker\nu_s\}.$$

We let $A(G/N)$ act on $S \setminus \setminus N$ as: $(a(gN), orb_N(s)) \mapsto (a(gN))(orb_N(s)) = orb_N(ags)$ for all $a(gN) \in A(G/N)$ and $orb_N(s) \in S \setminus \setminus N$.

Remark 6.14 *For any $A-$fibred $G-$set $S = AX$, $S \setminus \setminus N$ is an $A-$fibred $G/N-$set.*

Proof: If $orb_N(s) \in S\setminus\setminus N$ and $a(gN) \in A(G/N)$, then $orb_N(ags) \in S\setminus\setminus N$: We want to show that $stab_G(Aags) \cap N \leq Ker\nu_{ags}$. We have $stab_G(As) \cap N \leq Ker\nu_s$. From chapter 2 we know that $stab_G(Aags) = stab_G(Ags) = {}^g stab_G(As)$ and $\nu_{ags} = \nu_{gs} = {}^g\nu_s$. Hence by the normality of N, $stab_G(As) \cap N \leq Ker\nu_s$ implies $stab_G(Ags) \cap N \leq Ker\nu_{ags}$.
If $orb_N(s_1) = orb_N(s_2) \in S \setminus \setminus N$ and $a_1(g_1N) = a_2(g_2N) \in A(G/N)$, then $orb_N(a_1g_1s_1) = orb_N(a_2g_2s_2)$:
$orb_N(s_1) = orb_N(s_2)$ implies that $ns_1 = s_2$ for some $n \in N$. Also from $a_1(g_1N) = a_2(g_2N)$ it follows that $a_1 = a_2$ and $g_1n' = g_2$ for some $n' \in N$. Now, $a_2g_2s_2 =$

$a_1g_1n'ns_1 = a_1(g_1n'ng_1^{-1})g_1s = (g_1n'ng_1^{-1})a_1g_1s_1$. Since $g_1n'ng_1^{-1} \in N$ (N is normal), $orb_N(a_1g_1s_1) = orb_N(a_2g_2s_2)$.

The action properties are satisfied: It is obvious.

$S\setminus\setminus N$ is $A-$free: For any $orb_N(s) \in S\setminus\setminus N$ we find its $A-$stabilizer as follows. If a is in the $A-$stabilizer of $orb_N(s)$, then $orb_N(s) = orb_N(as)$. Then $ns = as$ for some $n \in N$ and so $n(As) = A(as) = As$ implying that $n \in stab_G(As)$. But we have already $n \in N$. So now $n \in stab_G(As) \cap N \leq Ker\nu_s$ implying that $as = ns = \nu_s(n)s = s$. Because S is $A-$free, $as = s$ implies $a = 1$. Thus, $S\setminus\setminus N$ is $A-$free.

Being an $A-$free $A(G/N)-$set, $S\setminus\setminus N$ is an $A-$fibred $G/N-$set.

$\square$

Remark 6.15 *Let $S = AX$ and $T = AY$ be $A-$fibred $G-$sets. Then*
(i) If $S \simeq_{AG} T$ then $(S\setminus\setminus N) \simeq_{A(G/N)} (S\setminus\setminus N)$.
(ii) $(S \uplus T)\setminus\setminus N = (S\setminus\setminus N) \uplus (T\setminus\setminus N)$.
(iii) If S is a transitive $A-$fibred $G-$set, then $S\setminus\setminus N$ is a transitive $A-$fibred $G/N-$set.

Proof: (i) Suppose $S \simeq_{AG} T$. Then there is a bijective $AG-$map $f : S \to T$. Define $\hat{f} : S\setminus\setminus N \to T\setminus\setminus N$ with the rule $\hat{f}(orb_N(s)) = orb_N(f(s))$ for all $orb_N(s) \in S\setminus\setminus N$.

(1) Since f is bijective and preserves the $AG-$action, we have $stab_G(As) = stab_G(Af(s))$ and $\nu_s = \nu_{f(s)}$. Hence, $orb_N(s) \in S\setminus\setminus N$ implies that $orb_N(f(s)) \in T\setminus\setminus N$.

(2) Suppose $\hat{f}(orb_N(s)) = \hat{f}(orb_N(s'))$ for some $orb_N(s)$ and $orb_N(s')$ in $S\setminus\setminus N$. Then $orb_N(f(s)) = orb_N(f(s'))$ and so $nf(s) = f(s')$ for some $n \in N$. Since f respects the $AG-$action, $nf(s) = f(s')$ implies that $f(ns) = f(s')$. Then $ns = s'$ by the injectivity of f, and so $orb_N(s) = orb_N(s')$. Thus $\hat{f}$ is injective.

(3) $\hat{f}$ is surjective because f is surjective.

(4) It is clear that $\hat{f}$ preserves the $A(G/N)-$action because f preserves the $AG-$action.

Hence we proved that $(S\setminus\setminus N) \simeq_{A(G/N)} (T\setminus\setminus N)$.

(ii) Obvious.

(iii) Suppose S is a transitive $A-$fibred $G-$set. Take any two elements $orb_N(s)$ and $orb_N(s')$ from $S \setminus \setminus N$. Since S is transitive, there is an $ag \in AG$ such that $ags = s'$. But then $(a(gN))orb_N(s) = orb_N(s')$ implying that $S \setminus \setminus N$ is a transitive $A-$fibred $G/N-$set.

□

Hence, by 6.15 we have a well-defined map, called the orbit map, $Orb^G_N : B(A,G) \to B(A,G/N)$ given by $Orb^G_N([S]) = [S \setminus \setminus N]$ for all $A-$fibred $G-$set S.

Remark 6.16 *$Orb^G_N : B(A,G) \to B(A,G/N)$ is a $\mathbb{Z}-$module homomorphism.*

Proof: It follows from 6.15.

□

Remark 6.17 *For any $(V,\nu) \in ch(A,G)$;*

$$(A_\nu G/V) \setminus \setminus N = \begin{cases} \textit{empty} & \textit{if} \quad V \cap N \not\leq Ker\nu \\ \textit{consist of all } N\textit{-orbits} & \textit{if} \quad V \cap N \leq Ker\nu. \end{cases}$$

Proof: Remember that $A_\nu G/V = AG/\triangle_{(V,\nu)}$ where $\triangle_{(V,\nu)} = \{\nu(v^{-1})v : v \in V\}$. Put $\triangle = \triangle_{(V,\nu)}$. In chapter 2 we showed that $stab_G(Aag\triangle) = {}^gV$ and $\nu_{ag\triangle} = {}^g\nu$. Hence;

$$(A_\nu G/V) \setminus \setminus N = \{orb_N(ag\triangle) : ag \in AG, {}^gV \cap N \leq Ker({}^g\nu)\}.$$

Since $Ker({}^g\nu) = {}^g(Ker\nu)$, using the normality of N we get

$$(A_\nu G/V) \setminus \setminus N = \{orb_N(ag\triangle) : ag \in AG, V \cap N \leq Ker\nu\}$$

which proves the desired result.

□

Let $N \trianglelefteq G$ and $V \leq G$. For any $\nu \in Hom(V,A)$ such that $V \cap N \leq Ker\nu$, we write $\hat{\nu}$ for the group homomorphism $(NV)/N \to A$ given by $\hat{\nu}(nvN) = \nu(v)$ for all $nvN \in (NV)/N$. Note that $\hat{\nu}$ is well-defined.

Remark 6.18 *Let $N \trianglelefteq G$ and $(V,\nu) \in ch(A,G)$. Then we have*

$$Orb_N^G([A_\nu G/V]) = \begin{cases} [A_{\hat{\nu}} \frac{(G/N)}{((NV)/N)}], & V \cap N \leq Ker\nu \\ 0, & V \cap N \nleq Ker\nu. \end{cases}$$

Proof: If $V \cap N \nleq Ker\nu$, from 6.17 $Orb_N^G([A_\nu G/V]) = 0$.

Suppose $V \cap N \leq Ker\nu$. $Orb_N^G([A_\nu G/V]) = [(A_\nu G/V) \setminus \setminus N]$ is a transitive $A-$fibred $G/N-$set from 6.15(iii). Also from 6.17, $orb_N(\triangle) \in (A_\nu G/V) \setminus \setminus N$ where $\triangle = 1.1\triangle \in AG/\triangle = A_\nu G/V$ and $\triangle = \{\nu(v^{-1})v : v \in V\}$. So by the transitivity of $(A_\nu G/V) \setminus \setminus N$;

$(A_\nu G/V) \setminus \setminus N = orb_{A(G/N)}(orb_N(\triangle)) \simeq_{A(G/N)} \frac{A(G/N)}{stab_{A(G/N)}(orb_N(\triangle))}$. Now, $a(gN) \in stab_{A(G/N)}(orb_N(\triangle))$ if and only if $orb_N(ag\triangle) = orb_N(\triangle)$ which is to say that $ag\triangle = n\triangle$ for some $n \in N$. But then the definition of $\triangle$ yields; $ag\triangle = n\triangle$ if and only if $n^{-1}g \in V$ and $\nu((n^{-1}g)^{-1}) = a$, or equivalently $a(gN) \in \{\hat{\nu}((gN)^{-1})(gN) : gN \in (NV)/N\} = \triangle_{((NV)/N,\hat{\nu})}$. Thus,

$(A_\nu G/V) \setminus \setminus N \simeq_{A(G/N)} \frac{A(G/N)}{\triangle_{((NV)/N,\hat{\nu})}} = A_{\hat{\nu}} \frac{(G/N)}{(NV)/N}$.

□

For the orbit map, we could not find the images of the primitive idempotents of $\mathbb{C}B(A,G)$.

There are three more maps, for two of which the images of primitive idempotents of $\mathbb{C}B(A,G)$ were found in [1]. The remaining one is the conjugation map that is very trivial. For this reason in the following three sections we give the definitions and results without proving them.

6.4 The Conjugation Map

Let $F \leq G$ and $g \in G$. For any $A-$fibred $F-$set $S = AX$, we define $g-$conjugate ${}^gS$ of S as the $A-$fibred ${}^gF-$set ${}^gS = S$ with the A^gF-action: $(a^gf)s = (af)s$ for all $a^gf \in A^gF$ and $s \in {}^gS$. Then we have a well-defined map, called the conjugation map,

$Con^g_F : B(A,F) \to B(A,{}^gF)$ given by $Con^g_F([S]) = [{}^gS]$ for any $A-$fibred $F-$set S.

Remark 6.19 *(i)* $Con^g_F : B(A,F) \to B(A,{}^gF)$ *is a* $\mathbb{Z}-$*algebra isomorphism.*
(ii) $Con^g_F([A_\nu F/V]) = [A_{{}^g\nu}{}^gF/{}^gV]$.
(iii) $Con^g_F(e^F_{K,k}) = e^{{}^gF}_{{}^gK,{}^gk}$.

Proof: Obvious.

$\square$

6.5 The Restriction Map

Let $F \leq G$, and $S = AX$ be an $A-$fibred $G-$set. By restricting the $AG-$action to AF, we get an $A-$fibred $F-$set $res^G_F(S) = S$. Then we have a well-defined map, called the restriction map,
$Res^G_F : B(A,G) \to B(A,F)$ given by $Res^G_F([S]) = [res^G_F(S)]$ for all $A-$fibred $G-$set $S = AX$.

Remark 6.20 *(i)* $Res^G_F : B(A,G) \to B(A,F)$ *is a* $\mathbb{Z}-$*algebra homomorphism.*
(ii)

$$Res^G_F([A_\nu G/V]) = \sum_{FgV \subseteq G} [A_{{}^g\nu}F/F \cap {}^gV].$$

(iii)

$$Res^G_F(e^G_{K,k}) = \sum_{(H,h) \in_F el(A,F), (H,h) =_G (K,k)} e^F_{H,h}.$$

Proof: See [1].

$\square$

6.6 The Induction Map

Let $H \leq G$ and $S = AX$ be an $A-$fibred $H-$set. Then the cartesian product $AG \times S$ becomes an $A-$fibred $H-$set with the action: $(ah, (bg, s)) \mapsto (a^{-1}bh^{-1}g, ahs)$ for all $ah \in AH$, $bg \in AG$ and $s \in S$.
Let $AG \times_{AH} S$ denote the set of $AH-$orbits of the $AH-$set $AG \times S$. That is,

$$AG \times_{AH} S = \{orb_{AH}(ag, s) : (ag, s) \in AG \times S\}.$$

Then $AG \times_{AH} S$ becomes an $A-$fibred $G-$set with the $AG-$action:
$(ag, orb_{AH}(bh, s)) \mapsto orb_{AH}(abgh, s)$ for all $orb_{AH}(bh, s) \in AG \times_{AH} S$ and $ag \in AG$.
So we have a well-defined map, called the induction map,
$Ind_H^G : B(A, H) \to B(A, G)$ given by $Ind_H^G([S]) = [AG \times_{AH} S]$ for any $A-$fibred $H-$set $S = AX$.

Remark 6.21 *(i)* $Ind_H^G : B(A, H) \to B(A, G)$ *is a* $\mathbb{Z}-$*module homomorphism.*
(ii) $Ind_H^G([A_\nu H/V]) = [A_\nu G/V]$.
(iii)

$$Ind_H^G(e_{K,k}^H) = \frac{|N_G(K, k)|}{|N_H(K, k)|} e_{K,k}^G.$$

Proof: Remember that $el(A, G)$ is a $G-$set by conjugation. The notations in (iii) are $N_H(K, k) = stab_H((K, k))$ and $N_G(K, k) = stab_G((K, k))$. For the proof see [1].

□

Chapter 7

Prime Ideals Of $B(A, G)$

In this chapter we will find the prime ideals of the monomial Burnside rings and try to get some consequences about the primitive idempotents of the monomial Burnside rings tensored over $\mathbb{Z}$ with an integral domain of characteristic 0. Some of the results we are going to obtain are already obtained by Dress [9] and Barker [1].

In [1], Barker found the primitive idempotents of $\mathbb{C}B(A, G)$ and gave a formula expressing the primitive idempotents of $\mathbb{C}B(A, G)$ in terms of the transitive basis of $\mathbb{C}B(A, G)$.

In [9], Dress found the prime spectrum of the monomial Burnside rings and gave some consequences including his celebrated characterization of solvable groups.

As stated in the introduction Dress introduced the monomial Burnside rings in [9] but his monomial Burnside ring is more general than the one we are considering here. Because of the full generality of the assumptions in [9], we follow a different approach some of whose parts inspired from [9].

Let ζ be a primitive n^{th} root of unity and A be a cyclic group of order n. So we can assume that $A \leq D^*$ where $D = \mathbb{Z}[\zeta]$. We first find the prime ideals of $DB(A, G)$. Some of the results that we will obtain for $DB(A, G)$ hold for $RB(A, G)$ where R is a ring which is more general than D. After finding the prime ideals and the spectrum of $DB(A, G)$, and some consequences we state (if possible) our results for $RB(A, G)$ where R is a ring satisfying weaker assumptions than D satisfies. To do so we need to remember some theory about integral ring extensions, prime ideals of product rings and Dedekind domains. Our rings will be commutative with identity elements.

A ring extension of a ring R is a ring S containing R as a unital subring. We write $R \leq S$ to denote that S is a ring extension of R.

Let $R \leq S$ be a ring extension, an element $s \in S$ is said to be integral over R if $f(s) = 0$ for some monic polynomial $f(x) \in R[x]$.

A ring extension $R \leq S$ is said to be integral if every element of S is integral over R.

The following six theorems are all well-known (See [11] and [13]).

Theorem 7.1 *Let $R \leq S$ be a ring extension and $s \in S$. The following conditions are equivalent;*
(i) s is integral over R.
(ii) $R[s]$ is a finitely generated $R-$module.
(iii) There exists a faithful $R[s]-$module which is finitely generated $R-$module.

Theorem 7.2 *Let $R \leq S$ be a ring extension. If S is finitely generated $R-$module, then $R \leq S$ is an integral extension.*

Theorem 7.3 *Let $R \leq S$ be a ring extension. For any prime ideal P of S, $R \cap P$ is a prime ideal of R.*

For a ring extension $R \leq S$, not all prime ideals of R is of the form $R \cap P$ where P is a prime ideal of S. However if $R \leq S$ is an integral extension then prime ideals of R are of the form $R \cap P$ where P is a prime ideal of S. Indeed,

Theorem 7.4 *Let $R \leq S$ be an integral ring extension. Then we have;*
(i)(Lying Over) Let P be a prime ideal of R. Then for any ideal I of S such that $R \cap I \subseteq P$ there exists a prime ideal Q of S such that $I \subseteq Q$ and $R \cap Q = P$. In particular, for any prime ideal P of R there exists a prime ideal Q of S with $R \cap Q = P$.

(ii)(Going Up) Given prime ideals $P \subsetneqq P_0$ of R and Q of S with $R \cap Q = P$, there exists a prime ideal Q_0 of S satisfying $Q \subsetneqq Q_0$ and $R \cap Q_0 = P_0$.
(iii)(Going Down) Given prime ideals $P \supsetneqq P_0$ of R and Q of S with $R \cap Q = P$, there exists a prime ideal Q_0 of S satisfying $Q \supsetneqq Q_0$ and $R \cap Q_0 = P_0$.
(iv)(Incomparability) Two different prime ideals of S having the same intersection with R cannot be comparable.
(v)(Maximality) Let P be a prime ideal of R and Q of S with $R \cap Q = P$. Then Q is a maximal ideal of S if and only if P is a maximal ideal of R.
(vi)(Minimality) Let P be a prime ideal of R and Q of S with $R \cap Q = P$. Then Q is a minimal prime ideal of S if and only if P is a minimal prime ideal of R.

Theorem 7.5 *Let R_i, $i \in I$, be a family of rings and π_i be the projection of the product ring $\prod_{i\in I} R_i$ onto the i^{th} term R_i. Then any prime ideal of $\prod_{i\in I} R_i$ is of the form $\pi_i^{-1}(P)$ for some $i \in I$ and prime ideal P of R_i. Moreover, for any $i \in I$ and prime ideal P of R_i $\pi_i^{-1}(P)$ is a prime ideal of $\prod_{i\in I} R_i$.*

An integral domain R is called Dedekind domain if the following three conditions hold:
(1) R is Noetherian,
(2) R is integrally closed,
(3) Every nonzero prime ideal of R is maximal.

Theorem 7.6 *Let R be a Dedekind domain. Then*
(i) If I is an ideal of R, then R/I is a principal ideal ring,
(ii) For any ideal I of R and nonzero $a \in I$, there exists $b \in I$ such that $I = (a, b)$,
(iii) For any prime ideal P of R the localized ring R_P is a principal ideal domain.

Let $K = \mathbb{Q}[\zeta]$ and we are assuming that $A \leq D^*$ where $D = \mathbb{Z}[\zeta]$. Note that D is a Dedekind domain and K is its field of fractions. For any $(H,h) \in_G el(A,G)$ we have a ring epimorphism

$$S^G_{H,h} : DB(A,G) \longrightarrow D \qquad \text{given} \qquad \text{by} \qquad [A_\nu G/V] \mapsto \sum_{gV \subseteq G, H \leq {}^gV} {}^g\nu(h),$$

and the product map $\psi = \prod_{(H,h)\in_G el(A,G)} S^G_{H,h}$ is injective. For any nonzero prime ideal P of D let π_P be the canonical epimorphism from D to D/P. We write $I(H,h,0)$ and $I(H,h,P)$ for the kernels of the maps $S^G_{H,h}$ and $\pi_P \circ S^G_{H,h}$ respectively. Hence,

$$I(H,h,0) = \{x \in DB(A,G) : S^G_{H,h}(x) = 0\},$$

$$I(H,h,P) = \{x \in DB(A,G) : S^G_{H,h}(x) \in P\}.$$

Remark 7.7 *Let P be a nonzero prime ideal of D and $(H,h) \in_G el(A,G)$. Then*
(i) $DB(A,G)/I(H,h,0) \simeq D$,
(ii) $DB(A,G)/I(H,h,P) \simeq D/P$,
(iii) $I(H,h,0)$ and $I(H,h,P)$ are prime ideals of $DB(A,G)$.

Proof: Because D and D/P are domains and the maps $S^G_{H,h}$, and $\pi_P \circ S^G_{H,h}$ are epimorphisms, the result follows.

□

Theorem 7.8 *(i) Any prime ideal of $DB(A,G)$ is of the form $I(H,h,0)$ or $I(H,h,P)$ for some nonzero prime ideal P of D and $(H,h) \in_G el(A,G)$.*
(ii) If P is a nonzero prime ideal of D, then $I(H,h,P)$ is a maximal ideal of $DB(A,G)$ for any $(H,h) \in_G el(A,G)$.
(iii) For any $(H,h) \in_G el(A,G)$, $I(H,h,0)$ is a minimal prime ideal of $DB(A,G)$.
(iv) For any nonzero prime ideal P of D and $(H,h) \in_G el(A,G)$ we have
$I(H,h,P) \cap D = P$, $I(H,h,0) \cap D = 0$ and $I(H,h,0) \subsetneqq I(H,h,P)$.
(v) For any $(H,h),(K,k) \in_G el(A,G)$ and prime ideals (possibly 0) P,Q of D
$I(H,h,P) = I(K,k,Q)$ implies that $P = Q$.
(vi) Let $(H,h),(K,k) \in_G el(A,G)$. Then
$I(H,h,0) = I(K,k,0)$ if and only if $(H,h) =_G (K,k)$.

Proof: (i) We know that the map ψ is injective. Hence we can write $DB(A,G) \leq \prod_{(H,h)\in_G el(A,G)} D$. Since both rings are finitely generated as

$D-$modules, the ring extension $DB(A,G) \leq \prod_{(H,h)\in_G el(A,G)} D$ is an integral extension.

Consequently by 7.4; any prime ideal of $DB(A,G)$ is of the form $\psi^{-1}(I)$ where I is a prime ideal of the product ring $\prod_{(H,h)\in_G el(A,G)} D$. Then by 7.5; $I = \pi^{-1}_{(H,h)}(P)$ for some $(H,h) \in_G el(A,G)$ and prime ideal P of D. Thus any prime ideal of $DB(A,G)$ is of the form

$$\psi^{-1}(\pi^{-1}_{(H,h)}(P)) = (S^G_{H,h})^{-1}(P).$$

Note that

if $P = 0$, then $(S^G_{H,h})^{-1}(P) = Ker(S^G_{H,h}) = I(H,h,0)$, and

if $P \neq 0$, then $(S^G_{H,h})^{-1}(P) = Ker(\pi_P \circ S^G_{H,h}) = I(H,h,P)$.

(ii) and (iii) Since D is a Dedekind domain, its nonzero prime ideals are maximal. Hence for any nonzero prime ideal P of D $\pi^{-1}_{(H,h)}(P)$ is a maximal ideal, and $\pi^{-1}_{(H,h)}(0)$ is a minimal prime ideal of $\prod_{(H,h)\in_G el(A,G)} D$. Now from 7.4 ((Maximality) and (Minimality)), the results follow.

(iv) Let P be a nonzero prime ideal of D. Since $I(H,h,P)$ is the kernel of the $D-$linear ring epimorphism

$$\pi_P \circ S^G_{H,h} : DB(A,G) \to D \to D/P$$

we have $P \subseteq I(H,h,P) \cap D$. Note that $I(H,h,P) \cap D$ is a prime ideal of D because $D \leq DB(A,G)$. As D is a Dedekind domain, P is a maximal ideal of D. Also it is clear that $I(H,h,P) \cap D \neq D$. Thus, maximality of P implies that $I(H,h,P) \cap D = P$.

The ring extension $D \leq DB(A,G)$ is an integral extension and $I(H,h,0)$ is a minimal prime ideal of $DB(A,G)$. Then by 7.4(Minimality) $I(H,h,0) \cap D$ must be a minimal prime ideal of D. Being Dedekind domain, 0 is the only minimal prime ideal of D.

It is clear that $I(H,h,0) \subseteq I(H,h,P)$. Because $S^G_{H,h}$ is $D-$linear, $S^G_{H,h}(P) = P$. As a result, $P \subseteq I(H,h,P)$ but $P \nsubseteq I(H,h,0)$.

(v) By part (iv).

(vi) Suppose $I(H,h,0) = I(K,k,0)$. Any primitive idempotents of $\mathbb{C}B(A,G)$ lies in $KB(A,G)$. So for $e^G_{H,h}$ we can find a nonzero $d \in D$(=product of denominators) such that $de^G_{H,h} \in DB(A,G)$. Then $S^G_{H,h}(de^G_{H,h} - d) = 0$. Hence $de^G_{H,h} - d \in$

$I(H,h,0) = I(K,k,0)$. We must have $S^G_{K,k}(de^G_{H,h} - d) = 0$. However $S^G_{K,k}(de^G_{H,h} - d) = 0$ if and only if $(H,h) =_G (K,k)$ (otherwise it is equal to $-d$; but in D, $-d$ cannot be equal to 0 since D is of characteristic 0).

$\square$

Lemma 7.9 *Let P be a nonzero prime ideal of D and $(H,h),(K,k) \in el(A,G)$. Then*
$I(H,h,0) \subseteq I(K,k,P)$ if and only if $I(H,h,P) = I(K,k,P)$.

Proof: ($\Rightarrow$) Suppose $I(H,h,0) \subseteq I(K,k,P)$. Take any $x \in I(H,h,P)$. Then $S^G_{H,h}(x) = d \in D = \mathbb{Z}[\zeta]$ and $d \in P$. Because $S^G_{H,h}$ is a $D-$linear ring epimorphism from $DB(A,G)$ to D, we have $S^G_{H,h}(x-d) = 0$ and so $x - d \in I(H,h,0) \subseteq I(K,k,P)$. Then $S^G_{K,k}(x-d) = S^G_{K,k}(x) - d \in P$. As $d \in P$, we must have $S^G_{K,k}(x) \in P$. Consequently, $x \in I(K,k,P)$. So we proved that $I(H,h,P) \subseteq I(K,k,P)$. But then maximality of $I(H,h,P)$ implies that $I(H,h,P) = I(K,k,P)$.
($\Leftarrow$) It is clear from 7.8(iv).

$\square$

The next lemma states that the maps $\pi_P \circ S^G_{H,h}$ and $\pi_P \circ S^G_{K,k}$ are equal if and only if their kernels are equal.

Lemma 7.10 *For a nonzero prime ideal P of D let π_P be the canonical epimorphism from D onto D/P. Suppose $(H,h),(K,k) \in el(A,G)$ and $I(H,h,P) = I(K,k,P)$. Then $\pi_P \circ S^G_{H,h} = \pi_P \circ S^G_{K,k}$.*

Proof: Since $I(H,h,P) = I(K,k,P)$, for any $x \in DB(A,G)$, it follows that $S^G_{H,h}(x) \in P$ if and only if $S^G_{K,k}(x) \in P$.
Take any $x \in DB(A,G)$ we know that both or none of $S^G_{H,h}(x)$ and $S^G_{K,k}(x)$ belong to P. If they are in P, then $\pi_P \circ S^G_{H,h}(x)$ and $\pi_P \circ S^G_{K,k}(x)$ are both equal to 0. Assume that they are not in P. Then write $S^G_{H,h}(x) = y_1$ and $S^G_{K,k}(x) = y_2$ for some $y_1, y_2 \in D$. Because D/P is a field there exists a $z \in D$ with $z \notin P$ such that

$$(z+P)(y_1+P) = (y_2+P).$$

Now $zy_1 - y_2 \in P$ and note that $zx - y_2 \in DB(A,G)$ with $S^G_{H,h}(zx - y_2) = zy_1 - y_2 \in P$. Thus we must also have $S^G_{K,k}(zx - y_2) = zy_2 - y_2 \in P$. But then from $zy_1 - y_2 \in P$ and $zy_2 - y_2 \in P$ we get

$$(zy_1 - y_2) - (zy_2 - y_2) = z(y_1 - y_2) \in P.$$

Since P is a prime ideal and $z \notin P$ it follows that $y_1 - y_2 \in P$. Therefore;
for any $x \in DB(A,G)$ $S^G_{H,h}(x) - S^G_{K,k}(x) \in P$. That is to say $\pi_P \circ S^G_{H,h} = \pi_P \circ S^G_{K,k}$.

□

We will study the prime spectrum of $DB(A,G)$, written as $Spec(DB(A,G))$. So we begin by some facts about spectrum of commutative rings (see [7]). Let R be a commutative ring. $Spec(R)$ is defined to be the set of prime ideals of R. For any ideal I of R we put $V(I) = \{P \in Spec(R) : I \subseteq P\}$. We can define a topology on $Spec(R)$ by calling sets of the form $V(I)$ closed. The closure of a point $P \in Spec(R)$ is $\overline{\{P\}} := \bar{P} = V(P)$. Suppose R is a Noetherian ring. Then two prime ideals P, Q of R are in the same connected component of $Spec(R)$ if and only if there is a sequence $P_1, ..., P_n$ of minimal prime ideals of R such that $P \in \bar{P}_1, Q \in \bar{P}_n, \bar{P}_i \cap \bar{P}_{i+1} \neq \varnothing$ for $i = 1, 2, ..., n-1$.
Being in the same connected component of $Spec(R)$ forms an equivalence relation on $Spec(R)$. We write $P \sim Q$ if the prime ideals P, Q of R are in the same connected component of $Spec(R)$.
For a Noetherian ring R, we have a bijective correspondence between the connected components of $Spec(R)$ and the primitive idempotents of R. The connected component of $Spec(R)$ corresponding to a primitive idempotent e of R consists of all prime ideals of R containing $1 - e$.

Recall that for $H \leq G$, $O(H) = \cap_{\nu \in Hom(H,A)} Ker\nu$. For any automorphism α of H and $\nu \in Hom(H,A)$, $\nu \circ \alpha \in Hom(H,A)$. So $O(H)$ is a characteristic subgroup of H. For any $g \in G$ and $\nu \in Hom(H,A)$, ${}^g\nu \in Hom({}^gH, A)$ implying that $O({}^gH) = {}^gO(H)$. We write $\bar{H}$ for $H/O(H)$, $\bar{h}$ for $hO(H)$ and $N(H)$ for $N_G(H)/H$.

$N_G(H)$ acts on $\bar{H}$ by conjugation;
$(gH, hO(H)) := {}^g\bar{h} \mapsto ghg^{-1}O(h)$, for $gH \in N_G(H)$ and $\bar{h} \in \bar{H}$.
The above action of $N_G(H)$ on $\bar{H}$ respects the multiplication of $\bar{H}$. That is;
${}^g(\overline{h_1 h_2}) = ({}^g\bar{h_1})({}^g\bar{h_2})$.
Note that $H \leq N_G(H)$ fixes $\bar{H}$ pointwise and so $\bar{H}$ is also $N(H)-$set with respect to conjugation action.

Lemma 7.11 *Let P be a nonzero prime ideal of D such that D/P has characteristic p. For $(H, h_1), (H, h_2) \in el(A, G)$ with $\overline{h_1^{-1}h_2} \in \bar{H}$ is of $p-$power order, we have $I(H, h_1, P) = I(H, h_2, P)$.*

Proof: Let $\nu \in Hom(H, A)$. Put $\nu(h_1) = d_1 \in A \leq D^*$ and $\nu(h_2) = d_2 \in A \leq D^*$. Since $\overline{h_1^{-1}h_2}$ has $p-$power order in $\bar{H}$, we must have $d_1^{p^m} = d_2^{p^m}$ for some natural number m. For any $d \in D$, let $\bar{d} = d + P = \pi_P(d)$ where π_P is the canonical ring epimorphism from D to D/P. Because D/P has characteristic p,
$\bar{0} = \bar{d_1}^{p^m} - \bar{d_2}^{p^m} = (\bar{d_1} - \bar{d_2})^{p^m}$.
So, $\bar{0} = \bar{d_1} - \bar{d_2}$ since a field has no nonzero nilpotent elements. Thus we have $\nu(h_1) + P = \nu(h_2) + P$ for any $\nu \in Hom(H, A)$. Therefore
$\pi_P \circ S^G_{H,h_1}([A_\nu G/V]) = \pi_P \circ S^G_{H,h_2}([A_\nu G/V])$ for any $[A_\nu G/V] \in B(A, G)$. That is; $I(H, h_1, P) = I(H, h_2, P)$.

□

Let G be a group and p be a prime number. For any $g \in G$ there are uniquely determined elements g_p and $g_{p'}$ of G satisfying; $g = g_p g_{p'} = g_{p'} g_p$, the order of g_p is a $p-$power, and the order of $g_{p'}$ is not divisible by p. Indeed, let $p^k m$ be the order of g where $(p, m) = 1$. Then there are integers u and v such that $p^k u + mv = 1$, and so $g = (g^{mv})(g^{p^k u}) = (g^{p^k u})(g^{mv})$ where $g_p = g^{mv}$ and $g_{p'} = g^{p^k u}$.
By the uniqueness of such representations, for any $H \leq G$ and $h \in H$ we have $h_p O(H) = (hO(H))_p$ and $h_{p'} O(H) = (hO(H))_{p'}$. 7.11 implies that $I(H, h, P) = I(H, h_{p'}, P)$ for any nonzero prime ideal of D such that D/P has characteristic p.

Lemma 7.12 *Let P be a prime ideal (possibly 0) of D and $(H, h) \in el(A, G)$.*

Then the prime ideals $I(H,h,P)$ and $I(H,1,0)$ are in the same connected component of $Spec(DB(A,G))$.

Proof: Since $\bar{H}$ is an abelian group, it is the direct product of its Sylow subgroups;

$$\bar{H} = S_{p_1}(\bar{H}) \times ... \times S_{p_r}(\bar{H}).$$

Then $\bar{h}$ can be written uniquely as $\bar{h} = \bar{t}_1...\bar{t}_r$ where $\bar{t}_i \in S_{p_i}(\bar{H})$ for $i = 1, ..., r$. Chose prime ideals $P_1, ..., P_r$ of D such that D/P_i has characteristic p_i. It is possible to choose such prime ideals because $D = \mathbb{Z}[\zeta]$.
For $i = 1, 2, ..., r-1$; $(\bar{t}_i...\bar{t}_r)^{-1}(\bar{t}_{i+1}...\bar{t}_r) \in S_{p_i}(\bar{H})$. So by 7.11 we have

$$I(H,h,P_1) = I(H,t_2t_3...t_r,P_1), I(H,t_2t_3...t_r,P_2) = I(H,t_3t_4...t_r,P_2),$$
$$I(H,t_3...t_r,P_3) = I(H,t_4...t_r,P_3), ..., $$
$$I(H,t_{r-1}t_r,P_{r-1}) = I(H,t_r,P_{r-1}), I(H,t_r,P_r) = I(H,1,P_r).$$

Also note that for any prime ideals $\mathfrak{A}, \mathfrak{B}$ (possibly 0) of D and $(K,k) \in el(A,G)$ we have $I(K,k,\mathfrak{A}) \supseteq I(K,k,0) \subseteq I(K,k,\mathfrak{B})$.
Thus, $I(K,k,\mathfrak{A})$ and $I(K,k,\mathfrak{B})$ are in the same connected component of $Spec(DB(A,G))$. Recall that being in the same connected component is an equivalence relation for which we use the notation $\sim$. Now what we have is

$$I(H,h,P) \sim I(H,h,P_1) = I(H,t_2t_3...t_r,P_1) \sim I(H,t_2t_3...t_r,P_2) =$$
$$I(H,t_3t_4...t_r,P_2) \sim I(H,t_3...t_r,P_3) = I(H,t_4...t_r,P_3) \sim$$
$$I(H,t_4...t_r,P_4) = I(H,t_5...t_r,P_4) \sim I(H,t_5...t_r,P_5) = ...$$
$$\sim I(H,t_{r-1}t_r,P_{r-1}) = I(H,t_r,P_{r-1}) \sim I(H,t_r,P_r) = I(H,1,P_r) \sim I(H,1,0).$$

Hence, $I(H,h,P) \sim I(H,1,0)$.

□

Lemma 7.13 *Let P be a nonzero prime ideal of D such that D/P has characteristic p. Let $H \trianglelefteq K$ be such that K/H is a $p-$group. Then we have $I(H,1,P) = I(K,1,P)$.*

Proof: Take any $[A_\nu G/V] \in B(A,G)$ and compute;

$$S^G_{H,1}([A_\nu G/V]) = \sum_{gV \subseteq G, H \leq {}^gV} 1;\ S^G_{K,1}([A_\nu G/V]) = \sum_{gV \subseteq G, K \leq {}^gV} 1.$$

For any $G-$set S and $H \leq G$, let S^H denote the $H-$fixed points of S. We have

$$S^G_{H,1}([A_\nu G/V]) - S^G_{K,1}([A_\nu G/V]) = |G/V^H - G/V^K|.$$

The set $G/V^H - G/V^K$ is a $K/H-$set whose $K/H-$orbits are nontrivial. Since K/H is a $p-$group, $|G/V^H - G/V^K|$ is divisible by p. Thus,

$$\pi_P \circ S^G_{H,1}([A_\nu G/V]) - \pi_P \circ S^G_{K,1}([A_\nu G/V]) = 0$$

where π_P is the canonical ring epimorphism from D to D/P. Therefore $\pi_P \circ S^G_{H,1} = \pi_P \circ S^G_{K,1}$ implying that $I(H,1,P) = I(K,1,P)$.

$\square$

Let $H \trianglelefteq K$ with K/H is a $p-$group. For any prime ideals $\mathfrak{A}$ and $\mathfrak{B}$ (possibly 0) of D we have $I(H,1,\mathfrak{A}) \sim I(K,1,\mathfrak{B})$. Because; choosing a prime ideal P of D such that D/P has characteristic p we have by 7.13
$I(H,1,\mathfrak{A}) \sim I(H,1,P) = I(K,1,P) \sim I(K,1,\mathfrak{B})$.
For a group G, let $S(G)$ denote the unique minimal normal subgroup of G such that $G/S(G)$ is solvable. Indeed; $S(G)$ is the intersection of all normal subgroups of G with solvable quotient groups. Thus, $S(G)$ is a characteristic subgroup of G.
For $g \in G$ and $H \leq G$, we have $S({}^gH) = {}^gS(H)$ and $S(S(H)) = S(H)$.
A group G is called perfect if $S(G) = G$, equivalently if $D(G) = G$ where $D(G)$ is the commutator subgroup of G.

Lemma 7.14 *Let P be a prime ideal (possibly 0) of D and $(H,h) \in el(A,G)$. Then the prime ideals $I(H,h,P)$ and $I(S(H),1,0)$ of $DB(A,G)$ are in the same connected component of $Spec(DB(A,G))$.*

Proof: Because $H/S(H)$ is solvable, it has a subnormal series whose factor groups have prime orders;

$$H = H_1 \trianglerighteq H_2 \trianglerighteq H_3 \trianglerighteq \ldots \trianglerighteq H_{n-1} \trianglerighteq H_n = S(H),$$

where for each i $|H_i/H_{i+1}|$ =a prime number p_i. Since $D = \mathbb{Z}[\zeta]$, we can choose prime ideals $P_1, ..., P_n$ of D such that D/P_i has characteristic p_i.
(A): By 7.13;

$$I(H, 1, P_1) = I(H_2, 1, P_1), I(H_2, 1, P_2) = I(H_3, 1, P_2),$$

$$I(H_3, 1, P_3) = I(H_4, 1, P_3), I(H_4, 1, P_4) = I(H_5, 1, P_4), ...,$$

$$I(H_{n-2}, 1, P_{n-2}) = I(H_{n-1}, 1, P_{n-2}),$$

$$I(H_{n-1}, 1, P_{n-1}) = I(H_n, 1, P_{n-1}) = I(S(H), 1, P_{n-1}).$$

(B): By 7.12;

$$I(H, h, P) \sim I(H, 1, 0) \sim I(H, 1, P_1), I(S(H), 1, P_{n-1}) \sim I(S(H), 1, 0).$$

(C): Also we know already that $I(K, k, \mathfrak{A}) \sim I(K, k, \mathfrak{B})$ for any prime ideals $\mathfrak{A}, \mathfrak{B}$ of D and $(K, k) \in el(A, G)$.
Now since $\sim$ is an equivalence relation on $Spec(DB(A, G))$, it follows from (A), (B) and (C) that $I(H, h, P) \sim I(S(H), 1, 0)$.

□

Theorem 7.15 *Let P and Q are prime ideals (possibly 0) of D and (H, h), $(K, k) \in el(A, G)$ with $S(H) =_G S(K)$. Then the prime ideals $I(H, h, P)$ and $I(K, k, Q)$ of $DB(A, G)$ are in the same connected component of $Spec(DB(A, G))$.*

Proof: By 7.14;

$$I(H, h, P) \sim I(S(H), 1, 0), I(K, k, Q) \sim I(S(K), 1, 0).$$

But $S(H) =_G S(K)$ implies that $I(S(H), 1, 0) = I(S(K), 1, 0)$.
Since $\sim$ is an equivalence relation on $Spec(DB(A, G))$ we have
$I(H, h, P) \sim I(K, k, Q)$.

□

For a group G and prime number p, let $O^p(G)$ be the unique minimal normal subgroup of G such that $G/O^p(G)$ is a $p-$group. Actually, $O^p(G)$ is the intersection of all normal subgroups of G having $p-$power indexes. So $O^p(G)$ is a characteristic subgroup of G.

Since $p-$groups are solvable, for a group G we have $S(G) \trianglelefteq O^p(G)$.

Being a subgroup of the solvable group $G/S(G)$, $O^p(G)/S(G)$ is solvable, and so $S(O^p(G)) \trianglelefteq S(G)$. As $S(G)$ is a normal subgroup of G and $S(O^p(G))$ is a characteristic subgroup of $S(G)$, we must have $S(O^p(G)) \trianglelefteq G$. Now the group $G/S(O^p(G))$ has a solvable normal subgroup $O^p(G)/S(O^p(G))$ whose quotient group (isomorphic to $G/O^p(G)$) is also solvable. Consequently, $G/S(O^p(G))$ is solvable and $S(G) \leq S(O^p(G))$. Therefore, $S(O^p(G)) = S(G)$ for any group G.

Let $H \trianglelefteq K$ be such that K/H is a $p-$group. Because $O^p(H)$ is a characteristic subgroup of H and H is a normal subgroup of K, we get $O^p(H) \trianglelefteq K$. Then $K/O^p(H)$ is a $p-$group and so $O^p(K) \trianglelefteq O^p(H)$. On the other hand; $O^p(K)$ is a normal subgroup of H of $p-$power index. Thus $O^p(H) \trianglelefteq O^p(K)$. As a result, $O^p(H) = O^p(K)$ for any normal subgroup H of K of $p-$power index.

Lemma 7.16 *Let P be a nonzero prime ideal of D such that D/P has characteristic p, and $(H,h),(K,k) \in el(A,G)$. Suppose $|N_G(H):H|$ and $|N_G(K):K|$ are not divisible by p. Then*
$I(H,h,P) \cap B(G) = I(K,k,P) \cap B(G)$ implies that $H =_G K$.

Proof: For any $T \leq G$, let τ denote the trivial elements of the groups $Hom(T,A)$.

For $[A_\tau G/H],[A_\tau G/K] \in B(G)$ we compute;

$$S^G_{H,h}([A_\tau G/H]) = \sum_{gH\subseteq G, H\leq {}^gH} 1 = |N_G(H):H|,$$

$$S^G_{K,k}([A_\tau G/K]) = \sum_{gK\subseteq G, K\leq {}^gK} 1 = |N_G(K):K|.$$

Since $|N_G(H):H|$ and $|N_G(K):K|$ are not divisible by p;

$$[A_\tau G/H] \notin I(H,h,P), [A_\tau G/K] \notin I(K,k,P).$$

Then from $I(H,h,P) \cap B(G) = I(K,k,P) \cap B(G)$ we have;

$$[A_\tau G/H] \notin I(K,k,P), [A_\tau G/K] \notin I(H,h,P).$$

Hence in particular;

$$S^G_{K,k}([A_\tau G/H]) = \sum_{gH\subseteq G, H\leq^g K} 1 \neq 0, S^G_{H,h}([A_\tau G/K]) = \sum_{gK\subseteq G, K\leq^g H} 1 \neq 0.$$

Therefore, we must have $H \leq_G K \leq_G H$.

$\square$

Lemma 7.17 *Let P be a prime ideal (possibly 0) of D and $(H,h) \in el(A,G)$. Then; $I(H,h,P) \cap B(G) = I(H,1,P) \cap B(G)$.*

Proof: Let τ be the trivial elements of the groups $Hom(T,A)$ for all $T \leq G$. Then for any $[A_\tau G/T] \in B(G)$, $S^G_{H,h}([A_\tau G/T])$ and $S^G_{H,1}([A_\tau G/T])$ are both equal to G/T^H where G/T^H is the set of $H-$fixed points of the $G-$set G/T. So the result follows.

$\square$

Theorem 7.18 *Let P be a nonzero prime ideal of D such that D/P has characteristic p, and $(H,h),(K,k) \in el(A,G)$.*
If $I(H,h,P) \cap B(G) = I(K,k,P) \cap B(G)$, then $O^p(H) =_G O^p(K)$.

Proof: For any subgroup T of G we define a subnormal series;

$$T = T_0 \trianglelefteq T_1 \trianglelefteq T_2 ... \trianglelefteq T_n \trianglelefteq ...$$

where T_{n+1}/T_n is a Sylow $p-$subgroup of $N(T_n) = N_G(T_n)/T_n$ for each n. Since we are considering only finite groups this series must stop in a finite number, say n. Then $T_{n+k} = T_n$ for each natural number k, and so $|N_G(T_n) : T_n|$ is not divisible by p.
For $i = 1,2,...,n-1$; $T_i \trianglelefteq T_{i+1}$ and T_{i+1}/T_i is a $p-$group. Hence by 7.13; $I(T,1,P) = I(T_n,1,P)$.
So finding the above series of H and K, we have

$$H = H_0 \trianglelefteq H_1 \trianglelefteq ... \trianglelefteq H_r;\ K = K_0 \trianglelefteq K_1 \trianglelefteq ... \trianglelefteq K_s.$$

Then $I(H,1,P) = I(H_r,1,P)$, $I(K,1,P) = I(K_s,1,P)$ implying by 7.17 that $I(H_r,1,P) \cap B(G) = I(K_s,1,P) \cap B(G)$.
Since $|N_G(H_r) : H_r|$ and $|N_G(K_s) : K_s|$ are not divisible by p, 7.16 gives that $H_r =_G K_s$.
Recall also that, for groups $M \trianglelefteq N$ with N/M is a $p-$ group we have $O^p(M) = O^p(N)$.
Thus $O^p(H) = O^p(H_1) = ... = O^p(H_r) =_G O^p(K_s) = ... = O^p(K_1) = O^p(K)$.

$\square$

Theorem 7.19 *Let P and Q be prime ideals (possibly 0) of D and $(H,h),(K,k) \in el(A,G)$. If the prime ideals $I(H,h,P)$ and $I(K,k,Q)$ of $DB(A,G)$ are in the same connected component of $Spec(DB(A,G))$, then $S(H) =_G S(K)$.*

Proof: By 7.14;
$I(H,h,P) \sim I(S(H),1,0)$ and $I(K,k,Q) \sim I(S(K),1,0)$.
Since $\sim$ is an equivalence relation, $I(S(H),1,0) \sim I(S(K),1,0)$.
So ($DB(A,G)$ is Noetherian as it is finitely generated $\mathbb{Z}-$module) there is a sequence of minimal prime ideals
$I(T_1,t_1,0),...,I(T_n,t_n,0)$ of $DB(A,G)$ such that
(A) $I(S(H),1,0) \supseteq I(T_1,t_1,0)$,
(B) $I(S(K),1,0) \supseteq I(T_n,t_n,0)$,
(C) Closures of the points $I(T_i,t_i,0)$ and $I(T_{i+1},t_{i+1},0)$ of $Spec(DB(A,G))$ intersect nontrivially for each $i = 1,2,...,n-1$.
From (A), (B) and 7.8(vi) it follows that $(S(H),1) =_G (T_1,t_1)$ and $(S(K),1) =_G (T_n,t_n)$. In particular, $S(H) =_G T_1$ and $S(K) =_G T_n$. Hence $S(H) =_G S(T_1)$ and $S(K) =_G S(T_n)$.
From (C); for each i there is a nonzero prime ideal R_i of D and $(L_i,l_i) \in el(A,G)$ such that
$I(T_i,t_i,0) \subseteq I(L_i,l_i,R_i) \supseteq I(T_{i+1},t_{i+1},0)$.
By 7.9, for each $i = 1,2,...n-1$ we have
$I(T_i,t_i,R_i) = I(T_{i+1},t_{i+1},R_i)$, and it implies from 7.18 that $O^{p_i}(T_i) =_G O^{p_i}(T_{i+1})$ where p_i is the characteristic of D/R_i. But for any group $\mathfrak{E}$ and prime number p,

$S(O^p(\mathfrak{E})) = S(\mathfrak{E})$.
Hence, $S(H) =_G S(T_1) =_G S(T_2) =_G ... =_G S(T_{n-1}) =_G S(T_n) =_G S(K)$.

$\square$

Consider the set $\mathfrak{F} = \{S(H) : H \leq G\}$. Let $=_G$ denote the conjugacy relation on the set of subgroups of G. We know from 7.15 and 7.19 that prime ideals $I(H, h, P)$ and $I(K, k, Q)$ of $DB(A, G)$ are in the same connected component of $Spec(DB(A, G))$ if and only if $S(H) =_G S(G)$. Hence, the number of connected components of $Spec(DB(A, G))$ is equal to the $|\mathfrak{F}/{=_G}|$ = the number of nonconjugate perfect subgroups of G. $\mathfrak{F}$ has only one nonconjugate element if and only if G is solvable. We state some simple consequences of what we proved up to this point in the following corollary which contains also the characterization of solvable groups given first by Dress in [9].

Corollary 7.20 *(i) The connected components of $Spec(DB(A, G))$ are in bijective correspondence with the conjugacy classes of perfect subgroups of G.*
(ii) The number of primitive idempotents of $DB(A, G)$ is equal to the number of nonconjugate perfect subgroups of G.
(iii) G is solvable if and only if $Spec(DB(A, G))$ is connected if and only if 0 and 1 are the only idempotents of $DB(A, G)$.
(iv) If $|(DB(A, G))^*| = 2$, then G is solvable.

Proof: (i) Follows from the above explanation.
(ii) It follows from the bijection between the connected components of the prime ideal spectrum of a Noetherian commutative ring R and its primitive idempotents.
(iii) By the above explanation we readily have that G is solvable if and only if $Spec(DB(A, G))$ is connected. What remains is the content of (ii).
(iv) Suppose G is not solvable. Then $Spec(DB(A, G))$ is not connected and hence $DB(A, G)$ has at least one idempotent e different from 0 and 1. Then since the nontrivial idempotents e and $1 - e$ are orthogonal and sum up to 1 we have a decomposition of $DB(A, G)$;

$DB(A,G) \simeq DB(A,G)e \times DB(A,G)(1-e)$.
Because both e and $1-e$ are different from $0,1$ the rings $DB(A,G)e$ and $DB(A,G)(1-e)$ are of characteristics 0. Then $|(DB(A,G))^*| \geq 2$ and $|(DB(A,G)(1-e))^*| \geq 2$. Hence $|(DB(A,G))^*| \geq 4$ which is a contradiction. Therefore G is solvable.

□

We know the connected components of $Spec(DB(A,G))$, but we do not know much about the equality of two maximal ideals of $DB(A,G)$. We proved that if $I(H,h,P)$ and $I(K,k,Q)$ are equal then $P = Q$ and $O^p(H) =_G O^p(K)$ where p is the characteristic of the field D/P. Before dealing with general case we examine some special cases including $B(G)$, for instance we will show for abelian groups G that the prime ideals $I(H,h,P)$ and $I(O^p(H), h_{p'}, P)$ of $DB(A,G)$ are equal. Note that for a group G, any element g of order coprime to p belongs to $O^p(G)$ because $gO^p(G)$ has order dividing the order of $g \in G$ and $|G/O^p(G)|$. Hence in particular $h_{p'} \in O^p(H)$ for all $h \in H$.

Lemma 7.21 *Let P be a nonzero prime ideal of D such that D/P has characteristic p. Then for any $H, K \leq G$;*
$I(H,1,P) = I(K,1,P)$ *if and only if* $O^p(H) =_G O^p(K)$.

Proof: ($\Rightarrow$) It is 7.18.
($\Leftarrow$) Suppose $O^p(H) =_G O^p(K)$. Then $I(O^p(H),1,P) = I(O^p(K),1,P)$. By 7.13, $I(O^p(H),1,P) = I(H,1,P)$ and $I(O^p(K),1,P) = I(K,1,P)$.
Thus, $I(H,1,P) = I(K,1,P)$.

□

If A is taken to be the trivial group then $DB(A,G)$ reduces to $B(G)$. The results we obtained for $DB(A,G)$ imply all the desired results about the prime ideals of $B(G)$ which is the content of the next theorem. All facts about the prime spectrum of the Burnside rings first obtained by Dress in [8].

Let p be a prime number and $H \leq G$. Define

$$I(H,p) = \{x \in B(G) : S_H^G(x) \equiv 0 \bmod(p)\},$$

$$I(H,0) = \{x \in B(G) : S_H^G(x) = 0\}$$

where S_H^G is the ring epimorphism from $B(G)$ to $\mathbb{Z}$ sending each $G-$set S to $|S^H|$. Note that $S_H^G = S_{H,h}^G|_{B(G)}$ for any $h \in H$.

Theorem 7.22 *Let p be any prime number and H, K be any subgroups of G. Then*
(i) $I(H,p)$ and $I(H,0)$ are prime ideals of $B(G)$. Moreover, any prime ideal of $B(G)$ is one of these forms.
(ii) $I(H,0)$ is a minimal prime ideal, and $I(H,p)$ is a maximal ideal of $B(G)$.
(iii) $I(H,0) \cap \mathbb{Z} = 0$, $I(H,p) \cap \mathbb{Z} = p\mathbb{Z}$.
(iv) Let m, n be prime numbers (possibly 0). Then
$I(H,m) = I(H,n)$ implies $m = n$.
(v) $I(H,0) = I(K,0)$ if and only if $H =_G K$.
(vi) $I(H,p) = I(K,p)$ if and only if $O^p(H) =_G O^p(K)$.
(vii) $I(H,0) \subsetneqq I(H,p)$, and $I(H,0) \subseteq I(K,p)$ if and only if $I(H,p) = I(K,p)$.
(viii) Let m, n be prime numbers (possibly 0). Then the prime ideals $I(H,m)$ and $I(H,n)$ of $B(G)$ are in the same connected component of Spec$(B(G))$ if and only if $S(H) =_G S(K)$.
(ix) The number of primitive idempotents of $B(G)$ is equal to the number of nonconjugate perfect subgroups of G.
(x) G is solvable if and only if Spec$(B(G))$ is connected if and only if 0 and 1 are the only idempotents of $B(G)$.

Proof: Let A be the trivial group. Then $D = \mathbb{Z}$ and $DB(A,G) = B(G)$. Moreover $I(H,h,P)$ and $I(H,h,0)$ reduce to $I(H,p)$ and $I(H,0)$, respectively. Now 7.8 gives (i), (ii), (iii), (iv) and (v); (vii) follows from 7.9; (vi) is 7.21; $(viii)$, (ix) and (x) follow from 7.20.

$\square$

The following result was obtained by Barker also in [1].

Corollary 7.23 *The primitive idempotents of $B(G)$, $B(A,G)$ and $DB(A,G)$ are all the same.*

Proof: Let $R \leq S$ be a ring extension. A primitive idempotent e of R is an idempotent of S. If e is not primitive in S then it must be a sum of primitive idempotents of S. Hence, the number of primitive idempotents of S is more than the number of primitive idempotents of R, and if they have the same number of primitive idempotents then their primitive idempotents must be the same. By 7.20 and 7.22, the number of primitive idempotents of $DB(A,G)$ and $B(G)$ are the same. The result follows because $B(G) \leq B(A,G) \leq DB(A,G)$.

$\square$

Lemma 7.24 *Let $(|A|,|G|)$ be a $p-$power (possibly 1), and P be a nonzero prime ideal of D such that D/P has characteristic p. Then for any (H,h) and $(K,k) \in el(A,G)$;*
$I(H,h,P) = I(K,k,P)$ if and only if $O^p(H) =_G O^p(K)$.

Proof: For any $T \leq G$, $O(T)$ is the minimal normal subgroup of T such that $T/O(T)$ is an abelian group of exponent dividing the order of A. Hence any element of $\bar{T}(= T/O(T))$ has order dividing both $|A|$ and $|G|$. Thus, $\bar{T}$ is a $p-$group. Now by 7.11, $I(H,h,P) = I(H,1,P)$ and $I(K,k,P) = I(K,1,P)$. But then the result follows from 7.21.

$\square$

Corollary 7.25 *Let A have $p-$power order, and P be a nonzero prime ideal of D such that D/P has characteristic p. Then for any $(H,h),(K,k) \in el(A,G)$;*
$I(H,h,P) = I(K,k,P)$ if and only if $O^p(H) =_G O^p(K)$.

Proof: Since $(|A|,|G|)$ is a $p-$power, the result follows from 7.24.

$\square$

Corollary 7.26 *Let G be a $p-$group, and P be a nonzero prime ideal of D such that D/P has characteristic p. Then $I(H,h,P) = I(K,k,P)$ for all $(H,h),(K,k) \in el(A,G)$.*

Proof: It is immediate from 7.24 because for any subgroup T of G, we have $O^p(T) = 1$.

□

Now we want to examine the prime ideals of $B(A,G)$ when it is tensored over some ring of fractions of D. Hence we begin with writing some basic facts about the ring of fractions of a commutative ring R with 1 (See [11]). A subset S of R is called multiplicative if $1 \in S$, and $st \in S$ for all $s,t \in S$. It is called proper if also $0 \notin S$. Given a proper multiplicative subset S of R we define an equivalence relation on $R \times S$ as follows: (a,s) is equivalent to (b,t) if and only if $atu = bsu$ for some $u \in S$. The equivalence class containing (a,s) is denoted by a/s, and the set of equivalence classes , denoted by $S^{-1}R$, becomes a commutative ring with 1 ($0 = 0/1$, $1 = 1/1$)with respect to the operations: $a/s + b/t = (at+bs)/st$ and $(a/s)(b/t) = (ab)/(st)$. We have a ring homomorphism $\iota : R \to S^{-1}R$ given by $\iota(r) = r/1$ for all $r \in R$. This homomorphism is injective if R is an integral domain. For an ideal $\mathfrak{A}$ of $S^{-1}R$, the ideal $\iota^{-1}(\mathfrak{A})$ of R is denoted by $\mathfrak{A}^C$ and called the contraction of $\mathfrak{A}$. For an ideal $\mathfrak{a}$ of R, the ideal $\{a/s \in S^{-1}R : a \in \mathfrak{a}\}$ of $S^{-1}R$ is denoted by $\mathfrak{a}^E$ and called the expansion of $\mathfrak{a}$. We have a bijective correspondence between the prime ideals of $S^{-1}R$ and the prime ideals of R not intersecting S given by $\mathfrak{A} \to \mathfrak{A}^C$, $\mathfrak{a}^E \leftarrow \mathfrak{a}$. Let P be a prime ideal of R. Then $R - P = S$ is a proper multiplicative subset of R and we write R_P for $S^{-1}R$. The ring R_P is called the localization of R at P, and it is a local ring with unique maximal ideal P^E, and also $R_P/P^E \simeq$the quotient field of the integral domain R/P. If R is a Dedekind domain and S is a proper multiplicative subset, then $R \leq S^{-1}R$ is an integral ring extension and the ring $S^{-1}R$ is also a Dedekind domain. Now let $R = D$ and P be a nonzero prime ideal of D such that D/P has characteristic p, we have $\mathbb{Z}_{p\mathbb{Z}} \leq D_P$, $D \leq D_P$, and both are integral ring extensions.

Theorem 7.27 *Let G be a $p-$group, and P be a nonzero prime ideal of D such that D/P has characteristic p. Then*
(i) The rings $D_PB(A,G)$ and $\mathbb{Z}_{p\mathbb{Z}}B(A,G)$ are local.
(ii) 0 and 1 are the only idempotents of the rings $D_PB(A,G)$ and $\mathbb{Z}_{p\mathbb{Z}}B(A,G)$.

Proof: (i) $\prod_{(H,h)\in_G el(A,G)} D_P$ can be seen as an integral ring extension of the ring $D_PB(A,G)$ because the product of the D_P-linear extensions of the maps $S^G_{H,h} : DB(A,G) \to D$, $(H,h) \in_G el(A,G)$, is still injective. Hence any prime ideal of $D_PB(A,G)$ is of the form $I_P(H,h,Q^E) = \{x \in D_PB(A,G) : S^G_{H,h}(x) \in Q^E\}$ where Q is prime ideal of D not intersecting $D-P$. But there are two possibilities for Q, namely 0 and P. Hence maximal ideals of $D_PB(A,G)$ are precisely $I_P(H,h,P^E)$ for $(H,h) \in el(A,G)$.
Let $d_1, d_2 \in D$ be such that $d_1 + P = d_2 + P$ in the field D/P. Then $d_1 - d_2 \in P$ and so $d_1/1 - d_2/1 \in P^E$ implying that $d_1/1 + P^E = d_2/1 + P^E$ in the field $D_P/P^E \simeq D/P$.
Note that $I_P(T,t,P^E)$ is the kernel of the map $\pi_{P^E} \circ S^G_{T,t} : D_PB(A,G) \to D_P \to D_P/P^E$ where π_{P^E} is the canonical ring epimorphism from D_P to D_P/P^E.
By 7.26 $I(H,h,P) = I(K,k,P)$ for all $(H,h),(K,k) \in el(A,G)$,
and so from 7.10 we have $\pi_P \circ S^G_{H,h} = \pi_P \circ S^G_{K,k}$ where π_P is the canonical ring epimorphism from D to D/P.
For any $x = [A_\nu G/V] \in B(A,G)$ we have $S^G_{H,h}(x) + P = S^G_{K,k}(x) + P$. Then $S^G_{H,h}(x)/1 + P^E = S^G_{K,k}(x)/1 + P^E$, and so $\pi_{P^E} \circ S^G_{H,h}(x) = \pi_{P^E} \circ S^G_{K,k}(x)$.
Therefore $I_P(H,h,P^E) = I_P(K,k,P^E)$ for all $(H,h),(K,k) \in el(A,G)$.
Consequently $D_PB(A,G)$ has only one maximal ideal(=it is local).
$\mathbb{Z}_{p\mathbb{Z}}B(A,G) \leq D_PB(A,G)$ is an integral ring extension and hence the maximal ideals of $\mathbb{Z}_{p\mathbb{Z}}B(A,G)$ can be obtained from the maximal ideals of $D_PB(A,G)$ by intersecting with $\mathbb{Z}_{p\mathbb{Z}}B(A,G)$. Thus $\mathbb{Z}_{p\mathbb{Z}}B(A,G)$ must be a local ring.
(ii) Follows from (i) and the bijective correspondence between the primitive idempotents and the connected components of spectrum.

□

Let π be a set of prime numbers and $\mathbb{Z}_{(\pi)}$ be the ring of fractions of $\mathbb{Z}$ with respect to the proper multiplicative subset $\mathbb{Z} - \cup_{p\in\pi} p\mathbb{Z}$. 7.27 has the following obvious generalization.

Theorem 7.28 *Let G be a $\pi-$group. Then $\mathbb{Z}_{(\pi)}B(A, G)$ is a local ring (0 and 1 are the only idempotents of $\mathbb{Z}_{(\pi)}B(A, G)$).*

Theorem 7.29 *Let A have $p-$power order. Then the primitive idempotents of $\mathbb{Z}_{p\mathbb{Z}}B(A, G)$ and $\mathbb{Z}_{p\mathbb{Z}}B(G)$ are the same.*

Proof: Let P be a nonzero prime ideal of D such that D/P has characteristic p. For $(H, h) \in el(A, G)$, let $I_P(H, h, P^E) = \{x \in D_P B(A, G) : S^G_{H,h}(x) \in P^E\}$ and $I_p(H, (p\mathbb{Z})^E) = \{x \in \mathbb{Z}_{p\mathbb{Z}}B(G) : S^G_H(x) \in (p\mathbb{Z})^E\}$. From 7.25 it follows that $I_P(H, h, P^E) = I_P(K, k, P^E)$ if and only if $O^p(H) =_G O^p(K)$, and from 7.22(vi) $I_p(H, (p\mathbb{Z})^E) = I_p(K, (p\mathbb{Z})^E)$ if and only if $O^p(H) =_G O^p(K)$. Hence spectrums of the rings $D_P B(A, G)$ and $\mathbb{Z}_{p\mathbb{Z}}B(G)$ have the same number of connected components, and so they have the same number of primitive idempotents. Since $\mathbb{Z}_{p\mathbb{Z}}B(G) \leq \mathbb{Z}_{p\mathbb{Z}}B(A, G) \leq D_P B(A, G)$, the result follows.

$\square$

As a generalization of 7.29 we state the following theorem.

Theorem 7.30 *Let π be a set of prime numbers and $\mathbb{Z}_{(\pi)}$ be the ring of fractions of $\mathbb{Z}$ with respect to $\mathbb{Z} - \cup_{p\in\pi} p\mathbb{Z}$. Then the rings $\mathbb{Z}_{(\pi)}B(G)$ and $\mathbb{Z}_{(\pi)}B(A, G)$ have the same primitive idempotents for all $\pi-$group A.*

As an implication of 7.24 we give the next result.

Theorem 7.31 *The primitive idempotents of the rings $\mathbb{Z}_{(\pi)}B(A, G)$ and $\mathbb{Z}_{(\pi)}B(G)$ are the same where π is the set of primes dividing $(|A|, |G|)$.*

For a given prime number p and subgroup H of G, let $H_p/O(H)$ be the Sylow $p-$subgroup of $H/O(H)$ and $H_{p'}/O(H)$ be the complement of $H_p/O(H)$.
We have the following immediate consequences; $H/O(H) = H_p/O(H) \times H_{p'}/O(H)$, $O(H) \trianglelefteq H_p \trianglelefteq H \trianglerighteq H_{p'} \trianglerighteq O(H)$, $H_p \cap H_{p'} = O(H)$. Take any automorphism f of H. Then $O(H) = f(O(H)) \leq f(H_p)$ (because $O(H)$ is a

characteristic subgroup of H) and so $f(H_p)/O(H)$ is a Sylow $p-$subgroup of the abelian group $H/O(H)$. Hence $f(H_p) = H_p$ and similarly $f(H_{p'}) = H_{p'}$. Therefore H_p and $H_{p'}$ are characteristic subgroups of H.

Lemma 7.32 *For a prime number p, let $(H,h),(K,k) \in el(A,G)$ satisfy the following conditions;*
(C1) $H \trianglelefteq K$ and K/H is a $p-$group,
(C2) $xhx^{-1} \equiv h \bmod H_p$ for all $x \in K$,
(C3) $h \equiv k \bmod K_p$.
Then we have $I(H,h,P) = I(K,k,P)$ for any nonzero prime ideal P of D such that D/P has characteristic p.

Proof: From (C3); $h^{-1}kO(K)$ has $p-$power order in $\bar{K}$ and so we have $I(K,h,P) = I(K,k,P)$ by 7.11. But this means $\pi_P \circ S^G_{K,h} = \pi_P \circ S^G_{K,k}$ from 7.10.
Now take any $[A_\nu G/V] \in B(A,G)$ and compute,

$$\pi_P \circ S^G_{H,h}([A_\nu G/V]) = \sum_{gV \subseteq G, H \leq {}^gV} ({}^g\nu(h) + P)$$

$$= \sum_{gV \subseteq G, K \leq {}^gV} ({}^g\nu(h) + P) + \sum_{gV \subseteq G, H \leq {}^gV, K \nleq {}^gV} ({}^g\nu(h) + P)$$

$$= \pi_p \circ S^G_{K,h}([A_\nu G/V]) + \sum_{gV \subseteq G, H \leq {}^gV, K \nleq {}^gV} ({}^g\nu(h) + P).$$

Thus, using $\pi_P \circ S^G_{K,h} = \pi_P \circ S^G_{K,k}$ we get

$$\pi_P \circ S^G_{H,h}([A_\nu G/V]) - \pi_P \circ S^G_{K,k}([A_\nu G/V]) = \sum_{gV \subseteq G, H \leq {}^gV, K \nleq {}^gV} ({}^g\nu(h) + P).$$

From (C2); $(h^{-1}(xhx^{-1}))O(H)$ has $p-$power order in $\bar{H}$ and so $\nu(h)^{-1}\nu(xhx^{-1})$ $+P$ is a $p-$power root of unity in the field D/P for any $\nu \in Hom(H,A)$. Since D/P has characteristic p, it has no nontrivial $p-$power roots of unity. Thus we must have $\nu(h) + P = \nu(xhx^{-1}) + P$ for all $x \in K$ and $\nu \in Hom(H,A)$.
Note that the indices of the last sum range in the $K/H-$set $G/V^H - G/V^K$. Because K/H is a $p-$group and $K/H-$orbits of $G/V^H - G/V^K$ have nontrivial sizes, any orbit has size divisible by p. Take a $gV \in G/V^H - G/V^K$ and consider

its orbit $orb_{K/H}(gV) = \{kgV : kH \in K/H\}$.
If $x^{-1}gV \in orb_{K/H}(gV)$ then we have
${}^{x^{-1}g}\nu(h) + P = {}^{g}\nu(xhx^{-1}) + P = {}^{g}\nu(h) + P$. Therefore,

$$\sum_{gV \subseteq G, H \leq {}^{g}V, K \nleq {}^{g}V} ({}^{g}\nu(h) + P) = \sum_{gV \in G/V^H - G/V^K} ({}^{g}\nu(h) + P)$$

$$= \sum_{gV \in_{K/H} G/V^H - G/V^K} |orb_{K/H}(gV)|({}^{g}\nu(h) + P) = P.$$

So we proved $\pi_P \circ S^G_{H,h} = \pi_P \circ S^G_{K,k}$ implying that $I(H,h,P) = I(K,k,P)$.

□

Remark 7.33 *Let G be an abelian group, and P be a nonzero prime ideal of D such that D/P has characteristic p. Then $I(H,h,P) = I(O^p(H), h_{p'}, P)$ for all $(H,h) \in el(A,G)$.*

Proof: It suffices to note that $(O^p(H), h_{p'})$ and (H,h) satisfy the conditions in 7.32. (C1) holds trivially , and the condition (C2) disappears because G is abelian.
Since $H/O(H) = H_p/O(H) \times H_{p'}/O(H)$, any element $hO(H)$ of $\bar{H}$ can be written uniquely as $hO(H) = (h_1O(H))(h_2O(H))$ where $h_1O(H) \in H_p/O(H)$ and $h_2O(H) \in H_{p'}/O(H)$. Since $h_1O(H)$ has $p-$power order, $h_2O(H)$ has order not divisible by p and $hO(H) = (h_1O(H))(h_2O(H)) = (h_2O(H))(h_1O(H))$ we must have $h_1O(H) = (hO(H))_p = h_pO(H)$ and $h_2O(H) = (hO(H))_{p'} = h_{p'}O(H)$. Thus $(h_{p'})^{-1}hO(H) = h_pO(H) = h_1O(H) \in H_p/O(H)$ implying that $h_{p'} \equiv h$ mod H_p. Therefore the condition (C3) holds.

□

Let p be a prime number and H be a subgroup of G. Then $(O^p(H))_{p'}$ is a characteristic subgroup of $O^p(H)$ of $p-$power index. Therefore, $(O^p(H))_{p'}$ is a normal subgroup of H having $p-$power index. Then by the minimality of $O^p(H)$ we must have $O^p(H) = (O^p(H))_{p'}$. Thus, $O^p(H)/O(O^p(H))$ is a $p'-$group.

Lemma 7.34 *Let $H \leq G$ be such that $\bar{H}$ $(= H/O(H))$ is a $p'-$group, and P be a nonzero prime ideal of D such that D/P has characteristic p. Let $|A| = p^\alpha m$ where $(p,m) = 1$. Then;*
(i) The field D/P contains a primitive m^{th} root of unity. In fact, $\zeta^{p^\alpha} + P$ is a primitive m^{th} root of unity in D/P where $A =<\zeta>$.
(ii) The group algebra $(D/P)\bar{H}$ is semisimple.
(iii) For any $\nu \in Hom(H,A)$ define a $(D/P)-$algebra map $ev(\nu) : (D/P)\bar{H} \rightarrow (D/P)$ given by $ev(\nu)(\sum_{\bar{h}\in\bar{H}} \lambda_{\bar{h}}\bar{h}) = \sum_{\bar{h}\in\bar{H}} \lambda_{\bar{h}}\nu(h)$ for all $\sum_{\bar{h}\in\bar{H}} \lambda_{\bar{h}}\bar{h} \in (D/P)\bar{H}$. The maps $ev(\nu)$, where ν ranges in $Hom(H,A)$, are all distinct.
(iv) The product map $\prod_{\nu\in Hom(H,A)} ev(\nu) : (D/P)\bar{H} \rightarrow \prod_{\nu\in Hom(H,A)}(D/P)$ is injective.

Proof: (i) The following three facts can be found in [12](page $57, 58$).
(a) Let θ be a $(p^\beta)^{th}$ primitive root of unity. Then the principal ideal $(1-\theta)\mathbb{Z}[\theta]$ is the unique prime ideal of $\mathbb{Z}[\theta]$ lying over $p\mathbb{Z}$.
(b) Let θ be a primitive m^{th} root of unity. If m is divisible by at least two different prime numbers, then $1-\theta$ is unit in $\mathbb{Z}[\theta]$.
(c) Let θ be a primitive m^{th} root of unity, and q be a prime number not dividing m. Then for any prime ideal Q of $\mathbb{Z}[\theta]$ lying over $q\mathbb{Z}$, $1-\theta^k \in Q$ if and only if $1-\theta^k = 0$. So $\theta + Q$ is a primitive m^{th} root of unity in $(\mathbb{Z}[\theta])/Q$.
We are assuming that $\zeta =$a primitive n^{th} root of unity, $A =<\zeta>$, $D = \mathbb{Z}[\zeta]$, and $A \leq D^*$.
(A) Any prime ideal P of D lying over $p\mathbb{Z}$ contains $1-\zeta^m$:
It is clear that ζ^m is a primitive $(p^\alpha)^{th}$ root of unity. So by the above fact (a), $(1-\zeta^m)\mathbb{Z}[\zeta^m]$ is the unique prime ideal of $\mathbb{Z}[\zeta^m]$ lying over $p\mathbb{Z}$. Now $P \cap \mathbb{Z}[\zeta^m]$ is a prime ideal of $\mathbb{Z}[\zeta^m]$ lying over $p\mathbb{Z}$ because $\mathbb{Z}[\zeta^m] \leq D$. Hence, $P \cap \mathbb{Z}[\zeta^m] = (1-\zeta^m)\mathbb{Z}[\zeta^m]$ implying that $1-\zeta^m \in P$.
(B) Let P be a prime ideal of D lying over $p\mathbb{Z}$. If $0 \neq 1-\zeta^k \in P$, then m divides k :
$1-\zeta^k \in P \cap \mathbb{Z}[\zeta^k]$ which is a prime ideal of $\mathbb{Z}[\zeta^k]$ because $\mathbb{Z}[\zeta^k] \leq D$. Let ζ^k has order t. Then since $1-\zeta^k$ is in a prime ideal of $\mathbb{Z}[\zeta^k]$, $1-\zeta^k$ is nonunit. Thus by the above fact (b), t cannot be divisible by more than one prime number. Also from the above fact (c) p must divide t, otherwise $1-\zeta^k = 0$. Hence $t = p^a$ for

some natural number a. Now the order of ζ^k is given by the formula; $p^a = t = p^\alpha m/(k, p^\alpha m)$. Hence k must divide m.
(C) Let P be a prime ideal of D lying over $p\mathbb{Z}$. The smallest integer k such that $1 - \zeta^k \in P$ is m :
Follows from (A) and (B).
Now we know from (C) that $1 - \zeta^m \in P$ and m is the smallest such number. Then $(\zeta + P)^m = 1 + P$, and so $\zeta + P$ is an m^{th} root of unity in D/P. On the other hand the minimality of m implies that $\zeta + P$ has order m. Thus $\zeta + P$ is a primitive m^{th} root of unity in D/P.
(ii) D/P is a field of characteristic not dividing $\bar{H}$. Thus $(D/P)\bar{H}$ is semisimple by Mascke's Theorem.
(iii) Any element $\bar{h}$ of $\bar{H}$ has order dividing m. So for any $\nu \in Hom(H, A)$, $\nu(h)$ is an m^{th} root of unity in $< \zeta > \leq D^*$. Thus $\nu(h)$ belongs to the unique subgroup of A of order m, namely $< \zeta^{p^\alpha} >$. Note that ζ^{p^α} is a primitive m^{th} root of unity. Since D/P contains a primitive m^{th} root of unity $1 + \zeta$ we have a group monomorphism $< \zeta^{p^\alpha} > \simeq < \zeta + P > \leq (D/P)^*$ given by $(\zeta^{p^\alpha})^r \mapsto \zeta^r + P$ for any integer r. Therefore, the maps $ev(\nu)$ are all distinct for all $\nu \in Hom(H, A)$.
(iv) Since $(D/P)\bar{H}$ is a semisimple algebra over the field (D/P) containing all a^{th} roots of unity for all a dividing the order of $\bar{H}$, $(D/P)\bar{H}$ must be isomorphic to the direct sum of $|\bar{H}|$ many (D/P) by Wedderburn Theorem. Hence there are exactly $|\bar{H}| = |Hom(H, A)|$ many distinct $(D/P)-$algebra maps from $(D/P)\bar{H}$ to D/P. However we know that the $(D/P)-$algebra maps $ev(\nu), \nu \in Hom(H, A)$, are all distinct. Thus they are precisely all the $(D/P)-$algebra maps. Consequently, the product map is injective.

□

Recall that $N(H)$ acts on $\bar{H}$ by conjugation where $N(H)$ denotes $N_G(H)/H$.

Lemma 7.35 *Let $H \leq G$ be such that $\bar{H}$ is a $p'-$group, and P be a nonzero prime ideal of D such that D/P has characteristic p. Given $\bar{h} \in \bar{H}$;*
$[A_\nu G/H] \in I(H, h, P)$ for all $\nu \in Hom(H, A)$ if and only if there is a $gH \in N(H)$ of order p such that ${}^g\bar{h} = \bar{h}$.

Proof: Let $x = \sum_{gH \subseteq N_G(H)} {}^{g^{-1}}\bar{h} \in (D/P)\bar{H}$. Take any $\nu \in Hom(H, A)$ and compute;

$$ev(\nu)(x) = \sum_{gH\subseteq N_G(H)} \nu({}^{g^{-1}}h) + P = \sum_{gH\subseteq N_G(H)} {}^{g}\nu(h) + P$$

$$= \pi_P \circ S^G_{H,h}([A_\nu G/H]) = 0$$

Since it is true for all $\nu \in Hom(H,A)$, x is in the kernel of the product map $\prod_{\nu\in Hom(H,A)} ev(\nu) : (D/P)\bar{H} \to (D/P)$. By the injectivity of the product map we must have,

$$0 = x = \sum_{gH\subseteq N_G(H)} {}^{g^{-1}}\bar{h} = |stab_{N(H)}(\bar{h})| \sum_{k=_{N(H)}h} k.$$

In the group algebra $(D/P)\bar{H}$, the different elements of $\bar{H}$ are linearly independent over (D/P). Thus, $x = 0$ if and only if the characteristic of the field (D/P) divides $|stab_{N(H)}(\bar{h})|$. Hence, $stab_{N(H)}(\bar{h})$ has an element of order p if and only if $[A_\nu G/H] \in I(H,h,P)$ for all $\nu \in Hom(H,A)$.

□

Lemma 7.36 *Let $H \leq G$ be such that $\bar{H}$ is a $p'-$group, and P be a nonzero prime ideal of D such that D/P has characteristic p. Let $\bar{h}_1, \bar{h}_2 \in \bar{H}$. Suppose that $\pi_P \circ S^G_{H,h_1}([A_\nu G/H]) = \pi_P \circ S^G_{H,h_2}([A_\nu G/H])$ for all $\nu \in Hom(H,A)$, but $\pi_P \circ S^G_{H,h_1}([A_\nu G/H]) \neq 0$ for at least one $\nu \in Hom(H,A)$. Then there is a $gH \in N(H)$ such that ${}^{g}\bar{h}_1 = \bar{h}_2$.*

Proof: Let $x = \sum_{gH\subseteq N_G(H)} {}^{g^{-1}}\bar{h}_1$ and $y = \sum_{gH\subseteq N_G(H)} {}^{g^{-1}}\bar{h}_2$. For any $\nu \in Hom(H,A)$ we evaluate the image of $x - y \in (D/P)\bar{H}$ under the map $ev(\nu) : (D/P)\bar{H} \to D/P$;

$$ev(\nu)(x-y) = \sum_{gH\subseteq N_G(H)} (\nu({}^{g^{-1}}h_1) - \nu({}^{g^{-1}}h_2)) + P$$

$$= \sum_{gH\subseteq N_G(H)} ({}^{g}\nu(h_1) + P) - \sum_{gH\subseteq N_G(H)} ({}^{g}\nu(h_2) + P)$$

$$= \pi_P \circ S^G_{H,h_1}([A_\nu G/H]) - \pi_P \circ S^G_{H,h_2}([A_\nu G/H]) = 0.$$

Since it is true for all $\nu \in Hom(H,A)$, $x-y$ is in the kernel of the product map $\prod_{\nu \in Hom(H,A)} ev(\nu)$ which is an injective map. Hence $x-y=0$ in the group algebra $(D/P)\bar{H}$. Since $N(H)$ acts on $\bar{H}$ by conjugation we can write $x-y$ as follows;

$$0 = x-y = |stab_{N(H)}(\bar{h}_1)| \sum_{\bar{k}_1 =_{N(H)} \bar{h}_1} \bar{k}_1 - |stab_{N(H)}(\bar{h}_2)| \sum_{\bar{k}_2 =_{N(H)} \bar{h}_2} \bar{k}_2.$$

By 7.35, p cannot divide the orders of $N(H)-$stabilizers of $\bar{h}_1$ and $\bar{h}_2$ because $\pi_P \circ S^G_{H,h_1}([A_\nu G/H])$ is nonzero for some $\nu \in Hom(H,A)$. Since $x-y=0$, by the linear independency of different elements of $\bar{H}$ over (D/P) we can conclude that $\bar{k}_1 = \bar{k}_2$ for some $\bar{k}_1 =_{N(H)} \bar{h}_1$ and $\bar{k}_2 =_{N(H)} \bar{h}_2$. Thus $\bar{h}_1 =_{N(H)} \bar{h}_2$, as desired.

$\square$

Lemma 7.37 *Let P be a nonzero prime ideal of D such that D/P has characteristic p, and $H \leq G$ be such that $\bar{H}$ is a $p'-$group For $h_1, h_2 \in H$ if $I(H,h_1,P) = I(H,h_2,P)$, then $\bar{h}_1 =_{N(H)} \bar{h}_2$ or $[A_\nu G/H] \in I(H,h_1,P)$ for all $\nu \in Hom(H,A)$.*

Proof: Suppose $I(H,h_1,P) = I(H,h_2,P)$. Then by 7.10 $\pi_P \circ S^G_{H,h_1} = \pi_P \circ S^G_{H,h_2}$. In particular, $\pi_P \circ S^G_{H,h_1}([A_\nu G/H]) = \pi_P \circ S^G_{H,h_2}([A_\nu G/H])$ for all $\nu \in Hom(H,A)$. Now if $[A_\nu G/H] \notin I(H,h_1,P)$ for some $\nu \in Hom(H,A)$ then by 7.36 we must have $\bar{h}_1 =_{N(H)} \bar{h}_2$ which completes the proof.

$\square$

Corollary 7.38 *Let P be a nonzero prime ideal of D such that D/P has characteristic p, and G be a $p'-$group. Then for any $(H,h),(K,k) \in el(A,G)$; $I(H,h,P) = I(K,k,P)$ if and only if $(H,h) =_G (K,k)$.*

Proof: $(\Rightarrow)$ Suppose $I(H,h,P) = I(K,k,P)$. We know from 7.18 that $I(H,h,P) = I(K,k,P)$ implies $O^p(H) =_G O^p(K)$. Since G is a $p'-$group, $O^p(T) = T$ for any $T \leq G$. Hence $H =_G K$. Suppose $H = {}^gK$. Then $I(H,h,P) = I({}^gK,h,P) = I(K,k,P) = I({}^gK,{}^gk,P)$ implying that $I(H,h,P) = I(H,{}^gk,P)$.

Since $\bar{H}$ is a $p'-$group, from 7.37 we get $\bar{h} =_{N(H)} {}^g\bar{k}$ or $[A_\nu G/H] \in I(H,h,P)$ for all $\nu \in Hom(H,A)$. However, according to 7.35 $[A_\nu G/H] \in I(H,h,P)$ for all $\nu \in Hom(H,A)$ if and only if $stab_{N(H)}(\bar{h}) \leq G$ contains an element of order p. Since G is a $p'-$group, $stab_{N(H)}(\bar{h})$ has no element of order p. Therefore, $\bar{h} =_{N(H)} {}^g\bar{k}$. So ${}^x\bar{h} = {}^g\bar{k}$ for some $x \in N_G(H)$. Then ${}^x(H,h) = ({}^xH, {}^xh) = (H, {}^xh) = ({}^gK, {}^gk) = {}^g(K,k)$. That is, $(H,h) =_G (K,k)$.
$(\Leftarrow)$ Trivially holds.

□

Theorem 7.39 *Let G be a $p'-$group, and P be a nonzero prime ideal of D such that D/P has characteristic p. Then the rings $D_PB(A,G)$ and $\mathbb{C}B(A,G)$ have the same primitive idempotents.*

Proof: By the integrality of the ring extension $D_PB(A,G) \leq \prod_{(H,h)\in_G el(A,G)} D_P$, it is clear that the prime ideals of $D_PB(A,G)$ are the kernels of the maps $S^G_{H,h} : D_PB(A,G) \to D_P$ and $\pi_{P^E} \circ S^G_{H,h} : D_PB(A,G) \to D_P \to D_P/P^E$. We denote these kernels by $I_P(H,h,0)$ and $I_P(H,h,P^E)$, respectively. The maximal ideals of $D_PB(A,G)$ are of the form $I_P(H,h,P^E)$ where $(H,h) \in el(A,G)$. By 7.38 we conclude that $I_P(H,h,P^E) \sim I_P(K,k,P^E)$ (equivalently, $I(H,h,P) = I(K,k,P)$) if and only if $(H,h) =_G (K,k)$. Hence, the number of primitive idempotents of $D_PB(A,G)$ is equal to the number of nonconjugate subelements $(H,h) \in el(A,G)$ which is equal to the number of primitive idempotents of $\mathbb{C}B(A,G)$. The result follows, because $D_PB(A,G) \leq \mathbb{C}B(A,G)$.

□

The previous theorem implies the following result.

Theorem 7.40 *The primitive idempotents of $\mathbb{Z}_{p\mathbb{Z}}B(A,G)$ and $\mathbb{Q}B(A,G)$ are the same for any prime number p not dividing the order of G.*

Proof: Let P be a nonzero prime ideal of D such that D/P has characteristic p where p is a prime number not dividing the order of G. From 7.39 we know that the primitive idempotents of $D_PB(A,G)$ are precisely the primitive idempotents of $\mathbb{C}B(A,G)$. Since $\mathbb{Q}\cap D_P = \mathbb{Z}_{p\mathbb{Z}}$, $\mathbb{Q}B(A,G)\cap D_PB(A,G) = \mathbb{Z}_{p\mathbb{Z}}B(A,G)$ which completes the proof.

$\square$

Next we find the prime ideals of $B(A,G)$ and their properties by using the facts we obtained so far. We use the integrality of the ring extensions $B(A,G) \leq DB(A,G)$ and $B(A,G) \leq \prod_{(H,h)\in_G el(A,G)} D$. We already found almost all facts appearing in the next theorem. For any $(H,h) \in el(A,G)$, and nonzero prime ideal P of D define

$$J(H,h,0) = \{x \in B(A,G) : S^G_{H,h}(x) = 0\},$$

$$J(H,h,P) = \{x \in B(A,G) : S^G_{H,h}(x) \in P\}.$$

Let $\mathcal{G}$ be the Galois group of the extension $\mathbb{Q} \leq \mathbb{Q}(\zeta)$. Then $\mathcal{G}$ acts transitively on the prime ideals of D lying over $p\mathbb{Z}$ where p is any prime number. Restricting any element $\sigma \in \mathcal{G}$ to A we get an automorphism of A. Thus by function composition $\mathcal{G}$ acts on $Hom(H,A) \simeq \bar{H}$ for any $H \leq G$. Note that $\mathcal{G}$ acts also on $DB(A,G)$ as follows;

$$(\sigma, \sum d_{V,\nu}[A_\nu G/V]) \mapsto {}^\sigma(\sum d_{V,\nu}[A_\nu G/V]) = \sum \sigma(d_{V,\nu})[A_\nu G/V].$$

For any prime ideal (possibly 0) $\mathfrak{A}$ of D, ${}^\sigma(I(H,h,\mathfrak{A})) = I(H,\sigma(h),\sigma(\mathfrak{A}))$.

Theorem 7.41 *Let P,Q be any nonzero prime ideals of D, and $(H,h),(K,k)$ be any elements of $el(A,G)$. Then*
(i) $J(H,h,0)$ and $J(H,h,P)$ are prime ideals of $B(A,G)$. Moreover, any prime ideal of $B(A,G)$ is one of these forms.
(ii) $J(H,h,0)$ is a minimal prime ideal of $B(A,G)$, and $J(H,h,P)$ is a maximal ideal of $B(A,G)$.
(iii) $J(H,h,0)\cap\mathbb{Z} = 0$, and $J(H,h,P)\cap\mathbb{Z} = P\cap\mathbb{Z} = p\mathbb{Z}$ where p is the characteristic of the field D/P.
(iv) $J(H,h,P) = J(K,k,Q)$ implies that $P\cap\mathbb{Z} = Q\cap\mathbb{Z}$.
(v) Let $\mathfrak{A},\mathfrak{B}$ be prime ideals (possibly 0) of D. Then $J(H,h,\mathfrak{A}) \sim J(K,k,\mathfrak{B})$ if and only if $S(H) =_G S(K)$.

(vi) The number of primitive idempotents of $B(A,G)$ is equal to the number of nonconjugate perfect subgroups of G.
(vii) G is solvable if and only if Spec$(B(A,G))$ is connected if and only if 0 and 1 are the only idempotents of $B(A,G)$.

Proof: By the integrality of the ring extension $B(A,G) \leq DB(A,G)$, any prime ideal of $B(A,G)$ is the intersection of a prime ideal of $DB(A,G)$ with $B(A,G)$. Note that $I(H,h,0) \cap B(A,G) = J(H,h,0)$ and $I(H,h,P) \cap B(A,G) = J(H,h,P)$. Then (i) and (ii) follows. (iii) and (iv) are obvious. For the rest; note that D is $\mathcal{G}-$stable and $(DB(A,G))^{\mathcal{G}} = B(A,G)$. Consequently, $I(H,h,\mathfrak{A}) \cap B(A,G) = I(K,k,\mathfrak{B}) \cap B(A,G)$ if and only if there is a $\sigma \in \mathcal{G}$ such that ${}^{\sigma}(I(H,h,\mathfrak{A})) = I(H,\sigma(h),\sigma(\mathfrak{A})) = I(K,k,\mathfrak{B})$. Therefore, $J(H,h,P) = J(K,k,Q)$ still implies that $O^p(H) =_G O^p(K)$, and so the results follow.

$\square$

As remarked at the beginning of this chapter some results we obtained so far can be extended to $RB(A,G)$ where R is not so specific as D. In the following three remarks we collect some of the results which we obtained for $DB(A,G)$ and state them for $RB(A,G)$.
Let R be an integral domain of characteristic 0 such that R contains a primitive n^{th} root of unity to ensure that we have a fixed embedding of A into the unit group of R. As before, for any $(H,h) \in el(A,G)$ we have a ring epimorphism

$$S^G_{H,h} : RB(A,G) \to R \qquad \text{where} \qquad [A_\nu G/V] \mapsto \sum_{gV \subseteq G, H \leq {}^gV} {}^g\nu(h).$$

The product map

$$\psi = \prod_{(H,h) \in_G el(A,G)} S^G_{H,h} : RB(A,G) \to \prod_{(H,h) \in_G el(A,G)} R$$

is still injective. For any nonzero prime ideal P of R we again define

$$I(H,h,0) = \{x \in RB(A,G) : S^G_{H,h}(x) = 0\},$$

$$I(H,h,P) = \{x \in RB(A,G) : S^G_{H,h}(x) \in P\}.$$

Let also $\pi_P : R \to R/P$ be the canonical ring epimorphism.

Remark 7.42 *Let R be an integral domain of characteristic 0 such that R contains a primitive n^{th} root of unity to ensure that $A \leq R^*$. Then;*
(i) Any prime ideal of $RB(A,G)$ is of the form $I(H,h,P)$ for some prime ideal P of R and element (H,h) of $el(A,G)$. Conversely, any $I(H,h,P)$ is a prime ideal of $RB(A,G)$
(ii) $I(H,h,0)$ is a minimal prime ideal of $RB(A,G)$
(iii) P is a maximal ideal of R if and only if $I(H,h,P)$ is a maximal ideal of $RB(A,G)$
(iv) For any prime ideal P of R, $I(H,h,P) \cap R = P$
(v) For any nonzero prime ideal P of R, $I(H,h,0) \subsetneqq I(H,h,P)$
(vi) $I(H,h,P) = I(K,k,Q)$ implies $P = Q$
(vii) $I(H,h,0) = I(K,k,0)$ if and only if $(H,h) =_G (K,k)$
(viii) Suppose P is a maximal ideal of R. Then $I(H,h,0) \subseteq I(K,k,P)$ if and only if $I(H,h,P) = I(K,k,P)$
(ix) $I(H,h,P) = I(K,k,P)$ if and only if $\pi_P \circ S^G_{H,h} = \pi_P \circ S^G_{K,k}$
(x) Let P be a prime ideal of R such that R/P has characteristic p. Then for any $(H,h_1),(H,h_2) \in el(A,G)$ with $h_1^{-1}h_2O(H) \in \bar{H}$ is of $p-$power order, we have $I(H,h_1,P) = I(H,h_2,P)$.

Proof: Since both rings $RB(A,G)$ and $\prod_{(H,h)\in_G el(A,G)} R$ are finite over R, the ring extension $RB(A,G) \leq \prod_{(H,h)\in_G el(A,G)} R$ is an integral ring extension. Hence, all parts of the remark follow by slight modifications of the proofs that we gave for 7.8, 7.9, 7.10 and 7.11 (indeed, word by word).

□

To study the prime spectrum of $RB(A,G)$ we assume R is Noetherian implying $RB(A,G)$ is Noetherian.

Remark 7.43 *Let R be a Noetherian integral domain of characteristic 0 such that R contains a primitive n^{th} root of unity to ensure that $A \leq R^*$. Then;*
(i) Let P be a prime ideal of R and $(H,h) \in el(A,G)$. If no prime divisor of $|\bar{H}|$ is invertible in R, then the prime ideals $I(H,h,P)$ and $I(H,1,0)$ of $RB(A,G)$ are in the same connected component of $Spec(RB(A,G))$

(ii) Let P be a prime ideal of R and $(H,h) \in el(A,G)$. If no prime divisor of $|\bar{H}|$ is invertible in R, then the prime ideals $I(H,h,P)$ and $I(S(H),1,0)$ of $RB(A,G)$ are in the same connected component of $Spec(RB(A,G))$
(iii) Let P and Q be prime ideals of R, and $(H,h),(K,k) \in el(A,G)$ Suppose no prime divisor of $|\bar{H}||\bar{K}|$ is invertible in R. Then $S(H) =_G S(K)$ implies that the prime ideals $I(H,h,P)$ and $I(K,k,Q)$ of $RB(A,G)$ are in the same connected component of $Spec(RB(A,G))$.

Proof: It is clear that the proofs given for 7.12, 7.13, 7.14 and 7.15 still work under the assumptions given in this remark.

□

Remark 7.44 *Let R be a Noetherian integral domain of characteristic 0 such that R contains a primitive n^{th} root of unity to ensure that $A \leq R^*$. Suppose further that any nonzero prime ideal of R is maximal, and no prime divisor of $|G|$ is invertible in R. Then for any prime ideals P and Q of R, the prime ideals $I(H,h,P)$ and $I(K,k,Q)$ of $RB(A,G)$ are in the same connected component of $Spec(RB(A,G))$ if and only if $S(H) =_G S(K)$.*

Proof: Under the hypothesis of this remark, evidently 7.19 is still true for $RB(A,G)$. Hence the result follows from 7.19 and 7.43.

□

We finish generalizations of our results obtained for $DB(A,G)$ after giving a generalization of one of the remarkable results of Dress appearing in [9].

Remark 7.45 *Let R be a Noetherian integral domain of characteristic 0 such that R contains a primitive n^{th} root of unity to ensure that $A \leq R^*$. Suppose further that any nonzero prime ideal of R is maximal, and no prime divisor of $|G|$ is invertible in R. Then G is solvable if and only if $Spec(RB(A,G))$ is connected.*

Proof: It is immediate from 7.44.

□

Now we collect the results we proved related to idempotents in the following corollary. The first three parts of the following corollary are already known ((ii) was obtained by Barker in [1], and (i) and (iii) were obtained by Dress in [9]). We also achieved to get these three known parts by our slightly different way.

Corollary 7.46 *Let $A =< \zeta >$, $\zeta =$a primitive n^{th} root of unity, $\pi =$a set of prime numbers, $\mathbb{Z}_{(\pi)} = \{a/b \in \mathbb{Q} : b \notin \cup_{p\in\pi} p\mathbb{Z}\}$, and for $\pi = \{p\}$ write $\mathbb{Z}_{(\pi)} = \mathbb{Z}_{p\mathbb{Z}}$. Then;*
(i) G is solvable if and only if 0 and 1 are the only idempotents of $B(A,G)$
(ii) The primitive idempotents of $B(A,G)$ and $B(G)$ are the same
(iii) If G is a $\pi-$group, 0 and 1 are the only idempotents of $\mathbb{Z}_{(\pi)}B(A,G)$
(iv) If A is a $\pi-$group, the primitive idempotents of $\mathbb{Z}_{(\pi)}B(A,G)$ and $\mathbb{Z}_{(\pi)}B(G)$ are the same
(v) If π is the set of primes dividing $(|A|,|G|)$, then the primitive idempotents of $\mathbb{Z}_{(\pi)}B(A,G)$ and $\mathbb{Z}_{(\pi)}B(G)$ are the same
(vi) For any prime number p not dividing $|G|$, the primitive idempotents of $\mathbb{Z}_{p\mathbb{Z}}B(A,G)$ and $\mathbb{Q}B(A,G)$ are the same
(vii) If G is $\pi'-$group, then the primitive idempotents of $\mathbb{Z}_{(\pi)}B(A,G)$ and $\mathbb{Q}B(A,G)$ are the same.

Proof: All of them except (vii) are obtained so far. However (vii) is trivial because it is an obvious generalization of (vi).

□

We will close this chapter after finding the primitive idempotents of $\mathbb{Z}_{(\pi)}B(A,G)$ where π is any set of prime numbers, and G is a nilpotent group. We will show that there is a bijection between the primitive idempotents of

$\mathbb{Z}_{(\pi)}B(A,G)$ and the primitive idempotents of $\mathbb{Q}B(A,K)$ where G is a nilpotent group, and K is the unique Hall $\pi'-$subgroup of G. I discovered this result, but the proof I suggested was not complete. A complete proof was given by my supervisor L. Barker.
For a $K-$algebra R, a $K-$algebra homomorphism from R to K is called species. We use the notation $Ipot(R)$ to denote the set of the primitive idempotents of R. We begin with giving a slight generalization of 7.46.

Remark 7.47 *Let R be an integral domain with characterisric 0, and π be a set of prime numbers that are not invertible in R. Then;*
(i) If G is a $\pi-$group, then $Ipot(RB(A,G)) = Ipot(B(G))$
(ii) If G is a $\pi'-$group, then $Ipot(RB(A,G)) = Ipot(\mathbb{K}B(A,G))$ where $\mathbb{K}$ is the field of fractions of R.

Proof: Both part follow from obvious generalizations of 7.46 (iii), (v) and (vii).

□

The next lemma and its proof was given by L. Barker.

Lemma 7.48 *Let R be an integral domain with characteristic 0. Let $\mathbb{K}$ be the field of fractions of R. Let A and B be $R-$algebras, finitely generated and free as $R-$modules. Suppose that $\mathbb{K}A$ and $\mathbb{K}B$ are direct sums of copies of $\mathbb{K}$ (Here, $\mathbb{K}A = \mathbb{K} \otimes_R A$ and $\mathbb{K}B = \mathbb{K} \otimes_R B$). Given primitive idempotents e and f of A and B, respectively, then $e \otimes f$ is a primitive idempotent of $A \otimes_R B$.*

Proof: By replacing A and B with eAe and fBf, we may assume that that $e = 1_A$ and $f = 1_B$. Write $e = \sum_{i\in I} e_i$ and $f = \sum_{j\in J} f_j$ as the sums of the primitive idempotents of $\mathbb{K}A$ and $\mathbb{K}B$. Thus

$$\mathbb{K}A = \oplus_{i\in I}\mathbb{K}e_i,\ \mathbb{K}B = \oplus_{j\in J}\mathbb{K}f_j.$$

Consider a nonzero idempotent ε of $A \otimes_R B$. We must show that $\varepsilon = 1$. We have

$$\varepsilon = \sum_{(i,j)\in T} e_i \otimes f_j$$

for some subset T of $I \times J$. We are to show that $T = I \times J$.

Fix an element $i \in I$, let $J(i) = \{j \in J : (i, j) \in T\}$, and suppose that $J(i)$ is nonempty. We claim that $J(i) = J$. Let s_i be the species $\mathbb{K}A \to \mathbb{K}$ such that $s_i(e_i) = 1$. Then $s_i(e_{i'}) = 0$ for $i' \in I - \{i\}$. Since $s_i(1_A) = 1$ and $s_i(R1_A) = R$, we have $R \subseteq s_i(A)$. But $s_i(A)$ is finitely generated as an $R-$module, so $R = s_i(A)$. Therefore, we have an $R-$algebra map

$$s_i \otimes id_B : A \otimes_R B \to B, \ a \otimes b \mapsto s_i(a)b.$$

The element

$$s_i \otimes id_B(\varepsilon) = \sum_{j \in J(i)} f_j$$

is an idempotent of B, nonzero because $J(i)$ is nonempty. But 1_B is a primitive idempotent of B, so $J(i) = J$, as claimed.

Now fix an element $j \in J$, and let $I(j) = \{k \in I : (k, j) \in T\}$. By what we have shown, $i \in I(j)$. In particular, $I(j)$ is nonempty. Interchanging A and B, the claim established above implies that $I(j) = I$. Therefore $T = I \times J$, as required.

$\square$

Let H and K be two groups of coprime order. Then any subgroup of $H \times K$ is of the form $T \times L$ where T and L are subgroups of H and K, respectively. For any $\nu \in Hom(T \times L, A)$, define $\nu_1 \in Hom(T, A)$ and $\nu_2 \in Hom(L, A)$ as $\nu_1(t) = \nu((t, 1))$ and $\nu_2(l) = \nu((1, l))$ for all $t \in T$ and $l \in L$. So we have a map $Hom(T \times L, A) \to Hom(T, A) \times Hom(L, A)$, given by $\nu \mapsto (\nu_1, \nu_2)$, which is a group isomorphism. This map induces the ring homomorphism given in the following lemma.

Lemma 7.49 *Let H and K be two groups of coprime order. Then the map $B(A, H \times K) \to B(A, H) \times B(A, K)$ given by $[A_\nu \frac{H \times K}{T \times L}] \mapsto ([A_{\nu_1} H/T], [A_{\nu_2} K/L])$ for any $(T \times L, \nu) \in ch(H \times K, A)$ is a unital ring epimorphism.*

Proof: Straightforward checking.

$\square$

By considering dimensions of $B(A, H \times K)$ and $B(A,H) \times B(A,K)$ over $\mathbb{Z}$, it is clear that the map in the previous lemma is not injective in general. To make it injective, an obvious way is the taking the tensor product of $B(A,H)$ and $B(A,K)$ over $\mathbb{Z}$ instead of taking the direct product of them.

Lemma 7.50 *Let H and K be two groups of coprime order. Then the map $\psi : B(A, H \times K) \to B(A,H) \otimes_{\mathbb{Z}} B(A,K)$ given by $\psi([A_\nu \frac{H\times K}{T\times L}]) = [A_{\nu_1} H/T] \otimes [A_{\nu_2} K/L])$ for any $(T \times L, \nu) \in ch(H \times K, A)$ is a ring isomorphism.*

Proof: It is clear.

$\square$

Note that for any integral domain R with characteristic 0, the $R-$linear extension of the map ψ given in 7.50 is also a ring isomorphism from $RB(A, H \times K)$ to $RB(A,H) \otimes_R RB(A,K)$.

The following theorem was obtained by L. Barker, and its proof below was suggested by him.

Theorem 7.51 *Suppose $G = P \times Q$ where P is a $\pi-$group and Q is a $\pi'-$group, and π is a set of prime numbers. Also suppose that $\mathfrak{O}$ is a ring extension of $\mathbb{Z}$ whose field of fractions $\mathbb{K}$ is a Galois extension of $\mathbb{Q}$. Let $R = \mathfrak{O}_{(\pi)} = \{x/y \in \mathbb{K} : y \notin \cup_{p\in\pi} p\mathbb{Z}\}$. Then there is a bijection*

$$Ipot(RB(A,G)) = Ipot(B(P)) \times Ipot(\mathbb{K}B(A,Q))$$

such that $\varepsilon \leftrightarrow e \otimes f$ provided $\varepsilon = e \otimes f$. Here, we make the identification $RB(A,G) = RB(A,P) \otimes_R RB(A,Q)$ given by the $R-$linear extension of the map ψ introduced in 7.50.

Proof: First suppose that R has enough roots of unity. By 7.48, there is a bijection

$$Ipot(RB(A,G)) \leftrightarrow Ipot(RB(A,P)) \times Ipot(RB(A,Q))$$

described by tensor products, $e \otimes f \leftrightarrow (e,f)$. The required conclusion now follows from 7.47 in this case.

For the general case, let $\mathbb{K} \leq \mathbb{K}'$ be a Galois extension such that $\mathbb{K}'$ has enough roots of unity. Let $\mathfrak{O}'$ be the ring of algebraic integers in $\mathbb{K}'$, and let $R' = \mathfrak{O}'_{(\pi)}$. The Galois group Γ of the extension $\mathbb{K} \leq \mathbb{K}'$ acts on R', and the $\Gamma-$fixed subalgebra of R' is R. Letting Γ act in the evident way on $R'B(A,G)$, then the $\Gamma-$fixed subalgebra is $RB(A,G)$. So

$$Ipot(RB(A,G)) = Ipot(R'B(A,G))^{\Gamma}.$$

Similarly, we have

$$Ipot(\mathbb{K}B(A,Q)) = Ipot(\mathbb{K}'B(A,Q))^{\Gamma}.$$

On the other hand, the bijection

$$Ipot(R'B(A,G)) \leftrightarrow Ipot(B(P)) \times Ipot(\mathbb{K}'B(A,G))$$

is invariant under Galois automorphisms. Therefore

$$\begin{aligned} Ipot(RB(A,G)) &= Ipot(R'B(A,G))^{\Gamma} \\ &\leftrightarrow (Ipot(B(P))) \times Ipot((\mathbb{K}'B(A,Q))^{\Gamma}) \\ &= Ipot((B(P))^{\Gamma}) \times Ipot((\mathbb{K}'B(A,Q))^{\Gamma}) \\ &= Ipot(B(P)) \times Ipot(\mathbb{K}B(A,Q)). \end{aligned}$$

□

Theorem 7.51 implies the following result which is about the primitive idempotents of $\mathbb{Z}_{(\pi)}B(A,G)$ where G is a nilpotent group and π is any set of prime numbers.

Theorem 7.52 *Let π be a set of prime numbers and $\mathbb{Z}_{(\pi)} = \{a/b \in \mathbb{Q} : b \notin \cup_{p\in\pi} p\mathbb{Z}\}$. For a nilpotent group G, there is a bijective correspondence between the primitive idempotents of $\mathbb{Z}_{(\pi)}B(A,G)$ and the primitive idempotents of $\mathbb{Q}B(A,Q)$ where Q is the unique Hall $\pi'-$subgroup of G.*

Proof: Sine G is nilpotent, $G = P \times Q$ where P is the unique Hall $\pi-$subgroup of G, and Q is the unique Hall $\pi'-$subgroup of G. Let $\mathfrak{O} = \mathbb{Z}$. Then $R = \mathbb{Z}_{(\pi)}$ and $\mathbb{K} = \mathbb{Q}$. So by theorem 7.51, there is a bijection

$$Ipot(\mathbb{Z}_{(\pi)}B(A,G)) = Ipot(B(P)) \times Ipot(\mathbb{Q}B(A,Q)).$$

Then the result follows because from 7.47, $Ipot(B(P)) = Ipot(\mathbb{Z}_{(\pi)}B(A,P)) = \{1\}$ (by 7.46, $\mathbb{Z}_{(\pi)}B(A,P)$ is a local ring).

$\square$

By the previous theorem, we know the primitive idempotents of $\mathbb{Z}_{(\pi)}B(A,G)$ where G is a nilpotent group and π is any set of prime numbers, because for any group H the primitive idempotents of $\mathbb{Q}B(A,H)$ are known, and obtained by Barker in [1].

Chapter 8

Some Further Maps

In this chapter, we collect together some miscellaneous further material on maps between monomial Burnside rings.

8.1 $B(G) \to B(A, G)$

We find the images of the primitive idempotents of $\mathbb{C}B(G)$ under the map ψ_1 defined in 3.4. We repeat its definition below.
For a $G-$set S, let $AS = \{as : a \in A, s \in S\}$ be the set of formal products. Thus, $a_1s_1 = a_2s_2$ if and only if $s_1 = s_2$ and $a_1 = a_2$. We let AG act on AS as: $(bg)(as) = (ab)(gs)$ for all $bg \in AG$ and $s \in S$. Then, AS becomes an $A-$fibred $G-$set and we have a well-defined map
$\psi_1 : B(G) \to B(A, G)$ given by $[S] \mapsto [AS]$ for any $G-$set S. In 3.4 we proved that ψ_1 is a unital ring monomorphism and $\psi_1([G/V]) = [A_\tau G/V]$ for any $V \leq G$ where τ is the trivial group homomorphism $V \to A$. In the following remark we consider the $\mathbb{C}-$linear extension of ψ_1 for which we still use the same notation.

Remark 8.1 *(i) For any $G-$set S and $(H, h) \in el(A, G)$ we have*

$$S^G_{H,h}(\psi_1([S])) = S^G_H([S]).$$

(ii) For any primitive idempotent e_H^G of $\mathbb{C}B(G)$;

$$\psi_1(e_H^G) = \sum_h e_{H,h}^G$$

where h runs over all distinct representatives of $N(H)-$orbits of the $N(H)-$set $H/O(H)$.

Proof: (i) It suffices to prove the desired result for transitive $G-$sets. Take $S = G/V$. Then using $\psi_1([G/V]) = [A_\tau G/V]$;

$$S_{H,h}^G(\psi_1([G/V])) = S_{H,h}^G([A_\tau G/V]) = \sum_{gV \subseteq G, H \leq {}^gV} 1 = S_H^G([G/V]).$$

(ii) For some complex numbers $\lambda_{K,k}$;

$$\psi_1(e_H^G) = \sum_{(K,k) \in_G el(A,G)} \lambda_{K,k} e_{K,k}^G,$$

where by part (i)

$$\lambda_{K,k} = S_{K,k}^G(\psi_1(e_H^G)) = S_K^G(e_H^G) = \begin{cases} 1, & H =_G K \\ 0, & \text{otherwise.} \end{cases}$$

Hence the result follows.

□

8.2 $B(A,G) \to B(G)$

We find the images of the primitive idempotents of $\mathbb{C}B(A,G)$ under the map ψ_2 defined in 3.5. Remember that ψ_2 is a unital ring epimorphism from $B(A,G)$ to $B(G)$ given by $\psi_2([A_\nu G/V]) = [G/V]$ for any $(V,\nu) \in ch(A,G)$.

Remark 8.2 *(i) For any $A-$fibred $G-$set $S = AX$ and $H \leq G$;*

$$S_H^G(\psi_2([S])) = S_{H,1}^G([S]).$$

(ii) For any primitive idempotent $e_{H,h}^G$ of $\mathbb{C}B(A,G)$;

$$\psi_2(e_{H,h}^G) = \begin{cases} e_H^G, & h \in O(H) \\ 0, & \textit{otherwise.} \end{cases}$$

Proof: (i) There is no loss in taking S to be transitive. Let $S = A_\nu G/V$. Then using $\psi_2([A_\nu G/V]) = [G/V]$,
$S^G_H(\psi_2([A_\nu G/V])) = S^G_H([G/V]) = S^G_{H,1}([A_\nu G/V])$.
(ii) For some complex numbers λ_K;

$$\psi_2(e^G_{H,h}) = \sum_{K \leq_G G} \lambda_K e^G_K.$$

Then using part (i);
$\lambda_K = S^G_K(\psi_2(e^G_{H,h})) = S^G_{K,1}(e^G_{H,h}) = \begin{cases} 1, & (H,h) =_G (K,1) \\ 0, & \text{otherwise.} \end{cases}$
Hence, the result follows.

□

8.3 $B(A,G) \to B(A',G)$

Let A and A' be two cyclic groups, and f be a group homomorphism from A to A'. So f is given by $f(a) = b^n$ for some natural number n where a and b are the respective generators of A and A'. Note that $f \circ \nu \in Hom(V, A')$ for any $\nu \in Hom(V, A)$. Thus, we can transform the transitive $A-$fibred $G-$set $A_\nu G/V$ to the transitive $A'-$fibred $G-$set $A'_{f\circ\nu}G/V$.

Lemma 8.3 *Let $(V,\nu), (W,\omega) \in ch(A,G)$ and $g \in G$. Then we have*
(i) ${}^g(f \circ \nu) = f \circ {}^g\nu \in Hom({}^gV, A')$.
(ii) $f \circ (\nu.\omega) = (f \circ \nu).(f \circ \omega) \in Hom(V \cap W, A')$.
(iii) If $A_\nu G/V \simeq_{AG} A_\omega G/W$, then $A'_{f\circ\nu}G/V \simeq_{A'G} A'_{f\circ\omega}G/W$.

Proof: (i) For any $v \in V$ we compute that

$${}^g(f \circ \nu)({}^gv) = f \circ \nu(v) = f \circ {}^g\nu({}^gv).$$

(ii) It is clear.
(iii) $A_\nu G/V \simeq_{AG} A_\omega G/W$ if and only if $(V,\nu) =_G (W,\omega)$. Then $({}^gV, {}^g\nu) = (W,\omega)$

for some $g \in G$. But then by part (i);

$$(W, f \circ \omega) = ({}^gV, f \circ {}^g\nu) = ({}^gV, {}^g(f \circ \nu))$$

implying that $A'_{f\circ\nu}G/V \simeq_{A'G} A'_{f\circ\omega}G/W$.

$\square$

Now by the above lemma (iii), we have a well-defined map $\psi : B(A,G) \to B(A',G)$ given by $\psi([A_\nu G/V]) = [A'_{f\circ\nu}G/V]$ for all $(V,\nu) \in ch(A,G)$.

Theorem 8.4 *(i) ψ is a (unital) ring homomorphism.*
(ii) For any $A-$fibred $G-$set $S = AX$ and $(H,h) \in el(A',G)$;

$${}_{A'}S^G_{H,h}(\psi([S])) = {}_AS^G_{H,h^n}([S]).$$

(iii) For any $(K,k) \in el(A,G)$;

$$\psi({}_Ae^G_{K,k}) = \sum_{h^n =_{N_G(K)} k} {}_{A'}e^G_{K,h}.$$

Proof: (i) By using the part (i) and (ii) of the above lemma;

$$\psi([A_\nu G/V][A_\omega G/W]) = \psi(\sum_{VgW \subseteq G} [A_{\nu.{}^g\omega}G/V \cap {}^gW])$$

$$= \sum_{VgW \subseteq G} [A'_{f\circ(\nu.{}^g\omega)}G/V \cap {}^gW]$$

$$= \sum_{VgW \subseteq G} [A'_{(f\circ\nu).{}^g(f\circ\omega)}G/V \cap {}^gW]$$

$$= [A'_{f\circ\nu}G/V][A'_{f\circ\omega}G/W] = \psi([A_\nu G/V])\psi([A_\omega G/W]).$$

So ψ is multiplicative and hence a ring homomorphism.
(ii) Since $f : A =< a >\to A' =< b >$ with $f(a) = b^n$, we have $f \circ \nu(v) = \nu(v^n)$ for any (V,ν) in $ch(A,G)$ and $v \in V$. By this observation;

$${}_{A'}S^G_{H,h}(\psi([A_\nu G/V])) = {}_{A'}S^G_{H,h}([A'_{f\circ\nu}G/V])$$

$$= \sum_{gV \subseteq G, H \leq {}^gV} {}^g(f \circ \nu)(h)$$
$$= \sum_{gV \subseteq G, H \leq {}^gV} {}^g\nu(h^n)$$
$$= {}_AS^G_{H,h^n}([A_\nu G/V]).$$

(iii) For some elements $\lambda_{H,h}$ of $\mathbb{C}$;

$$\psi({}_Ae^G_{K,k}) = \sum_{(H,h) \in_G el(A',G)} \lambda_{H,h\,A'}e^G_{H,h}.$$

Then by part (ii);

$$\lambda_{H,h} = {}_{A'}S^G_{H,h}(\psi({}_Ae^G_{K,k})) = {}_AS^G_{H,h^n}({}_Ae^G_{K,k}) = \begin{cases} 1, & (H,h^n) =_G (K,k) \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\psi({}_Ae^G_{K,k}) = \sum_{(H,h) \in_G el(A',G), (H,h^n) =_G (K,k)} {}_{A'}e^G_{H,h} = \sum_{h^n =_{N_G(K)} k} {}_{A'}e^G_{K,h}.$$

□

8.4 $B(A,H) \to B(A,G)$

Let G and H be finite groups and $\alpha : G \to H$ be a group homomorphism. Then any $H-$set S can be viewed as a $G-$set with the G action on S:

$$gs \mapsto \alpha(g)s.$$

Hence any $A-$fibred $H-$set S can be viewed as an $A-$fibred $G-$set with the same action of A and the above action of G. Let $\alpha^*(S)$ denote this new fibred set. Of course, as a set $\alpha^*(S) = S$.
It is clear that if $S \simeq_{AH} T$ then $\alpha^*(S) \simeq_{AG} \alpha^*(T)$. Therefore, we have a well-defined map

$$\psi : B(A,H) \to B(A,G) \qquad \text{where} \qquad [S] \mapsto [\alpha^*(S)]$$

for any $A-$fibred $H-$set S.

Theorem 8.5 *(i) The map ψ is a unital ring homomorphism.*
(ii) If α is surjective, then for any $V \leq H$ and $\nu \in Hom(V, A)$

$$\psi([A_\nu H/V]) = \frac{|\alpha^{-1}(V)|}{|Ker(\alpha)||V|}[A_{\nu\circ\alpha}G/\alpha^{-1}(V)].$$

(iii) For any $A-$fibred $H-$set $S = AX$ and $(W, w) \in el(A, G)$;

$$S^G_{W,w}(\psi([S])) = S^H_{\alpha(W),\alpha(w)}([S]).$$

(iv) For any primitive idempotent $e^H_{V,v}$ of $\mathbb{C}B(A, H)$

$$\psi(e^H_{V,v}) = \sum_{(W,w)\in_G el(A,G),(\alpha(W),\alpha(w))=_H(V,v)} e^G_{W,w}.$$

Proof: (i) It is clear because $\psi(S) = S$ for any $A-$fibred $H-$set.
(ii) Remember $[A_\nu H/V]$ represents the $AH-$isomorphism class of the $A-$free $AH-$set $AH/\triangle_{(V,\nu)}$ where $\triangle_{(V,\nu)}$ is the subgroup $\{\nu(v^{-1})v : v \in V\}$ of AH.
Find the $AG-$stabilizer of an element $ah\triangle_{(V,\nu)}$ of $\alpha^*(AH/\triangle_{(V,\nu)}) = AH/\triangle_{(V,\nu)}$: bk is in the stabilizer if and only if $(bk)ah\triangle_{(V,\nu)} = ah\triangle_{(V,\nu)}$. But then since $b\alpha(k)h\triangle_{(V,\nu)} = h\triangle_{(V,\nu)}$ is equivalent to $bh^{-1}\alpha(k)h \in \triangle_{(V,\nu)}$, we have $h^{-1}\alpha(k)h = v \in V$ and $b = \nu(v^{-1})$. So $k \in \alpha^{-1}({}^hV)$. If α is surjective then there exists an $x \in G$ such that $\alpha(x) = h$. Then $b = \nu(v^{-1}) = \nu(h^{-1}\alpha(k^{-1})h) = \nu(\alpha(x^{-1}k^{-1}x)) = {}^x\nu \circ \alpha(k^{-1})$ and $k \in {}^x\alpha^{-1}(V)$. Hence;
$stab_{AG}(ah\triangle_{(V,\nu)}) = \{{}^x\nu \circ \alpha(k^{-1})k : k \in {}^x\alpha^{-1}(V)\} = \triangle_{(\alpha^{-1}(V),\nu\circ\alpha)}$. Therefore we must have $\psi([A_\nu H/V]) = n[A_{\nu\circ\alpha}G/\alpha^{-1}(V)]$ for some natural number n because the $AG-$stabilizer of $ah\triangle_{(V,\nu)}$ does not depend on ah up to isomorphism of $AG-$sets. Moreover n can be determined by counting the sizes of both sides

$$|A_\nu H/V]| = n|[A_{\nu\circ\alpha}G/\alpha^{-1}(V)]|,$$

$$|A||H : V| = n|A||G : \alpha^{-1}(V)|.$$

(iii) Since $\alpha^*(AX) = AX$ for any $A-$fibred $H-$set $S = AX$, we have;
(a) $g \in stab_G(Ax)$ if and only if $gAx = Ax$ which is to say that $\alpha(g)Ax = Ax$, or equivalently $\alpha(x) \in stab_H(Ax)$. So $\alpha(stab_G(Ax)) = stab_H(Ax) \cap \alpha(G)$.
(b) For $W \leq G$; $W \leq stab_G(Ax)$ if and only if $\alpha(W) \leq stab_H(Ax)$.

(c) For any $x \in X$, let ϑ_x^G and ϑ_x^H denote the uniquely determined elements of $Hom(stab_G(Ax), A)$ and $Hom(stab_H(Ax), A)$ by the conditions $gx = \vartheta_x^G(g)x$ and $hx = \vartheta_x^H(h)x$. We can compute that for any $w \in W \leq stab_G(Ax)$ we have;
$\vartheta_x^G(w)x = wx = \alpha(w)x = \vartheta_x^H(\alpha(w))x$ implying that $\vartheta_x^G(w) = \vartheta_x^H(\alpha(w))$. Now

$$S^G_{W,w}(\psi([S])) = S^G_{W,w}(\alpha^*(AX)) = \sum_{x \in X, W \leq stab_G(Ax)} \vartheta_x^G(w)$$

$$= \sum_{x \in X, \alpha(W) \leq stab_H(Ax)} \vartheta_x^H(\alpha(w)) = S^H_{\alpha(W),\alpha(w)}(AX) = S^H_{\alpha(W),\alpha(w)}([S]).$$

(iv) Clearly for some complex numbers $\lambda_{W,w}$ we have

$$\psi(e^H_{V,v}) = \sum_{(W,w) \in_G el(A,G)} \lambda_{W,w} e^G_{W,w},$$

where (by using part (iii))
$\lambda_{W,w} = S^G_{W,w}(\alpha^*(e^H_{V,v})) = S^H_{\alpha(W),\alpha(w)}(e^H_{V,v})$.
Then the result follows because $S^H_{\alpha(W),\alpha(w)}(e^H_{V,v})$ takes only two values 0 or 1 (takes value 1 if and only if $(\alpha(W), \alpha(w)) =_H (V, v)$).

□

8.5 $B(A,G) \to B(G)$

Suppose $S = AX$ is an $A-$fibred $G-$set. Hence, in particular S is a $G-$set. Moreover, for $A-$fibred $G-$sets S and T, if $S \simeq_{AG} T$ then $S \simeq_G T$. Thus, we have a well-defined map
$\psi : B(A,G) \to B(G)$ given by $\psi([S]) = [_GS]$ for any $A-$fibred $G-$set S where the notation $_GS$ means that we regard S as a $G-$set.

Remark 8.6 *(i) ψ is a $\mathbb{Z}-$module homomorphism.*
(ii) For any $(V,\nu) \in ch(A,G)$

$$\psi([A_\nu G/V]) = \frac{|A|}{|V : Ker\nu|}[G/Ker\nu].$$

(iii) ψ is not multiplicative, not injective, not surjective.

Proof: (i) Obvious.
(ii) $A_\nu G/V = AG/\triangle_{(V,\nu)}$ where $\triangle_{(V,\nu)} = \{\nu(v^{-1})v : v \in V\}$ which is a subgroup of AG. Put $\triangle_{(V,\nu)} = \triangle$.
We find the $G-$stabilizer of an element $ag\triangle$ of $A_\nu G/V$;
$h \in G$ is in the stabilizer if and only if $ahg\triangle = ag\triangle$ which is equivalent to $g^{-1}hg \in \triangle$. By the definition of $\triangle$, $g^{-1}hg \in \triangle$ if and only if $g^{-1}hg \in V$ and $\nu((g^{-1}hg)^{-1}) = 1$ which is to say that $h \in {}^gV$ and $h \in Ker^g\nu = {}^g(Ker\nu)$, or equivalently $h \in {}^g(Ker\nu)$. Hence, $orb_G(ag\triangle) \simeq_G G/Ker\nu$ which does not depend on ag. Therefore $A_\nu G/V = nG/Ker\nu$ for some natural number n which can be determined by counting of the elements of both sets.
(iii) Obvious from part (ii).

□

8.6 $B(A,G) \to B(A,AG)$

Suppose $S = AX$ is an $A-$fibred $G-$set. Thus S is an $AG-$set which is $A-$free implying that it is also an $A-$fibred $AG-$set. It is obvious that if S and T are isomorphic $A-$fibred $G-$sets, then they are also isomorphic as $A-$fibred $AG-$sets. Hence, we have a well-defined map
$\psi : B(A,G) \to B(A,AG)$ given by $\psi([S]) = [S]$ for any $A-$fibred $G-$set S.

Remark 8.7 *(i) ψ is a $\mathbb{Z}-$module monomorphism.*
(ii) For any $(V,\nu) \in ch(A,G)$;

$$\psi([A_\nu G/V]) = [A_{\nu_A}\frac{AG}{AV}]$$

where $\nu_A \in Hom(AV,A)$ is given by $\nu_A(av) = a\nu(v)$ for all $av \in AV$.
(iii) ψ is not multiplicative, not surjective.

Proof: (i) Additivity of ψ is clear. It is injective because $A(AG)-$isomorphism implies $AG-$isomorphism.

(ii) $A_\nu G/V = AG/\triangle_{(V,\nu)}$ where $\triangle_{(V,\nu)} = \{\nu(v^{-1})v : v \in V\}$ which is a subgroup of AG. Put $\triangle_{(V,\nu)} = \triangle$.
It is clear that $A_\nu G/V$ is a transitive $A-$fibred $AG-$set and so

$$A_\nu G/V = orb_{A(AG)}(\triangle) \simeq_{A(AG)} \frac{A(AG)}{stab_{A(AG)}(\triangle)}$$

where $\triangle = 1.1\triangle \in AG/\triangle$. Now;
$a(bg) \in A(AG)$ is in the $A(AG)-$stabilizer of $\triangle$ if and only if $a(bg)\triangle = \triangle$ which is equivalent to $g \in V$ and $\nu(g^{-1}) = ab$. But this holds if and only if $bg \in AV$ and $\nu_A((bg)^{-1}) = a$, or equivalently $a(bg) \in \{\nu_A((bg)^{-1})(bg) : bg \in AV\} = \triangle_{(AV,\nu_A)}$.
Hence,
$A_\nu G/V \simeq_{A(AG)} A_{\nu_A} AG/AV$ which completes the proof.
(iii) It is clear from part (ii).

□

8.7 $B(A,G) \to B(A)$

Suppose $S = AX$ be an $A-$fibred $G-$set. Let $S \setminus G$ be the set of $G-$orbits of S. Thus

$$S \setminus G = \{orb_G(s) : s \in S\}.$$

We let A act on $S \setminus G$ as: $(a, orb_G(s)) \mapsto a(orb_G(s)) = orb_G(as)$. It is clear that isomorphic $A-$fibred $G-$sets have isomorphic (as $A-$sets) $G-$orbit sets. Therefore, we have well-defined map
$\psi : B(A,G) \to B(A)$ given by $\psi([S]) = [S \setminus G]$ for any $A-$fibred $G-$set S.

Remark 8.8 *(i) If S is a transitive $A-$fibred $G-$set, then $S \setminus G$ is a transitive $A-$set.*
(ii) For any $(V,\nu) \in ch(A,G)$;

$$\psi([A_\nu G/V]) = [A/\nu(V)].$$

(iii) ψ is a $\mathbb{Z}-$module homomorphism.
(iv) ψ is not multiplicative, not injective, not surjective.

Proof: (i) Suppose S is a transitive $A-$fibred $G-$set. Take any two elements $orb_G(s), orb_G(s')$ from $S \setminus G$. By the transitivity of S there is an $ag \in AG$ such that $ags = s'$ implying that $a(orb_G(s)) = orb_G(s')$. Thus, $S \setminus G$ is a transitive $A-$set.
(ii) Since $\psi([A_\nu G/V])$ is transitive, we must have $\psi([A_\nu G/V]) = [orb_A(orb_G(\triangle))]$ where $\triangle = \{\nu(g^{-1})g : g \in G\} \leq AG$ and $A_\nu G/V = AG/\triangle$. Now, $a \in A$ is in the $A-$stabilizer of $orb_G(\triangle)$ if and only if $orb_G(a\triangle) = orb_G(\triangle)$ which is to say that $ag\triangle = \triangle$ for some $g \in G$. But $ag\triangle = \triangle$ for some $g \in G$ if and only if $g \in V$ and $\nu(g^{-1}) = a$, or equivalently $a \in \nu(V)$. Therefore,
$\psi([A_\nu G/V]) = [orb_A(orb_G(\triangle))] = [A/stab_A(orb_G(\triangle))] = [A/\nu(V)]$.
(iii) and (iv) They are clear from part (ii).

$\square$

8.8 $B(A, G) \to B(AG)$

Let $\nu : V \to A$ be a group homomorphism. Then observing ${}^g\nu({}^gV) = \nu(V)$ we can see the map
$\psi : B(A, G) \to B(AG)$ given by $[A_\nu G/V] \mapsto [\frac{AG}{\nu(V)V}]$ is a well-defined map which is not multiplicative.

8.9 $B(A_1 \times A_2, G) \to B(A_1, G) \times B(A_2, G)$

Since our fibre group A is abelian, it is of interest to consider a direct product decomposition $A = A_1 \times A_2$. Let π_1 and π_2 be the respective projections from A to A_1 and A_2. Define a map
$\psi : B(A, G) \to B(A_1, G) \times B(A_2, G)$ given for all $(V, \nu) \in ch(A, G)$ by $\psi([A_\nu G/V]) = ([A_{1_{\pi_1 \circ \nu}} G/V], [A_{2_{\pi_2 \circ \nu}} G/V])$.

Remark 8.9 *(i)* $\pi_i \circ (\nu.\mu) = (\pi_i \circ \nu).(\pi_i \circ \mu)$ *for any* $H, K \leq G$, $\nu \in Hom(H, A)$,

and $\mu \in Hom(K, A)$ *where* $i = 1, 2$.
(ii) For any $g \in G$, $H \leq G$, *and* $\nu \in Hom(H, A)$ *we have* $\pi_i \circ ({}^g\nu) = {}^g(\pi_i \circ \nu)$ *where* $i = 1, 2$.
(iii) For any $H \leq G$, $\nu \in Hom(H, A)$ *we have* $Ker\nu \leq Ker(\pi_i \circ \nu)$ *and* $Ker\nu = Ker(\pi_1 \circ \nu) \cap Ker(\pi_2 \circ \nu)$ *where* $i = 1, 2$.
(iv) ψ *is a unital ring homomorphism.*

Proof: (i), (ii) and (iii) follow from easy calculations.
(iv) Using (i) and (ii) it is clear that ψ respects the multiplication of two transitive $A-$fibred $G-$sets. Hence, the result follows.

□

8.10 The Number Of Orbits

Let $S = AX$ be an $A-$fibred $G-$set. In particular S is an $AG-$set, a $G-$set, and an $A-$set. So we have the following three maps:
$O_{AG} : B(A, G) \to \mathbb{Z}$, $O_{AG}([S])$ =the number of $AG-$orbits of S
$O_G : B(A, G) \to \mathbb{Z}$, $O_G([S])$ =the number of $G-$orbits of S
$O_A : B(A, G) \to \mathbb{Z}$, $O_A([S])$ =the number of $A-$orbits of S.

Remark 8.10 *(i)* O_{AG} *is a* $\mathbb{Z}-$*module homomorphism.*
(ii) $O_{AG}([AX]) = \frac{1}{|A||G|} \sum_{ag \in AG} |AX^{<ag>}|$.
(iii) $O_{AG}([A_\nu G/V][A_\omega G/W])$ *=the number of double coset representatives of* (V, W) *in* G.

Proof: For (ii) we use the result of Burnside which counts the number of orbits (it is stated at the beginning of chapter 1). The other parts are trivial.

□

Remark 8.11 *(i)* O_G *is a* $\mathbb{Z}-$*module homomorphism.*
(ii) $O_G([A_\nu G/V]) = \frac{|A|}{|V:Ker\nu|}$.

Proof: (i) is trivial, and (ii) follows from the result of Burnside which counts the number of orbits.

□

Remark 8.12 *(i)* O_A *is a* $\mathbb{Z}-$*algebra homomorphism.*
(ii) $O_A([AX]) = |X|$.
(iii) $O_A([A_\nu G/V]) = |G : V|$.

Proof: Since any $A-$fibred $G-$set is an $A-$free set, the results are straightforward.

□

Since O_A is multiplicative, it is more important than the first two maps. Some properties of the map O_A is given in the next remark.

Remark 8.13 *(i)* $KerO_A = J(G, 1, 0) = \{x \in B(A, G) : S^G_{G,1}(x) = 0\}$.
(ii) If $x \in B(A, G)$ *is unit in* $B(A, G)$, *then* $O_A(x) \in \{-1, +1\}$.

Proof: (i) $KerO_A$ is a prime ideal of $B(A, G)$ which is not maximal. Hence, we know from chapter 7 that $KerO_A = J(H, h, 0)$ for some $(H, h) \in el(A, G)$. However, it is clear that $[A_\nu G/V] - [A_\mu G/V] \in KerO_A$ for all $V \leq G$ and $\nu, \mu \in Hom(V, A)$. Thus $KerO_A$ must be $J(G, 1, 0)$.
(ii) It is obvious.

□

Chapter 9

The Ring $B(A, G)$

In this short chapter we study some ring theoretic properties of the monomial Burnside rings. We are assuming that A is a finite cyclic group regarded as a subgroup of $\mathbb{C}^*$. As with the previous chapter, this is a compendium of further results and observations, recorded with a view to subsequent development.

Recall that for any $(H, h) \in el(A, G)$, $S^G_{H,h} : B(A, G) \to \mathbb{C}$ is the ring homomorphism given for any $(V, \nu) \in ch(A, G)$ by $S^G_{H,h}([A_\nu G/V]) = \sum_{gV \subseteq G, H \leq {}^gV} {}^g\nu(h)$. Also the injectivity of the product map $\prod_{(H,h)\in_G el(A,G)} S^G_{H,h} : B(A, G) \to \prod_{(H,h)\in_G el(A,G)} \mathbb{C}$ implies that two elements x and y of $B(A, G)$ are equal if and only if $S^G_{H,h}(x) = S^G_{H,h}(y)$ for all $(H, h) \in el(A, G)$.

Let R be a unital subring of S. We write ${}_RS$ to imply that we are regarding S as an $R-$module. We have $\mathbb{Z} \leq B(G) \leq B(A, G)$ via the embeddings $1 \mapsto [G/G]$, $[G/V] \mapsto [A_\tau G/V]$. So for instance if we write ${}_{B(G)}B(A, G)$ this means that we are regarding $B(A, G)$ as a $B(G)-$module.

Remark 9.1 *${}_\mathbb{Z}B(G)$, ${}_{B(G)}B(G)$, ${}_\mathbb{Z}B(A, G)$, ${}_{B(G)}B(A, G)$, ${}_{B(A,G)}B(A, G)$ are all Noetherian modules but not Artinian.*

Proof: Since both $B(G)$ and $B(A, G)$ are finite over $\mathbb{Z}$, the results follow because $\mathbb{Z}$ is a Noetherian but not Artinian (as a ring, or equivalently as a module over itself).

$\square$

Remark 9.2 *The only nilpotent element of $B(A,G)$ is 0. So, the nilradical of the ring $B(A,G)$ is the 0 ideal.*

Proof: Let $x \in B(A,G)$ be a nilpotent element. Then $x^n = 0$ for some natural number n. We know from 5.1 that

$$x = \sum_{(H,h)\in_G el(A,G)} S^G_{H,h}(x) e^G_{H,h}.$$

Then it follows from $x^n = 0$ that

$$0 = \sum_{(H,h)\in_G el(A,G)} (S^G_{H,h}(x))^n e^G_{H,h}$$

which implies $S^G_{H,h}(x) = 0$ for all $(H,h) \in el(A,G)$ because $S^G_{H,h}(x)$ is a complex number for any $(H,h) \in el(A,G)$. Hence, $x = 0$.

$\square$

Remark 9.3 *The Jacobson radical of $B(A,G)$ is the 0 ideal.*

Proof: Let $R \leq S$ be a ring extension of commutative rings. Then from 7.4(Maximality) it is clear that $J(R) = R \cap J(S)$ where for any ring T, $J(T)$ denotes the Jacobson radical of the ring T. Let D be as in chapter 7. Using the integrality of the ring extension $DB(A,G) \leq \prod_{(H,h)\in_G el(A,G)} D$ we conclude that $J(DB(A,G)) = 0$ because $J(D) = 0$. Hence, $J(B(A,G)) = 0$ because $B(A,G) \leq DB(A,G)$ is an integral extension.

$\square$

Remark 9.4 *If $x \in B(A,G)$ is a zero divisor, then $S^G_{H,h}(x) = 0$ for some $(H,h) \in el(A,G)$. Hence the zero divisors of $B(A,G)$ belong to the union of the minimal prime ideals of $B(A,G)$.*

Proof: Let x be a nonzero zero divisor of $B(A,G)$. Then there is a nonzero y in B(A,G) such that $xy = 0$. Now using (see 5.1)

$$x = \sum_{(H,h)\in_G el(A,G)} S^G_{H,h}(x)e^G_{H,h},\ y = \sum_{(H,h)\in_G el(A,G)} S^G_{H,h}(y)e^G_{H,h}$$

we see from $xy = 0$ that $S^G_{H,h}(x)S^G_{H,h}(y) = 0$ for all $(H,h) \in el(A,G)$. Since y is nonzero, the result follows.

□

Remark 9.5 *Any ring homomorphism $\psi : B(A,G) \to \mathbb{C}$ is of the form $S^G_{H,h}$ for some $(H,h) \in el(A,G)$.*

Proof: The map ψ extends by $\mathbb{C}-$linear extension to a $\mathbb{C}-$algebra map from $\mathbb{C}B(A,G)$ to $\mathbb{C}$. So the result follows from 4.11 for which we referred [1]. The same result appears also in [9].

□

Remark 9.6 *An element $x \in B(A,G)$ belongs to $B(G)$ if and only if $S^G_{H,h}(x) = S^G_{H,1}(x)$ for all $(H,h) \in el(A,G)$.*

Proof: See [1].

□

Remark 9.7 *Let $\psi : B(A,G) \to B(A,G)$ be a ring homomorphism. Then for any $(H,h) \in el(A,G)$ there exists a $(K,k) \in el(A,G)$ such that $S^G_{H,h} \circ \psi = S^G_{K,k}$. Hence any ring endomorphism ψ of $B(A,G)$ induces a map $\hat{\psi} : el(A,G) \to el(A,G)$ given by the condition: $S^G_{H,h} \circ \psi = S^G_{\hat{\psi}((H,h))}$ for all $(H,h) \in el(A,G)$.*

Proof: It is obvious since any ring homomorphism from $B(A,G)$ to $\mathbb{C}$ is of the form $S^G_{H,h}$ for some $(H,h) \in el(A,G)$.

□

Let $G\backslash el(A,G)$ denote the set of $G-$orbit representatives of the $G-$set $el(A,G)$. By the previous remark any ring endomorphism of $B(A,G)$ induces a map from

$G \setminus el(A,G)$ to $G \setminus el(A,G)$. However, the converse may not be true since given any map $\hat{\psi} : G \setminus el(A,G) \to G \setminus el(A,G)$ the image of the map ψ is in general belongs to $\mathbb{C}B(A,G)$.

Remark 9.8 *Let $\hat{\psi} : G \setminus el(A,G) \to G \setminus el(A,G)$ be any map. Define $\psi : \mathbb{C}B(A,G) \to \mathbb{C}B(A,G)$ by the condition: $S^G_{H,h} \circ \psi = S^G_{\hat{\psi}((H,h))}$ for all $(H,h) \in el(A,G)$. Then ψ is a ring homomorphism.*

Proof: Take any $(H,h) \in el(A,G)$. Then for any $x, y \in B(A,G)$

$$S^G_{H,h}(\psi(x+y)) = S^G_{\hat{\psi}((H,h))}(x+y) = S^G_{\hat{\psi}((H,h))}(x) + S^G_{\hat{\psi}((H,h))}(y)$$

$$= S^G_{H,h}(\psi(x)) + S^G_{H,h}(\psi(y)) = S^G_{H,h}(\psi(x)+\psi(y)).$$

Since it is true for all $(H,h) \in el(A,G)$, $\psi(x+y) = \psi(x) + \psi(y)$. Similar calculations shows that $\psi(xy) = \psi(x)\psi(y)$.

$\square$

The previous two remarks imply that there are finitely many (at most $|G \setminus el(A,G)|$) ring endomorphisms of $B(A,G)$.

Theorem 9.9 *Let $\psi : B(A,G) \to B(A,G)$ be a ring homomorphism. Then ψ is injective if and only if it is surjective. In fact, it is almost true also for $\mathbb{Z}-$module endomorphisms of $B(A,G)$. If $\psi : B(A,G) \to B(A,G)$ is a surjective $\mathbb{Z}-$module homomorphism, then ψ is injective.*

Proof: ($\Rightarrow$) Suppose ψ is injective. Since there are finitely many ring endomorphisms of $B(A,G)$, not all of $\psi, \psi^2, ..., \psi^n, ...$ can be distinct. So there are natural numbers n_1, n_2 such that $\psi^{n_1} = \psi^{n_2}$. To show that ψ is surjective, take any $x \in B(A,G)$. Then $\psi^{n_1}(x) = \psi^{n_2}(x)$ implying by the injectivity of ψ that $\psi^{n_1-1}(x) = \psi^{n_2-1}(x)$. Using injectivity of ψ inductively we get $x = \psi^m(x) = \psi(\psi^{m-1}(x))$. So ψ is surjective.
($\Leftarrow$) Suppose ψ is surjective. Since $B(A,G)$ is Noetherian (as both $\mathbb{Z}-$module and $B(A,G)-$module), the chain $Ker\psi \subseteq Ker\psi^2 \subseteq ...Ker\psi^n \subseteq ...$ cannot be

infinite. So there is a natural number n such that $Ker\psi^n = Ker\psi^{n+1}$. We show that $Ker\psi^{n-1} = Ker\psi^n$. Let $x \in Ker\psi^n$. Then by the surjectivity of ψ, there is a $y \in B(A,G)$ such that $\psi(y) = x$. Then $\psi^{n+1}(y) = \psi^n(x)$ and so $y \in Ker\psi^{n+1} = Ker\psi^n$ implying that $\psi^{n-1}(x) = \psi^{n-1}(\psi(y)) = \psi^n(y)$. So $x \in Ker\psi^{n-1}$, and hence $Ker\psi^{n-1} = Ker\psi^n$. Proceeding in this way we can show that $Ker\psi^2 = Ker\psi$. To show that ψ is injective, take any $z \in Ker\psi$. Then since ψ is surjective, there is a $t \in B(A,G)$ such that $z = \psi(t)$. Now from $\psi(z) = \psi^2(t)$ it follows that $t \in Ker\psi^2 = Ker\psi$ and so $z = \psi(t) = 0$. Thus ψ is injective.

$\square$

Remark 9.10 *B(A,G) has no minimal ideals.*

Proof: It is trivial because $\mathbb{Z}$ is embeddable in $B(A,G)$.

$\square$

Lastly we show below that we can extend some maps from $B(G)$ to $B(A,G)$. The following result is immediate from 8.1. We give an alternative proof that does not require the classification of the species of the monomial Burnside algebras.

Remark 9.11 *Let $\mathbb{K}$ be an algebraically closed field. Then any ring homomorphism $\psi : B(G) \to \mathbb{K}$ can be extended to a ring homomorphism $\tilde{\psi} : B(A,G) \to \mathbb{K}$.*

Proof: Let $\psi : B(G) \to \mathbb{K}$ be a ring homomorphism which is nonzero (otherwise the result is trivial). Then $B(G)/Ker\psi$ is a subring of $\mathbb{K}$, and so it is an integral domain. Put $P = Ker\psi$ which is a prime ideal of $B(G)$. Let $S = B(G) - P$. Then S is a proper multiplicative subset of both $B(G)$ and $B(A,G)$. We consider the ring of fractions of $B(G)$ and $B(A,G)$ with respect to S. For notations and details about the ring of fractions of commutative rings see chapter 7. So now we have two new rings $B(G)_P$ and $S^{-1}B(A,G)$.
Define $\psi_1 : B(G)_P \to \mathbb{K}$ as $\psi_1(\frac{a}{s}) = \psi(a)\psi(s)^{-1}$ for all $\frac{a}{s} \in B(G)_P$. It can be

checked easily that ψ_1 is a ring homomorphism. Now $Ker\psi_1$ is a proper ideal of $B(G)_P$, and so it is contained in a maximal ideal of $B(G)_P$. However $B(G)_P$ is a local ring with its unique maximal ideal P^E. Hence, $Ker\psi_1 \subseteq P^E$. On the other hand, if $\frac{a}{s} \in P^E$ then there is a $b \in P$ and $t \in S$ such that $\frac{a}{s} = \frac{b}{t}$ and so $\psi_1(\frac{a}{s}) = \psi_1(\frac{b}{t})$ implying that $\psi(b)\psi(t)^{-1} = \psi(a)\psi(s)^{-1}$. But $b \in P = Ker\psi$ gives that $\frac{a}{s} \in Ker\psi_1 = P^E$. Hence, $P^E = Ker\psi_1$. As a result, $B(G)_P/P^E = B(G)_P/Ker\psi_1$ is a field.

Note that $B(G)_P \leq S^{-1}B(A,G)$ is an integral ring extension. Also $Ker\psi_1 = P^E$ is a maximal ideal of $B(G)_P$. Then by 7.4(Lying Over) there is a maximal ideal $\mathfrak{a}$ of $S^{-1}B(A,G)$ such that $Ker\psi_1 = P^E = \mathfrak{a} \cap B(G)_P$.

Define $\phi : B(G)_P/P^E \to S^{-1}B(A,G)/\mathfrak{a}$ as $\phi(\frac{a}{s} + P^E) = \frac{a}{s} + \mathfrak{a}$ for all $\frac{a}{s} + P^E \in B(G)_P/P^E$. Then from $P^E = \mathfrak{a} \cap B(G)_P$ it follows that ϕ is well-defined, and it can be checked that ϕ is a ring monomorphism. Hence, $B(G)_P/P^E \leq S^{-1}B(A,G)/\mathfrak{a}$ is a field extension. Moreover it must be an algebraic field extension because $B(G)_P \leq S^{-1}B(A,G)$ is an integral ring extension.

Define $\psi_2 : B(G)_P/P^E \to \mathbb{K}$ as $\psi_2(\frac{a}{s} + P^E) = \psi_1(\frac{a}{s})$ for all $\frac{a}{s} + P^E \in B(G)_P/P^E$. It is a well-known fact from the field theory that any ring homomorphism from a field $\mathbb{F}$ to an algebraically closed field $\mathbb{K}$ can be extended to a ring homomorphism from $\mathbb{F}'$ to $\mathbb{K}$ if $\mathbb{F} \leq \mathbb{F}'$ is an algebraic field extension.

So, there is a ring homomorphism $\psi_3 : S^{-1}B(A,G)/\mathfrak{a} \to \mathbb{K}$ extending the ring homomorphism $\psi_2 : B(G)_P/P^E \to \mathbb{K}$.

Define $\psi_4 : S^{-1}B(A,G) \to \mathbb{K}$ as $\psi_4(\frac{x}{s}) = \psi_3(\frac{x}{s} + \mathfrak{a})$ for all $\frac{x}{s} \in S^{-1}B(A,G)$. It is clear that ψ_4 is a ring homomorphism.

Define $\psi_5 : B(A,G) \to \mathbb{K}$ as $\psi_5(x) = \psi_4(\frac{x}{1})$ for all $x \in B(A,G)$. It can checked that $\tilde{\psi} = \psi_5$ is a ring homomorphism extending ψ.

$\square$

By the extension procedure given in the last proof we can extend the ring homomorphisms $S_H^G : B(G) \to \mathbb{Z} \leq \mathbb{C}$, $S_H^G([S]) = |S^H|$ to ring homomorphisms from $B(A,G)$ to $\mathbb{C}$. For example, by using the above procedure if we extend S_H^G we get $S_{H,h}^G$ where $h \in H$ is arbitrary, as we already know from 8.1.

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