PHASE-ONLY BEAM SYNTHESIS BY ITERATIVE SEMIDEFINITE RELAXATIONS WITH RANK REFINEMENT

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ABSTRACT

In phased array antennas, by varying the complex element weights beam patterns with desired shapes can be synthesized and/or steered to desired directions. These complex weights can be implemented by using amplitude controllers and phase shifters at the system level. Since controlling the phase of an RF signal is much easier than controlling its power, many systems do not have an individual amplitude controller for each element. Hence, beam shaping and steering are to be achieved by varying only the element phases. In this work, a new approach is proposed for phase-only beam synthesis problem. In this approach, the phase-only beam synthesis is formulated as a non-convex quadratically constrained quadratic problem (QCQP). Then, it is relaxed to a convex semidefinite problem (SDP), which generally provides an undesired high rank solution. An iterative technique is developed to obtain a rank-1 solution to the relaxed convex SDP. Conducted experiments show that, proposed method can successfully synthesize beam shapes with desired characteristics and steering directions by using only the element phases.

Index Terms— phased array antenna, beam pattern, quadratically constraint quadratic problem, semidefinite problem, convex relaxation

1. INTRODUCTION

Phased array antennas are used in many applications such as airport surveillance, missile detection and tracking, magnetic resonance imaging, etc., because of their electronic scanning capabilities [1]. Operating frequency and positions of the array elements define the main characteristics of the antenna pattern. By applying different complex weights to the array elements, the beam pattern can be steered to different directions. Moreover, its shape can also be modified, i.e., sidelobe levels can be suppressed, mainlobe beamwidht can be reduced, etc. These complex weights are implemented as amplitude controllers and phase shifters at the system level. Since controlling the phase of an RF signal is much easier than controlling its power, many systems do not have an individual amplitude controller for each element. Hence, beam synthesis by only varying the element phases assuming that all the elements are operating at the same power level is desired.

Since the beam pattern is a non-linear function of the element phases, there is no previously proposed method for approaching the problem from the convex optimization perspective. Generally, ant colony based optimization methods (particle swarm optimization, genetic algorithm, vs.) are used to minimize a certain cost function of element phases [2, 3]. Null insertion to the undesired spatial directions by varying element phases are studied in [4, 5, 6, 7]. An iterative method based on generalized projections is proposed in [8], resulting in a common amplitude and various phases distributions for different steering directions.

In this work, different from the previous approaches, we first constructed a non-convex quadratically constraint quadratic problem (QCQP) to model the problem. Then, we relaxed it to a convex semidefinite problem (SDP), which can be solved at the global optimum point in polynomial time. Although the resulting SDP is convex, its optimal solution is generally not a rank-1 matrix [9]. To achieve a rank-1 solution, we propose a novel iterative method, where in each step a SDP with additional convex constraints are solved. We show that, after a few iterations, the optimal solution of the constructed SDP has very fast decaying singular values, converging to a rank-1 solution. Conducted experiments show that, proposed method can successfully design beam patterns with desired characteristics and steering directions by using only element phases.

In Section-2, mathematical definition of the problem is given. In Section-3, proposed method is detailed. Section-4 is reserved for experimental results. Concluding remarks are provided in Section-5. Through out the paper, bold characters will denote vectors for minuscules and matrices for capitals. $(.)^T$ will denote the transposition operation and $\|.\|$ will denote the L_2 norm of its argument.

2. PROBLEM DEFINITION

Let $\mathbf{p}_n,\ n = 1, ..., N$ denote the positions of antenna elements, where $\mathbf{p}_n = [p_{n,x}, p_{n,y}, p_{n,z}]^T$. The beam pattern is given by

$$B(\theta, \phi) = \sum_{n=1}^{N} \alpha_n v_n(\theta, \phi), \tag{1}$$

where $v_n(\theta, \pi) = \exp\{j \frac{2\pi}{\lambda} \mathbf{p}_n^T \mathbf{a}\}$. The directional cosines is defined as $\mathbf{a} = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]^T$, λ is the wavelength of transmission and $\alpha_n, n = 1, ..., N$ are the complex antenna weights. By changing the weights, antenna beam can be steered to different directions, its sidelobe levels, mainlobe power and beam width can be controlled. For phase-only beam synthesis problem, all the antenna weights are constrained to have the same magnitude. Hence, phase-only beam synthesis problem can be described by the following feasibility problem:

$$\begin{aligned}
&\text{find } \alpha_1, ..., \alpha_N \\
&\text{ s.t. } |B(\theta_m, \phi_m)|^2 \geq \delta_m, \\
&\qquad |B(\theta_{s,k}, \phi_{s,k})|^2 \leq \delta_s, \ \forall k = 1, ..., K, \\
&\qquad |\alpha_n|^2 = \delta_p, \ \forall n = 1, ..., N, \\
&\qquad |B(\theta_m, \phi_m)|^2 > |B(\theta_{m,h}, \phi_{m,h})|^2, \ \forall h = 1, ..., H.
\end{aligned} \tag{2}$$

Here, (θ_m, ϕ_m) is the steering direction, δ_m is the allowed minimum power level at the steering direction. $(\theta_{s,k}, \phi_{s,k}), k = 1, ..., K$ are the sidelobe directions for which the maximum allowed power level is δ_s and δ_p is the operating power level of the all antenna elements. The last constraint is to force the power pattern to have its highest peak at the steering direction, which is critical especially for direction finding applications. In Fig.1, these constraints are shown. The power pattern given in this figure belongs to a uniform linear array with 21 elements, where the antenna weights are chosen as $\alpha_n = v_n^\star(\theta_m, \phi_m)$ to maximize the power at the steering direction. The feasible set for constraints in (2) is generally empty. To ensure a non-empty feasible set, we transform the weight design problem in (2) to the following optimization problem:

$$\begin{aligned}
&\max_{\boldsymbol{\alpha} \in \mathbb{C}^N} \ \|\boldsymbol{\alpha}\|^2 \\
&\text{ s.t. } |\boldsymbol{\alpha}^T \mathbf{v}_m|^2 \geq \delta_m, \\
&\qquad |\boldsymbol{\alpha}^T \mathbf{v}_{s,k}|^2 \leq \delta_s, \ \forall k = 1, ..., K, \\
&\qquad |\alpha_n|^2 \leq \delta_p, \ \forall n = 1, ..., N, \\
&\qquad |\boldsymbol{\alpha}^T \mathbf{v}_m|^2 > |\boldsymbol{\alpha}^T \mathbf{v}_{m,h}|^2, \ \forall h = 1, ..., H,
\end{aligned} \tag{3}$$

where $\mathbf{v}_m = [v_1(\theta_m, \phi_m), v_2(\theta_m, \phi_m), .., v_N(\theta_m, \phi_m)]^T$, $\mathbf{v}_{s,k} = [v_1(\theta_{s,k}, \phi_{s,k}), v_2(\theta_{s,k}, \phi_{s,k}), .., v_N(\theta_{s,k}, \phi_{s,k})]^T$, $\mathbf{v}_{m,h} = [v_1(\theta_{m,h}, \phi_{m,h}), v_2(\theta_{m,h}, \phi_{m,h}), .., v_N(\theta_{m,h}, \phi_{m,h})]^T$ and $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, .., \alpha_N]^T$. In this formulation, sum of the energies of antenna weights is to be maximized, sidelobe and mainlobe constraints of (2) are preserved and '=' constraints on the energy of the antenna weights are replaced with '$\leq$' constraints. Hence, the feasible set of (3) is guaranteed to be non-empty for reasonable choices of δ_m, δ_s and δ_p. If the feasibility problem in (2) has a solution, then it would also be an optimal solution for (3).

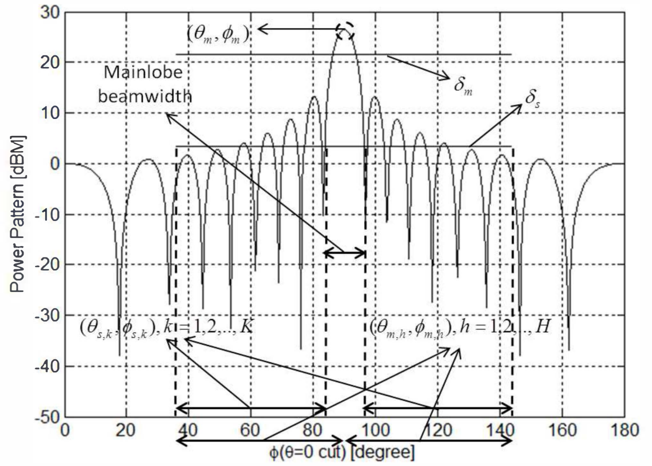

Fig. 1. Design constraints in (2). δ_m and δ_s define the minimum allowed mainlobe power and maximum allowed sidelobe power, respectively. (θ_m, ϕ_m) is the steering direction. $(\theta_{s,k}, \phi_{s,k}), k = 1, 2, .., K$ define the sidelobe constraints. $(\theta_{m,h}, \phi_{m,h}), h = 1, 2, .., H$ define the 'highest peak at the steering direction' constraint (last constraint in (2)).

The optimization problem in (3) has dimension N where the optimization variables are complex numbers. It can equivalently be formulated as a $2N$ dimensional optimization problems in real variables:

$$\begin{aligned}
&\min_{\boldsymbol{\beta} \in \mathbb{R}^{2N}} \ -\boldsymbol{\beta}^T \boldsymbol{\beta} \\
&\text{ s.t. } \boldsymbol{\beta}^T \mathbf{V}_m \mathbf{V}_m^T \boldsymbol{\beta} \geq \delta_m, \\
&\qquad \boldsymbol{\beta}^T \mathbf{V}_{s,k} \mathbf{V}_{s,k}^T \boldsymbol{\beta} \leq \delta_s, \ \forall k = 1, ..., K, \\
&\qquad \boldsymbol{\beta}^T \mathbf{W}_n^T \mathbf{W}_n \boldsymbol{\beta} \leq \delta_p, \ \forall n = 1, ..., N, \\
&\qquad \boldsymbol{\beta}^T \left(\mathbf{V}_m \mathbf{V}_m^T - \mathbf{V}_{m,h} \mathbf{V}_{m,h}^T \right) \boldsymbol{\beta} \geq \epsilon, \ \forall h = 1, ..., H,
\end{aligned} \tag{4}$$

where $\boldsymbol{\beta} = \begin{bmatrix} \Re\{\boldsymbol{\alpha}\} \\ \Im\{\boldsymbol{\alpha}\} \end{bmatrix}$, $\mathbf{V}_m = \begin{bmatrix} \Re\{\mathbf{v}_m^T\}, & -\Im\{\mathbf{v}_m^T\} \\ \Im\{\mathbf{v}_m^T\}, & \Re\{\mathbf{v}_m^T\} \end{bmatrix}$, $\mathbf{V}_{s,k} = \begin{bmatrix} \Re\{\mathbf{v}_{s,k}^T\}, & -\Im\{\mathbf{v}_{s,k}^T\} \\ \Im\{\mathbf{v}_{s,k}^T\}, & \Re\{\mathbf{v}_{s,k}^T\} \end{bmatrix}$, $\mathbf{V}_{m,h} = \begin{bmatrix} \Re\{\mathbf{v}_{m,h}^T\}, & -\Im\{\mathbf{v}_{m,h}^T\} \\ \Im\{\mathbf{v}_{m,h}^T\}, & \Re\{\mathbf{v}_{m,h}^T\} \end{bmatrix}$, $\mathbf{W}_n$ is an $2 \times 2N$ matrix composed of all zeros except $\mathbf{W}_n(1, n) = 1$ and $\mathbf{W}_n(2, N + n) = 1$, and ϵ is a positive number very close to zero. Note that the maximization in (3) is converted to a minimization in (4).

For notational simplicity, we further define the following matrices $\mathbf{A} = \mathbf{V}_m \mathbf{V}_m^T$, $\mathbf{B}_k = \mathbf{V}_{s,k} \mathbf{V}_{s,k}^T$, $\mathbf{C}_n = \mathbf{W}_n^T \mathbf{W}_n$, $\mathbf{D}_h = \mathbf{V}_m \mathbf{V}_m^T - \mathbf{V}_{m,h} \mathbf{V}_{m,h}^T$ and rewrite (4) as the following quadratically constrained quadratic problem (QCQP) with non-convex cost function and non-convex constraints, which can not be solved at the global optimum point in polynomial time:

$$\begin{aligned}\min_{\boldsymbol{\beta}\in\mathrm{R}^{2N}} \quad & -\boldsymbol{\beta}^T\boldsymbol{\beta} \\ \text{s.t.} \quad & \boldsymbol{\beta}^T\mathbf{A}\boldsymbol{\beta} \geq \delta_m, \\ & \boldsymbol{\beta}^T\mathbf{B}_k\boldsymbol{\beta} \leq \delta_s,\ \forall k = 1,...,K, \\ & \boldsymbol{\beta}^T\mathbf{C}_n\boldsymbol{\beta} \leq \delta_p,\ \forall n = 1,...,N, \\ & \boldsymbol{\beta}^T\mathbf{D}_h\boldsymbol{\beta} \geq \epsilon,\ \forall h = 1,...,H. \end{aligned} \tag{5}$$

In the next section proposed method for solving the optimization problem in (5) will be detailed.

3. PROPOSED METHOD: ITERATIVE SEMIDEFINITE RELAXATIONS WITH RANK REFINEMENT

Since the matrices in (5) $\mathbf{A}$, $\mathbf{B}_k, k = 1,...,K$, $\mathbf{C}_n, n = 1,...,N$ and $\mathbf{D}_h, h = 1,...,H$ are all symmetric, the QCQP in (5) can be equivalently written as:

$$\begin{aligned}\min_{\boldsymbol{\Lambda}\in\mathrm{R}^{2N\times 2N}} \quad & Tr\{-\boldsymbol{\Lambda}\} \\ \text{s.t.} \quad & Tr\{\mathbf{A}\boldsymbol{\Lambda}\} \geq \delta_m, \\ & Tr\{\mathbf{B}_k\boldsymbol{\Lambda}\} \leq \delta_s,\ \forall k = 1,...,K, \\ & Tr\{\mathbf{C}_n\boldsymbol{\Lambda}\} \leq \delta_p,\ \forall n = 1,...,N, \\ & Tr\{\mathbf{D}_h\boldsymbol{\Lambda}\} \geq \epsilon,\ \forall h = 1,...,H, \\ & \boldsymbol{\Lambda} \text{ is symetric and positive-semidefinite}, \\ & rank(\boldsymbol{\Lambda}) = 1. \end{aligned} \tag{6}$$

Note that the optimization variable in (6) is a matrix $\boldsymbol{\Lambda} \in \mathrm{R}^{2N\times 2N}$. If $\boldsymbol{\beta}_{opt}$ is an optimal solution for (5), then $\boldsymbol{\beta}_{opt}\boldsymbol{\beta}_{opt}^T$ is an optimal solution for (6). However, (6) is still an NP hard problem because of the rank constraint. By removing the rank constraint, it can be relaxed to a convex SDP which can be solved efficiently in polynomial time [9]:

$$\begin{aligned}\min_{\boldsymbol{\Lambda}\in\mathrm{R}^{2N\times 2N}} \quad & Tr\{-\boldsymbol{\Lambda}\} \\ \text{s.t.} \quad & Tr\{\mathbf{A}\boldsymbol{\Lambda}\} \geq \delta_m, \\ & Tr\{\mathbf{B}_k\boldsymbol{\Lambda}\} \leq \delta_s,\ \forall k = 1,...,K, \\ & Tr\{\mathbf{C}_n\boldsymbol{\Lambda}\} \leq \delta_p,\ \forall n = 1,...,N, \\ & Tr\{\mathbf{D}_h\boldsymbol{\Lambda}\} \geq \epsilon,\ \forall h = 1,...,H, \\ & \boldsymbol{\Lambda} \text{ is symetric and positive-semidefinite}. \end{aligned} \tag{7}$$

However, optimal solution $\boldsymbol{\Lambda}_{opt}$ of (7) is in general not rank-1. A rank-1 approximate of $\boldsymbol{\Lambda}_{opt}$ can be formed as

$$\tilde{\boldsymbol{\Lambda}}_{opt} = \sigma_1\mathbf{u}_1\mathbf{u}_1^T, \tag{8}$$

where λ_1 is the largest singular value of $\boldsymbol{\Lambda}_{opt}$ and $\mathbf{u}_1$ is the corresponding left singular vector. Then a candidate solution

Algorithm 1 Iterative semidefinite relaxations with rank refinement:

%Initializations
$i \leftarrow 0$.
$\zeta_i = 1$.
$r(i) = 1$.
Find $\boldsymbol{\Lambda}^i_{opt}$ by solving (7).
Apply SVD to $\boldsymbol{\Lambda}^i_{opt}$ and find its singular values $\sigma^i_1 \geq \sigma^i_2 \geq .. \geq \sigma^i_{2N}$ and the corresponding left singular vectors $\mathbf{u}^i_1, \mathbf{u}^i_2, .., \mathbf{u}^i_{2N}$.
$\tilde{\boldsymbol{\beta}}^i = \sqrt{\sigma^i_1}\mathbf{u}^i_1$.
Compute $r(i)$ by using (11).
while $i \leq N_{iter}$ and $r(i) \geq \nu$ **do**
  Attach $(\mathbf{u}^i_k)^T\boldsymbol{\Lambda}(\mathbf{u}^i_k)^T \leq \zeta_i\frac{1}{2N}\sum_{n=1}^{2N}\sigma^i_n\ \forall k = 2,3,..,2N$ constraints to (7) and resolve it for finding $\boldsymbol{\Lambda}^{i+1}_{opt}$.
  Apply SVD to $\boldsymbol{\Lambda}^{i+1}_{opt}$ and find its singular values $\sigma^{i+1}_1 \geq \sigma^{i+1}_2 \geq .. \geq \sigma^{i+1}_{2N}$ and the corresponding left singular vectors $\mathbf{u}^{i+1}_1, \mathbf{u}^{i+1}_2, .., \mathbf{u}^{i+1}_{2N}$.
  $\tilde{\boldsymbol{\beta}}^{i+1} = \sqrt{\sigma^{i+1}_1}\mathbf{u}^{i+1}_1$.
  Compute $r(i+1)$ by using (11).
  if $Tr\{-\boldsymbol{\Lambda}^{i+1}_{opt}\} \leq O_t$ **then**
    $\zeta_{i+1} \leftarrow \mu\zeta_i$.
  end if
  $i \leftarrow i + 1$
end while
Form complex weights: $\tilde{\boldsymbol{\alpha}}^i = \hat{\mathbf{W}}\tilde{\boldsymbol{\beta}}^i$
Normalize complex weights: $\tilde{\alpha}^i_n = \delta_p\tilde{\alpha}^i_n|\tilde{\alpha}^i_n|, n = 1,...,N$.

for the QCQP in (5) can be constructed as

$$\tilde{\boldsymbol{\beta}} = \sqrt{\sigma_1}\mathbf{u}_1. \tag{9}$$

However, since optimal solution $\boldsymbol{\Lambda}_{opt}$ of (7) is not rank-1, the candidate solution $\tilde{\beta}$ can be an infeasible point or a non-optimal solution for (5). Since the QCQP in (5) and its equivalent formulation in (7) are NP hard, the convex semidefinite relaxation in (7) can not be forced to have a strictly rank-1 optimal solution. However, it can iteratively be forced to have optimal solution matrix with fast decaying singular values, hence approximating to a rank-1 solution. Let $\boldsymbol{\Lambda}^i_{opt}$ be the optimal solution of (7) at the i^{th} step of the iterative algorithm. Assume $\sigma^i_1 \geq \sigma^i_2 \geq ... \geq \sigma^i_{2N}$ are the singular values and $\mathbf{u}^i_1, \mathbf{u}^i_2, ..., \mathbf{u}^i_{2N}$ are the corresponding left singular vectors of $\boldsymbol{\Lambda}^i_{opt}$. Then, the following $2N-1$ convex quadratic constraints

$$(\mathbf{u}^i_k)^T\boldsymbol{\Lambda}(\mathbf{u}^i_k)^T \leq \zeta_i\frac{1}{2N}\sum_{n=1}^{2N},\ \forall k = 2,...,2N \tag{10}$$

are attached to (7) and it is resolved. Here, ζ_i is the predefined multiplier which we initially choose as $\zeta_i = 1$. If the

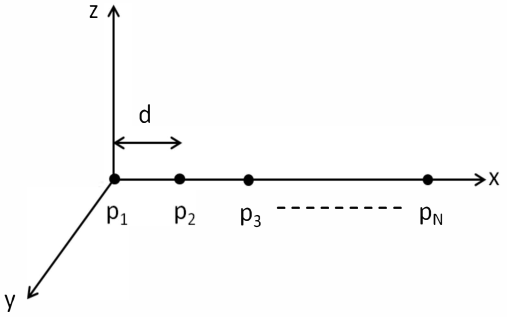

Fig. 2. Uniform linear array with $N = 21$ elements

objective value $-Tr\{\mathbf{\Lambda}_{opt}^{i+1}\}$ is less than a predefined target objective value O_t, then the multiplier at iteration $i+1$ is updated as $\zeta_{i+1} \leftarrow \mu\zeta_i$, where $0 < \mu < 1$ is the parameter controlling the convergence rate of the algorithm. After finite number of iterations N_{iter} or the difference between energy ratio of highest singular value of $\mathbf{\Lambda}_{opt}^{i}$ between two consecutive iterations, i.e.,

$$r(i) = \left| \sigma_1^i / \sum_{n=1}^{2N} \sigma_n^i - \sigma_1^{i-1} / \sum_{n=1}^{2N} \sigma_n^{i-1} \right| \tag{11}$$

is smaller than a certain threshold ν, iterations are terminated and the final solution of (5) is obtained as:

$$\tilde{\boldsymbol{\beta}}^i = \sqrt{\sigma_1^i}\mathbf{u}_1^i. \tag{12}$$

Corresponding complex antenna weight vector is given by

$$\tilde{\boldsymbol{\alpha}}^i = \hat{\mathbf{W}}\tilde{\boldsymbol{\beta}}^i, \tag{13}$$

where $\hat{\mathbf{W}}$ is an $N \times 2N$ matrix composed of all zeros except $\hat{\mathbf{W}}(n,n) = 1, \hat{\mathbf{W}}(n, N+n) = j, \forall n = 1, ..., N$. If the value of the cost function in (7) evaluated at the optimal solution $\mathbf{\Lambda}_{opt}^{i}$ at the final iteration is greater than $-N$, then the complex antenna weights $\tilde{\alpha}_n^i, n = 1, ..., N$ do not satisfy the power constraint in (2). Hence, weights are finally normalized as:

$$\hat{\alpha}_n^i = \delta_p \tilde{\alpha}_n^i / |\tilde{\alpha}_n^i|, n = 1, ..., N. \tag{14}$$

In Algorithm-1, proposed iterative method is summarized. In the next section experimental results demonstrating the performance of the proposed method will be provided.

4. EXPERIMENTAL RESULTS

To investigate the performance of the proposed method, we used a uniform linear array with $N = 21$ elements shown in Fig.2. Element positions are $\mathbf{p}_n = [d(n-1), 0, 0]^T, n = 1, ..., N$. Inter element spacing is $d = 0.4\lambda$, where λ is the wavelength, and operating frequency is chosen as $f =$ 2GHz. As design constraints, we allow 5dB power reduction in the steering direction ($\delta_m = 10 \log N^2 - 5$ dB) and require 23 dB sidelobe suppression ($\delta_s = 10 \log N^2 - 23$ dB). The beamwidth measured at 23 dB below the maximum power level ($10 \log N^2$) around the steering direction is constrained to be less than 15 degree in azimuth. All the antenna elements are required to operate at 1 Watt power level ($\delta_p = 1$). The proposed method in Algorithm-1 is initialized with parameters $N_{iter} = 20$, $\nu = 0.01$ for steering direction in azimuth $\phi_p = 90$ degree and in elevation $\theta_p = 0$ degree. For solving the SDP in (6), we used CVX, a package for specifying and solving convex programs [10].

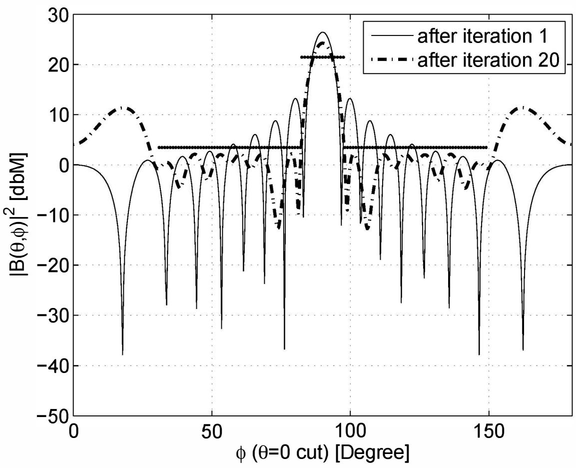

Fig. 3. $\theta = 0$ degree cut of the power pattern for steering direction ($\theta_p = 0, \phi_p = 90$)degree computed using the weights found after iteration $i = 1$ (solid) and $i = 20$(dashed-dotted)

After the first iteration, optimal value of the SDP in (7) is found to be -21. However, since the provided solution is not rank-1, the total power of the antenna elements is $\|\tilde{\boldsymbol{\alpha}}^1\|^2 = 8$, much smaller than 21. Hence the normalized coefficients $\hat{\boldsymbol{\alpha}}^1$ differ from the computed ones $\tilde{\boldsymbol{\alpha}}^1$ much. In Fig.3, elevation $\theta = 0$ cut of the power pattern generated by using the normalized complex weight vector after iteration $i = 1$ ($\hat{\boldsymbol{\alpha}}^1$) is plotted (solid). As observed, resulting beam pattern do not satisfy the design constraints. After 20 iterations, still the optimal value of the SDP in (7) is computed to be -21, the optimal solution matrix $\mathbf{\Lambda}_{opt}^{20}$ is nearly rank-1 and the total power of the antenna elements is $\|\tilde{\boldsymbol{\alpha}}^{20}\|^2 = 20.88$. Hence the normalized coefficients $\hat{\boldsymbol{\alpha}}^{20}$ are nearly the same with $\tilde{\boldsymbol{\alpha}}^{20}$. The resulting pattern after iteration 20 is plotted (dashed-dotted). As observed, all the design constraints are satisfied.

In Fig.4, the ratio of the largest singular value of the optimal solution matrix $\mathbf{\Lambda}_{opt}^{i}$ of (7) to the sum of all its singular values as a function of iteration number i, i.e., $\sigma_1^i / \sum_{n=1}^{2N} \sigma_n^i$, is plotted. As observed, at iteration $i = 20$, the solution matrix $\mathbf{\Lambda}^i$ is nearly rank one, since the largest singular value

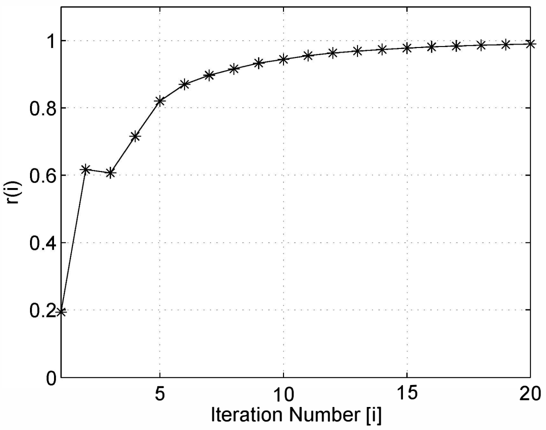

Fig. 4. Ratio of the largest singular value of the optimal solution matrix $\mathbf{\Lambda}_{opt}^{i}$ of (7) to the sum of all its singular values as a function of iteration number i.

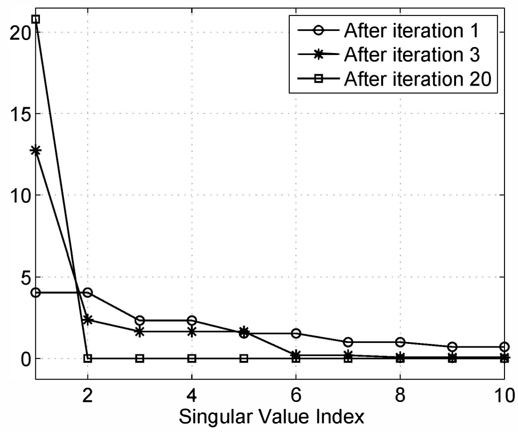

Fig. 5. 10 largest singular values of $\mathbf{\Lambda}_{opt}^{i}$ at iteration $i = 1$, $i = 3$, $i = 10$.

occupies most of its energy. Note that, as the iteration number increases, this ratio increases, demonstrating the converge behaviour of the proposed iterations. In Fig.5, 10 largest singular values of $\mathbf{\Lambda}_{opt}^{i}$ at iteration $i = 1$, $i = 3$, $i = 10$ are plotted. In the first iteration, singular values $\mathbf{\Lambda}_{opt}^{1}$ have a small decay rate. After iteration 3, singular values have a faster decay. At iteration 20, most of the energy is accumulated in the first singular value and the remaining ones are very close to 0. Hence, the proposed iterations provided a rank-1 solution to (7).

5. CONCLUSIONS

In this work, we proposed a novel iterative method for the phase-only beam synthesis problem. First, desired weights are formulated to be the solution of a non-convex QCQP. Then, the QCQP is relaxed to a convex SDP. Proposed iterations constrain the optimal solution of the SDP to have fast decaying singular values. After a few iterations, obtained solution is observed to be nearly rank-1. Conducted experiments indicate that, proposed method has a certain convergence behaviour and can successfully design beam shapes with desired characteristics by only using element phases.

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