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Stabilization and Disturbance Rejection For the Wave Equation

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Abstract

We consider a system described by the one dimensional linear wave equation in a bounded domain with appropriate boundary conditions. To stabilize the system, we propose a dynamic boundary controller applied at the free end of the system. We also consider the case where the output of the controller is corrupted by a disturbance and show that it may be possible to attenuate the effect of the disturbance at the output if we choose the controller transfer function appropriately.

1 Introduction

We consider a system whose behaviour is modeled by the following wave equation :

$$y_{tt}(x,t) = y_{xx}(x,t) \qquad x \in (0,1) \qquad t \geq 0 \qquad (1)$$

$$y(0,t) = 0 \qquad y_x(1,t) = -f(t) \qquad t \geq 0 \qquad (2)$$

where a subscript, as in y_t denotes a partial differential with respect to the corresponding variable, and $f(t)$ is the boundary control force applied at the free end.

It is well known that if we apply the following boundary controller

$$f(t) = dy_t(1,t) \quad , \quad d > 0 \qquad (3)$$

then the closed loop-system given by (1)-(3) is exponentially stable, see [1]. However, we will show later that when the system is subjected to a disturbance, due to measurements and actuation, this choice may not be a good one.

The problem we consider in this paper is to choose the controller which generates $f(t)$ appropriately to make the closed-loop system stable in some sense. Later we will analyze the effect of this controller to the output of the system, $(y_t(1,t))$, when the controller is corrupted by disturbance.

In this paper we assume that $f(t)$ is generated by a dynamic controller whose relation between its input $y_t(1,t)$, and its output $f(t)$ is given by the following :

$$\dot{z}_1 = Az_1 + by_t(1,t) \qquad (4)$$

$$\dot{x}_1 = \omega_1 x_2 \quad , \quad \dot{x}_2 = -\omega_1 x_1 + y_t(1,t) \qquad (5)$$

$$f(t) = c^T z_1 + dy_t(1,t) + k_1 y(1,t) + k_2 x_2 \qquad (6)$$

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where $z_1 \in \mathbf{R}^n$, for some natural number n, is the actuator state, $A \in \mathbf{R}^{n\times n}$ is a constant matrix, $b, c \in \mathbf{R}^n$ are constant column vectors, $d \in \mathbf{R}$, and the superscript T denotes transpose.

We make the following assumptions concerning the actuator given by (4)-(6) thoroughout this work :

Assumption 1 : All eigenvalues of $A \in \mathbf{R}^{n\times n}$ have negative real parts.

Assumption 2 : (A, b) is controllable and (c, A) is observable.

Assumption 3 : $d \geq 0, k_1 \geq 0, k_2 \geq 0$; moreover there exists a constant γ, $d \geq \gamma \geq 0$, such that the following holds

$$d + \mathcal{R}e\{c^T(j\omega I - A)^{-1}b\} > \gamma, \qquad , \qquad \omega \in \mathbf{R} \qquad (7)$$

Moreover for $d > 0$, we assume $\gamma > 0$ as well. □

2 Stability Results

Let the assumptions 1-3 stated above hold. Then it follows from the Meyer-Kalman-Yakubovich Lemma that given any symmetric positive definite matrix $Q \in \mathbf{R}^{n\times n}$, there exists a symmetric positive definite matrix $P \in \mathbf{R}^{n\times n}$, a vector $q \in \mathbf{R}^n$ and a constant $\epsilon > 0$ satisfying : (see [4, p. 133].)

$$A^T P + PA = -qq^T - \epsilon Q \qquad (8)$$

$$Pb - c = \sqrt{2(d-\gamma)}q \qquad (9)$$

To analyze the system given by (1)-(2), (4)-(6), we define the following "energy" function :

$$E(t) = \tfrac{1}{2}\int_0^1 y_t^2 dx + \tfrac{1}{2}\int_0^1 y_x^2 dx + \tfrac{1}{2}k_1 y^2(1,t) + \tfrac{1}{2}z_1^T P z_1 + \tfrac{1}{2}k_2(x_1^2 + x_2^2) \quad . \qquad (10)$$

Theorem 1 : Consider the system given by (1)-(2), (4)-(6).

i : The energy $E(t)$ given by (10) is a nonincreasing function of time along the solutions of this system.

ii : If $\omega_1 \neq m\pi$ for some natural number $m \in \mathbf{N}$, then solutions of this system asymptotically converge to zero.

Proof : i : We differentiate (10) with respect to time. Then by using (1)-(2), (4)-(6), integrating by parts and using (8), (9), we obtain :

$$\dot{E} = -\gamma y_t^2(1,t) - \frac{1}{2}[\sqrt{2(d-\gamma)}y_t(1,t) - z_1^T q]^2 - \frac{\epsilon}{2}z_1^T Q z_1 \qquad (11)$$

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Since $\dot{E} \leq 0$, it follows that $E(t)$ is a nonincreasing function of time.

ii : To prove the assertion ii, we use LaSalle's invariance principle, extended to infinite dimensional systems. According to this principle, all solutions asymptotically tend to the maximal invariant subset of the following set : $\mathcal{S} = \{\dot{E} = 0\}$ provided that the solution trajectories for $t \geq 0$ are *precompact* in the underlying space. By casting the equations in operator form, it can be shown that the above system generates a C_0-semigroup in an appropriate space, see [2] for similar results. It could also be shown that this operator has a compact resolvent, which, together with (11), implies that the solutions are precompact in the space considered.

To prove that $\mathcal{S}$ contains only the zero solution, we set $\dot{E} = 0$ in (11), which results in $z_1 = 0$. This implies that $\dot{z}_1 = 0$, hence by using (4) and (6) we obtain $y_t(1,t) = 0$, $f(t) = k_1 y(1,t) + k_2 x_2$. By using these it can be shown that to have a nontrivial solution for the system considered, we must have $\omega_1 = m\pi$ for some natural number $m \in \mathbf{N}$. Therefore if $\omega_1 \neq m\pi$ for some natural number $m \in \mathbf{N}$, we conclude that the only solution of this system which lies in the set $\mathcal{S}$ is the zero solution, hence, by LaSalle's invariance principle, we conlude that the solutions asymptotically tend to the zero solution. □

3 Disturbance Rejection

In this section we show the effect of the proposed control law given by (4)-(6) on the solutions of the system given by (1)-(2), when the output of the controller is corrupted by a disturbance $d(t)$, that is (6) has the following form :

$$f(t) = c^T z_1 + d y_t(1,t) + k_1 y(1,t) + k_2 x_2 + d(t) \qquad (12)$$

or equivalently

$$\hat{f}(s) = g(s)\hat{y}_t(1,s) + \hat{d}(s) \qquad (13)$$

where $\hat{d}(s)$ is the Laplace transform of the disturbance $d(t)$. For another type of disturbance acting on the system, see [3].

To find the transfer function from $d(t)$ to $y_t(1,t)$, we take the Laplace transform of (1)-(2) and set initial conditions to zero. Then, the solution of (1), becomes :

$$y(x,s) = c \sinh xs \qquad (14)$$

where c is a constant and sinh is the hyperbolic sine function. By using (2) and (13), we obtain :

$$c = -\frac{1}{s(\cosh s + g(s)\sinh s)}\hat{d}(s) \qquad (15)$$

Now, consider the controller given by (3). It is known that, without disturbance, this system is exponentially stable, and that by choosing d appropriately, one can achieve arbitrary decay rates. Moreover $d = 1$ is the best choice since in this case all solutions become zero for $t \geq 2$. However, from (15) one can easily see that the case $d = 1$ is not a good choice for disturbance rejection. To see this, first note that in this case the controller transfer function $g(s)$ is given by $g(s) = d = 1$, (see (3), and (13)). Hence, we obtain

$$y_t(1,t) = \frac{1}{2}(d(t-2) - d(t)) \qquad (16)$$

In case $d(t)$ is sinusoidal, from (16) it follows that $y_t(1,t)$ is sinusoidal as well. Hence the case $d = 1$ is not a good choice for disturbance rejection. It can be shown that $d \neq 1$, $d \in \mathbf{R}$ yields similar results.

Another choice for disturbance rejection is the use of dynamic controllers proposed here. From (15) we can also derive a procedure to design $g(s)$ if we know the structure of $d(t)$. For example if $d(t)$ has a band-limited frequency spectrum, (i.e. has frequency components in an interval of frequencies $[\Omega_1, \Omega_2]$), then we can choose $g(s)$ to minimize $| c(j\omega)$ for $\omega \in [\Omega_1, \Omega_2]$. As a simple example, assume that $d(t) = a\cos\omega_0(t)$. Then we may choose $g(s)$ in the form with $\omega_1 = \omega_0$. Provided that the assumptions 1-2 are satisfied and that $\omega_0 \neq m\pi$ for some natural number $m \in \mathbf{N}$, the closed-loop system is asymptotically stable, (see Theorem 1). Moreover, if $k_2 > 0$, then $c(\omega)$ given above satisfies $c(\omega_0) = 0$. From (15) we may conclude that this eliminates the effect of the disturbance at the output $y_t(1,t)$.

4 Conclusion

In this note, we considered a linear time invariant system which is represented by one-dimensional wave equation in a bounded domain. We assumed that the system is fixed at one end and the boundary control input is applied at the other end. For this system, we proposed a finite dimensional *dynamic* boundary controller. This introduces extra degrees of freedom in designing controllers which could be exploited in solving a variety of control problems, such as disturbance rejection, pole assignment, etc., while maintaining stability. The transfer function of the controller is a proper rational function of the complex variable s, and may contain a single pole at $s = 0$ and another one $s = j\omega_1$, $\omega_1 \neq 0$, provided that the residues corresponding to these poles are nonnegative; the rest of the transfer function is required to be a strictly positive real function. We then proved that the closed-loop system is asymptotically stable provided that $\omega_1 \neq m\pi$ for some natural number $m \in \mathbf{N}$. We also studied the case where the output of the controller is corrupted by a disturbance. We showed that, if the frequency spectrum of the controller is known, then by choosing the controller appropriately we can obtain better disturbance rejection.

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