Integral Action Controllers for Systems with Time Delays*

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Summary. Consider a stabilizing controller C_1 for a given plant P. If C_1 and P do not have any zeros at the origin, then one can use a cascade connected PI (proportional plus integral) controller C_{pi} with C_1 and keep the feedback system stable. In this work we examine the allowable range of the integral action gain in C_{pi}, and discuss how C_1 should be chosen to maximize this range for systems with time delays.

1 Introduction

In the design of feedback controllers it is often desirable to use an integrator to be able to track constant reference signals. For example, internal model principle says that the controller must include a copy of the reference signal (or disturbance) generator in order to have a robust tracking (or disturbance rejection), see e.g. [2, 4, 7]. Typically, the reference generator $G_r(s)$ is an unstable system: an integrator (resp. oscillator) if the reference is a constant (resp. a sinusoidal signal). One way to achieve robust asymptotic tracking (or disturbance rejection) is to append G_r to the plant P and then design a controller C_o for the combined "plant" G_rP. Thus $C = C_oG_r$ is a stabilizing controller for P and it achieves the performance objectives, see e.g. [1, 16] for more details.

In this chapter we consider the dual problem: first design a stabilizing controller for the plant, then append a PI term to this controller. A similar problem has been discussed in [3] for finite dimensional systems. Briefly, the problem we deal with can be stated as follows: let C_1 be a stabilizing controller for a time delay system P, and append (in the form of a cascade connection) $C_{pi}(s) = \frac{(s+k_i)}{s}$ to C_1. Note that the proportional gain of the PI controller is set to unity; this is without loss of generality since a non-unity gain can be absorbed into C_1. Assume that P and C_1 do not have any zeros at the origin. Then, there exists k_i such that the feedback system is stable. We examine the range of allowable k_i, and discuss the problem of designing an optimal C_1 so that this range is maximized.

We should indicate that rather than the cascade PI-controller connection to be discussed here, a two-stage parallel connection of controllers is also possible. For example, as before, let C_1 be a a stabilizing feedback controller for a given plant P. If the PI part of the controller,

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C_{pi}, is a stabilizing controller for the new plant $P(I + C_1P)^{-1}$, then the parallel connection of the controller, $C_{pi} + C_1$, is a stabilizing controller for the original plant P, see [9, 14, 16]. One can study the problem choosing the best C_1 so that the allowable range of k_i is maximum. We leave this problem aside, because the techniques to be used in such a study would be similar to the approach taken in this chapter for the cascade connection of the controllers.

This chapter is organized as follows: stability of the feedback system under cascade connection of the PI controller is investigated in Section 2. Design of C_1, maximizing the allowable range of the integral gain, is discussed in Section 3. Concluding remarks are made in Section 4.

Notation used here is standard. In particular, the norm sign $\|\cdot\|$ stands for the $\mathcal{H}_\infty$ norm $\|\cdot\|_\infty$ whenever the argument is in $\mathcal{H}_\infty$.

2 Feedback System Stability Under Cascade Connection of the PI Controller

Consider the feedback system shown in Figure 1 with an r input r output plant whose $r \times r$ transfer matrix is $P(s)$. The $r \times r$ controller transfer matrix is $C(s)$. Assume that P is full rank. The feedback system is said to be stable if $C(1 + PC)^{-1}, PC(1 + CP)^{-1}, C(I + PC)^{-1}P, P(1 + CP)^{-1}$ are in $\mathcal{H}_\infty^{r\times r}$. In this case, we say $C \in \mathcal{S}(P)$, where $\mathcal{S}(P)$ is the set of all controllers stabilizing the feedback system with plant P.

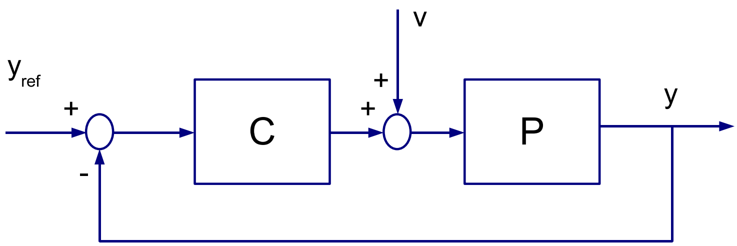

Fig. 1. Feedback System

Let C_1 be in $\mathcal{S}(P)$ and consider the cascade connection $C = C_1C_x$ for some C_x. The result stated below as Theorem 1 addresses the following question: Is the closed-loop system still stable if $C = C_1C_x$, i.e. do we have $C \in \mathcal{S}(P)$?

Theorem 1. *Let P be a given $r \times r$ plant and let $C_1 \in \mathcal{S}(P)$. Assume that P and C_1 are full rank and define the complementary sensitivity function for the feedback system with $C = C_1$ as $T_1 := PC_1(I + PC_1)^{-1}$. Then, we have the following two results:*

a) *If $C_y := C_x - I$ stabilizes $T_1 \in \mathcal{H}_\infty^{r\times r}$, then*

$$C = C_1C_x \in \mathcal{S}(P)\,. \tag{1}$$

b) *Let P and C_1 have no transmission-zeros at the origin. Choose any $\hat{K}_P$, $\hat{K}_D \in \mathbb{R}^{r\times r}$, and $\tau \in \mathbb{R}_+$. Define $\Psi \in \mathcal{H}_\infty^{r\times r}$ as*

$$\Psi(s) = \frac{T_1(s)T_1(0)^{-1} - I}{s} + T_1(s)\,(\hat{K}_P + \frac{\hat{K}_D\, s}{\tau s + 1}). \tag{2}$$

Then for $\rho \in \mathbb{R}_+$ *satisfying*

$$\rho \;<\; \|\,\Psi\,\|^{-1} =: \psi^{-1} \tag{3}$$

the controller $C = C_1 C_{pid} \in \mathcal{S}(P)$*, where* C_{pid} *is a PID-controller given by* $C_{pid} = I + \hat{C}_{pid}$*, where*

$$\hat{C}_{pid} = \rho \left(\hat{K}_P + \frac{T_1(0)^{-1}}{s} + \frac{\hat{K}_D\, s}{\tau s + 1} \right). \tag{4}$$

For $\hat{K}_D = 0$*, (4) becomes a PI-controller.* ■

Proof of Theorem 1 is given in the Appendix. By part (a) of this theorem the stabilizing controller $C = C_1 C_x$ gives rise to the following complementary sensitivity $T_x = PC(I + PC)^{-1}$:

$$\begin{aligned} T_x &= PC_1(I + C_x PC_1)^{-1} C_x = PC_1(I + PC_1 + C_y PC_1)^{-1} C_x \\ &= T_1(I + C_y T_1)^{-1}(I + C_y) = (I + T_1(I - C_x))^{-1} T_1 C_x. \end{aligned} \tag{5}$$

If $C_x = C_{pid}$ as in (4) of Theorem 1-(b), then T_x in (5) becomes

$$T = (I + T_1 \hat{C}_{pid})^{-1} T_1 C_{pid}\ , \tag{6}$$

which can be expressed as

$$\begin{aligned} T &= (\frac{s}{s+\rho} I + T_1(s)\hat{C}_{pid}(s)\frac{s}{s+\rho})^{-1} T_1(s)(\frac{s}{s+\rho} I + \frac{s}{s+\rho}\hat{C}_{pid}) \\ &= (I + \frac{\rho s}{s+\rho}\,\Psi(s)\,)^{-1} T_1(s) \frac{\rho s}{s+\rho}(I + \rho\, s(\hat{K}_P + \frac{T_1(0)^{-1}}{s} + \frac{\hat{K}_D s}{\tau s + 1})). \end{aligned} \tag{7}$$

Therefore, $T(0) = I$ and

$$\|T\| \le \|(I + \frac{\rho s}{s+\rho}\Psi)^{-1}\| \cdot \|T_1(s)\frac{s}{s+\rho}(I + \hat{C}_{pid})\|\ .$$

Writing $(I + \frac{\rho s}{s+\rho}\,\Psi\,)^{-1} = I - (I + \frac{\rho s}{s+\rho}\Psi)^{-1}\frac{\rho s}{s+\rho}\Psi$, we obtain

$$\|(I + \frac{\rho s}{s+\rho}\,\Psi(s)\,)^{-1}\| \le 1 + \rho\,\psi\,\|\,(I + \frac{\rho s}{s+\rho}\Psi)^{-1}\|$$

and hence, $\|\,(I + \frac{\rho s}{s+\rho}\,\Psi(s)\,)^{-1}\| \le (1 - \rho\,\psi)^{-1}$, and

$$\|\,T\,\| \;\le\; \frac{1}{1 - \rho\,\psi}\;\|\,T_1(s)\,\frac{s}{s+\rho}(I + \hat{C}_{pid}(s)\,)\|. \tag{8}$$

Now suppose that in the PID-controller $\hat{C}_{pid}$ we choose $\hat{K}_P = 0$, $\hat{K}_D = 0$. Then by (4), the PI-controller is

$$C_{pi}(s) = I + \frac{\rho\, T_1(0)^{-1}}{s}\ ,$$

where $\rho \in \mathbb{R}_+$ satisfies (3), i.e.,

$$\rho < \|\,\frac{T_1(s)T_1(0)^{-1} - I}{s}\,\|^{-1} =: \psi_o^{-1}\ . \tag{9}$$

In this case, the upper-bound on $\| T \|$ given in (8) becomes

$$\|T\| \leq \frac{1}{1-\rho\,\psi_o}\,\|T_1\,T_1(0)^{-1}\|\cdot\|\frac{sT_1(0)+\rho I}{s+\rho}\|.$$

In particular, if $T_1(0) = I$, i.e. C_1 and/or P contain a pole at $s = 0$, then

$$\|T\| \leq \frac{1}{1-\rho\,\psi_o}\,\|T_1\|.$$

>From the above discussion we see that if $\rho\,\psi_o \ll 1$ then the upper bound of $\|T\|$ is close to $\|T_1\|$.

3 Design of C_1 Maximizing the Integral Action Gain

In this section we discuss the design of C_1 for the largest allowable range of ρ, (9), for a class of single input single output (SISO) plants with time delays. In this case, from (9) we see that C_1 should be designed to minimize

$$\psi = \|\frac{T_1(s)T_1^{-1}(0)-1}{s}\|_\infty, \tag{10}$$

where $T_1 = PC_1(1+PC_1)^{-1}$ and $C_1 \in \mathcal{S}(P)$.

Solution of this problem will be obtained below in two steps: (i) first we solve the problem for stable plants, then (ii) we extend this solution to cover unstable plants case. In both steps we begin with inner-outer and coprime factorizations of given P, then we solve an $\mathcal{H}_\infty$ optimization problem. Inner-outer factorizations require finding $\mathbb{C}_+$ roots of a quasi-polynomial, for which several algorithms exist by now, see e.g. [5, 12, 17] and their references. Using these algorithms and the methods developed for the $\mathcal{H}_\infty$ control of general infinite dimensional systems, (see e.g. [6] and [8]) we can solve the problem in step (i) for a large class of time delay systems. We will see that the extension (ii) to unstable plants, with finitely many poles in $\mathbb{C}_+$, involves a parameterization of all suboptimal solutions of the problem in (i), and the use of Nevanlinna-Pick interpolation. The mathematical tools for these problems can be found in [6, 11, 15, 18].

3.1 Stable plants

In this section we consider stable SISO plants whose inner-outer factorizations are in the form $P = P_iP_o$ where P_i is inner (all-pass) with $P_i(0) = 1$, and P_o is outer (minimum-phase).

Example. Consider the plant with input/output delay, $h > 0$, and internal delays

$$P(s) = \left(\frac{e^{-hs}}{s+2}\right)\,\frac{(s+1)+2(s-1)e^{-2s}}{(s+3)+e^{-3s}}. \tag{11}$$

Then, the following is an inner-outer factorization:

$$P_i(s) = -e^{-hs}\,\frac{(s+1)+2(s-1)e^{-2s}}{2(s+1)+(s-1)e^{-2s}}$$

$$P_o(s) = \frac{-1}{s+2}\,\frac{2(s+1)+(s-1)e^{-2s}}{(s+3)+e^{-3s}}$$

Clearly, $P_i(0) = 1$, the poles and zeros of P_i are symmetric around the Im-axis, and P_o contains no poles or zeros in the right half plane. ■

The set of all stabilizing controllers is parameterized as

$$\mathcal{S}(P) = \{Q/(1-PQ) \;:\; Q \in \mathcal{H}_\infty \text{ and } PQ \neq 1\}.$$

Therefore, C_1 must be in the form $C_1 = Q_1/(1-PQ_1)$, where $Q_1 \in \mathcal{H}_\infty$ is free. Let

$$Q_1(s) = P_o^{-1}(s)\,\frac{Q_i(s)}{(1+\varepsilon s)^\ell} \tag{12}$$

where $Q_i \in \mathcal{H}_\infty$ is the free parameter, $\varepsilon > 0$, and ℓ is the relative degree of P_o. Then we have

$$T_1(s) = P(s)Q_1(s) = P_i(s)\,\frac{Q_i(s)}{(1+\varepsilon s)^\ell}.$$

Hence the problem of maximizing the allowable range of ρ reduces to finding

$$\begin{aligned}\psi_o &= \inf_{Q_i \in \mathcal{H}_\infty} \|(\frac{P_i(s)}{(1+\varepsilon s)^\ell} Q_i(s)Q_i^{-1}(0) - 1)/s\|_\infty \\ &= \|(\frac{P_i(s)}{(1+\varepsilon s)^\ell} Q_{i,\text{opt}}(s)Q_{i,\text{opt}}^{-1}(0) - 1)/s\|_\infty\end{aligned} \tag{13}$$

and the corresponding optimal $Q_{i,\text{opt}} \in \mathcal{H}_\infty$ solving this problem. Note that optimal solution is not unique: if $Q_{i,o}$ is a solution of (13), then so is $KQ_{i,o}$, for any non-zero constant K. Therefore, we define the normalized free parameter $\widetilde{Q}_i(s) = Q_i(s)Q_i^{-1}(0)$, and try to find optimal $\widetilde{Q}_i$ in the problem (14) defined below. First let $\mathcal{H}_\infty^o = \{\widetilde{Q}_i \in \mathcal{H}_\infty \;:\; \widetilde{Q}_i(0) = 1\}$. Note that

$$\begin{aligned}\psi_o &= \inf_{\widetilde{Q}_i \in \mathcal{H}_\infty^o} \|(\frac{P_i(s)}{(1+\varepsilon s)^\ell} \widetilde{Q}_i(s) - 1)/s\|_\infty \\ &\geq \inf_{\widetilde{Q}_i \in \mathcal{H}_\infty} \|(\frac{P_i(s)}{(1+\varepsilon s)^\ell} \widetilde{Q}_i(s) - 1)/s\|_\infty =: \widetilde{\psi}_o.\end{aligned}$$

But the optimal solution of the problem defining $\widetilde{\psi}_o$ must lie in $\mathcal{H}_\infty^o$, because $(\frac{P_i(s)}{(1+\varepsilon s)^\ell}\widetilde{Q}_i(s) - 1)/s$ is in $\mathcal{H}_\infty$ only if $\widetilde{Q}_i(0) = 1$. Therefore, $\widetilde{\psi}_o = \psi_o$ and $\widetilde{Q}_{i,\text{opt}}$ is the optimal solution of

$$\psi_o = \inf_{\widetilde{Q}_i \in \mathcal{H}_\infty} \|(\frac{P_i(s)}{(1+\varepsilon s)^\ell} \widetilde{Q}_i(s) - 1)/s\|_\infty. \tag{14}$$

The problem (14) is a one-block $\mathcal{H}_\infty$ optimization problem, which can be seen as equivalent to a weighted sensitivity minimization for a stable plant with the sensitivity weigh being an integrator. For a general inner function P_i, the problem (14) can be solved using the techniques developed for the $\mathcal{H}_\infty$ control of infinite dimensional systems, see e.g. [6, 10, 13] and their references. It turns out that the optimal $\widetilde{Q}_i$ is in the form

$$\widetilde{Q}_{i,\text{opt}}(s) = \frac{(1+\varepsilon s)^\ell}{(1+\delta s)^{(\ell+1)}} \left(\frac{1+\psi_o^2 s^2}{P_i(s) + \psi_o s}\right) \tag{15}$$

where $\delta \to 0$ and ψ_o is the largest value of $\psi > 0$ for which we have

$$P_i(j/\psi) = -j. \tag{16}$$

Choosing Q_1 as in (12) with $Q_i = K_1 \widetilde{Q}_{i,\mathrm{opt}}$, for an arbitrary $K_1 \neq 0$, and defining the controller $C_1 = Q_1/(1 - PQ_1)$, we obtain

$$C_1(s) = \frac{K_1 P_o^{-1}(s)}{\frac{(1+\delta s)^{\ell+1}(\psi_o s + P_i(s))}{1+\psi_o^2 s^2} - K_1 P_i(s)}. \tag{17}$$

$\rho = k_i\, K_1 < \psi_o^{-1}$. Hence, depending on the gain K_1 used in C_1 we get an allowable range for k_i,

$$|k_i| < \psi_o^{-1}/|K_1|\ .$$

Note that ψ_o is invariant and completely determined by the inner part $P_i(s)$ of the plant.

Another interesting problem in this context is to investigate PD (proportional plus derivative) type of $\widetilde{Q}_i(s) = (1 + k_d\ s)$ in (14). More precisely,

$$\psi_{pd} := \inf_{k_d \in \mathbb{R}} \|(\frac{P_i(s)}{(1+\varepsilon s)^\ell}(1 + k_d\ s) - 1)/s\|_\infty = \inf_{k_d \in \mathbb{R}} \|\frac{f(s) - 1}{s} + k_d f(s)\|_\infty . \tag{18}$$

where $f(s) = \frac{P_i(s)}{(1+\varepsilon s)^\ell}$. The function f is in $\mathcal{H}_\infty$ and $f(0) = 1$. This problem has been studied in the context of resilient PD controller design in [14] and a closed form expression is obtained for the optimal k_d.

Example (Revisited) For the example given in (11), the equation (16) can be written as

$$e^{-jh/\psi}\ m(j/\psi)\ \left(\frac{1 - m(-j/\psi)/2}{1 - m(j/\psi)/2}\right) = -j$$

where $m(s) = e^{-2s}\ \frac{(1-s)}{(1+s)}$. Since $m(s)$ is inner we have $m(j/\psi) = e^{-j\theta_m}$, where $\theta_m = 2/\psi + 2\tan^{-1}(1/\psi)$. We also have $\dfrac{1 - m(-j/\psi)/2}{1 - m(j/\psi)/2} = e^{-j\theta}$,
where $\theta = 2\tan^{-1}\left(\frac{\sin(\theta_m)}{2-\cos(\theta_m)}\right)$. Therefore ψ_o^{-1} is the smallest x satisfying

$$(1 + \frac{h}{2})x + \tan^{-1}(x) + \tan^{-1}\mathit{\Omega}(x) = \frac{\pi}{4} \tag{19}$$

where $\mathit{\Omega}(x) = \dfrac{\sin(2(x + \tan^{-1}(x)))}{2 - \cos(2(x + \tan^{-1}(x)))}$.

We should also note that if we change the inner part of the plant to an input/output delay, $P_i(s) = e^{-hs}$ (i.e. consider $m(s) = 1$), then from (16) we get $\psi_o^{-1} = \pi/2h$, which is precisely the gain margin of the feedback system whose open loop transfer function is e^{-hs}/s.

When we consider a PD type of $\widetilde{Q}_i(s) = (1 + k_d\ s)$, the solution of (18) gives optimal ψ_{pd}^{opt} and the corresponding k_d^{opt}. Figure 2 illustrates how the optimal ψ_o, ψ_{pd}^{opt} and k_d^{opt} vary with h. Note from this figure that the use of PD term does not lead to significant performance degradation (reduction in the largest allowable k_i range) compared to the use of optimal $\widetilde{Q}_i$ of (15).

3.2 Extension to Unstable Plants

Now consider unstable SISO plants factorized as

$$P(s) = \frac{N(s)}{D_i(s)}\ , \quad N(s) = N_i(s) N_o(s) \tag{20}$$

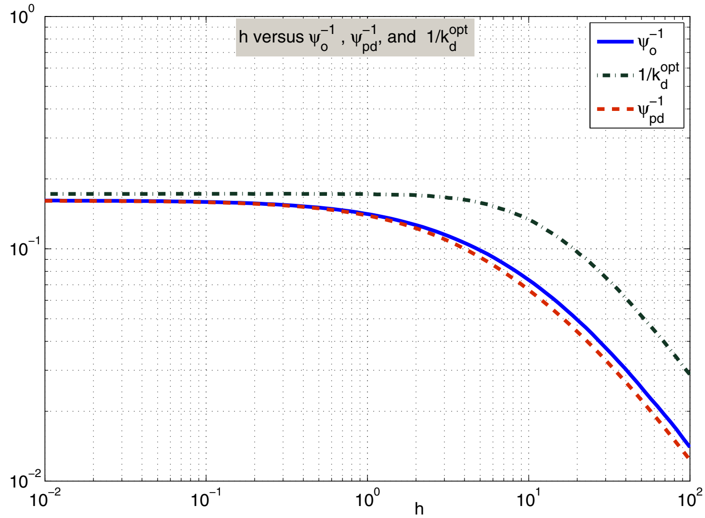

Fig. 2. h versus ψ_o^{-1}, ψ_{pd}^{-1} and $1/k_d^{\rm opt}$.

where $N_o(s)$ is outer, $N_i(s)$ and $D_i(s)$ are inner with D_i being a finite Blaschke product (i.e. the plant has finitely many poles in $\mathbb{C}_+$, and it has no poles on the Im-axis). As before, we will assume that $N_i(0) = 1$. For this type of plants $C_1 \in \mathcal{S}(P)$ if and only if

$$C_1(s) = \frac{X(s) + D_i(s)Q_1(s)}{Y(s) - N(s)Q_1(s)} \text{ for some } Q_1 \in \mathcal{H}_\infty \tag{21}$$

where $X, Y \in \mathcal{H}_\infty$ satisfy

$$Y(s) = \frac{1 - N(s)X(s)}{D_i(s)} \, . \tag{22}$$

Let $p_1, \ldots, p_n$ be the zeros of $D_i(s)$, i.e. poles of $P(s)$ in $\mathbb{C}_+$, and for simplicity of the exposition assume that they are distinct. Then, $Y \in \mathcal{H}_\infty$ if and only if the function $X \in \mathcal{H}_\infty$ satisfy $X(p_i) = 1/N(p_i)$, $i = 1, \ldots, n$. If we use C_1 in the form of (21) as the initial stabilizing controller for the plant P, then

$$T_1(s) = N(s)(X(s) + D_i(s)Q_1(s)).$$

Therefore $\psi = \|(T_1(s)T_1^{-1}(0) - 1)/s\|_\infty$ is obtained as

$$\psi = \| \frac{N(s)N(0)^{-1}Q_{1X}(s)Q_{1X}(0)^{-1} - 1}{s} \|_\infty, \tag{23}$$

where $Q_{1X}(s) = (X(s) + D_i(s)Q_1(s))$. Thus the optimal ψ_o is the smallest ψ over $Q_{1X}(s) = (X(s) + D_i(s)Q_1(s))$ for $Q_1 \in \mathcal{H}_\infty$. Define

$$Q_{1X}(s) =: \frac{N_o^{-1}(s)}{(1 + \varepsilon s)^\ell} Q_X(s) \text{ where } \varepsilon \searrow 0,$$

and ℓ is the relative degree of $N_o(s)$. Then, we have an invertible relation between the free parameters Q_{1X} and Q_X in $\mathcal{H}_\infty$. Note that the problem (23) is exactly in the form (13) except that $Q_{1X}(s)$ is restricted to have $Q_{1X}(p_i) = X(p_i) = 1/N(p_i)$, whereas in (13) there is no such restriction on the free parameter $Q_i \in \mathcal{H}_\infty$. In summary, we have the following

$$\psi(Q_X) := \|s^{-1}(\frac{N_i(s)}{(1+\varepsilon s)^\ell}\,\frac{Q_X(s)}{Q_X(0)} - 1)\|_\infty \tag{24}$$

$$\psi_o = \inf\{\ \psi(Q_X)\ :\ Q_X \in \mathcal{H}_\infty \ \text{ and } Q_X(p_i) = \frac{(1+\varepsilon p_i)^\ell}{N_i(p_i)}\ , i = 1\ldots, n\}. \tag{25}$$

As in Section 3.1, we will be restricting ourselves to $Q_X \in \mathcal{H}_\infty$ such that $Q_X(0) = 1$, because $\psi(KQ_X) = \psi(Q_X)$ for any non-zero K. Thus, in the unstable plants case the problem is modified to finding

$$\psi_o = \inf_{\widetilde{Q}_X} \left\| s^{-1}\left(\frac{N_i(s)}{(1+\varepsilon s)^\ell}\,\widetilde{Q}_X(s) - 1\right)\right\|_\infty$$
$$\text{subject to } \widetilde{Q}_X \in \mathcal{H}_\infty \text{and } \widetilde{Q}_X(p_i) = \frac{(1+\varepsilon p_i)^\ell}{N_i(p_i)}\ , i = 1\ldots, n.$$

For a given $\gamma > \psi_o$, the set of all $\widetilde{Q}_X \in \mathcal{H}_\infty$ satisfying

$$\left\| s^{-1}\left(\frac{N_i(s)}{(1+\varepsilon s)^\ell}\,\widetilde{Q}_X(s) - 1\right)\right\|_\infty \le \gamma \tag{26}$$

can be characterized as

$$\mathcal{Q}_\gamma = \left\{\widetilde{Q}_X(s) = \frac{F_1(s)+F_2(s)U(s)}{F_3(s)+F_4(s)U(s)}\quad : \ U \in \mathcal{H}_\infty\ ,\ \|U\|_\infty \le 1\right\} \tag{27}$$

where $F_1, \ldots, F_4$ are computed explicitly from the problem data, see e.g. [6]. Therefore, the problem at hand can be transformed to finding the smallest γ for which there exists $U \in \mathcal{H}_\infty$, $\|U\|_\infty \le 1$ such that

$$\frac{F_1(p_i)+F_2(p_i)U(p_i)}{F_3(p_i)+F_4(p_i)U(p_i)} = \frac{(1+\varepsilon p_i)^\ell}{N_i(p_i)} =: \alpha_i\ , \tag{28}$$

for $i = 1\ldots, n$. This leads to a set of interpolation conditions on U

$$U(p_i) = \frac{\alpha_i\ F_3(p_i) - F_1(p_i)}{F_2(p_i) - \alpha_i\ F_4(p_i)} =: \beta_i \tag{29}$$

for $i = 1\ldots, n$. For each fixed γ we can find β_i using for example [6]. Now we need to check whether there exists $U \in \mathcal{H}_\infty$ with $\|U\|_\infty \le 1$ such that $U(p_i) = \beta_i$. This is a Nevanlinna-Pick interpolation problem and it can be solved from the given problem data $\{(p_1\ldots, p_n), (\beta_1, \ldots, \beta_n)\}$, see e.g. [6, 11, 18].

In summary, for unstable plants the problem is solved in two steps:

1. Given $\gamma > \psi_o$, solve the suboptimal version (26) of the problem (13) studied in Section 3.1; characterize all suboptimal solutions in the form (27), i.e. find F_1, F_2, F_3, F_4.
2. Given $p_1, \ldots, p_n$, determine $\beta_1, \ldots, \beta_n$ from the first step. Use this data to check if the Nevanlinna-Pick interpolation problem has a feasible solution. If yes decrease γ, if no increase γ, and repeat Steps 1 and 2; using a bisection in this iteration find the optimal γ_o. For $\gamma = \gamma_o + \epsilon$, where $\epsilon > 0$, the Nevanlinna-Pick problem gives a solution U, which in turn gives our suboptimal Q_X, from which we get Q_{1X} and hence C_1.

Example. Let $h > 0$, and consider $P(s) = \dfrac{e^{-hs}}{s+2}\left(\dfrac{s+1+2(s-1)e^{-2s}}{s+3-5e^{-0.5s}}\right)$. This plant is unstable with single pole $p_1 = 0.6367$ in $\mathbb{C}_+$. Therefore, its factorization can be done as (20) where

$$D_i(s) = \frac{s-0.6367}{s+0.6367}$$
$$N_i(s) = e^{-hs}\frac{(s+1)+2(s-1)e^{-2s}}{(1-s)e^{-2s}-2(s+1)}$$
$$N_o(s) = \frac{(1-s)e^{-2s}-2(s+1)}{(s+2)(s+3-5e^{-0.5s})}\,D_i(s).$$

For $h = 3$ we have $\psi_o = 8.6744$. This gives $\alpha_1 = N_i(p_1)^{-1} = -14.945$. Applying the procedure described above we find $F_1, \ldots, F_4$ for each fixed $\gamma > \psi_o$, and compute β_1 defined by (29). Since we have single interpolation condition, the solution of Nevanlinna-Pick problem is rather trivial: it is solvable if and only if $|\beta_1| \leq 1$, and as a solution we can take $U(s) = \beta_1$. By using a bi-section search we find that smallest $\gamma > \psi_o$ leading to $|\beta_1| \leq 1$ is $\gamma_o = 13.4485$, which leads to $\beta_1 = -1$, see Figure 3. Thus if we choose $U(s) = -1$, we get

$$\widetilde{Q}_X(s) = \frac{F_1(s) - F_2(s)}{F_3(s) - F_4(s)},$$

where $F_1, \ldots, F_4$ are computed from the solution of the suboptimal one-block $\mathcal{H}_\infty$ problem with $\gamma = 13.45 > \gamma_o$.

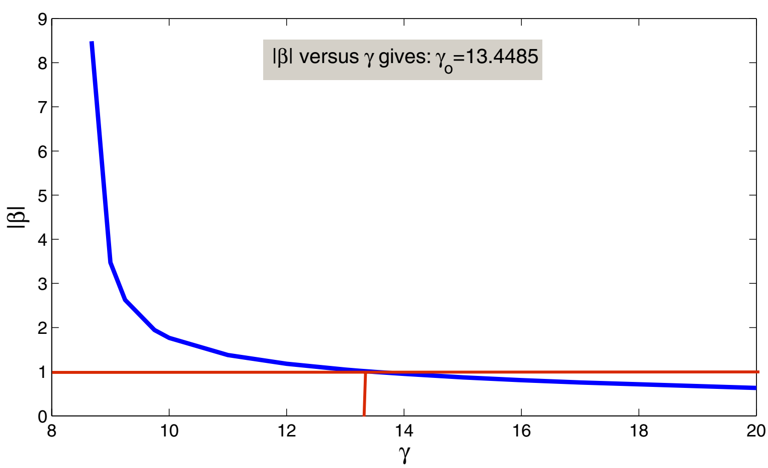

Fig. 3. $|\beta|$ versus γ.

4 Conclusions

A sufficient condition is derived for $C = C_1 C_{pi}$, cascade connection of a PI controller C_{pi}, and an initial stabilizing controller C_1, to stabilize a given plant P. Design of C_1 for the largest allowable range of the integral action gain interval is investigated for stable plants, including systems with internal and input-output delays. We used parametrization of all stabilizing controller to characterize C_1. Then we have seen that the problem at hand reduces to a weighted sensitivity minimization for a stable plant whose inner part is infinite dimensional and the weight is an integrator. When we consider a PD-like $\tilde{Q}_i(s)$ in the parametrization of C_1, the problem becomes finding optimal k_d in (18), which is solved in [14].

For unstable plants the problem of finding the largest allowable range of the integral action gain is solved in two steps. First the a suboptimal one-block problem is solved and in the second step a Nevanlinna-Pick interpolation problem is solved.

We should also point out that the result stated in Section 2 is a sufficient condition. Therefore the largest allowable integral action gain found in Section 3 is within the set of allowable gains characterized by this sufficient condition, which may be conservative. It would be interesting to investigate the level of conservatism in this approach. We leave this open problem to a future study.

APPENDIX

Proof of Theorem 1: **a)** Let $P = \tilde{Y}^{-1}\tilde{X}$ be a left-coprime-factorization (LCF) of P and let $N_1 D_1^{-1}$ be a right-coprime-factorization (RCF) of C_1. Since C_1 stabilizes P, $M_1 := \tilde{Y}D_1 + \tilde{X}N_1$ is unimodular in $\mathcal{H}_\infty^{r\times r}$. With $C_1 \in \mathcal{S}(P)$, we have $Q_1 := C_1(I+PC_1)^{-1} \in \mathcal{H}_\infty^{r\times r}$ and $T_1 := PC_1(I+PC_1)^{-1} = PQ_1 \in \mathcal{H}_\infty^{r\times r}$. Now $C_y = I - C_x$ stabilizes T_1 if and only if $C_y(I+T_1C_y)^{-1} \in \mathcal{H}_\infty^{r\times r}$, which implies $(I+T_1C_y)^{-1} \in \mathcal{H}_\infty^{r\times r}$. Define $D_c := (I+TC_y)^{-1}D_1$, $N_c = N_1 + Q_1C_yD_c$; then N_c, $D_c \in \mathcal{H}_\infty^{r\times r}$. Write $C = C_1C_x$ as $C = C_1 + C_1C_y = N_cD_c^{-1}$. Then $\tilde{Y}D_c + \tilde{X}N_c = \tilde{Y}D_c + \tilde{X}[N_1 + Q_1C_yD_c] = \tilde{Y}D_c + \tilde{X}N_1 + \tilde{Y}PQ_1C_yD_c = \tilde{Y}(I+T_1C_y)D_c + \tilde{X}N_1 = \tilde{Y}(I+T_1C_y)(I+TC_y)^{-1}D_1 + \tilde{X}N_1 = M_1$ is unimodular and hence, $C = C_1C_x \in \mathcal{S}(P)$.
b) Let $P = XY^{-1}$ be an RCF and $C_1 = \tilde{D}_1^{-1}\tilde{N}_1$ be an LCF. Then $C_1 \in \mathcal{S}(P)$ if and only if $\tilde{M}_1 := \tilde{D}_1Y + \tilde{N}_1X$ is unimodular; hence, $\det \tilde{M}_1(0) \neq 0$. Since $P, C_1 \in \mathcal{S}(P)$ do not have transmission-zeros at $s = 0$, $\det X(0) \neq 0$ and $\det \tilde{N}_1(0) \neq 0$. Since $\det T_1(0) = \det X(0)\tilde{M}_1(0)\tilde{N}_1(0) \neq 0$, we conclude that $T_1 \in \mathcal{H}_\infty^{r\times r}$ does not have transmission-zeros at $s = 0$. It follows from [9], Proposition 2, that the PID-controller $\hat{C}_{pid}$ in (4) stabilizes $T_1 \in \mathcal{H}_\infty^{r\times r}$. Therefore, by (1), $C_1C_x \in \mathcal{S}(P)$, where $C_x = I + \hat{C}_{pid}$.

References

1. Bélanger, P.R.: Control engineering: A modern approach. Saunders College Pub., Philadelphia (1995)
2. Davison, E.J.: The robust control of a servomechanism problem for linear time-invariant multivariable systems. IEEE Transactions on Automatic Control 21, 25–34 (1976)
3. Davison, E.J., Özgüner, Ü.: Expanding system problem. Systems and Control Letters 1, 255–260 (1982)
4. Desoer, C.A., Wang, Y.T.: On the minimum order of a robust servocompensator. IEEE Transactions on Automatic Control 23, 70–73 (1978)

5. Engelborghs, K., Luzyanina, T., Roose, D.: Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL. ACM Transactions on Mathematical Software 28, 1–21 (2002)
6. Foias, C., Özbay, H., Tannenbaum, A.: Robust control of infinite dimensional systems: Frequency domain methods. LNCIS, vol. 209. Springer, London (1996)
7. Francis, B.A., Wonham, W.M.: The internal model principle of control theory. Automatica 12, 457–465 (1976)
8. Gümüşsoy, S., Özbay, H.: Remarks on H^∞ controller design for SISO plants with time delays. In: Proc. 5th IFAC Symposium on Robust Control Design, Toulouse, France (2006)
9. Gündeş, A.N., Özbay, H., Özgüler, A.B.: PID controller synthesis for a class of unstable MIMO plants with I/O delays. Automatica 43, 135–142 (2007)
10. Kashima, K., Özbay, H., Yamamoto, Y.: A Hamiltonian-based solution to the mixed sensitivity optimization problem for stable pseudorational plants. Systems & Control Letters 54, 1063–1068 (2005)
11. Krein, M.G., Nudel'man, A.A.: The Markov moment problem and extremal problems. Translations of Mathematical Monographs 50 (1977)
12. Olgaç, N., Sipahi, R.: A practical method for analyzing the stability of neutral type LTI-time delayed systems. Automatica 40, 847–853 (2004)
13. Özbay, H.: A simpler formula for the singular values of a certain Hankel operator. Systems & Control Letters 15, 381–390 (1990)
14. Özbay, H., Gündeş, A.N.: Resilient PI and PD controller designs for a class of unstable plants with I/O delays. Applied and Computational Mathematics 6, 18–26 (2007)
15. Toker, O., Özbay, H.: H^∞ optimal and suboptimal controllers for infinite dimensional SISO plants. IEEE Transactions on Automatic Control 40, 751–755 (1995)
16. Vidyasagar, M.: Control system synthesis: A factorization approach. MIT Press, Cambridge (1985)
17. Vyhlidal, T., Zitek, P.: Mapping the spectrum of a retarded time-delay system utilizing root distribution features. In: Proc. IFAC Workshop on Time-Delay Systems, L'Aquila, Italy (2006)
18. Zeren, M., Özbay, H.: Comments on Solutions to combined sensitivity and complementary sensitivity problem in control systems. IEEE Transactions on Automatic Control 43, 724 (1998)