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DECENTRALIZED STRONG STABILIZATION PROBLEM*

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Abstract

In the decentralized strong stabilization problem for linear time-invariant finite-dimensional systems, the objective is to stabilize a given plant using a *stable* decentralized controller. A solvability condition for this problem is given in terms of a parity interlacing property which is to be satisfied among the real unstable poles and real unstable *decentralized blocking zeros* of the plant. The problem of synthesizing decentralized stabilizing controllers with minimum number of unstable poles is also solved.

1. Introduction

In the strong stabilization problem the objective is to design a *stable* controller which internally stabilizes a given plant [18]. It is well-known that the strong stabilization problem is solvable if and only if the given plant has an even number of real poles between each pair of its unstable blocking zeros on the extended real axis [18], [16]. Although there are various procedures for constructing stable stabilizing controllers for a strongly stabilizable plant, these methods are not directly applicable to the controllers with feedback constraints, such as decentralized controllers.

In this paper we consider the Decentralized Strong Stabilization Problem (DSSP), where the objective is to solve the decentralized stabilization problem with stable local controllers. A motivation for DSSP is its close relation to some reliable decentralized stabilization problems ([8], see also [15], [9], [3], [6]). In many cases, the reliable decentralized stabilization problem can be transformed to an equivalent problem of decentralized strong stabilization problem (see e.g. [15], [11, Sections 2, 3]). Another motivation for DSSP is that the solution to the problem yields an understanding of how the total number of unstable poles of the overall controller can be distributed between the local controllers. This requires an extension of the solution of DSSP similarly to Theorem 5.3.1 of [16]. The problem DSSP also plays a primary role in the solution of *Decentralized Concurrent Stabilization Problem* (see Conclusions). In case of 2×2 plants DSSP has been previously considered in [8] and some partial results have been obtained.

In the next section we give the solution of DSSP in Theorem 1. We show that the problem is solvable if and only if the plant has no unstable decentralized fixed modes and it satisfies a certain interlacing property among its real unstable poles and real unstable decentralized blocking zeros. In Section 3, the solution of DSSP is extended to obtain the minimum number of unstable poles that any decentralized stabilizing controller should have. The distribution of these poles between the controllers has also been considered. The main result of Section 3 is stated in Theorem 2 which is a decentralized counterpart of Theorem 5.3.1 of [16]. Some concluding remarks are given in Conclusions.

We restrict the exposition in this paper to 2-channel systems. All the results stated in Theorems 1 and 2 can be extended to general N-channel systems.

Notation and Terminology: We denote by $\mathbf{P}$ and $\mathbf{S}$, the rings of proper rational functions and its subring, stable proper rational functions, respectively. The set of complex numbers and the set of extended complex numbers including infinity are denoted by $\mathcal{C}$ and $\mathcal{C}_e$, respectively. The closed right half complex plane including infinity (occasionally referred to as the unstable region) is denoted by $\mathcal{C}_{+e}$. We define $\mathcal{R}$ to be the set of real numbers and let $\mathcal{R}_{+e}$ denote the nonnegative real numbers including infinity. For all other definitions and terminology in the paper we refer the reader to [7]. The algebraic and topological properties of the ring $\mathbf{S}$ can be found in [16].

2. Decentralized Strong Stabilization Problem

We consider a 2-channel linear time-invariant finite-dimensional system with the following input/output relation:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Z \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

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where $Z_{ij} \in \mathbf{P}^{p_i \times r_j}$, $i,j = 1,2$, and Z_{11}, Z_{22} are strictly proper.

Decentralized Stabilization Problem (DSP). *Given Z above, determine a controller $Z_c = diag\{Z_{c1}, Z_{c2}\}$, $Z_{ci} \in \mathbf{P}^{r_i \times p_i}$, $i = 1,2$, such that (Z, Z_c) is internally stable.*

It is known that DSP is solvable if and only if Z has no unstable *decentralized fixed modes* [17]. These are the unstable open loop eigenvalues of a minimal state-space realization of Z, that remain unchanged under all constant decentralized output feedback controllers. An alternative solvability condition for DSP can be given in terms of the notion of completeness [4], [7], [5]. Let

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} Q^{-1} [R_1 \; R_2] \quad (1)$$

be a bicoprime fraction of Z over $\mathbf{S}$, where $Q \in \mathbf{S}^{q \times q}$, $P_i \in \mathbf{S}^{p_i \times q}$, and $R_i \in \mathbf{S}^{q \times r_i}$, $i = 1,2$. Then, DSP is solvable if and only if

$$\begin{bmatrix} Q & R_1 \\ -P_2 & 0 \end{bmatrix}(z) \geq q \;\; and \;\; \begin{bmatrix} Q & R_2 \\ -P_1 & 0 \end{bmatrix}(z) \geq q, \quad (2)$$

for all $z \in \mathcal{C}_+$ ([7], [10]). Any unstable $z \in \mathcal{C}_+$ for which the rank condition in (2) fails is an unstable decentralized fixed mode of Z ([4], [1]).

We now define the main problem considered in this paper.

Decentralized Strong Stabilization Problem (DSSP). *Given Z above, determine a* stable *decentralized controller $Z_c = diag\{Z_{c1}, Z_{c2}\}$, $Z_{ci} \in \mathbf{S}^{r_i \times p_i}$, $i = 1,2$, such that (Z, Z_c) is internally stable.*

In the solution of DSSP we restrict our attention to strongly connected systems [4], i.e., to those systems for which Z_{12} and Z_{21} are both nonzero. If Z is not strongly connected, DSSP can be easily shown to have a solution if and only if Z has no unstable decentralized fixed modes and the subplants Z_{11} and Z_{22} are both strongly stabilizable (see [10, Lemma 4.1]).

In order to state the main result of the paper define the *decentralized blocking zeros* of Z to be the elements of the set

$$\mathcal{S}_Z := \{z \in \mathcal{C}_e | \begin{bmatrix} Z_{11} & 0 \\ Z_{21} & Z_{22} \end{bmatrix}(z) = 0 \; or \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{bmatrix}(z) = 0\}.$$

Also let $\Psi = \mathcal{S}_Z \cap \mathcal{R}_{+e}$. Thus, Ψ is the set of real unstable decentralized blocking zeros of Z. Note that

$\Psi =$ *{intersection of unstable real blocking zeros of Z_{11} and Z_{22}}$\cap$*
{union of unstable real blocking zeros of Z_{12} and Z_{21}}.

Theorem 1. *Let Z satisfy*

$$rank\, Z_{12} \geq 2 \;\; or \;\; rank Z_{21} \geq 2. \quad (3)$$

Then, DSSP is solvable if and only if (i) *Z is free of unstable decentralized fixed modes and* (ii) *Z has an even number of real poles counting with multiplicities between each pair of real unstable decentralized blocking zeros of Z.*

Proof. [Only If] By the problem definition, the solvability of DSP is necessary for the solvability of DSSP. Hence, Z is free of unstable decentralized fixed modes. The set of unstable zeros of $det(Q)$ is precisely the set of unstable poles of Z. As a consequence of these, it can be shown that the set of unstable zeros of $det(Q)$ and Ψ are disjoint. Also, if $z \in \Psi$ is such that $[Z_{11}' \; Z_{21}']'(z) = 0$, then

$$rank \begin{bmatrix} Q & R_2 \\ -P_2 & 0 \\ -P_1 & 0 \end{bmatrix}(z) = q. \quad (4)$$

If $z \in \Psi$ satisfies $[Z_{11} \; Z_{12}](z) = 0$, on the other hand, it holds that

$$rank \begin{bmatrix} Q & R_2 & R_1 \\ -P_1 & 0 & 0 \end{bmatrix}(z) = q. \quad (5)$$

Let $Z_{ci} \in \mathbf{S}^{r_i \times p_i}$, $i = 1,2$, be such that $(Z, diag\{Z_{c1}, Z_{c2}\})$ is internally stable. Then, from Theorem 3.2 of [7]

$$\hat{Z} := [P_1 \; 0] \begin{bmatrix} Q & R_2 Z_{c2} \\ -P_2 & I \end{bmatrix}^{-1} \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \quad (6)$$

is a bicoprime fraction over $\mathbf{S}$. Moreover, $(\hat{Z}, Z_{c1})$ is internally stable. For any $z \in \mathcal{R}_{+e}$, for which (4) or (5) holds, it is easy to show, using the fact th fraction in (6) is bicoprime, that

$$rank \begin{bmatrix} Q & R_2 Z_{c2} & R_1 \\ -P_2 & I & 0 \\ -P_1 & 0 & 0 \end{bmatrix}(z) = q + p_2,$$

i.e., every $z \in \Psi$ is an $\mathcal{R}_{+e}$ blocking zero of $\hat{Z}$ of (6). Let $\Psi := \{\sigma_1, \sigma_2, ..., \sigma_t\}$, where $\sigma_i < \sigma_{i+1}$, $i = 1,2,...,t-1$. From Theorem 5.3.1 of [16], Z_{c1} internally stabilizes $\hat{Z}$ just in case $\hat{Z}$ has even number of poles between each pair of elements in the set $\{\sigma_1, \sigma_2, ..., \sigma_t\}$, or equivalently the determinant of the matrix

$$K := \begin{bmatrix} Q & R_2 Z_{c2} \\ -P_2 & I \end{bmatrix}$$

has no sign changes in the sequence σ_1, σ_2, ..., σ_t. On the other hand, for any $z \in \Psi$, one has $Z_{22}(z) = 0$. Therefore, $det(K)$ and $det(Q)$ take the same sign at the sequence $\sigma_1, \sigma_2, ..., \sigma_t$. Hence, for DSSP to be solvable $det(Q)$ should take the same sign at the sequence $\sigma_1, \sigma_2, ..., \sigma_t$. This holds if and only if Z has even number of real poles between each pair of elements in the set $\{\sigma_1, \sigma_2, ..., \sigma_t\}$.

[If] Using the assumption that Z has no unstable decentralized fixed modes, it is straightforward to show that the set of unstable zeros of $det(Q)$ and Ψ are disjoint. Let some left and right coprime fractions of Z_{22} over S be given by $Z_{22} = D_l^{-1}N_l = N_rD_r^{-1}$. Let $\Omega_l := gclf(Q, R_2)$, so that $Q = \Omega_l\tilde{Q}$, $R_2 = \Omega_l\tilde{R}_2$, for a left coprime pair of matrices $(\tilde{Q}, \tilde{R}_2)$. Also let $\Omega_r := gcrf(\tilde{Q}, P_2)$ so that $\tilde{Q} = \bar{Q}\Omega_r$, $P_2 = \bar{P}_2\Omega_r$, for a right coprime pair of matrices $(\bar{Q}, \bar{P}_2)$. Then, a bicoprime fraction of Z_{22} over S is given by $\bar{P}_2\bar{Q}^{-1}\tilde{R}_2$. Also note that $det(D_l) = det(D_r) = det(\bar{Q})$. Let

$$\Omega := \{z \in \mathcal{R}_{+e} | det(\Omega_l)det(\Omega_r)(z) \neq 0\},$$
$$\mathbf{D} := \{z \in \mathcal{R}_{+e} | det(D_l)(z) = 0\},$$

$$\hat{\Psi} := \Omega \cap \{\mathbf{D} \cup \{z \in \mathcal{R}_{+e} | \ [Z_{11}\ Z_{12}](z) = 0 \ or \ [Z_{11}'\ Z_{21}']'(z) = 0\}\},$$

$$\hat{\Psi}_1 := \{z \in \hat{\Psi} | Z_{22}(z) = 0\}, \ and$$
$$\hat{\Psi}_2 := \hat{\Psi} - \hat{\Psi}_1.$$

Note that Ω is the set of extended real numbers excluding the input-output decoupling zeros of (P_2, Q, R_2), and $\mathbf{D}$ is the set of unstable real poles of Z_{22}. Observe that $z \in \hat{\Psi}_1$ implies $[Z_{11}\ Z_{12}](z)$ or $[Z_{11}'\ Z_{21}']'(z)$ is zero, i.e., $z \in \hat{\Psi}_1$ implies $z \in \Psi$. Also for any $z \in \hat{\Psi}_2$, $N_l(z)$ is nonzero. Without loss of generality assume that $det(Q)$ takes positive sign at the sequence $\sigma_1, \sigma_2, ..., \sigma_t$. Now, construct Z_{c2} using known interpolation techniques and the genericity properties of the ring S, to satisfy

(a) $(I + Z_{22}Z_{c2})^{-1}$ is well-defined,

(b) $det(\Omega_l)det(\Omega_r)det(D_l + N_lZ_{c2})$ takes nonzero values with positive sign on the elements of $\hat{\Psi}_2$ (see the proof of Theorem 2.2 in [11]),

(c) The pairs (D_r, Z_{c2}) and (D_l, Z_{c2}) are left and right coprime respectively ([16, Lemma 7.6.31]),

(d) The transfer matrix defined by (6) is bicoprime ([10, Theorem 4.1]).

Note that the property (b) yields that $det(\Omega_l)\ det(\Omega_r)\ det(D_l + N_l\ Z_{c2})$ takes positive sign at each element of $\hat{\Psi}$. (This can be more clearly seen as follows: If $z \in \hat{\Psi}$, it belongs to either $\hat{\Psi}_1$ or $\hat{\Psi}_2$. If $z \in \hat{\Psi}_1$ then $z \in \Psi$ also, and $det(\Omega_l)det(\Omega_r)det(D_l + N_lZ_{c2})(z) = det(Q)(z) > 0$. If $z \in \hat{\Psi}_2$, on the other hand, the construction of Z_{c2} ensures that $det(\Omega_l)det(\Omega_r)det(D_l + N_lZ_{c2})(z) > 0$.) Also the properties (a), (b), (c), (d) above still continue to hold under sufficiently small perturbations on Z_{c2}, with respect to the graph topology [16]. *We will now show that by an arbitrarily small perturbation on Z_{c2} the set of $\mathcal{R}_{+e}$ blocking zeros of $\hat{Z}$ above can be made to be contained in the set $\hat{\Psi}$.* Since $det(\Omega_l)det(\Omega_r)det(D_l + N_lZ_{c2})$ is equal to

$$det(\begin{bmatrix} Q & R_2Z_{c2} \\ -P_2 & I \end{bmatrix}),$$

the italicized statement implies by [16, Theorem 5.3.1] that $\hat{Z}$ can be internally stabilized by some stable compensator Z_{c1}. Consequently, there exist Z_{c1}, Z_{c2} such that $(Z, diag\{Z_{c1}, Z_{c2}\})$ is internally stable.

We now prove the italicized statement above. Let $T := Z_{c2}(I + Z_{22}Z_{c2})^{-1}$ and let $T_1^{-1}T_2 = T$ be a left coprime fraction of T. It holds that $Z_{c2} = D_r(T_1D_r - T_2N_r)^{-1}T_2$. Since (D_r, Z_{c2}) is left coprime, $(T_1D_r - T_2N_r)D_r^{-1}$ is over S, i.e., $T_2 = \tilde{T}_2D_l$ for some matrix $\tilde{T}_2$ over S. Let $T_1^{-1}\tilde{T}_2 = \hat{T}_2\hat{T}_1^{-1}$, for a right coprime pair of matrices $(\hat{T}_2, \hat{T}_1)$. It follows that $Z_{c2} = \hat{T}_2(\hat{T}_1 - N_l\hat{T}_2)^{-1}D_l$. By the left coprimeness $(\hat{T}_1 - N_l\hat{T}_2, \hat{T}_2)$ and by the right coprimeness of (D_l, Z_{c2}) it easily follows that $D_l = (\hat{T}_1 - N_l\hat{T}_2)V$ for some unimodular V over S. Observe that for any $\Delta \in \mathbf{S}^{r_2 \times p_2}$, satisfying $||\Delta|| < 1/||VN_r||$, $V^{-1} - N_r\Delta$ is unimodular. Using Lemma A.2 in [11] and the connectivity assumption (3), it can be shown that there exists an open and dense subset $\mathcal{X}$ of $\mathbf{S}^{r_2 \times p_2}$ such that for any fixed but otherwise arbitrary $\Delta \in \mathcal{X}$, the implication

$$(Z_{11} - Z_{12}(\hat{T}_2 + D_r\Delta)\hat{T}_1^{-1}D_lZ_{21})(z) = 0$$
$$\Longrightarrow$$
$$\begin{bmatrix} Z_{11}\ Z_{12} \end{bmatrix}(z) = 0 \ or \ [Z_{11}'\ Z_{21}']'(z) = 0, \ \forall z \in \mathcal{R}_{+e} - \mathbf{D} \quad (7)$$

holds. Now choose $\Delta \in \mathcal{X}$ with sufficiently small norm such that when Z_{c2} is replaced by $Z_{c2\Delta} := (\hat{T}_2 + D_r\Delta)(V^{-1} - \hat{N}_r\Delta)^{-1}$ (a), (b), (c) and (d) above still hold. Now $\hat{Z} = Z_{11} - Z_{12}Z_{c2\Delta}(I + Z_{22}Z_{c2\Delta})^{-1}Z_{21}$ $= Z_{11} - Z_{12}(\hat{T}_2 + D_r\Delta)\hat{T}_1^{-1}D_lZ_{21}$. By the fact that (d) holds, it follows that the unstable blocking zeros of $\hat{Z}$ and the set of unstable zeros of $det(\Omega_l)det(\Omega_r)$ are disjoint. Then, the implication in (7) shows that the $\mathcal{R}_{+e}$ blocking zeros of $\hat{Z}$ are contained in $\hat{\Psi}$. This proves the italicized statement.□

Remark. Note that, the connectivity assumption (3) is used only in the sufficiency part of the proof. Thus, every plant for which DSSP is solvable satisfies the interlacing property indicated in Theorem 1.

3. Distribution of the Unstable Poles Between the Local Compensators

The result stated in Theorem 1 can be extended to investigate the design of decentralized stabilizing controllers with minimum number of unstable poles. The following result can be proved similarly to Theorem 1 above, [12]. For an analogy with the full-feedback case see Theorem 5.3.1 of [16].

Given a strongly connected plant Z where (3) holds and Z is free of unstable decentralized fixed modes, let $\sigma_1, \sigma_2, ..., \sigma_t$ denote the elements of Ψ arranged in the ascending order. Also let η_i denote the number of poles of Z counted with multiplicities in the interval (σ_i, σ_{i+1}), $i \in \{1, 2, ..., t-1\}$. Assume that η denotes the number of odd integers in the sequence $\eta_1, \eta_2, ..., \eta_{t-1}$.

Theorem 2.

I. *If a solution $diag\{Z_{c1}, Z_{c2}\}$ to DSP is such that the number of unstable poles of Z_{ci} counted with multiplicities is equal to n_i, $i = 1, 2$, then $\eta \le n_1 + n_2$.*

II. *Given two nonnegative integers n_1, n_2 such that $\eta = n_1 + n_2$, there exists a solution $diag\{Z_{c1}, Z_{c2}\}$ to DSP such that the number of unstable poles of Z_{ci} counted with multiplicities is equal to n_i, $i = 1, 2$.*

An interesting feature of Theorem 2 is that the unstable poles of the overall controller can be arbitrarily distributed between the local controllers. The reader is referred to [2] where some related problems are investigated.

4. Conclusions

In this paper we have introduced the notion of decentralized blocking zeros and obtained the solution of Decentralized Strong Stabilization Problem where the objective is to stabilize a system using a stable decentralized controller. It is shown that DSSP is solvable if and only if the real unstable poles and the real unstable decentralized blocking zeros of the plant satisfy a parity interlacing property. The synthesis of decentralized stabilizing controllers with minimum number of unstable poles is also investigated. The constructive parts of the results in the paper are stated under the mild connectivity assumption (3).

We finally note that DSSP is the core problem of Decentralized Concurrent Stabilization Problem which is defined as follows.

Decentralized Concurrent Stabilization Problem (DCSP). *Let the two-channel plants Z and $T = diag\{T_1, T_2\}$ be given, where the sizes of T_i and Z_{ii} are compatible, $i = 1, 2$. Determine a decentralized controller $Z_c = diag\{Z_{c1}, Z_{c2}\}$ such that (Z, Z_c) and (T, Z_c) are both internally stable.*

The problem DCSP is a special decentralized simultaneous stabilization problem [14]. In [12], [13] it is shown that a solution to DCSP exists if and only if DSSP is solvable for an auxilary plant the decentralized blocking zeros of which can be explicitly described.

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