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Persistent Perfect Entanglement in Atomic Systems

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Abstract—It is shown that the system of an even number of three-level atoms in the Λ configuration in a cavity can evolve into a persistent maximum entangled state. The time of formation of such an entangled state is estimated.

Atoms and ions, interacting with cavity photons, are the basic building blocks of quantum information processing. At least, they represent a useful tool for testing quantum algorithms in communications, cryptography, and computing [1, 2]. Realization of different quantum information processes, such as teleportation [3], requires perfect (maximum) and long-lived entangled states. Unfortunately, the lifetime of entanglement in atomic systems is mostly specified by the lifetime of excited atomic states with respect to dipole transitions and therefore is quite short.

Stabilization of perfect entanglement in the general case using the variational principle for maximum entanglement has been proposed recently [4]. According to this principle, maximum entanglement is a property of states that show the maximum scale of quantum fluctuations of all essential measurements responsible for the manifestation of entanglement. To stabilize the maximum entangled state, it is necessary to achieve the maximum level of fluctuation and a (local) minimum of energy.

As an illustrative example, consider a system of two two-level atoms interacting with a single cavity photon described by the Hamiltonian

$$\begin{aligned} H &= H_0 + H_{\text{int}}, \\ H_0 &= \omega_p a_p^+ a_p + \omega_{21} \sum_{f=1}^{2} R_{22}(f), \\ H_{\text{int}} &= g \sum_f [R_{21}(f) a_p + \text{H.c.}]. \end{aligned} \tag{1}$$

Here, a_p denotes the photon annihilation operator and $R_{ij} = |i\rangle\langle j|$ is the atomic operator; $i = 1$ denotes the ground level, and $i = 2$ marks the excited level. Assume that both atoms are initially in the ground state, while the cavity contains a single photon

$$|\psi_0\rangle = |1, 2\rangle \otimes |1_p\rangle. \tag{2}$$

Then, the system will periodically elapse over the state [5]

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|2, 1\rangle + |1, 2\rangle)|0_p\rangle, \tag{3}$$

which manifests maximum two-qubit entanglement in the atomic subsystem. In the case of qubits, the essential measurements are given by the infinitesimal generators of the $SL(2, \mathbb{C})$ algebra [6]:

$$\begin{cases} \sigma_1 = |2\rangle\langle 1| + \text{H.c.} \\ \sigma_2 = i|1\rangle\langle 2| + \text{H.c.} \\ \sigma_3 = |2\rangle\langle 2| - |1\rangle\langle 1|. \end{cases} \tag{4}$$

It is seen that the total variance

$$\mathbb{V} = \sum_{f=1}^{3} \sum_{f=1}^{2} \langle (\Delta\sigma_i^{(f)})^2 \rangle, \tag{5}$$

describing the remoteness of a two-qubit state from "classical reality" [4], achieve the maximum $\mathbb{V}_{\max} = 6$ in state (3) and minimum $\mathbb{V}_{\min} = 4$ in state (2). Unfortunately, this maximum entangled state (3) has the maximum energy $E = \omega_{21}$ and is unstable with respect to the dipole decay of the atomic excitation.

Assume now that, instead of the two-level atoms, we have three-level atoms in Λ configuration (see Fig. 1).

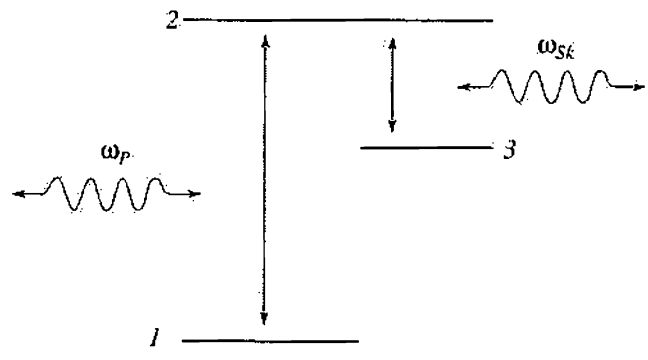

Fig. 1. Structure of a three-level atom in the Λ configuration.

Then, the atom-field Hamiltonian takes the form

$$H = H_0 + H_{\text{int}},$$

$$H_0 = \omega_P a_P^+ a_P + \omega_S a_S^+ a_S + \sum_{f=1}^{2} [\omega_{21} R_{22}(f) + \omega_{31} R_{33}(f)], \quad (6)$$

$$H_{\text{int}} = \sum_f [g_P R_{21}(f) a_P + g_S R_{23}(f) a_S + \text{H.c.}].$$

Beginning with the same initial state (2) as above, the system can achieve one more maximum entangled two-qubit state,

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|3, 1\rangle + |1, 3\rangle) \otimes |0_P\rangle \otimes |1_S\rangle. \quad (7)$$

Discarding the Stokes photon [7, 8], we can turn unstable state (7) into the following two-qubit state:

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|3, 1\rangle + |1, 3\rangle) \otimes |0_P\rangle \otimes |0_S\rangle. \quad (8)$$

Let us note that the local measurements (4) should be replaced by

$$\begin{cases} \sigma_1 = |3\rangle\langle 1| + \text{H.c.} \\ \sigma_2 = -i|3\rangle\langle 1| + \text{H.c.} \\ \sigma_3 = |3\rangle\langle 3| - |1\rangle\langle 1| \end{cases} \quad (9)$$

in this case and that state (8) again provides the maximum total variance for the two-qubit entanglement $\mathbb{V}_{\max} = 6$. At the same time, this state (8) has an energy $E' = \omega_{31} \approx \omega_{21} - \omega_S$ that is much lower than $E = \omega_{21}$ and corresponds to a local minimum because the radiative transition $3 \longrightarrow 1$ is dipole-forbidden. Thus, state (8) is a persistent maximum entangled atomic state of two qubits.

The discard of Stokes photons can be realized in different manners [8]. Either they abandon the cavity or they are absorbed by the cavity walls. In both cases, the atomic system evolves towards the persistent state (8) with reliability.

In this paper, we extend our previous results [7, 8] on the multiatom case. It should be emphasized that the problem of multipartite entanglement in cavity QED has attracted a great deal of interest recently (e.g., see [9] and references therein).

Consider first the system of three Λ-type atoms in the ground state interacting with a single cavity photon with frequency ω_P. Then, the irreversible evolution leads to the three-qubit state

$$|\psi_5\rangle = \frac{1}{\sqrt{3}}(|3, 1, 1\rangle + |1, 3, 1\rangle + |1, 1, 3\rangle) \otimes |0_P, 0_S\rangle, \quad (10)$$

which does not manifest perfect entanglement because $\mathbb{V}(\psi_5) = 8 + 2/3$, while $\mathbb{V}_{\max} = 9$ in the case of three qubits. Moreover, this state (10) belongs to the class of the so-called W states [10]. Therefore, it does not manifest entanglement at all [4, 11].

If instead we begin with the state of three unexcited atoms interacting with two cavity photons, the final state

$$|\psi_6\rangle = \frac{1}{\sqrt{3}}(|3, 3, 1\rangle + |3, 1, 3\rangle + |1, 3, 3\rangle) \otimes |0_P, 0_S\rangle \quad (11)$$

also belongs to the class of W states. Finally, if we start with the state with three cavity photons, the final state takes the form

$$|\psi_7\rangle = |3, 3, 3\rangle \otimes |0_P, 0_S\rangle \quad (12)$$

and definitely is not entangled as well. This agrees with our previous result [12] that the maximum entanglement in an atom–photon system can be achieved if initially we have $2N$ atoms in the ground state and N pumping photons. In particular, this means that the so-called GHZ state

$$|\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}}(|3, 3, 3\rangle \pm |1, 1, 1\rangle) \otimes |0_P, 0_S\rangle \quad (13)$$

cannot be achieved when the system is initially prepared in the state

$$|1, 1, 1\rangle \otimes |3_P\rangle \otimes |0_S\rangle. \quad (14)$$

State (13) can be generated in the way proposed in [13], by sending three atoms, one after another, with specially selected velocities through a cavity containing the field in the superposition state

$$\frac{1}{\sqrt{2}}(|0_P\rangle - |3_P\rangle).$$

For other proposals, see [14–17].

Consider the case when the Stokes photons are allowed to escape from the cavity and initially all the atoms are in the ground state while the cavity contains n_P pump photons. If there are N atoms in the cavity, n_P excitations in the third state will be created and these excitations will be distributed equally over the N atoms:

$$|\Psi(t=0)\rangle = \bigotimes_{i=1}^{N} |1\rangle_i \otimes |n_P\rangle_P \longrightarrow \frac{1}{\sqrt{C_{n_P}(N)}} \sum_{\wp} \prod_{i=1}^{n_P} \otimes |3\rangle_{\wp i} \prod_{i=n_P+1}^{N} \otimes |1\rangle_{\wp i} \otimes |0\rangle_P, \quad (15)$$

where $\wp$ denotes all possible permutations over N atoms and $C_{n_P}(N) = \binom{N}{n_P}$. In the case $n_P \geq N$, all the atoms will evolve to the third state, thus leading to an unentangled state. The Hamiltonian of the system in the interaction picture has the form

$$H_0 = \Delta_P a_P^\dagger a_P + \sum_k \Delta_k a_k^\dagger a_k + g_P a_P \mathcal{R}_{21} + g_P^* a_P^\dagger \mathcal{R}_{12},$$
$$H_{\text{int}} = \sum_k g_k \mathcal{R}_{23} a_k + g_k^* a_k^\dagger \mathcal{R}_{32}, \tag{16}$$

where $\mathcal{R}_{ij} = \sum_{f=1}^{N} R_{ij}(f)$ constitute the collective atomic operators. The Stokes modes make up the environment and lead to spontaneous decay from the second level to the third level. Upon elimination of the Stokes modes, the master equation for the reduced density matrix of atoms and pump photons in a thermal environment is as follows:

$$\dot{\rho}(t) = -i[H_0, \rho(t)] + (\bar{n}+1)\Gamma\{2\mathcal{R}_{32}\rho(t)\mathcal{R}_{23} - \rho(t)\mathcal{R}_{23}\mathcal{R}_{32} - \mathcal{R}_{23}\mathcal{R}_{32}\rho(t)\} + \bar{n}\Gamma\{2\mathcal{R}_{23}\rho(t)\mathcal{R}_{32} - \rho(t)\mathcal{R}_{32}\mathcal{R}_{23} - \mathcal{R}_{32}\mathcal{R}_{23}\rho(t)\}, \tag{17}$$

where Γ is the spontaneous decay rate for the $2 \longrightarrow 3$ transition and $\bar{n}$ is the mean number of thermal Stokes photons at the resonant frequency E_{23}. Consider for simplicity the case when the temperature is much smaller than the resonant energy E_{23}, so that $\bar{n} \sim 0$ and the master equation (17) reduces to

$$\dot{\rho}(t) = -i[H_0, \rho(t)] + \Gamma\{2\mathcal{R}_{32}\rho(t)\mathcal{R}_{23} - \rho(t)\mathcal{R}_{23}\mathcal{R}_{32} - \mathcal{R}_{23}\mathcal{R}_{32}\rho(t)\}. \tag{18}$$

Initially, all the atoms are supposed to be in the first ground state. Then, because of the coupling between the first and second levels mediated by the pump photons, an excitation in the second level will appear. Since the spontaneous decay rate Γ for the $2 \longrightarrow 3$ transition is supposed to be much greater than the coupling constant for the $1 \longrightarrow 2$ transition, the state with one excitation in the second level will rapidly decay to the third state preceding the $1 \longrightarrow 2$ transitions. As a result, the evolution can be approximately described through the use of the Hilbert space spanned by the vectors

$$|\Psi_n\rangle = \frac{1}{\sqrt{C_n(N)}} \sum_{\wp} \bigotimes_{i=1}^{n} |3\rangle_{\wp i} \bigotimes_{i=n+1}^{N} |1\rangle_{\wp i} \otimes |n_P - n\rangle_P,$$

$$|\Phi_n\rangle = \frac{1}{\sqrt{C_1(N-n)C_n(N)}} \sum_{\wp} \bigotimes_{i=1}^{n} |3\rangle_{\wp i} \bigotimes_{i=n+1}^{n+1} |2\rangle_{\wp i} \bigotimes_{i=n+2}^{N} |1\rangle_{\wp i} \otimes |n_P - n - 1\rangle_P, \tag{19}$$

$$|\Phi_n'\rangle = \frac{1}{\sqrt{C_2(N-n)C_n(N)}} \sum_{\wp} \bigotimes_{i=1}^{n} |3\rangle_{\wp i} \bigotimes_{i=n+1}^{n+2} |2\rangle_{\wp i} \bigotimes_{i=n+3}^{N} |1\rangle_{\wp i} \otimes |n_P - n - 2\rangle_P,$$

where $n = 0, 1, 2, \ldots$. In our approximate picture, first, the transition $|\Psi_n\rangle \longrightarrow |\Phi_n\rangle$ takes place. It is accompanied by the transitions $|\Phi_n\rangle \longrightarrow |\Phi_n'\rangle$ and $|\Phi_n\rangle \longrightarrow |\Psi_{n+1}\rangle$ at a time scale of $t \sim 1/\Gamma$. Hence, the population of $|\Phi_n'\rangle$ to that of $|\Psi_{n+1}\rangle$ is of the order $g_P^2/\Gamma^2 \ll 1$. So, we can confine ourselves to the subspace spanned by the states $\{|\Psi_n\rangle, |\Phi_n\rangle, n = 0, 1, 2, \ldots\}$.

In a sense, this is equivalent to the extension of the effective picture of Raman–Stokes processes in three-level atoms that was proposed in [18, 19] for the multiatom case.

Now, the density matrix can be chosen as follows:

$$\rho = \sum_{n=0}^{\infty} a_n |\Psi_n\rangle\langle\Psi_n| + b_n |\Psi_n\rangle\langle\Phi_n| + b_n^* |\Phi_n\rangle\langle\Psi_n| + c_n |\Phi_n\rangle\langle\Phi_n|. \tag{20}$$

From Eq. (18), keeping in mind that we restrict our consideration to the subspace spanned by $|\Psi_n\rangle, |\Phi_n\rangle$, $n = 0, 1, 2, \ldots$, we get the following equations for coefficients in (20):

$$\dot{a}_n = i\sqrt{(N-n)(n_P-n)}(g_P b_n - g_P^* b_n^*) + 2n\Gamma c_{n-1},$$
$$\dot{b}_n = i\sqrt{(N-n)(n_P-n)}(g_P^* a_n - g_P^* c_n) - i\Delta b_n - (n+1)\Gamma b_n,$$
$$\dot{b}_n^* = -i\sqrt{(N-n)(n_P-n)}(g_P a_n - g_P c_n^*) + i\Delta b_n^* - (n+1)\Gamma b_n^*, \tag{21}$$
$$\dot{c}_n = -i\sqrt{(N-n)(n_P-n)}(g_P b_n - g_P^* b_n^*) - 2(n+1)\Gamma c_{n-1}.$$

Given the initial condition $a_0(0) = 1$, we will assert that $\dot{a}_n/\Gamma \ll 1$, $\dot{c}_n/\Gamma \ll 1$ and solve the equations of motion (21) accordingly; then, the assertions can be checked for consistency. The coefficients b_n and b_n^* can be eliminated from the equations of motion through the use of the relations

$$b_n(t) = i\sqrt{(N-n)(n_P-n)}g_P^*$$

$$\times \int_0^t d\tau e^{-[(n+1)\Gamma + i\Delta]\tau}(a_n(t-\tau) - c_n(t-\tau))$$

$$= i\sqrt{(N-n)(n_P-n)}g_P^* \frac{1}{(n+1)\Gamma + i\Delta}[a_n(t) - c_n(t)],$$

where it is assumed that $\dot{a}_n/\Gamma \ll 1$, $\dot{c}_n/\Gamma \ll 1$. Then, the coupled equations for a_n and c_n are obtained:

$$\begin{aligned} \dot{a}_n &= -\gamma_n(a_n - c_n) + 2n\Gamma c_{n-1}, \\ \dot{c}_n &= \gamma_n(a_n - c_n) - 2(n+1)\Gamma c_n, \\ \gamma_n &= 2(N-n)(n_P-n)(n+1)\frac{\lambda^2\Gamma}{(n+1)^2\Gamma^2 + \Delta^2}. \end{aligned} \tag{22}$$

Thus, a_n and c_n can be obtained in terms of each other,

$$\begin{aligned} c_n(t) &= \gamma_n \int_0^t d\tau e^{-2(n+1)\Gamma\tau}(a_n(t-\tau) - c_n(t-\tau)) \\ &\approx \frac{\gamma_n}{2(n+1)\Gamma}(a_n(t) - c_n(t)) \\ &\approx \frac{\gamma_n}{2(n+1)\Gamma}a_n(t), \end{aligned} \tag{23}$$

so that the equation governing a_n's takes the form

$$\dot{a}_n = -\gamma_n a_n + \gamma_{n-1}a_{n-1}. \tag{24}$$

Under the above initial condition, we get the solution

$$\begin{aligned} a_0(t) &= e^{-\gamma_0 t} \\ a_n(t) &= \gamma_{n-1}\int_0^t d\tau e^{-\gamma_n\tau}a_{n-1}(t-\tau), \quad n = 1, 2, 3, \end{aligned} \tag{25}$$

In general, the solution for a_n and c_n's will be a linear superposition of terms of the form $\exp(-\gamma_i t)$, $i \le n$, which are in line with the assumption that $\dot{a}_n/\Gamma \ll 1$, $\dot{c}_n/\Gamma \ll 1$. When $n = \min(n_P, N)$, $\gamma_n = 0$, so that the final value is $n_f = \min(n_P, N)$ and the system evolves to the persistent state $|\Psi_{n_f}\rangle\langle\Psi_{n_f}|$. The time dependence of a_{n_f} has the form

$$a_{n_f}(t) = 1 - \sum_{i=0}^{n_f-1} e^{-\gamma_i t}\prod_{j\neq i}^{n_f-1}\frac{\gamma_j}{\gamma_j - \gamma_i}. \tag{26}$$

Thus, the characteristic time scale needed in order to obtain the final state is $1/\gamma_{n_f-1}$, because $\{\gamma_n\}$ is a monotonically decreasing sequence.

In the case of an even number of atoms $N = 2m$ and $n_P = m$, when the final state is a maximum entangled state, the characteristic time scale for obtaining a persistent perfect entangled state is

$$\begin{aligned} \tau &= \frac{m^2\Gamma^2 + \Delta^2}{2\lambda^2\Gamma m(m+1)} \\ &= \frac{\Gamma}{\lambda^2}\frac{m}{2(m+1)} + \frac{\Delta^2}{\lambda^2\Gamma}\frac{1}{2m(m+1)}. \end{aligned} \tag{27}$$

Here, λ denotes an effective coupling constant (a certain function of g_P and g_S [18, 19]). It is seen that the second term in (27) vanishes as $1/m^2$ as m increases, so that detuning influences the characteristic time only for small numbers of atoms. In turn, the first term in (27) grows with an increase in m, achieving a maximum value $\tau_{max} = \Gamma/2\lambda^2$ at $m \gg 1$. This time τ_{max} can be considered as an estimation from above of the time scale corresponding to the formation of perfect persistent entanglement in the system of $N = 2m$ Λ-type atoms interacting with $n_P = m$ cavity photons.

In summary, we have shown that a system consisting of an even number $2m$ of Λ-type atoms in a cavity initially prepared in the ground state, while the cavity field contains m pumping photons resonant with the transition $1 \longrightarrow 2$, evolves with reliability to the persistent maximum entangled atomic state if the Stokes photons created by the transition $2 \longrightarrow 3$ are discarded. We have shown that the time required to prepare such a state can be estimated from above through the use of an effective model that neglects the population of the intermediate atomic level 2. At $m \gg 1$, this time is completely determined by the spontaneous decay rate Γ for the $2 \longrightarrow 3$ transition and effective coupling constant λ. Detuning is important only at small m. The result obtained above should be considered as an estimation from above. A more detailed consideration of correlations between the atoms caused by photon exchange can lead to a smaller time (e.g., see [20]).

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