

## General entanglement

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**Abstract.** The paper contains a brief review of an approach to quantum entanglement based on analysis of dynamic symmetry of systems and quantum uncertainties, accompanying the measurement of mean value of certain basic observables. The latter are defined in terms of the orthogonal basis of Lie algebra, corresponding to the dynamic symmetry group. We discuss the relativity of entanglement with respect to the choice of basic observables and a way of stabilization of robust entanglement in physical systems.

### 1. Introduction

Entanglement, which is considered nowadays as the main physical resource of quantum information processing and quantum computing, has been discovered as a physical phenomenon representing “the characteristic trait of quantum mechanics” (Schrödinger 1935).

According to the modern point of view, entangled states form a special class of quantum states closed under SLOCC (Stochastic Local Operations assisted by Classical Communications) (Dür *et al* 2000, Verstraete *et al* 2002, Miyake 2003). Two states belong to the same class iff they are converted into each other by SLOCC. Mathematically SLOCC amounts to action of the complexified dynamic symmetry group  $G^c$  of the system (Verstraete *et al* 2002). This description puts entanglement in general framework of geometric invariant theory and allows extend it to arbitrary quantum systems (Klyachko 2002).

SLOCC cannot transform entangled state into unentangled one and vice versa (Dür *et al* 2000). We define *completely entangled* (CE) states, manifesting maximal entanglement in their SLOCC class, such that all entangled states of a given system can be constructed from them by means of SLOCC.

It was shown recently that CE states manifest the maximal amount of quantum fluctuations (Can *et al* 2002(a), Klyachko and Shumovsky 2003, Klyachko and Shumovsky 2004). This property can be used as a physical definition of CE states.

It should be stressed that quantum fluctuations caused by the representation of observables in terms of Hermitian operators is an undoubted “characteristic trait” of quantum systems. Within the classical description, the observables should be associated with c-numbers and hence are incapable of manifestation of quantum fluctuations.

We now note that characterization of quantum states with respect to quantum fluctuation is a common way in quantum optics. Coherent (Glauber 1963, Perelomov 1986) and squeezed (Stoler 1970, Dodonov 2002) states provide an important examples. In particular, it has been recognized recently that coherent states can in general be associated with the unentangled

(separable) states (Klyachko 2002, Barnum *et al* 2003). In turn, there are also attempts to characterize entanglement in terms of quantum fluctuations.

The aim of this article is to discuss the corollaries coming from the physical definition of CE states via quantum fluctuations. We mostly concentrate on the *relativity of entanglement* and on the creation of *robust entanglement*. Let us emphasize once more that as soon as CE states are defined, all other entangled states of the same system can be obtained from CE states by means of SLOCC.

The paper is arranged as follows. In Sec. 2 we briefly discuss the specification of basic observables based on the consideration of the dynamic symmetry properties of quantum systems and express the definition of CE states in terms of a variational principle. Then, in Sec. 3 the relativity of entanglement with respect to the choice of basic observables is considered. In Sec. 4 we discuss the stabilization of entanglement. Finally, in Sec. 5 we briefly summarize the obtained results.

## 2. Basic observables

Quantum entanglement as well as any other quantum phenomenon manifests itself via measurement of physical observables (Bell 1966). In von Neumann approach (von Neumann 1996) all observables are supposed to be equally accessible. However physical nature of the system often imposes inevitable constraints.

For example, the components of composite system  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$  may be spatially separated by tens of kilometers, as in EPR pairs used in quantum cryptography. In such circumstances only local observations  $X_A$  and  $X_B$  are available.

As another example, consider a system of  $N$  identical particles, each with space of internal degrees of freedom  $\mathcal{H}$ . By Pauli principle the state space of such system shrinks to *symmetric tensors*  $S^N \mathcal{H} \subset \mathcal{H}^{\otimes N}$  for bosons, and to *skew symmetric tensors*  $\wedge^N \mathcal{H} \subset \mathcal{H}^{\otimes N}$  for fermions. This superselection rule imposes severe restriction on manipulation with quantum states, effectively reducing the accessible measurements to that of a single particle.

This consideration led many researchers to the conclusion, that available observables should be included in description of any quantum system from the outset, see Hermann 1966, Emch 1984. Robert Hermann 1966 stated this thesis as follows

*“The basic principles of quantum mechanics seem to require the postulation of a Lie algebra of observables and a representation of this algebra by skew-Hermitian operators.”*

We denote this *Lie algebra of observables* by  $\mathfrak{L}$ . The corresponding Lie group

$$G = \exp(i\mathfrak{L})$$

will be called *dynamic symmetry group* of the system. We'll refer to unitary representation of the dynamical group  $G$  in state space  $\mathcal{H}_S$  as *quantum dynamical system*.

Note finally that there is no place for entanglement in von Neumann picture, where full dynamical group  $SU(\mathcal{H})$  makes all states equivalent. Entanglement is an effect caused by superselection rules or symmetry breaking which reduce the dynamical group to a subgroup  $G \subset SU(\mathcal{H})$  small enough to create intrinsic difference between states. For example, entanglement in two component system  $\mathcal{H}_A \otimes \mathcal{H}_B$  comes from reduction of the dynamical group to  $SU(\mathcal{H}_A) \times SU(\mathcal{H}_B) \subset SU(\mathcal{H}_A \otimes \mathcal{H}_B)$ . Entanglement essentially depends on the dynamical group and *must* be discussed in framework of a given quantum dynamical system  $G : \mathcal{H}$ . This *relativity of entanglement* is one of the topics of this paper.

For calculations we choose an arbitrary orthonormal basis  $X_i, i = 1 \dots N$  of  $\mathfrak{L} = \text{Lie}(G)$  and call its elements  $X_i$  *basic observables* (Klyachko 2002 and Klyachko and Shumovsky 2003).

For example, in the case of a qubit (spin- $\frac{1}{2}$  “objects”) the dynamic symmetry group is  $G = \text{SU}(2)$  and  $\mathfrak{L} = \mathfrak{su}(2)$  is algebra of traceless Hermitian  $2 \times 2$  matrices. One can choose spin projector operators  $J_x, J_y, J_z$  (or the Pauli matrices) as the basic observables.

The level of quantum fluctuations of a basic observable  $X_i$  in state  $\psi \in \mathcal{H}_S$  of system  $S$  is given by the variance

$$\mathbb{V}(X_i, \psi) = \langle \psi | X_i^2 | \psi \rangle - \langle \psi | X_i | \psi \rangle^2 \geq 0. \quad (1)$$

Summation over all basic observables of the quantum dynamic system gives the *total uncertainty* (total variance) peculiar to the state  $\psi$ :

$$\mathbb{V}(\psi) = \sum_i \mathbb{V}(X_i, \psi) = \sum_i \langle \psi | X_i^2 | \psi \rangle - \langle \psi | X_i | \psi \rangle^2. \quad (2)$$

This quantity is independent of the choice the basic observables and measures the total level of quantum fluctuations in the system.

Recall that the Casimir operator

$$\hat{C} = \sum_i X_i^2,$$

which appears in Eq. (2) is independent of the choice of the basis  $X_i$  and acts as a multiplication by scalar  $C$  if representation  $G : \mathcal{H}_S$  is irreducible. In this case Eq. (2) takes the form

$$\mathbb{V}(\psi) = C - \sum_i \langle \psi | X_i | \psi \rangle^2. \quad (3)$$

It has been observed that completely entangled (CE) states of an arbitrary number  $n \geq 2$  of qubits obey a certain conditions. Namely, expectation values of all three spin-projection operators for all parties of the system have zero value in CE state  $|\psi_{CE}\rangle$  (Can *et al* 2002). In general, the condition

$$\forall i \quad \langle \psi_{CE} | X_i | \psi_{CE} \rangle = 0, \quad |\psi_{CE}\rangle \in \mathcal{H}_S, \quad (4)$$

can be used as a general physical definition of CE (Klyachko and Shumovsky 2004). This is an *operational* definition of CE (definition in terms of what can be directly measured).

From Eq. (3) it follows that the total variance attains its maximal value equal to Casimir in the case of CE states:

$$\mathbb{V}(\psi_{CE}) = \max_{\psi \in \mathcal{H}_S} \mathbb{V}(\psi) = C. \quad (5)$$

This Eq. (5) is, in a sense, equivalent to the maximum of entropy principle, defining the equilibrium states in quantum statistical mechanics (Landau and Lifshitz 1980).

From operational point of view state  $\psi \in \mathcal{H}$  is entangled if one can prepare a completely entangled state  $\psi_{CE}$  from it using SLOCC operations. It should be emphasized that SLOCC transformations have been identified with action of the *complexified dynamical group*

$$G^c = \exp(\mathfrak{L} \otimes \mathbb{C}),$$

of the system (Verstraete 2003). This leads us to the following definition of general entangled states  $\psi_E$  of the system

$$\psi_E = g^c \psi_{CE}, \quad \text{for some } g^c \in G^c.$$

Thus, *the general entangled states can be defined as that obtained from the states, manifesting maximum total uncertainty, by action of the complexified dynamic group.*

Let's stress that the above definition of basic observables and equations of CE (4) do not assume the composite nature of the system  $S$ . In other words, a single-particle system can manifest entanglement if its state obeys the conditions (4) (Can *et al* 2005).

### 3. Relativity of entanglement

Physics of quantum system  $S$  with given Hilbert state space  $\mathcal{H}_S$  may implies different dynamical groups.

An important example is provided by a *qutrit* (three-state quantum system), which is widely discussed in the context of quantum ternary logic (Bechman-Pasquinucci and Peres 2000, Bruß and Macchiavello 2002, Kaszlikowski *et al* 2003). In this case, the general symmetry is given by  $G = \text{SU}(3)$ , so that the local basic observables are given by the eight independent Hermitian generators of the  $\mathfrak{L} = \mathfrak{su}(3)$  algebra (see Caves and Milburn 2000). In the special case of spin-1 system, the symmetry is reduced to the  $G' = \text{SU}(2)$  group, and the corresponding local basic observables coincide with the three spin-1 operators (Can *et al* 2005). Since  $\mathfrak{su}(2) \subset \mathfrak{su}(3)$ , the qutrit entanglement with respect to  $\mathfrak{su}(3)$  observables implies entanglement in the  $\mathfrak{su}(2)$  domain but not vice versa.

For example, a single spin-1 object can be entangled with respect to the  $\mathfrak{su}(2)$  basic observables but not in the  $\mathfrak{su}(3)$  sector (Can *et al* 2005). A general spin-1 state has the form

$$|\psi\rangle = \sum_{s=-1}^1 \psi_s |s\rangle, \quad \sum_s |\psi_s|^2 = 1, \quad (6)$$

where  $s = 0, \pm 1$  denotes the spin projection. In the basis  $|s\rangle$ , the spin-1 operators have the form

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Using CE condition (4) with the basic observables  $S_i$  and taking into account the normalization condition in (7), we obtain four equations for six real parameters  $\text{Re}(\psi_s)$  and  $\text{Im}(\psi_s)$ . In particular, the state with zero projection of spin  $|0\rangle$  manifests CE. This state  $|0\rangle$  together with the states

$$\frac{1}{\sqrt{2}}(|1\rangle \pm |-1\rangle),$$

form the basis of CE states in the three-dimensional Hilbert space of spin-1 states. The possibility of the single spin-1 entanglement was also discussed by Viola *et al* (Viola *et al* 2004).

To understand the physical meaning of this CE, we note that there is a certain correspondence between the states of two qubits and single qutrit provided by the Clebsch-Gordon decomposition

$$\mathcal{H}_{\frac{1}{2}} \otimes \mathcal{H}_{\frac{1}{2}} = \mathcal{H}_1 \oplus \mathcal{H}_0.$$

Here  $\mathcal{H}_{\frac{1}{2}}$  denotes the two-dimensional Hilbert space of a single qubit. The three-dimensional Hilbert space  $\mathcal{H}_1$  contains the symmetric states of two qubits

$$|1\rangle = |\uparrow\uparrow\rangle, \quad |0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |-1\rangle = |\downarrow\downarrow\rangle \quad (7)$$

while  $\mathcal{H}_0$  corresponds to the antisymmetric state

$$|A\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \quad (8)$$

It is now seen that the state  $|0\rangle$  of spin-1 is CE in terms of a certain pare of spin- $\frac{1}{2}$  “particles”, which can be interpreted as intrinsic degrees of freedom for the spin-1 object.

A vivid physical example is provided by the  $\pi$ -mesons. It is known that three  $\pi$ -mesons form an isotriplet (Bogolubov and Shirkov 1982)

$$\pi^+ = |1\rangle, \quad \pi^0 = |0\rangle, \quad \pi^- = |-1\rangle, \quad (9)$$

where  $|\ell\rangle$  ( $\ell = 0, \pm 1$ ) denotes the states of isospin  $I = 1$ . From the symmetry point of view, isospin is also specified by the SU(2) group. Thus, in view of our discussion one can conclude that  $\pi^0$  meson is CE with respect to internal degrees of freedom.

The internal structure of mesons is provided by the quark model (Huang 1982). Namely, the fundamental representation of the isospin symmetry corresponds to the two doublets (qubits) that contain the so-called up ( $u$ ) and down ( $d$ ) quarks and anti-quarks ( $\bar{u}$  and  $\bar{d}$ ). In terms of quarks, the isotriplet (10) has the form

$$\pi^+ = u\bar{d}, \quad \pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}), \quad \pi^- = \bar{u}d.$$

It is now clearly seen that  $\pi^0$  meson represents CE state with respect to quark degrees of freedom. An oblique corroboration of this fact is given by the high instability of  $\pi^0$  meson in comparison with  $\pi^\pm$ . Such an instability may result from the much higher amount of quantum fluctuations peculiar to CE state.

Another example is given by a single dipole photon, which is emitted by a dipole transition in atom or molecule and carries total angular momentum  $J = 1$  (Berestetskii *et al* 1982). In the state with projection of the total angular momentum  $m = 0$  it is completely entangled. In fact, such photon carries two qubits. One of them is the polarization qubit, which is usually considered in the context of quantum information processing. Another qubit is provided by the orbital angular momentum, which can be observed (Padgett *et al* 2002) and used for the quantum information purposes (Mair *et al* 2001). Like in the case of  $\pi^0$  meson, these two qubits correspond to the intrinsic degrees of freedom of the photon.

As one more example, let us consider the so-called *biphoton*, which consists of two photons of the same frequency, created at once, and propagating in the same direction (Burlakov *et al* 1999, Chechova *et al* 2004). Before splitting, biphoton can be interpreted as a single “particle”. In the basis of linear polarizations, the states of biphoton have the form

$$|1\rangle = |x, x\rangle, \quad |0\rangle = \frac{1}{\sqrt{2}}(|x, y\rangle + |y, x\rangle), \quad |-1\rangle = |y, y\rangle \quad (10)$$

(the propagation direction is chosen as the  $z$ -axis). Thus, formally they coincide with the spin-1 states. It should be stressed that the antisymmetric state

$$|A\rangle = \frac{1}{\sqrt{2}}(|x, y\rangle - |y, x\rangle),$$

is forbidden (Berestetskii *et al* 1982). The CE of the state  $|0\rangle$  in (11) is evident.

The antisymmetric state is also forbidden in a system of two two-level atoms with dipole interaction in the Lamb-Dicke limit of short distances (Çakır *et al* 2005), so that this system can also be considered as a single spin-1 object.

Although a single qutrit can be prepared in CE state in the SU(2) sector, it does not manifest entanglement in the SU(3) sector, where the local observables are given by the eight independent Hermitian generators of the  $\mathfrak{su}(3)$  algebra (Caves and Milburn 2000):

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned} \quad (11)$$

It is easily seen that conditions (6) cannot be realized for the state (7) with the basic observables (12).

Thus, a single three-state quantum system (qutrit) may or may not manifest entanglement, depending on what kind of basic observables is accessible. Hence, there is a relativity of entanglement with respect to choice of basic observables.

#### 4. Stabilization of entanglement

Numerous applications of quantum entanglement require not an arbitrary entangled state but a *robust* one. This assumes the high amount of entanglement together with the long lifetime of the entangled state. This lifetime is usually determined by interaction with a dissipative environment, which causes the decoherence in the system.

The approach under discussion reveals a way of obtaining robust entanglement. In conformity with the definition (5), we should first prepare a state of a given system with the maximal amount of quantum fluctuations of all basic observables. As the second step, we should decrease the energy of the system up to a minimum (local minimum) to stabilize the state, keeping the level of quantum fluctuations. Thus obtained state would be stable (metastable) and CE.

As an example, consider atomic entanglement caused by photon exchange between two atoms in a cavity. In the simplest case of two-level atoms in an ideal cavity, containing a single photon, the CE state in atomic subsystem arises and decays periodically due to the Rabi oscillations (Plenio *et al* 1999). The above stabilization scheme can be used if instead we consider three-level atoms with the  $\Lambda$ -type transitions (Can *et al* 2002(b), Can *et al* 2003).

In this case, the two three-level atoms with allowed dipole transitions  $1 \leftrightarrow 2$  and  $2 \leftrightarrow 3$  and dipole forbidden transition  $1 \leftrightarrow 3$  are located in a cavity (Fig. 1) tuned to the resonance with transition  $1 \leftrightarrow 2$ .

If initially both atoms are in the ground state and cavity contains one photon, than absorption of the photon by either atom leads to creation of CE atomic state

$$|\psi_{CE}^{(12)}\rangle = \frac{1}{\sqrt{2}}(|2\rangle_I \otimes |1\rangle_{II} + |1\rangle_I \otimes |2\rangle_{II}), \quad (12)$$

where  $|n\rangle_j$  denotes the state of  $j$ -th atom. This state manifests maximal amount of quantum fluctuations of the local basic observables (Pauli operators)

$$\sigma_x^{(j)} = |2\rangle_j \langle 1| + H.c., \quad \sigma_y^{(j)} = -i|2\rangle_j \langle 1| + H.c., \quad \sigma_z^{(j)} = |2\rangle_j \langle 2| - |1\rangle_j \langle 1|,$$

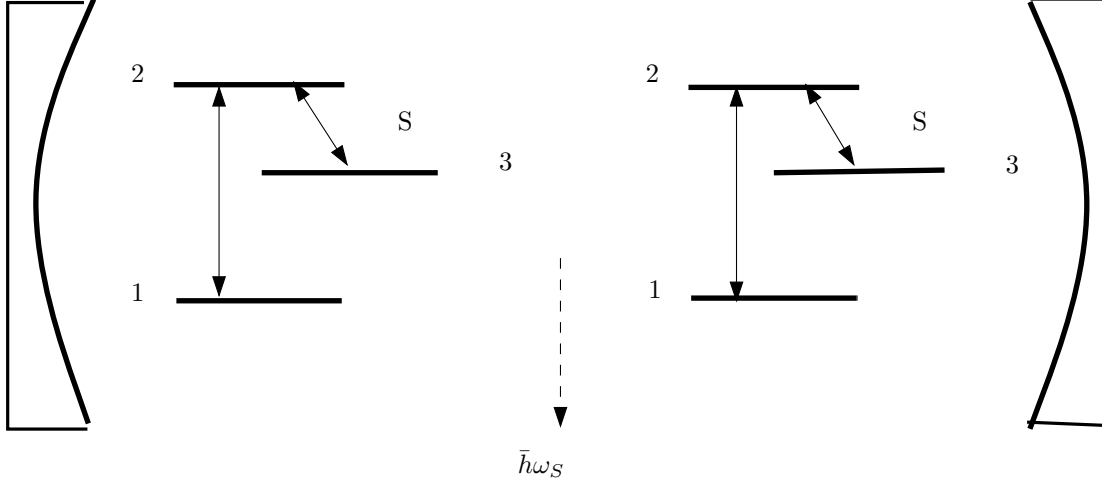
so that

$$\mathbb{V}(\psi_{CE}^{(12)}) = 6.$$

The corresponding energy of the system is

$$E^{(12)} = \epsilon_2 \sim \hbar\omega_C, \quad (13)$$

where  $\epsilon_j$  denotes the energy of the corresponding atomic level with respect to the ground state ( $\epsilon_1 = 0$ ) and  $\omega_C$  is the cavity mode frequency.



**Figure 1.** Scheme of transitions in two three-level  $\Lambda$ -type atoms in a cavity. Here P denotes transition resonant with the cavity (Pumping) field, while S corresponds to the transition with creation of Stokes photon

This state (13) is unstable. There are the two channels of decay of the excited atomic state:

$$|2\rangle_j \rightarrow \begin{cases} |1\rangle_j & \text{with creation of cavity photon} \\ |3\rangle_j & \text{with creation of Stokes photon} \end{cases}$$

The first way returns the system into the initial state. After that, the process would be repeated. The second decay channel creates the new CE state

$$|\psi^{(13)}\rangle = \frac{1}{\sqrt{2}}(|3\rangle_I \otimes |1\rangle_{II} + |1\rangle_I \otimes |3\rangle_{II}), \quad (14)$$

which manifests the same amount of quantum fluctuations as (12) but with respect to the new local basic observables

$$\sigma_x^{(j)} = |3\rangle_j \langle 1| + H.c., \quad \sigma_y^{(j)} = -i|3\rangle_j \langle 1| + H.c., \quad \sigma_z^{(j)} = |3\rangle_j \langle 3| - |1\rangle_j \langle 1|.$$

The corresponding energy is

$$E^{(13)} = \epsilon_3 + \hbar\omega_S \sim \epsilon_2,$$

where  $\omega_S$  denotes the frequency of Stokes photon. This is the same energy as for the (12) configuration (13). If the Stokes photon is now discarded, the energy is decreased

$$E^{(13)} \rightarrow E_{min}^{(13)} = \epsilon_3$$

and the state (15) becomes stable (at least, with respect to the dipole transitions). To discard the Stokes photon, we can think either about its absorption by the cavity walls or about its free leakage out of the cavity. In the latter case, detection of the Stokes photon outside the cavity signals the creation of the robust atomic entangled state (15). For further discussion of the above scheme, see Biswas and Agarwal 2004, Çakır *et al* 2004, Çakır *et al* 2005.

## 5. Conclusion

Summarizing, we should stress the generality of definition of CE states has been discussed in Sec. 2. Physically it associates CE with special behavior of expectation values of basic observables and, in that way, with the maximal amount of quantum fluctuations. In a sense, it follows Bell's ideology (Bell 1966) that entanglement manifests itself in local measurements and their correlations. The possible role of quantum fluctuations in formation of entangled states was also noticed by Gühne *et al* (Gühne *et al* 2002) and Hofmann and Takeuchi (Hofmann and Takeuchi 2003).

Since the classical level of description of physical systems neglects existence of quantum fluctuations, the total variance (1) can be chosen as a certain measure of *remoteness* of quantum reality from classical picture. Thus, the coherent states with minimal amount of quantum fluctuations are the closest states to classical picture, while CE states represent the most nonclassical states. In particular, Klyachko (Klyachko 2002) and Barnum *et al* (Barnum *et al* 2003) have associated generalized coherent states with the separable states of multipartite systems.

From the physical point of view, the definition, connecting CE with quantum fluctuations reveals the way of preparing robust entanglement (Sec. 4).

The general approach to quantum entanglement has been discussed in Sec. 2 is based on the consideration of the symmetry properties of physical systems. In particular, it associates definition of CE with the orthogonal basis of the Lie algebra, corresponding to the Lie group of the dynamical symmetry of the system. As it has been shown in Sec. 3, this causes a certain relativity of quantum entanglement with respect to the choice of the dynamic symmetry. As an example, a single-qutrit entanglement was considered.

In the case of a two-qutrit system, the entanglement takes place both in the SU(3) and SU(2) sectors. Since the set of the  $\mathfrak{su}(3)$  basic observables (12) contains the spin-1 operators, CE of two qutrits in the SU(3) sector involves CE in the SU(2) sector but not vice versa. The CE states of two spin-1 objects can be examined through the use of the following symmetry relation

$$SU(2) \times SU(2) \simeq SO(4).$$

The symmetry based approach to quantum entanglement leads to a certain "stratification" of possible states of quantum systems (Klyachko 2002). Namely, if  $G$  is the dynamic symmetry group, the SLOCC are defined by the action of complexified group  $G^c$ . Then, the different classes of states are given by the *orbits* of the action of  $g^c \in G^c$  in the Hilbert space  $\mathcal{H}_S$ .

For example, in the case of three qubits, the dynamic symmetry of the system is described by the Lie group

$$SU(2) \times SU(2) \times SU(2).$$

Thus, SLOCC belong to the group  $SL(2, \mathbb{C})$ . The orthogonal basis of the corresponding Lie algebra  $sl(2, \mathbb{C})$  is given by the Pauli operators. It was shown by Miyake (Miyake 2003) through the use of the mathematical analysis of multidimensional matrices and determinants by Gelfand *et al* (Gelfand *et al* 1994) that there are only four SLOCC nonequivalent classes of states shown in the Table below.

Table 1.

$\frac{1}{\sqrt{2}}( 000\rangle +  111\rangle)$	GHZ state
$\frac{1}{\sqrt{3}}( 001\rangle +  010\rangle +  100\rangle)$	W-state
$\frac{1}{\sqrt{2}} \times \begin{cases} ( 001\rangle +  010\rangle) \\ ( 001\rangle +  100\rangle) \\ ( 010\rangle +  100\rangle) \end{cases}$	biseparable states
$ 000\rangle$	completely separable states



Similar classification was proposed by Aćin *et al* (Aćin *et al* 2001) through the use of tripartite witnesses.

Besides the classification, the notion of complex orbits allows to introduce a proper measure  $\mu$  of entanglement as the length of minimal vector in the complex orbit (Klyachko 2002, Klyachko and Shumovsky 2004). Note that all natural measures of entanglement should be represented by the *entanglement monotones*, i.e. by functions decreasing under SLOCC (Vidal 2000, Eisert *et al* 2003, Verstraete *et al* 2003) and that the above measure obeys this condition.

In the case of an arbitrary pure two-qubit state

$$|\psi_{2,2}\rangle = \sum_{\ell,\ell'=0}^1 \psi_{\ell\ell'} |\ell\rangle \otimes |\ell'\rangle, \quad \sum_{\ell,\ell'} |\psi_{\ell\ell'}|^2 = 1,$$

the measure of entanglement  $\mathcal{C} = \det[\psi]$ , where  $[\psi]$  is the  $(2 \times 2)$  matrix of the coefficients of the state  $|\psi_{2,2}\rangle$ . This determinant represents the only entanglement monotone in this case. To within a factor, this measure coincides with the *concurrence*  $\mathcal{C}(\psi)$  (Wootters 1998), which is usually used to quantify entanglement in two-qubit systems:

$$\mathcal{C}(\psi) = 2|\det[\psi]|.$$

In the case of three qubits, the measure is given by the absolute value of Cayley hyperdeterminant multiplied by four (Miyake 2003) also known as *3-tangle* (Coffman *et al* 2000)

$$\begin{aligned} \tau = 4 & |\psi_{000}^2 \psi_{111}^2 + \psi_{001}^2 \psi_{110}^2 + \psi_{010}^2 \psi_{101}^2 + \psi_{100}^2 \psi_{011}^2 - 2(\psi_{000} \psi_{001} \psi_{110} \psi_{111} \\ & + \psi_{000} \psi_{010} \psi_{101} \psi_{111} + \psi_{000} \psi_{100} \psi_{011} \psi_{111} + \psi_{001} \psi_{010} \psi_{101} \psi_{110} \\ & + \psi_{001} \psi_{100} \psi_{011} \psi_{110} + \psi_{010} \psi_{100} \psi_{011} \psi_{101}) + 4(\psi_{000} \psi_{011} \psi_{101} \psi_{110} \\ & + \psi_{001} \psi_{010} \psi_{100} \psi_{111})|, \end{aligned} \quad (15)$$

where  $\psi_{i,j,k}$  are the coefficients of the normalized state

$$|\psi_{2,3}\rangle = \sum_{i,j,k=0}^1 \psi_{ijk} |i\rangle \otimes |j\rangle \otimes |k\rangle. \quad (16)$$

This is the again the only entangled monotone for the states (17). In the case of GHZ (Greenberger-Horne-Zeilinger) state (the first row in Table 1), 3-tangle (16) has the maximal value  $\tau(GHZ) = 1$ . For all other states in the Table 1, it has zero value, so that these states are unentangled.

This fact allows us to separate essential from the accidental in the definition of quantum entanglement. For example, violations of Bell's inequalities is often considered as a definition of entanglement. The so-called W-states (the second row in Table 1) violate Bell's inequalities (Cabello 2002). But as we have seen, these states do not manifest entanglement (at least in the tripartite sector).

In fact, violation of Bell's inequalities means the absence of hidden variables (Bell 1966) and can be observed even in the case of generalized coherent states (Klyachko 2002), which are unentangled by definition.

Other definitions based on the nonseparability and nonlocality of states also have a limited application. For sure, they are meaningless in the case of a single spin-1 particle entanglement have been considered in Sec. 3.

In this paper, we have considered entanglement of pure states. The generalization of the approach on the case of mixed states meets certain complications. The point is that the density

matrix contains classical fluctuations caused by the statistical nature of the state together with quantum fluctuations. Their separation represents a hard problem of extremely high importance. One of the possible approaches consists in the use of the methods of thermo-field dynamics (Takahashi Y and Umezawa H 1996), which allows to represent a mixed state in terms of a pure state of doubled dimension.

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