ON THE USE OF DISCONTINUOUS EXPANSION AND TESTING FUNCTIONS IN THE METHOD OF MOMENTS FOR ELECTROMAGNETIC PROBLEMS

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I. Introduction

The choice of the expansion and testing functions is the most important step in the application of the method of moments (MoM), because an improper choice can lead to nonconvergent integrals [1] and, consequently erroneous results [2]. Although this issue has been well-examined in the context of linear spaces and operators [3]-[4] from a mathematical point of view, there has been a little study of the choice of expansion and testing functions used in the electromagnetic problems [5]. Recently, it was shown mathematically that there is an admissible class of functions to be employed as the expansion and testing functions in the electromagnetic problems [1]. According to this study, the functional form of the expansion and testing functions in the direction of the polarization of the current must be such that the sum of the order of the differentiability of the expansion and testing functions along the direction of the polarization of the current is equal to or greater than one. However, it is sometimes essential to use the expansion functions with finite discontinuities to represent the current at the source and load junctions [6]. This leads us to employ the testing functions from, at least, first-order differentiable functions like rooftops, piecewise sinusoids, etc., if the above criteria is followed strictly, resulting in a non-symmetric MoM matrix which is usually not desirable.

The purpose of this paper is to show that there is an additional constraint to be satisfied by the current density when discontinuous expansion functions are used, and the discontinuous expansion and testing functions together can be employed in the application of the MoM in the spatial and spectral domain with the use of this additional constraint.

II. Discontinuous expansion functions

A microstrip line terminated in the complex load impedances and fed by a probe is considered as a typical example to demonstrate the use of the discontinuous expansion functions for the current density, as shown in Figs. 1(a) and 1(b).

The discontinuous expansion functions at the load terminals ($x=-x_l$, x_r) represent the non-zero values of the currents which flow into the loads, while the discontinuities of the current densities J_x and J_z at the source terminal (x, $z=0$) is due to the branching of the geometry. Since there is no real source or sink at these terminals, that is, the total current entering to the terminals are equal to the total current flowing out of the terminals, the conservation of charge has to be satisfied. This is equivalent to saying that the charge density anywhere on the line must be finite, which is the additional constraint to be satisfied in the application of the MoM

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to the problems necessitating the use of discontinuous expansion functions. This constraint can be written as

$$\nabla \cdot \mathbf{J} = -j\omega\rho \text{ must be bounded} \tag{1}$$

where $\mathbf{J} = J_x\mathbf{x} + J_y\mathbf{y} + J_z\mathbf{z}$ and ρ is the charge density.

The implementation of (1) in the MoM applications is straightforward in the spatial domain because the derivatives of the current density are explicit in the expressions of the tangential electric fields on the conducting lines:

$$E_x = -j\omega G^A_{xx} * J_x + \frac{1}{j\omega}\frac{\partial}{\partial x}\left[G^q_x * \frac{\partial}{\partial x}J_x + G^q_z * \frac{\partial}{\partial z}J_z \right] \tag{2a}$$

$$E_z = -j\omega G^A_{zx} * J_x - j\omega G^A_{zz} * J_z + \frac{1}{j\omega}\frac{\partial}{\partial z}\left[G^q_x * \frac{\partial}{\partial x}J_x + G^q_z * \frac{\partial}{\partial z}J_z \right] \tag{2b}$$

where G^A and G^q are the Green's functions of the vector and scalar potentials, respectively, and J_y is assumed, with no loss of generality, to be zero in this example. However, the tangential electric fields in the spectral domain are defined, in general, in terms of the spectral domain Green's functions of the electric fields, G^E, as

$$\tilde{E}_x(k_x,k_y) = \tilde{G}^E_{xx}\,\tilde{J}_x(k_x,k_y) + \tilde{G}^E_{xz}\,\tilde{J}_z(k_x,k_y) \tag{3a}$$

$$\tilde{E}_z(k_x,k_y) = \tilde{G}^E_{zx}\,\tilde{J}_x(k_x,k_y) + \tilde{G}^E_{zz}\,\tilde{J}_z(k_x,k_y) \tag{3b}$$

where the contributions of the spatial derivatives of the current densities are absorbed into the spectral representations of the corresponding Green's functions. Thus, the effect of a non-physical singularity in the charge density, caused by using the discontinuous current density, cannot be distinguished as easily as that in the spatial domain. Moreover, the validity of the spectral domain Green's functions of the electric fields is questionable because they are obtained either from the Fourier transforms of the terms in front of the corresponding current densities in (2), that is,

$$\tilde{G}^E_{xx} = \mathbf{F}\left\{ -j\omega G^A_{xx} + \frac{1}{j\omega}\frac{\partial^2}{\partial x^2}G^q_x \right\} \; ; \qquad \tilde{G}^E_{xz} = \mathbf{F}\left\{ \frac{1}{j\omega}\frac{\partial^2}{\partial x \partial z}G^q_z \right\} \tag{4a,b}$$

$$\tilde{G}^E_{zx} = \mathbf{F}\left\{ -j\omega G^A_{zx} + \frac{1}{j\omega}\frac{\partial^2}{\partial z \partial x}G^q_x \right\} ; \quad \tilde{G}^E_{zz} = \mathbf{F}\left\{ -j\omega G^A_{zz} + \frac{1}{j\omega}\frac{\partial^2}{\partial z^2}G^q_z \right\} \tag{4c,d}$$

where the derivatives of the current densities are transferred over to the Green's functions of the scalar potentials by using the integration by parts, or equivalently from

$$\tilde{G}^E = \mathbf{F}\left\{ -j\omega\left[\bar{\mathbf{I}} + \frac{\nabla\nabla}{k^2} \right] \cdot \mathbf{G}^A \right\} \tag{5}$$

In both cases, the electric field Green's functions in the spatial domain are not absolutely integrable, and hence their Fourier transforms cannot exist in the classical sense. Then, the electric field Green's functions in the spectral domain must be considered as distributions defined with some test functions (different from the testing functions used in the MoM) satisfying at least the Holder condition. Although the use of the testing functions in the MoM eases the constraint on the expansion functions (equivalent to the test functions defined for the distribution above), the use of the discontinuous expansion and testing functions together with these Green's functions is not possible.

III. Calculations of the MoM matrix elements

The (mn)-th element of the MoM matrix in the spatial domain is obtained from Eqn.(2) as

$$(\text{mn})\text{–th element} = \langle T_{xm}, G^A_{xx} * J_{xn}\rangle + \frac{1}{\omega^2}\langle T_{xm}, \frac{\partial}{\partial x}\left[G^q_x * \frac{\partial}{\partial x}J_{xn}\right]\rangle \tag{6}$$

where T_{xm} and J_{xn} are the testing and expansion functions for the current density J_x, respectively. Since the first term has no derivatives of the current density it can not introduce any singularity and the corresponding inner product is convergent. But, the second term, when a discontinuous expansion function is used, introduces an infinite charge density causing the corresponding convolution integral to be divergent. In the example given in Fig. 1, the discontinuous expansion functions are used at the load and source terminals where the conservation of charge has to be satisfied, and consequently the singularities generated by the differentiation of the current densities must be ignored. In doing so, the expansion functions used at the source and load terminals, half rooftops in this example, are made to be piecewise differentiable functions so that they satisfy the criteria of the admissible class of functions [1]. The differentiation in front of the convolution integral can be transferred over to the testing function by the use of the integration by parts, and the resulting singularity, if the testing function is discontinuous, is ignored.

In the spectral domain, the (mn)-th element of the MoM matrix can be written, by using Eqns. (3a) and (4a), as

$$\begin{aligned}(\text{mn})\text{–th element} &= \langle \tilde{T}_{xm}, \tilde{G}^E_{xx}\,\tilde{J}_{xn}\rangle \\ &= -j\omega\langle \tilde{T}_{xm}, \tilde{G}^A_{xx}\,\tilde{J}_{xn}\rangle + \frac{1}{j\omega}\langle \tilde{T}_{xm}, -k_x^2\,\tilde{G}^q_x\,\tilde{J}_{xn}\rangle\end{aligned} \tag{7}$$

where $\delta/\delta x^2$ is replaced by $-k_x{}^2$ in the spectral domain. Since the first term on the right-hand side is convergent, the second term is the contribution of the charge density and divergent, the additional constraint on the current density (1) has to be applied to the second term:

$$\begin{aligned}\langle \tilde{T}_{xm}, -k_x^2\,\tilde{G}^q_x\,\tilde{J}_{xn}\rangle &= \langle -jk_x\tilde{T}_{xm}, \tilde{G}^q_x\,(-jk_x\tilde{J}_{xn})\rangle \\ &= \langle \mathbf{F}\left\{\frac{\partial}{\partial x}T_{xm} \pm \delta(x-x_{dis})\right\}, \tilde{G}^q_x\,\mathbf{F}\left\{\frac{\partial}{\partial x}J_{xn} \pm \delta(x-x_{dis})\right\}\rangle\end{aligned} \tag{8}$$

where x_{dis} is the position of the discontinuity in the corresponding function, and + (-) sign is chosen if the sign of the singularity generated by the derivative of the corresponding function is - (+). The transfer of $-k_x{}^2$ onto the testing and expansion functions is equivalent to the transfer of the differentiations in the spatial domain. It should be noted that, as it was discussed in the previous section, we cannot apply the condition of the charge conservation if the Green's functions of the electric fields are used.

IV. Conclusions

An additional constraint on the current density has been obtained in order to use the discontinuous expansion functions in the MoM for the electromagnetic problems, and its implementation in the spatial and spectral domain MoM has been formulated.

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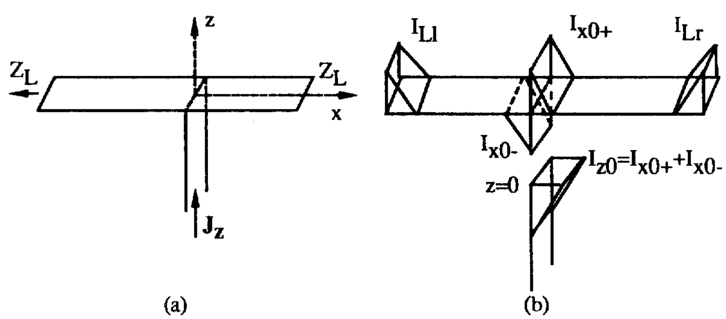

Fig.1. (a) Microstrip line fed by a probe and terminated in complex impedances, and(b) discontinuous expansion functions at the source and load junctions.