Ordinal Covering Using Block Designs*

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***Abstract*—A frequently encountered problem in peer review systems is to facilitate pairwise comparisons of a given set of documents by as few experts as possible. In [7], it was shown that, if each expert is assigned to review k documents then $\lceil n(n-1)/k(k-1)\rceil$ experts are necessary and $\lceil n(2n-k)/k^2\rceil$ experts are sufficient to cover all $n(n-1)/2$ pairs of n documents. In this paper, we show that, if $\sqrt{n} \le k \le n/2$ then the upper bound can be improved using a new assignnment method based on a particular family of balanced incomplete block designs. Specifically, the new method uses $\lceil n(n+k)/k^2\rceil$ experts where n/k is a prime power, n divides k^2, and $\sqrt{n} \le k \le n/2$. When $k = \sqrt{n}$, this new method uses the minimum number of experts possible and for all other values of k, where $\sqrt{n} < k \le n/2$, the new upper bound is tighter than the general upper bound given in [7].**

***Keywords*—assignment problems, balanced incomplete block design, combinatorial assignment, document evaluation, ordinal ranking, peer review.**

I. INTRODUCTION

Ordinal or relative evaluations of documents have been suggested as a more reliable alternative in identifying high quality documents over cardinal evaluations such as using average scores assigned to the documents by a set of experts [1-5]. Ordinal and cardinal strengths of preferences have also been advocated in [6,8] as natural extensions of ordinal comparison models. A set covering integer programming approach was introduced in [4] to obtain as many comparisons as possible between the documents reviewed by a fixed set of experts, where each expert is assigned to review a predetermined number of documents, called its *capacity*.

This paper is concerned with the ordinal covering of a set of documents, i.e., assignments of documents to experts in such a way that each pair of documents is compared and ranked by one or more experts. Let $r(n,k)$ denote the number of experts used in an ordinal covering of n documents with each expert having a capacity of k documents. Such a covering is called a *minimal ordinal covering* if

$$\frac{r(n,k)k(k-1)}{n(n-1)} \to 1 \text{ as } n \to \infty$$

Here, the underlying assumption is that experts can compare and rank the documents to which they are assigned in a pairwise manner. Thus, if documents a, b, and c are assigned to an expert then it is assumed that the pairs of documents ab, ac, and bc are covered by that expert. For n documents, the problem is then to cover all $n(n-1)/2$ pairs of documents, where each expert can review no more than k documents and the number of experts approaches the lower bound of $n(n-1)/k(k-1)$ for large n.

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It was shown in [7] that the lower bound of $\lceil n(n-1)/k(k-1)\rceil$ experts is tight for $k = n/2$, and almost tight for $k = n/3$ (10 versus 12 experts), and $k = n/4$ (17 versus 20 experts). In these two cases, the lower bounds were improved to 11 and 18 through additional results. It was further established that $\lceil n(2n-k)/k(k-1)\rceil$ experts are sufficient to generate all pairs of n documents under the same capacity constraint of k documents per expert for any k that divides n. The main contribution of this paper is a new assignment that covers all pairs of n documents using $\lceil n(n+k)/k^2\rceil$ experts, each with a capacity of k, whenever n/k is a prime power, n divides k^2, and $\sqrt{n} \le k \le n/2$. The new assignment relies on a balanced incomplete block design and mutually orthogonal set of Latin squares and results in an expert complexity that improves the expert complexity of the assignment described in [7] when $\sqrt{n} \le k \le n/2$. Furthermore, it provides a minimal ordinal covering when $k = \sqrt{n}$.

The rest of the paper is organized as follows. In Section II, we review the block design and Latin square concepts needed in the rest of the paper. In Section III, we present the new assignment and prove that it covers all pairs of n documents with $\lceil n(n+k)/k^2\rceil$ experts, each with a capacity of k. The paper is concluded in Section IV with the comparison of the new assignment with that presented in [7] and discussion of remaining problems.

II. PRELIMINARIES

A block design is a pair (G,B) where $G = \{G_1,G_2,\ldots,G_u\}$ is a set of v elements, and $B = \{B_1,B_2,..,B_v\}$ is a collection of u subsets of G, called blocks. For example, if $G = \{G_1,G_2,G_3,G_4,G_5,G_6\}$, then $(G,\{(G_1,G_3,G_5),(G_2,G_4,G_5,G_6)\})$ is a block design with the two blocks, (G_1,G_3,G_5) and (G_2,G_4,G_5,G_6).

A block design is called balanced if all blocks are of equal size and all pairs of elements of G occur in all of the blocks an equal number of times. It is called incomplete if the number of elements in every block is less than u. Let λ, t, and r be positive integers, where $2 \le t < u$. A block design (G,B) is called a (u,v,r,t,λ)-balanced and incomplete block design (BIBD) if (1) $|G| = u$, $|B| = v$, (2) each element in G appears in exactly r blocks, (3) all blocks in B have t elements, and (4) each pair of elements in G appears in exactly λ blocks. We have $ur = vt$ since each of the u elements in G appears r times in all the blocks and the union of the blocks as a multiset contains exactly vt elements. It can further be shown that $\lambda(u-1) = r(t-1)$. Solving for v and r in terms of λ and t, we have

$$v = \frac{u(u-1)\lambda}{t(t-1)}, \quad r = \frac{(u-1)\lambda}{t-1}$$

and thus a (u,v,r,t,λ)-BIBD design is often referred to as a (u,t,λ)-BIBD as will be done here as well [5]. In this paper, we will be concerned with BIBDs with $\lambda = 1$.

Proposition 1: Let $G = \{G_i: 1 \le i \le 9\}$ and[1]

$$B = \{(G_1, G_2, G_3), (G_4, G_5, G_6), (G_7, G_8, G_9), (G_1, G_4, G_7), (G_1, G_5, G_8), (G_1, G_6, G_9), (G_2, G_4, G_9), (G_2, G_5, G_7), (G_2, G_6, G_8), (G_3, G_4, G_8), (G_3, G_5, G_9), (G_3, G_6, G_7)\}.$$

(G,B) is a (9,3,1)-BIBD design with each G_i appearing in exactly 4 of 12 blocks, and each pair of G_i's appearing in exactly one block.

Proof: It follows from a direct inspection of the blocks. ||

Proposition 2: Let $G = \{ G_i: 1 \le i \le 16\}$ and

$$B=\{(G_1,G_2,G_3,G_4),(G_5,G_6,G_7,G_8),(G_9,G_{10},G_{11},G_{12}),(G_{13},G_{14},G_{15},G_{16}), (G_1,G_5,G_9,G_{13}),(G_1,G_6,G_{11},G_{16}),(G_1,G_7,G_{12},G_{14}),(G_1,G_8,G_{10},G_{15}), (G_2,G_6,G_{10},G_{14}),(G_2,G_5,G_{12},G_{15}),(G_2,G_8,G_{11},G_{13}),(G_2,G_7,G_9,G_{16}), (G_3,G_7,G_{11},G_{15}),(G_3,G_5,G_{10},G_{16}),(G_1,G_6,G_{12},G_{13}),(G_3,G_8,G_9,G_{14}), (G_4,G_8,G_{12},G_{16}),(G_4,G_5,G_{11},G_{14}),(G_4,G_7,G_{10},G_{13}),(G_4,G_6,G_9,G_{15})\}.$$

(G,B) is a (16,4,1)-BIBD with each G_i appearing in exactly 5 of 20 blocks and each pair of G_i's appearing in exactly one block.

Proof: It follows from a direct inspection of the blocks. ||

Certain BIBDs can be constructed using Latin squares and this fact will play a critical role in the sequel. A Latin square of order q is a $q \times q$ matrix in which each row and each column is a permutation of a set of q symbols [5]. Two Latin squares L_1 and L_2 are said to be orthogonal if the matrix obtained by juxtaposing L_1 and L_2 entry by entry contains each of the possible q^2 ordered pairs exactly once. For example, the following Latin squares are orthogonal.

Orthogonal Latin squares:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 4 & 5 & 6 \\ 6 & 4 & 5 \\ 5 & 6 & 4 \end{bmatrix} = \begin{bmatrix} (1,4) & (2,5) & (3,6) \\ (2,6) & (3,4) & (1,5) \\ (3,5) & (1,6) & (2,4) \end{bmatrix}$$

It should be noted that not all Latin squares are orthogonal as illustrated by the following example.

Non-orthogonal Latin squares (each pair of entries in the matrix on the right appears three times):

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 6 & 5 & 4 \\ 5 & 4 & 6 \\ 4 & 6 & 5 \end{bmatrix} = \begin{bmatrix} (1,6) & (2,5) & (3,4) \\ (2,5) & (3,4) & (1,6) \\ (3,4) & (1,6) & (2,5) \end{bmatrix}$$

A set of Latin squares, $L_1, L_2, ..., L_m$, is said to be mutually orthogonal if each pair of Latin squares is orthogonal, i.e., L_i and L_j are orthogonal, $1 \le i \ne j \le m$. We call such a set of Latin squares mutually orthogonal Latin squares and denote it by MOLS. It is known that the maximum number of Latin squares of order q that can be mutually orthogonal to one another cannot exceed q-1 [5]. Accordingly, a set of mutually orthogonal Latin squares, $L_1, L_2, ..., L_{q-1}$ is said to be a complete MOLS, and the $q \times q$ matrix obtained by juxtaposing them entry by entry will be denoted by $L_1 \times L_2 \times \ldots \times L_{q-1}$.

A number of methods are known for constructing a complete set of q-1 MOLS of order q. We state the following theorem [5] without a proof:

Theorem 1: If q is a prime power, a complete set of q-1 MOLS can be found by taking the nonzero elements of a finite field of order q, and setting the entry in the x^{th} row and y^{th} column in the a^{th} Latin square to $f_a(x, y) = ax + y \pmod q$, $1 \le a \le q$. ||

Example 1: For $n = 3$, we compute the Latin squares as follows[2] :

$$\begin{matrix} f_1(1,1)=2, & f_1(1,2)=3, & f_1(1,3)=1 \\ f_1(2,1)=3, & f_1(2,2)=1, & f_1(2,3)=2 \\ f_1(3,1)=1 & f_1(3,2)=2, & f_1(3,3)=3 \end{matrix} \qquad L_1 = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\begin{matrix} f_2(1,1)=3, & f_2(1,2)=1, & f_2(1,3)=2 \\ f_2(2,1)=2, & f_2(2,2)=3, & f_2(2,3)=1 \\ f_2(3,1)=1 & f_2(3,2)=2, & f_2(3,3)=3 \end{matrix} \qquad L_2 = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

III. COVERING WITH BALANCED INCOMPLETE BLOCK DESIGNS

The (9,3,1) and (16,4,1) BIBDs described in Propositions 1 and 2 above can be used to obtain a covering of n documents with 12 experts, each with a of capacity of $n/3$ and with 20 experts, each with a capacity of $k = n/4$. Here, we will only describe how such a covering is obtained for $k = n/3$ as the case of $k = n/4$ is very similar.

Divide the set of n documents into 9 subsets of $n/9$ documents[3], and call them G_i, $1 \le i \le 9$. Identify these subsets of documents with the elements of G in the (9,3,1)-BIBD as described in Proposition 1, and let B_j denote the j^{th} block in this (9,3,1)-BIBD, $1 \le j \le 12$. Assign the $n/3$ documents in the union of the subsets of documents in block B_j to the j^{th} expert, $1 \le j \le 12$. To prove that all $n(n-1)/2$ pairs of documents are covered by the 12 experts, we note that the experts assigned to blocks B_1, B_2, and B_3 cover

$$3\binom{n/3}{2}$$

distinct pairs of documents in G_i, $1 \le i \le 9$. This leaves

$$\binom{n}{2} - 3\binom{n/3}{2} = \frac{n^2}{3}$$

pairs of documents to be covered. Given that each pair (G_i,G_j) $1 \le i \ne j \le 9$ appears in exactly one of the blocks in the (9,3,1)-BIBD design, the remaining pairs of documents are covered by at least one of the 9 experts that are assigned to remaining blocks B_j, $4 \le j \le 12$. More precisely, each such expert covers

$$3 \times \frac{n}{9} \times \frac{n}{9} = \frac{n^2}{27}$$

distinct pairs of documents to the assignment, or all 9 experts covering the $n^2/3$ missing pairs of documents as desired.

Example 2: If $n = 9$, i.e., $k = n/k = 3$ then the (9,3,1)-design results in a minimal ordinal covering of 9 documents as illustrated below:

Let

$G_1=\{d_1\}$, $G_2=\{d_2\}$, $G_3=\{d_3\}$, $G_4=\{d_4\}$, $G_5=\{d_5\}$, $G_6=\{d_6,\}$, $G_7=\{d_7\}$, $G_8=\{d_8\}$, $G_9=\{d_9\}$.

The assignment in Table I covers all 36 pairs of 9 documents

[1] In this paper, $G_1, G_2, \ldots, G_v$ implicitly represent subsets of documents and $B_1, B_2, \ldots, B_v$ implicitly represent experts.

[2] Note that $0 \equiv 3 \pmod 3$.

[3] Here, we assume that $k^2 = n^2/9$ is divisible by n, i.e., $k^2/n = n/9$ is an integer.

with 12 experts with each assigned to review three documents. The number of experts in this assignment matches the lower bound $\lceil n(n-1)/k(k-1)\rceil = 9(9-1)/3(3-1) = 12$. ||

Table I. Assignment of 9 documents to 12 experts with a capacity of 3.

r_1	d_1	d_2	d_3						
r_2				d_4	d_5	d_6			
r_3							d_7	d_8	d_9
r_4	d_1			d_4			d_7		
r_5	d_1				d_5			d_8	
r_6	d_1					d_6			d_9
r_7		d_2		d_4					d_9
r_8		d_2			d_5		d_7		
r_9		d_2				d_6		d_8	
r_{10}			d_3	d_4				d_8	
r_{11}			d_3		d_5				d_9
r_{12}			d_3			d_6	d_7		

It should be emphasized that the (9,3,1)-BIBD can be used to obtain a covering of n documents by 12 experts, each with a capacity of k, for any n that is divisible by 9 and where $k = n/3$. In all such assignments, the number of experts is fixed to 12, and the capacity of each expert is fixed to $n/3$. The assignment in Table II uses 12 experts, each with a capacity of 9 to cover all 351 pairs of 27 documents. Likewise, the same (9,3,1)-BIBD can be used to cover all 1431 pairs of 54 documents using 12 experts, each with a capacity of 18. However, for $n > 9$, the lower bound on the number of experts becomes[7]:

$$\lceil n(n-1)/k(k-1)\rceil = \lceil n(n-1)/n/3(n/3-1)\rceil = \lceil 9(n-1)/(n-3)\rceil \geq 10.$$

Hence it cannot be said that either of the last two assignments generated by the (9,3,1)-BIBD is a minimal ordinal covering.

Similarly, the (16,4,1)-BIBD design results in a minimal ordinal covering of 16 documents by 20 experts each with a capacity of 4 since $\lceil 16(16-1)/4(4-1)\rceil = 20$ but again, for $n > 16$, the assignment is no longer minimal given that

$\lceil n(n-1)/k(k-1)\rceil = \lceil n(n-1)/n/4(n/4-1)\rceil = \lceil 16(n-1)/(n-4)\rceil \geq 17$, for $n > 16$.

Our main result is a generalization of the (9,3,1) and (16,4,1)-BIBD designs to a $(q^2,q,1)$-BIBD design for any prime power $q \geq 2$, and efficient assignments that result from this BIBD design. We note that the assignments with 12 and 20 experts were presented in [7] without using any connection to the (9,3,1) and (16,4,1)-BIBD designs described here.

Proposition 3: The intersection of any two columns in any given row or the intersection of any two rows in any given column in a $q\times q$ matrix obtained by juxtaposing the entries in a complete set of q-1 MOLS is empty.

Proof: The entries in any column or any row of a $q\times q$ Latin square form a permutation of the q elements used to construct the Latin square. Therefore the intersections of the juxtapositions of the entries across the rows or columns of the Latin squares in the MOLS must be empty. ||

Example 3: In the MOLS below, $\{1,4\}\cap\{2,5\}=\varnothing$, $\{1,4\}\cap\{3,6\}=\varnothing$, $\{2,5\}\cap\{3,6\}=\varnothing$ in the first row, and it can be verified the same holds for any other row or column. ||

$$\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 3 & 1 & 2\end{bmatrix}\times\begin{bmatrix}4 & 5 & 6\\ 6 & 4 & 5\\ 5 & 6 & 4\end{bmatrix}=\begin{bmatrix}(1,4) & (2,5) & (3,6)\\ (2,6) & (3,4) & (1,5)\\ (3,5) & (1,6) & (2,4)\end{bmatrix}$$

Theorem 2: For any prime power q, there exists a $(q^2,q,1)$-BIBD.

Proof: By Theorem 1, we can construct a complete set of MOLS using q-1 Latin squares of order q, L_i, $1 \leq i \leq q-1$. Let

$$U_i=\{(i-1)q+1,(i-1)q+2,\ldots,(i-1)q+q\}$$

denote the set of elements used in $L_i, 1\leq i \leq q-1$, and let

$$U_q= \{(q-1)q+1,(q-1)q+2,\ldots,(q-1)q+q\}.$$

Let $M = L_1\times L_2\times\ldots\times L_{q-1}$ denote the $q\times q$ matrix obtained by juxtaposing $L_1, L_2, \ldots, L_{q-1}$ as defined in Section 2. Suppose that M is modified by concatenating $(q-1)q + i$ to all the columns in the i^{th} row of M, $1 \leq i \leq q$. Denote this new matrix by M_a. Let the entries of matrix M_a represent the blocks of a block design. Each element in $U_1\cup U_2\cup\ldots\cup U_{q-1}$ clearly appears exactly q times among these blocks. Since each element in U_q is inserted into the columns of a distinct row, each element in U_q must also appear exactly q times among the blocks. Moreover, $U_q\cap U_i=\varnothing$ for i=1,2,...,q-1. Therefore, by Proposition 3, the intersection of any two rows in any given column must be empty. Furthermore, the intersection of any two columns in any given row cannot have more than one element in common. It follows that the pairs of elements that are formed by juxtaposing the elements in the blocks of matrix M_a must all be distinct.

Now to complete this block design to a $(q^2,q,1)$-BIBD, it suffices to add U_i, $1 \leq i \leq q$ as blocks to it and note that (a) the resulting block design consists of q^2+q blocks, each comprising q elements, (b) each element in $U_1\cup U_2\cup\ldots\cup U_q$ appears exactly in q+1 blocks, and (c) each pair of elements appears in exactly one block. ||

Remark 1: The equation

$$q^2\binom{q}{2}+q\binom{q}{2}=\binom{q^2}{2}$$

captures the fact that the set of all pairs of q^2 elements in $U_1\cup U_2\cup\ldots\cup U_q$ is obtained by the union of the set of all pairs of all elements in the q^2 blocks identified with the entries of M_a plus all pairs of elements generated by the elements in U_i, $1\leq i \leq q$. We also note that the (9,3,1) and (16,4,1)-BIBD designs are special cases of this $(q^2,q,1)$-BIBD construction with $q = 3$ and $q = 4$, respectively. ||

Example 4: Let $q = 5$. The following four Latin squares are constructed using Theorem 1 and form a complete set of MOLS of order 5.

$$L_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \\ 3 & 5 & 4 & 2 & 1 \\ 4 & 1 & 2 & 5 & 3 \\ 5 & 4 & 1 & 3 & 2 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 6 & 7 & 8 & 9 & 10 \\ 8 & 10 & 9 & 7 & 6 \\ 9 & 6 & 7 & 10 & 8 \\ 10 & 9 & 6 & 8 & 7 \\ 7 & 8 & 10 & 6 & 9 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 11 & 12 & 13 & 14 & 15 \\ 14 & 11 & 12 & 15 & 13 \\ 15 & 14 & 11 & 13 & 12 \\ 12 & 13 & 15 & 11 & 14 \\ 13 & 15 & 14 & 12 & 11 \end{bmatrix}, L_4 = \begin{bmatrix} 16 & 17 & 18 & 19 & 20 \\ 20 & 19 & 16 & 18 & 17 \\ 17 & 18 & 20 & 16 & 19 \\ 18 & 20 & 19 & 17 & 16 \\ 19 & 16 & 17 & 20 & 18 \end{bmatrix}$$

Matrix M is constructed by juxtaposing these Latin squares as follows:

$$M = \begin{bmatrix} (1,6,11,16) & (2,7,12,17) & (3,8,13,18) & (4,9,14,19) & (5,10,15,20) \\ (2,8,14,20) & (3,10,11,19) & (5,9,12,16) & (1,7,15,18) & (4,6,13,17) \\ (3,9,15,17) & (5,6,14,18) & (4,7,11,20) & (2,10,13,16) & (1,8,12,19) \\ (4,10,12,18) & (1,9,13,20) & (2,6,15,19) & (5,8,11,17) & (3,7,14,16) \\ (5,7,13,19) & (4,8,15,16) & (1,10,14,17) & (3,6,12,20) & (2,9,11,18) \end{bmatrix}$$

Matrix M_a is formed by inserting 21 to the 1st row of M, 22 to its 2nd row, 23 to its 3rd row, 24 to its 4th row and 25 to its 5th row as shown below:

$$M_a = \begin{bmatrix} (1,6,11,16,21) & (2,7,12,17,21) & (3,8,13,18,21) & (4,9,14,19,21) & (5,10,15,20,21) \\ (2,8,14,20,22) & (3,10,11,19,22) & (5,9,12,16,22) & (1,7,15,18,22) & (4,6,13,17,22) \\ (3,9,15,17,23) & (5,6,14,18,23) & (4,7,11,20,23) & (2,10,13,16,23) & (1,8,12,19,23) \\ (4,10,12,18,24) & (1,9,13,20,24) & (2,6,15,1924,) & (5,8,11,17,24) & (3,7,14,16,24) \\ (5,7,13,19,25) & (4,8,15,16,25) & (1,10,14,17,25) & (3,6,12,20,25) & (2,9,11,18,25) \end{bmatrix}$$

The blocks of the (25,5,1)-BIBD are obtained by combining the entries of M_a with the sets

$U_1 = \{1, 2, 3, 4, 5\}$,

$U_2 = \{6, 7, 8, 9, 10\}$,

$U_3 = \{11,12,13,14,15\}$,

$U_4 = \{16,17,18,19,20\}$,

$U_5 = \{21, 22, 23, 24, 25\}$ as

{(1, 2, 3, 4, 5) (6, 7, 8, 9, 10) (11, 12, 13, 14, 15) (16, 17, 18, 19, 20) (21, 22, 23, 24, 25) (1, 6, 11, 16, 21) (2, 7, 12, 17, 21) (3, 8, 13, 18, 21) (4, 9, 14, 19, 21) (5, 10, 15, 20, 21) (2, 8, 14, 20, 22) (3, 10, 11, 19, 22) (5, 9, 12, 16, 22) (1, 7, 15, 18, 22) (4, 6, 13, 17, 22) (3, 9, 15, 17, 23) (5, 6, 14, 18, 23) (4, 7, 11, 20, 23) (2, 10, 13, 16, 23) (1, 8, 12, 19, 23) (4, 10, 12, 18, 24) (1, 9, 13, 20, 24) (2, 6, 15, 19, 24) (5, 8, 11, 17, 24) (3, 7, 14, 16, 24) (5, 7, 13, 19, 25) (4, 8, 15, 16, 25) (1, 10, 14, 17, 25) (3, 6, 12, 20, 25) (2, 9, 11, 18, 25)}

As mentioned in Remark 1, this BIBD construction satisfies the equation:

$$25\binom{5}{2} + 5\binom{5}{2} = 300 = \binom{25}{2}$$

and thus generates all pairs of elements in the set {1,2,…,25}.

Now to obtain a minimal ordinal covering of a set of n documents from this $(q^2,q,1)$-BIBD, we set $n = q^2$, and set the capacity of each expert, k to q. Then the lower bound on the number of experts becomes

$$\lceil n(n-1)/k(k-1)\rceil = \lceil q^2(q^2-1)/q(q-1)\rceil = q(q+1) = q^2 + q$$

matching the upper bound on the number of experts obtained by the $(q^2,q,1)$-BIBD described in Theorem 2. In the above example, setting $n = 25$ and $k = 5$ gives 30 experts and this matches the lower bound 25(25-1)/5(5-1) = 30.

For other values of k, i.e., when $k \neq q$, the $(q^2,q,1)$-BIBD still produces an ordinal covering but such an assignment will no longer be minimal. In particular, for any prime power, $q = n/k$, where $\sqrt{n} < k \leq n/2$ and n divides k^2, the $(q^2,q,1)$-BIBD gives an ordinal covering with $q(q+1) = n/k\ (n/k+1) = n(n+k)/k^2$ experts, each with a capacity of k.

IV. Concluding Remarks

We presented a method to obtain a minimal ordinal covering of a set of n documents using an $(n,\sqrt{n},1)$-BIBD. The method works for any n that is a square of a prime power and requires $n+\sqrt{n}$ experts, each with a capacity of $\sqrt{n}$. By adjusting the parameters in the BIBD design, i.e., using an $(n^2/k^2,n/k,1)$-BIBD, and requiring that n/k be a prime power, the number of experts, each with a capacity of k, can be reduced to $\lceil (n/k)(n/k+1)\rceil$, where n divides k^2 and $\sqrt{n} \leq k \leq n/2$ or equivalently, $2 \leq n/k \leq \sqrt{n}$. The condition that n should divide k^2 ensures that the set of k documents assigned to each expert can be partitioned into n/k sets of documents to form the entries of the blocks in the $(n^2/k^2,n/k,1)$-BIBD. For example, if $n = 32$, $k = 8$, each of the $(n/k)(n/k+1) = 20$ experts is assigned k=8 documents that can be partitioned into n/k=4 groups of k^2/n =2 documents to form the entries of the 20 blocks in a (16,4,1)-BIBD. In this case, the number of experts used is quite close to the lower bound of $\lceil n(n-1)/k(k-1)\rceil = \lceil 32\times31/8\times7\rceil = 18$ experts but clearly not minimal.

When $\sqrt{n} \leq k \leq n/2$ and n divides k^2, the ordinal covering given in this paper results in a smaller number of experts than the ordinal covering described in [7] as can be seen from the inequality:

$$\overbrace{\frac{n(n+k)}{k^2}}^{\text{This paper's upper bound}} \leq \overbrace{\frac{n(2n-k)}{k^2}}^{\text{upper bound in [7]}} \quad \text{when } k \leq n/2.$$

Furthermore, we have

$$\overbrace{\frac{n(n+k)}{k^2}}^{\text{This paper's upper bound}} \Big/ \overbrace{\frac{n(n-1)}{k(k-1)}}^{\text{lower bound in [7]}} = \frac{(n+k)(k-1)}{(n-1)k} \leq 3/2 \text{ when } k \leq n/2.$$

Table III provides a comparison of the two bounds for some values of n and k, where we used q = 2, 3, and 4 for the assignments. The upper bound of this paper coincides with the upper bound in [7] when $q = 2$, and is smaller when $q = 3$ and 4. In fact, the difference between the two bounds is given by

$$\frac{n(2n-k) - n(n+k)}{k^2} = \frac{n^2 - 2nk}{k^2} = q^2 - 2q \geq 0 \text{ when } q = \frac{n}{k} \geq 2.$$

The capacity of experts in each of the three cases can be varied to account for a desired number of documents. For example, for q = 2, we set the capacity of each expert to $2i$+2 for any number of documents between $4i$ + 1 and $4i$ + 4. Similarly, for q = 3, we set the capacity of each expert to $3i$+3 for any number of documents between $9i$ + 1 and $9i$ + 9, and for q = 4, we set the capacity of each expert to $4i$+4 for any number of documents between $16i$ + 1 and $16i$ + 16. The capacities and number of experts can be traded using the three tables for a desired number of documents. For example, for 22 documents, the first table uses 6 experts, each with a capacity of 12 whereas the second table uses 12 experts, each with a capacity of 9. In this case, capacities are fairly close together and so it makes sense to use 6 experts. Other choices can be made similarly.

It remains open if the assignments described in this paper and in [7] can be improved further. The assignments that have just been described result in efficient assignments only when the number of documents is evenly divisible by a selected capacity. In particular, it is not known if there exists a minimal ordinal covering of a set of n documents with an expert capacity of k, for all $k \leq n$. Another direction that remains unexplored is to investigate ordinal covering of documents by experts with different capacities and/or stronger requirements of pairwise comparisons. These problems will be explored elsewhere.

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TABLE II. Assignment of 27 documents to 12 experts, each with a capacity of 9.

e_1	$d_1d_2d_3$	$d_4d_5d_6$	$d_7d_8d_9$						
e_2				$d_{10}d_{11}d_{12}$	$d_{13}d_{14}d_{15}$	$d_{16}d_{17}d_{18}$			
e_3							$d_{19}d_{20}d_{21}$	$d_{22}d_{23}d_{24}$	$d_{25}d_{26}d_{27}$
e_4	$d_1d_2d_3$			$d_{10}d_{11}d_{12}$			$d_{19}d_{20}d_{21}$		
e_5	$d_1d_2d_3$				$d_{13}d_{14}d_{15}$			$d_{22}d_{23}d_{24}$	
e_6	$d_1d_2d_3$					$d_{16}d_{17}d_{18}$			$d_{25}d_{26}d_{27}$
e_7		$d_4d_5d_6$		$d_{10}d_{11}d_{12}$					$d_{25}d_{26}d_{27}$
e_8		$d_4d_5d_6$			$d_{13}d_{14}d_{15}$		$d_{19}d_{20}d_{21}$		
e_9		$d_4d_5d_6$				$d_{16}d_{17}d_{18}$		$d_{22}d_{23}d_{24}$	
e_{10}			$d_7d_8d_9$	$d_{10}d_{11}d_{12}$				$d_{22}d_{23}d_{24}$	
e_{11}			$d_7d_8d_9$		$d_{13}d_{14}d_{15}$				$d_{25}d_{26}d_{27}$
e_{12}			$d_7d_8d_9$			$d_{16}d_{17}d_{18}$	$d_{19}d_{20}d_{21}$		

e_i : expert i, d_i : document i.

TABLE III. Comparison of the assignment method of this paper with that given in [7].

# documents	capacity(k)	# experts (This paper)	# experts (Ref. [7])
4	2	6	6
5-8	4	6	6
9-12	6	6	6
13-16	8	6	6
17-20	10	6	6
21-24	12	6	6

(a) $q = n/k = 2$.

# documents	capacity(k)	# experts (This paper)	# experts (Ref. [7])
9	3	12	15
10-18	6	12	15
19-27	9	12	15
28-36	12	12	15
37-45	15	12	15
46-54	18	12	15

(b) $q = n/k = 3$.

# documents	capacity(k)	# experts (This paper)	# experts (Ref. [7])
16	4	20	28
17-32	8	20	28
33-48	12	20	28
49-64	16	20	28
65-80	20	20	28
81-96	24	20	28

(c) $q = n/k = 4$.