COMPARATIVE STUDY OF ACCELERATION TECHNIQUES FOR INTEGRALS AND SERIES IN ELECTROMAGNETIC PROBLEMS

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Abstract

Most of the electromagnetic problems can be reduced down to either integrating oscillatory integrals or summing up complex series. However, limits of the integrals and the series usually extend to infinity. In addition, they may be slowly convergent. Therefore, numerically efficient techniques for evaluating the integrals or for calculating the sum of infinite series have to be used to make the numerical solution feasible and attractive. In the literature, there are a wide range of applications of such methods to various EM problems. In this paper, our main aim is to critically examine the popular series transformation (acceleration) methods which are used in electromagnetic problems and compare them by numerical examples.

I. Introduction Numerical techniques used in the solution of electromagnetic problems require, in general, either evaluating oscillatory integrals over infinite domain or calculating the sums of infinite complex series. For example, the Method of Moments (MoM) in the spectral domain for two-dimensional geometry requires double-infinite integration of complex highly oscillatory functions; the MoM in the spatial domain employs the spatial domain Green's functions which are defined as the Hankel transform of the spectral domain Green's function; in the analysis of a periodic structure one needs to employ periodic Green's function which are double infinite summations; or in the analysis of a microstrip patch antenna via cavity model, the input impedance or field distribution are written in terms of a infinite sum of modes in the cavity. If these summations and integrals given in the examples above are evaluated by brute force as they appear in the problems, the corresponding methods could be computationally very inefficient rendering these problems impractical. To overcome this computational burden, special acceleration techniques, also called transformation techniques, for both integrals and summations have been proposed and successfully employed. Since these techniques have been studied for specific problems and compared to only few other techniques, the potentials of these techniques with their advantages and disadvantages have not been examined entirely for electromagnetic problems.

II. The Transformation Methods The principle of a series acceleration method is to transform the original slowly convergent or asymptotic sequence, by using a linear or non-linear mapping, to a new sequence which is assumed to have faster convergence than the original one. The acceleration methods used in this processes

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can be divided into two main groups: the *general* methods and the *specific* methods. A general transformation method can be applied to any sequence which can be obtained from an infinite series or an infinite oscillatory integral. Examples of such methods are *Euler Transformation*, *Shanks' Transformation*, *Wynn's ε-algorithm*, *Chebyshev-Toeplitz Algorithm* and *Θ-algorithm.* On the other hand, specific methods are derived by making analytical work on the kernel of a series or an integration. Therefore, they can be applied only on their own types but they usually work better than the general methods. *Method of averages*, *Poisson Transformation*, *Ewald's Transformation*, *Method of Exponentials* and *Kummer's Transformation* are the examples. The transformation methods can be summarized as follows:

2.1 Euler's Transformation: Euler's transformation is a transformation method which can only be used for alternating series [1]. In Euler's transformation some of the previous terms are added directly to improve the accuracy.

2.2 Shanks' Transformation: The idea behind the Shanks' transformation is that the partial sums of a sequence can be treated as a mathematical transient, and it gives an approximation to the *base* of the transient which is the result of the infinite summation [2]. Wynn's ε-algorithm is an alternate way to evaluate the Shanks' transformation [3].

2.3 Method of Averages: This technique is especially suitable in evaluating the integrals which have a special form [4]. This method can also be used in the integration of suitable Bessel functions whose asymptotic forms can be expressed as a sinusoidal function.

2.4 The Θ-algorithm: The Θ-algorithm was derived from the Wynn's ε-algorithm [3].

2.5 The Chebyshev-Toeplitz Algorithm: The Chebyshev-Toeplitz algorithm uses *Toeplitz arrays* to accelerate a series [5].

2.6 The Poisson Transformation: The main idea behind the Poisson transformation is the reciprocal spreading property of the Fourier transformation. That is, if a function has a narrower support in one domain, then it would have a wider support in the other domain and vice versa.

2.7 Ewald's Transformation: This is a very powerful transformation method utilizing the complementary error function [6]. It is especially useful for double infinite summations.

2.8 Kummer's transformation: Kummer's transformation is based on extracting the asymptotic behavior of slowly converging series. When the asymptotic form is extracted, the series converges faster.

2.9 Method of Exponentials: This method is primarily used in evaluating the integrals which results from the Green's function evaluation of the planarly stratified media [7]. Although the numerical integration is possible utilizing the appropriate acceleration techniques discussed so far, the end result will be independent of r which means that for each different r one need to re-integrate the function. Therefore, it will be better if we can take the integral analytically approximating the spectral domain Green's function by complex exponentials. Then, the resultant integral can be evaluated using the Sommerfeld identity. The method is based on approximating a decaying function by complex exponentials by the GPOF method [7].

III. Results and Discussion In this section, some numerical examples on the transformations are going to be given. For all kind of transformations, relative error is defined in the following form

$$\varepsilon_r = \left| \frac{\tilde{S} - S}{S} \right| \tag{1}$$

where S and $\tilde{S}$ are the exact (or calculated up to sufficient precision) and approximated results respectively.

3.1 Integration of Bessel Functions: Here, acceleration of the integration of Bessel functions will be investigated. Now, consider the following integral [4]

$$\int_0^\infty x J_1(x)dx = 1 \tag{2}$$

The oscillatory integral is transformed into a sequence by considering each cycle, then the series transformation methods are used to find the result. Results are given in Figure 4. Here, the *weighted averages* seems the best transformation for this kind of integral. However, the weighted averages has the disadvantage that it needs the asymptotic behavior of the integrand to determine the optimum weights.

3.2 Free-Space Periodic Green's Functions: In this part, acceleration of the infinite summations appear in free-space Greens' functions are going to be investigated. At first, one-dimensional case will be demonstrated. Consider the following Green's function

$$G = \sum_{m=-\infty}^{\infty} \frac{e^{-j(k_x+2m\pi/d)(x-x')}e^{-jk_{ym}|y-y'|}}{j2dk_{ym}} \tag{3}$$

This equation appears in Green's function evaluation of one-dimensional line sources spaced d and located at (x', y'). The series in (3) converges rapidly when $y \neq y'$. Figures 1 and 2 show the results.

Next, the free-space periodic Green's function for the two-dimensional case will be demonstrated for this part.

$$G_p = \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-j\vec{k}_0 \cdot \vec{\rho}_{mn}} \frac{e^{jk_0 R_{mn}}}{R_{mn}} \tag{4}$$

This equation appears in the Green's function evaluation of two dimensional periodic structures [8]. Results are given in Figure 3. In the plots, the numbers near the graphics are the relative errors at the given convergence rate.

As a conclusion, the Euler transformation has a limitation that it needs steadily alternating sequences. For that reason, its application areas are limited. Even in the applicable cases, other transformation methods may work better than the Euler transformation. The ε-algorithm is advised instead of direct application of higher order Shanks' transformations. The θ-algorithm, ε-algorithm and Chebyshev-Toeplitz transformation work satisfactorily on most of the examples. For doubly infinite sums, the Ewald's transformation seems the best one.

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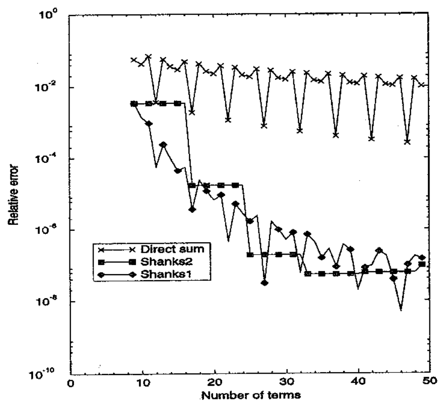

Figure 1: Relative errors for the summation in (3).

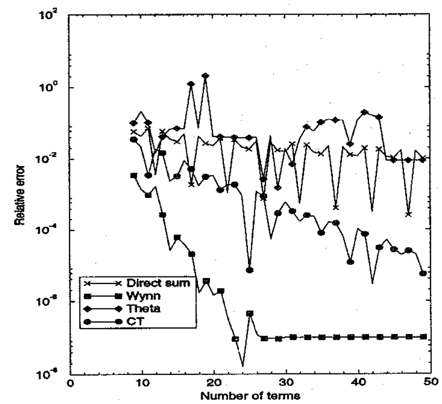

Figure 2: Relative errors for the summation in (3).

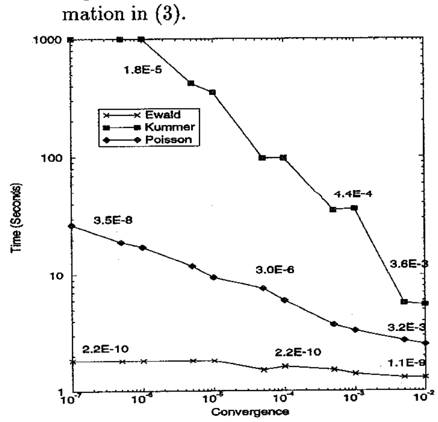

Figure 3: Relative errors and convergence times for the summation in (4).

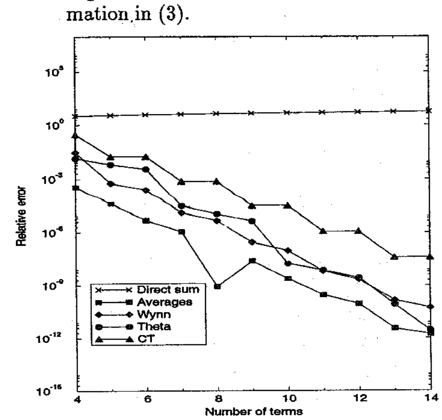

Figure 4: Relative errors for the integral in (2).