H^∞-Performance Analysis of Robust Controllers Designed for AQM[1]

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Abstract

It has been shown that the TCP connections through the congested routers with the Active Queue Management (AQM) can be modeled as a nonlinear feedback system. In this paper, we design H^∞ robust controllers for AQM based on the linearized TCP model with time delays. For the linear system model exhibiting LPV nature, we investigate the H^∞-performance with respect to the uncertainty bound of *RTT* (round trip time). The robust controllers and the corresponding analysis of H^∞-performance are validated by simulations in different scenarios.

1 Introduction

Active Queue Management has recently been proposed in [1] to support the end-to-end congestion control for TCP traffic regulation on the Internet. For the purpose of alleviating congestion for IP networks and providing some notion of quality of service (QoS), the AQM schemes are designed to improve the Internet applications. Earliest efforts on AQM (e.g. RED in [2]) are essentially heuristic without systematic analysis. The dynamic models of TCP ([9, 12]) make it possible to design AQM in the literature of feedback control theory. We refer to [11] for a general review of Internet congestion control.

In [12], an TCP/AQM model was derived using delay differential equations. They further provided a control theoretic analysis for RED where the parameters of RED can be tuned as an AQM controller [4]. In [5], a Proportional-Integral controller was developed based on the linearized model of [12]. Their controller could ensure robust stability of the closed loop system in the sense of gain-phase margin of the PI AQM [5, 6]. A challenging nature in the design of AQM is the presence of a time delay, which is called *RTT* (round trip time). To further complicate the situation, the linearized TCP/AQM model is linear parameter varying (LPV), with *RTT* being the scheduling parameter. In the present paper, robust AQM controllers are developed based on the H^∞ control techniques for SISO infinite dimensional systems [3, 15]. We also analyze the H^∞ performance for the robust controllers with respect to the uncertainty bound of the scheduling parameter *RTT*. Our results show that a smaller operating range of *RTT* results in better H^∞ performance of the AQM controller, which indicates that switching control among a set of robust controllers designed at selected smaller operating ranges can have better performance than a single H^∞ controller for the whole range.

The paper is organized as follows. The mathematical model of TCP/AQM is stated in Section 2, where the linearized LPV system with time delays is described. In Section 3, An H^∞ optimization problem is formulated, where the parametric uncertainties are modeled and the robust controllers are obtained. We investigate in Section 4 the H^∞ performance of the robust AQM controllers. MATLAB simulations are given in Section 5 to validate our design and analysis, followed by concluding remarks in Section 6.

2 Mathematical Model of TCP/AQM

In [12], a nonlinear dynamic model for TCP congestion control was derived, where the network topology was assumed to be a single bottleneck with N homogeneous TCP flows sharing the link. The congestion avoidance phase of TCP can be modeled as AIMD (additive-increase and multiplicative-decrease), where each positive ACK increases the TCP window size $W(t)$ by one per *RTT* and a congestion indication reduces $W(t)$ by half. Aggregating N TCP flows through one congested router results in the following TCP dynamics [12, 6]:

$$\begin{aligned} \dot{W}(t) &= \frac{1}{R(t)} - \frac{W(t)}{2}\frac{W(t-R(t))}{R(t-R(t))}p(t-R(t)) \\ \dot{q}(t) &= [\frac{N(t)}{R(t)}W(t) - C(t)]^+ \end{aligned} \quad (1)$$

where $R(t)$ is the *RTT*, $0 \le p(t) \le 1$ is the marking probability, $q(t)$ is the queue length at the router, and C is the link

[1] This work is supported by the National Science Fundation under grant number ANI-0073725.

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capacity. Note

$$R(t) = T_p + \frac{q(t)}{C}$$

where T_p is the propagation delay and $q(t)/C$ is the queuing delay.

Assume $N(t) = N$ and $C(t) = C$, the operating point of (1) is defined by $\dot{W} = 0$

$$R_0 = T_p + \frac{q_0}{C} \quad (2)$$

$$W_0 = \frac{R_0 C}{N} \quad (3)$$

$$p_0 = \frac{2}{W_0^2}. \quad (4)$$

Let $\delta q := q - q_0$ and $\delta p := p - p_0$, the linearization of (1) results in the following LPV time delay system, [6],

$$\frac{\delta q(s)}{\delta p(s)} := P_\theta(s) = \frac{K(\theta)e^{-h(\theta)s}}{(T_1(\theta)s+1)(T_2(\theta)s+1)} \quad (5)$$

where

$$K(\theta) = \frac{C^3\theta^3}{4N^2} \quad (6)$$

$$T_1(\theta) = \theta \quad (7)$$

$$T_2(\theta) = \frac{C\theta^2}{2N} \quad (8)$$

$$h(\theta) = \theta \quad (9)$$

and $\theta = R(t) \in [T_p,\ T_p + q_{max}/C]$ is the scheduling parameter of (5) where q_{max} is the buffer size. Note that we employ $L\{f(t,\theta)|_{\theta=\theta_0}\} = f_{\theta_0}(s)$ to describe the LPV dynamic equations in Laplace domain at fixed parameter values.

3 H^∞ Controller Design for AQM

Consider the nominal system

$$P_0(s) := P_\theta(s)|_{\theta=\theta_0} = \frac{K(\theta_0)e^{-h(\theta_0)s}}{(T_1(\theta_0)s+1)(T_2(\theta_0)s+1)} \quad (10)$$

where $\theta_0 = R_0$ is the nominal *RTT*. We would like to design a robust AQM controller $C_0(s)$ for the nominal plant (10) so that

(i) $C_0(s)$ robustly stabilizes $P_\theta(s)$ for $\forall\theta \in \Theta := [\theta_0 - \Delta\theta,\ \theta_0 + \Delta\theta]$;

(ii) The closed loop nominal system has good tracking of the desired queue length q_0 which is a step-like signal.

Notice that the plant (5) can be written as

$$P_\theta(s) = P_0(s)(1 + \Delta P_\theta(s)) \quad (11)$$

where $\Delta P_\theta(s)$ is the multiplicative plant uncertainty.

It can be shown that an uncertainty bound $W_2^{(\theta_0,\Delta\theta)}$ satisfying

$$|\Delta P_\theta(s)|_{s=j\omega} \le |W_2^{(\theta_0,\Delta\theta)}(s)|_{s=j\omega} \quad \forall\omega \in \mathbb{R}^+ \quad (12)$$

is

$$W_2^{(\theta_0,\Delta\theta)}(s) = a + bs + cs^2 \quad (13)$$

where a, b and c are defined in (35) (see the appendix for the details of derivation). Note that once θ_0 and $\Delta\theta$ are fixed, these coefficients are fixed.

Combining the robust stability and the nominal tracking performance condition, we come up with a two block infinite dimensional H^∞ optimization problem as follows:

Minimize γ, such that robust controller $C_0(s)$ is stabilizing $P_0(s)$ and

$$\left\| \begin{bmatrix} W_1(s)S_0(s) \\ W_2^{(\theta_0,\Delta\theta)}(s)T_0(s) \end{bmatrix} \right\|_\infty \le \gamma \quad (14)$$

where

$$S_0(s) = (1 + P_0(s)C_0(s))^{-1}$$

$$T_0(s) = 1 - S_0(s) = P_0(s)C_0(s)(1 + P_0(s)C_0(s))^{-1},$$

and $W_1(s) = 1/s$ is for good tracking of step-like reference inputs.

By applying the formulae given in [15] and [3], the optimal solution to (14) can be determined as follows:

$$C_0(s) = \frac{\gamma(T_1(\theta_0)s+1)(T_2(\theta_0)s+1)}{cK(\theta_0)s^2}\,\frac{1}{1 + A(s) + F(s)} \quad (15)$$

where

$$A(s) = \frac{\beta\xi\gamma^2}{s} \quad (16)$$

and $F(s)$ is a finite impulse response (FIR) filter with time domain response

$$f(t) = \begin{cases} (\alpha + \xi - \beta\xi\gamma^2)\cos(\frac{t}{\gamma}) & \\ +(\alpha\xi\gamma + \beta\gamma - \frac{1}{\gamma})\sin(\frac{t}{\gamma}) & \text{for } t < h(\theta_0) \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

where

$$\beta = \sqrt{x}$$

$$\xi = \frac{1}{c\gamma}\sqrt{\frac{\gamma^2 - a^2}{x}}$$

$$\alpha = \sqrt{\frac{(b^2 - 2ac)\gamma^2 - c^2}{c^2\gamma^2} + 2\sqrt{x} - \frac{\gamma^2 - a^2}{c^2\gamma^2 x}} \quad (18)$$

with x the unique positive root of

$$x^3 + \frac{b^2 - 2ac - a^2\gamma^2}{c^2\gamma^2}x^2 - (\gamma^2 - a^2)\frac{(2ac - b^2)\gamma^2 + c^2}{c^4\gamma^4}x - \frac{(\gamma^2 - a^2)^2}{c^4\gamma^4} = 0 \quad (19)$$

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Denver, Colorado June 4-6, 2003

The optimal H^∞ performance cost γ is determined as the largest root of

$$1-\frac{\gamma}{c}e^{-h(\theta_0)s}\frac{s}{(s+\xi)(s^2+\alpha s+\beta)}|_{s=\frac{i}{\gamma}}=0 \qquad (20)$$

Note that an internally robust digital implementation of the H^∞ AQM controller (15) includes a second-order term which is cascaded with a feedback block containing an FIR filter $F(s)$. The length of the FIR filter is $h(\theta_0)/T_s$, where T_s is the sampling period.

4 H^∞-Performance Analysis

As shown in Section 3, the H^∞ AQM controller (15) is designed for $P_\theta(s)|_{\theta=\theta_0}$ and allows for $\theta\in\Theta=[\theta-\Delta\theta,\ \theta+\Delta\theta]$. In this section, we would like to investigate the H^∞-performance for the corresponding closed loop system, which indicates the system robustness and system response.

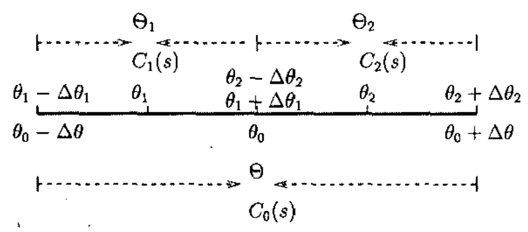

Figure 1: Partition of Θ by Θ_1 and Θ_2

Define the H^∞-performance of controller $C_0(s)$ with respect to $P_\theta(s)$ as follows:

$$\gamma_{C_0}(\theta)=\left\|\begin{bmatrix} W_1(s)S(s)\\ W_2^{(\theta_0,\Delta\theta)}(s)P_0(s)C_0(s)S(s)\end{bmatrix}\right\|_\infty \qquad (21)$$

for any $\theta\in\Theta=[\theta_0-\Delta\theta,\ \theta_0+\Delta\theta]$, where

$$S(s)=(1+P_\theta(s)C_0(s))^{-1}, \qquad (22)$$

here the term $|W_2^{(\theta_0,\Delta\theta)}(j\omega)P_0(j\omega)|$ can be seen as a bound on the additive plant uncertainty.

Furthermore, we define

$$\gamma_{C_0}^{\Delta\theta}:=\sup_{\theta\in\Theta}\{\gamma_{C_0}(\theta)\} \qquad (23)$$

which corresponds to the worst system response of controller $C_0(s)$ for plant $P_\theta(s)$ with $\forall\theta\in[\theta_0-\Delta\theta,\ \theta_0+\Delta\theta]$. Notice that a smaller $\gamma_{C_0}^{\Delta\theta}$ means better performance of the robust controller within the operating range Θ.

Particularly, we are interested in the scenario depicted in Fig.1, where Θ is equally partitioned by $\Theta_1=[\theta_1-\Delta\theta_1,\ \theta_1+\Delta\theta_1]$ and $\Theta_2=[\theta_2-\Delta\theta_2,\ \theta_2+\Delta\theta_2]$, with $\Delta\theta_1=\Delta\theta_2=\frac{\Delta\theta}{2}$. For $\theta\in\Theta_i$, $i=1,2$, we design H^∞ controller $C_i(s)$ obeying (15) with the nominal plant $P_i(s):=P_\theta(s)|_{\theta=\theta_i}$. Similar to (21) and (23), we have

$$\gamma_{C_i}(\theta)=\left\|\begin{bmatrix} W_1(s)S_i(s)\\ W_2^{(\theta_i,\Delta\theta_i)}(s)P_i(s)C_i(s)S_i(s)\end{bmatrix}\right\|_\infty \qquad (24)$$

for any $\theta\in\Theta_i$ $i=1,2$, and

$$\gamma_{C_i}^{\Delta\theta_i}:=\sup_{\theta\in\Theta_i}\{\gamma_{C_i}(\theta)\} \quad i=1,2 \qquad (25)$$

where $S_i(s)=(1+P_\theta(s)C_i(s))^{-1}$ is defined similarly to (22).

In what follows, we provide numerical analysis of the H^∞ performance with respect to the operating ranges and corresponding controllers shown in Fig.1. Assume $N=150$, $C=500$, $\Delta\theta=0.2$, and $\theta_0=0.5$, the H^∞ performance $\gamma_{C_0}(\theta)$ and $\gamma_{C_i}(\theta)$, $i=1,2$ can be numerically obtained from (21) and (24). As depicted in Fig.2, it is straightforward to have

$$\max(\gamma_{C_1}^{\Delta\theta_1},\gamma_{C_2}^{\Delta\theta_2})=24.4<\gamma_{C_0}^{\Delta\theta}=104.4$$

which means that the partition of Fig.1 can improve system performance in the sense of smaller H^∞ performance cost. In fact, it is a general trend that

$$\max(\gamma_{C_1}^{\Delta\theta_1},\gamma_{C_2}^{\Delta\theta_2})<\gamma_{C_0}^{\Delta\theta}, \qquad (26)$$

which can be further verified by Fig.3, Fig.4, and Fig.5, where N is chosen from 100 to 200, C from 400 to 600.

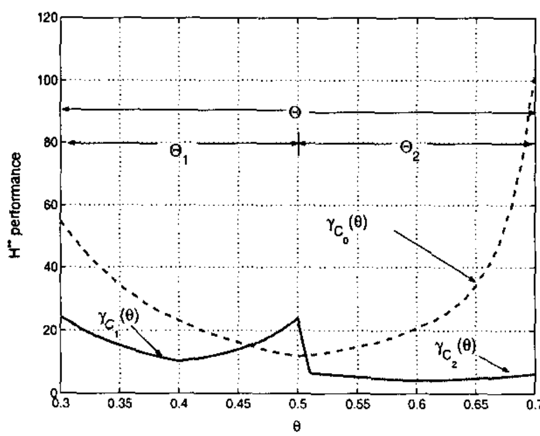

Figure 2: H^∞ performance with respect to θ

Remark: Based on the observation of better performance obtained by the partition shown in Fig.1, it is natural to consider switching robust control among a set of H^∞ controllers, each of which is designed for a smaller operating range. This will be an interesting extension of the present work. ■

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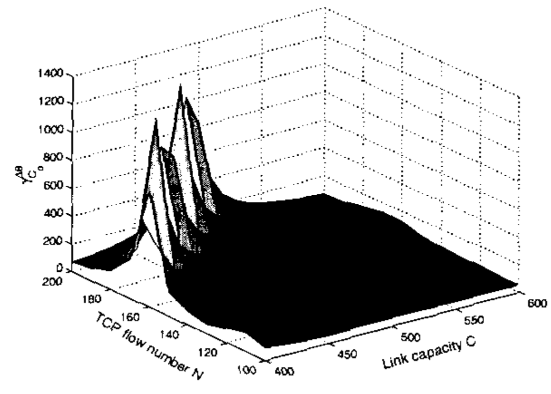

Figure 3: Performance cost $\gamma_{C_0}^{\Delta\theta}$ w.r.t. N and C

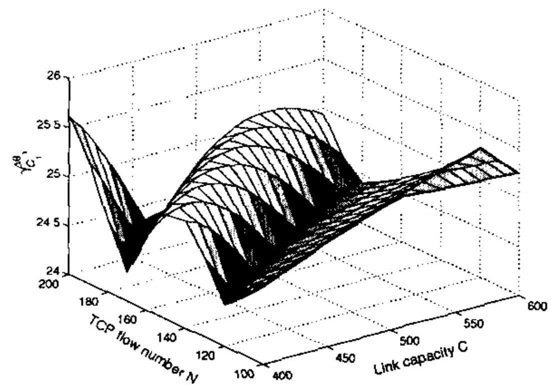

Figure 4: Performance cost $\gamma_{C_1}^{\Delta\theta_1}$ w.r.t. N and C

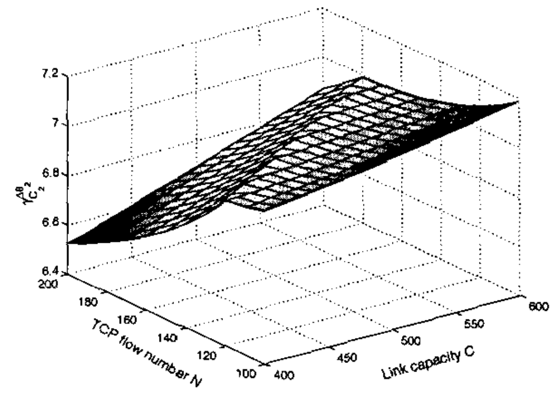

Figure 5: Performance cost $\gamma_{C_2}^{\Delta\theta_2}$ w.r.t. N and C

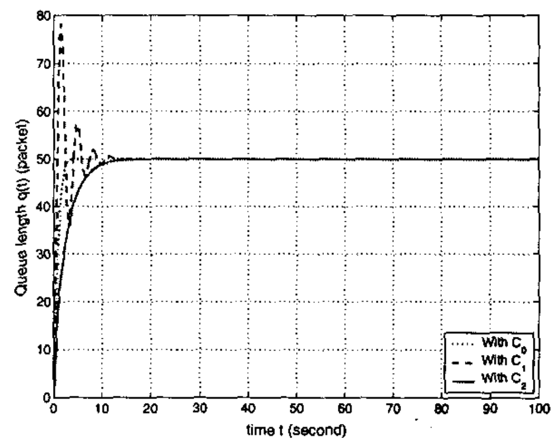

Figure 6: System responses of C_0, C_l and C_2 at $\theta = \theta_0 = 0.5$

5 Simulations

The closed loop system with the determined controllers is implemented in MATLAB/*simulink* to validate the controller design as well as the H^∞ performance analyzed in Section 4. We assume the TCP flow number $N = 150$, the link capacity $C = 500$ packets/sec. The propagation delay T_p is set to be 0.3 sec and the desired queue size is $q_0 = 100$ packets. Therefore, the nominal *RTT* is 0.5 sec ($\theta_0 = 0.5$), which is straightforward from (2). We use $\Delta\theta = 0.2$ in the design of $C_0(s)$ and $\Delta\theta_1 = \Delta\theta_2 = 0.1$ in $C_1(s)$ and $C_2(s)$. The following three scenarios are considered:

- Assuming the plant is the nominal one, i.e. $P_\theta(s) = P_0(s)$, we implement controller $C_0(s)$ as well as $C_1(s)$ and $C_2(s)$. It is shown in Fig.6 that the three controllers can stabilize the queue length because the nominal value θ_0 is within the operating range of Θ, Θ_1, and Θ_2. Note that the system response of $C_0(s)$ is better than the other two due to the fact that it achieves the optimal H^∞ performance at θ_0.
- Assuming $\theta = \theta_0 - \Delta\theta = 0.3$, we implement controller C_0 and C_1 (C_2 is not eligible in this scenario). As depicted in Fig.7, C_0 and C_1 can robustly stabilize the queue length. Observe that the system response of C_1 is better because it has much smaller H^∞ performance cost, which has been shown in Section 4.
- Similarly, we choose $\theta = \theta_0 + \Delta\theta = 0.7$ and repeat the simulation for controller C_0 and C_2 (C_1 is not eligible). As depicted in Fig.8, the two controllers can robustly stabilize the queue length and their system responses coincide with the H^∞ performance analysis given previously.

The above simulations show that our robust AQM controllers have good performance and robustness in the presence of parameter uncertainties. Meanwhile, the system responses also affirm a good coincidence with the H^∞ performance analysis in Section 4.

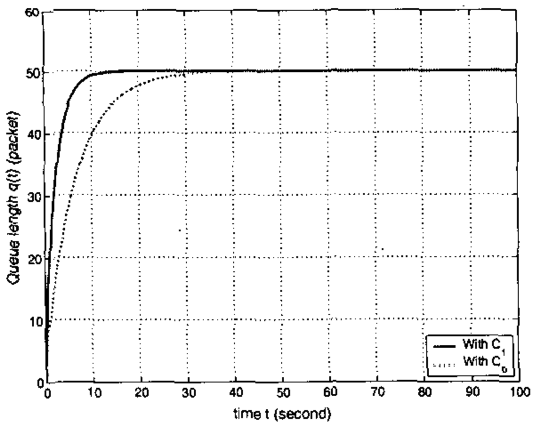

Figure 7: System responses of C_0 and C_1 at $\theta = \theta_0 - \Delta\theta = 0.3$

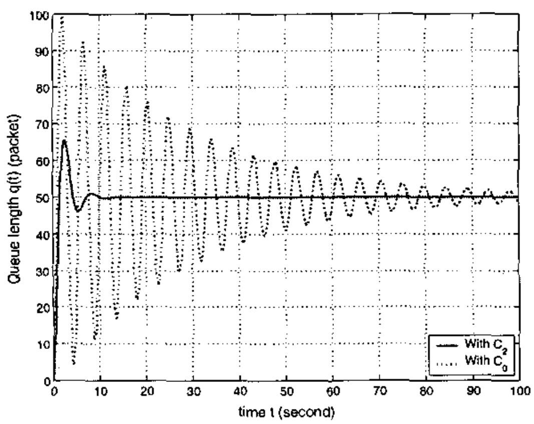

Figure 8: System responses of C_0 and C_2 at $\theta = \theta_0 + \Delta\theta = 0.7$

6 Concluding Remarks

We provided in this paper the guidelines of designing robust controllers for AQM, where the H^∞ techniques for infinite dimensional systems were implemented. The H^∞-performance was numerically analyzed with respect to the bound of the scheduling parameter θ. It was shown that smaller uncertainty bound could result in better H^∞-performance of the corresponding closed loop systems. Simulations were conducted to validate the design and analysis. A challenging extension of the present work is to consider switching H^∞ control, where the system performance can be improved in a larger operating range.

Appendix

Recall (5) and (10), we have

$$
\begin{aligned}
&|P_\theta(s) - P_0(s)|_{s=j\omega} \\
&= \left| \frac{K(\theta)e^{-h(\theta)s}}{(T_1(\theta)s+1)(T_2(\theta)s+1)} - \frac{K(\theta_0)e^{-h(\theta_0)s}}{(T_1(\theta_0)s+1)(T_2(\theta_0)s+1)} \right|_{s=j\omega} \\
&= \left| \frac{K(\theta)e^{-\Delta hs}}{(T_1(\theta)s+1)(T_2(\theta)s+1)} - \frac{K(\theta_0)}{(T_1(\theta_0)s+1)(T_2(\theta_0)s+1)} \right|_{s=j\omega} \\
&\leq \left| \frac{|K(\theta)e^{-\Delta hs} - K(\theta)| + |K(\theta) - K(\theta_0)|}{|(T_1(\theta)s+1)(T_2(\theta)s+1)|} \right|_{s=j\omega} \\
&\quad + \left| \frac{K(\theta_0) - \frac{K(\theta_0)(T_1(\theta)s+1)(T_2(\theta)s+1)}{(T_1(\theta_0)s+1)(T_2(\theta_0)s+1)}}{(T_1(\theta)s+1)(T_2(\theta)s+1)} \right|_{s=j\omega} \\
&\leq K(\theta) \left| \frac{\frac{e^{\Delta hs}-1}{s}}{\frac{(T_1(\theta)s+1)(T_2(\theta)s+1)}{s}} \right|_{s=j\omega} + \left| \frac{\Delta K}{(T_1(\theta)s+1)(T_2(\theta)s+1)} \right|_{s=j\omega} \\
&\quad + K(\theta_0) \left| \frac{(T_1(\theta)T_2(\theta) - T_1(\theta_0)T_2(\theta_0))s^2 + (\Delta T_1 + \Delta T_2)s}{T(s)} \right|_{s=j\omega}
\end{aligned} \tag{27}
$$

where

$$T(s) = (T_1(\theta)s+1)(T_2(\theta)s+1)(T_1(\theta_0)s+1)(T_2(\theta_0)s+1),$$

$\Delta h = h(\theta) - h(\theta_0)$, $\Delta K = K(\theta) - K(\theta_0)$, $\Delta T_1 = T_1(\theta) - T_1(\theta_0)$ and $\Delta T_2 = T_2(\theta) - T_2(\theta_0)$.

Note that

$$\left| \frac{e^{-\Delta hs} - 1}{s} \right|_{s=j\omega} \leq |\Delta h|$$

and

$$\left| \frac{(T_1(\theta)s+1)(T_2(\theta)s+1)}{s} \right|_{s=j\omega} \geq \max(T_1^-, T_2^-)$$

where

$$
\begin{aligned}
T_1^- &:= \min\{T_1(\theta),\ \theta \in [T_p,\ T_p + q_{max}/C]\} = T_p, \\
T_2^- &:= \min\{T_2(\theta),\ \theta \in [T_p,\ T_p + q_{max}/C]\} = \frac{CT_p^2}{2N}
\end{aligned}
$$

which are straightforward from (7) and (8). Thus

$$\left| \frac{\frac{e^{\Delta hs}-1}{s}}{\frac{(T_1(\theta)s+1)(T_2(\theta)s+1)}{s}} \right|_{s=j\omega} \leq \frac{|\Delta h|}{\max(T_1^-, T_2^-)}. \tag{28}$$

Recall

$$
\begin{aligned}
\Delta T_{12} &:= T_1(\theta)T_2(\theta) - T_1(\theta_0)T_2(\theta_0) \\
&= (T_1(\theta_0) + \Delta T_1)(T_2(\theta_0) + \Delta T_2) - T_1(\theta_0)T_2(\theta_0) \\
&= \Delta T_1 \Delta T_2 + T_1(\theta_0)\Delta T_2 + T_2(\theta_0)\Delta T_1.
\end{aligned} \tag{29}
$$

We have

$$\left|\frac{\Delta T_{12}s^2+(\Delta T_1+\Delta T_2)s}{T(s)}\right|_{s=j\omega}$$
$$\leq \left|\frac{|\Delta T_{12}s^2|+|(\Delta T_1+\Delta T_2)s|}{|T(s)|}\right|_{s=j\omega}$$
$$\leq \left|\frac{|\Delta T_1\Delta T_2|+|T_1(\theta_0)\Delta T_2|+|T_2(\theta_0)\Delta T_1|}{|\frac{(T_1(\theta)s+1)(T_2(\theta)s+1)(T_1(\theta_0)s+1)(T_2(\theta_0)s+1)}{s^2}|}\right|_{s=j\omega}$$
$$+\frac{|\Delta T_1+\Delta T_2|}{\max(T_1(\theta_0),T_2(\theta_0))}$$
$$\leq \frac{|\Delta T_1|}{T_1(\theta_0)}+\frac{|\Delta T_2|}{T_2(\theta_0)}+\frac{|\Delta T_1\Delta T_2|}{T_1(\theta_0)T_2(\theta_0)}+\frac{|\Delta T_1|+|\Delta T_2|}{\max(T_1(\theta_0),T_2(\theta_0))}$$

Invoking (27) and (28), we have

$$|P_\theta(s)-P_0(s)|_{s=j\omega}$$
$$\leq K(\theta)\frac{|\Delta h|}{\max(T_1^-,T_2^-)}+|\Delta K|+K(\theta_0)(\frac{|\Delta T_1|}{T_1(\theta_0)}+\frac{|\Delta T_2|}{T_2(\theta_0)}$$
$$+\frac{|\Delta T_1\Delta T_2|}{T_1(\theta_0)T_2(\theta_0)}+\frac{|\Delta T_1|+|\Delta T_2|}{\max(T_1(\theta_0),T_2(\theta_0))}) \qquad (30)$$

Defining

$$K^+:=\max\{K(\theta),\ \theta\in[T_p,\ T_p+q_{max}/C]\}=\frac{(CT_p+q_{max})^3}{4N^2},$$

and assuming

$$\begin{aligned}|\frac{dh(\theta)}{d\theta}| &\leq \beta_h \quad |\frac{dT_1(\theta)}{d\theta}|\leq\beta_{T_1}\\ |\frac{dT_2(\theta)}{d\theta}| &\leq \beta_{T_2} \quad |\frac{dK(\theta)}{d\theta}|\leq\beta_K,\end{aligned} \qquad (31)$$

the additive uncertainty (30) can be rewritten as

$$|P_\theta(s)-P_0(s)|_{s=j\omega}\leq\Delta_{(\theta_0,\Delta\theta)}$$
$$:=\frac{K(\theta_0)\beta_{T_1}\beta_{T_2}}{T_1(\theta_0)T_2(\theta_0)}(\Delta\theta)^2+(\frac{K^+\beta_h}{\max(T_1^-,T_2^-)}+\beta_K$$
$$+\frac{K(\theta_0)\beta_{T_1}}{T_1(\theta_0)}+\frac{K(\theta_0)\beta_{T_2}}{T_2(\theta_0)}+\frac{K(\theta_0)(\beta_{T_1}+\beta_{T_2})}{\max(T_1(\theta_0),T_2(\theta_0))})\Delta\theta. \qquad (32)$$

With (11) and (32), the multiplicative uncertainty $\Delta P_\theta(s)$ can be bounded by

$$|\Delta P_\theta(s)|_{s=j\omega}\leq\Delta_{(\theta_0,\Delta\theta)}|P_0(s)^{-1}|_{s=j\omega}=|W_2^{(\theta_0,\Delta\theta)}(s)|_{s=j\omega} \qquad (33)$$

where

$$W_2^{(\theta_0,\Delta\theta)}(s)=a+bs+cs^2 \qquad (34)$$

with

$$\begin{aligned}a &= \frac{\Delta_{(\theta_0,\Delta\theta)}}{K(\theta_0)}\\ b &= \frac{\Delta_{(\theta_0,\Delta\theta)}(T_1(\theta_0)+T_2(\theta_0))}{K(\theta_0)}\\ c &= \frac{\Delta_{(\theta_0,\Delta\theta)}T_1(\theta_0)T_2(\theta_0)}{K(\theta_0)}.\end{aligned} \qquad (35)$$

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