

Quantum marginal problem and N-representability

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Abstract. A fermionic version of the quantum marginal problem was known from the early sixties as *N-representability problem*. In 1995 it was mentioned by the National Research Council of the USA as one of ten most prominent research challenges in quantum chemistry. In spite of this recognition the progress was very slow, until a couple of years ago the problem came into focus again, now in the framework of quantum information theory. In the paper I give a survey of the recent development.

1. Introduction

The quantum marginal problem is about relation between reduced states ρ_A, ρ_B, ρ_C of a pure state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ of three (or multi) component quantum system. In plain language it can be stated as follows:

Under what conditions three Hermitian matrices ρ_A, ρ_B, ρ_C of orders ℓ, m, n coincide with the Gram matrices formed by Hermitian dot products of the parallel slices of a complex cubic matrix $\psi = [\psi_{\alpha\beta\gamma}]$ of format $\ell \times m \times n$?

Clearly the compatibility depends only on the spectra

$$\lambda^A = \text{Spec}(\rho_A), \quad \lambda^B = \text{Spec}(\rho_B), \quad \lambda^C = \text{Spec}(\rho_C). \quad (1.1)$$

An equivalent version of the problem seeks for relation between spectra of Hermitian operator $\rho_{AB} : \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow \mathcal{H}_A \otimes \mathcal{H}_B$ and its reduced operators $\rho_A : \mathcal{H}_A \rightarrow \mathcal{H}_A$ and $\rho_B : \mathcal{H}_B \rightarrow \mathcal{H}_B$. The reduction $\rho_{AB} \mapsto \rho_A$ is known to mathematicians as *contraction*, e.g. Ricci curvature operator $\text{Ric} : T \rightarrow T$ is the contraction of Riemann curvature $R : T \wedge T \rightarrow T \wedge T$.

The problem has a long history. Its fermionic version dealing with skew symmetric state $\psi \in \wedge^N \mathcal{H}$ of N fermions, e.g. electrons in an atom or a molecule, was known from the early 60s as *N-representability problem* [13, 10]. In mid 90s it was included in the list of ten most prominent research challenges in quantum chemistry [27]. A couple of years ago the problem came into focus again, now in the framework of quantum information theory. Here we outline a solution of the problem in terms of linear inequalities on the spectra (1.1) governed by topology of flag varieties.

The quantum marginal problem for overlapping reduced states like $\rho_{AB}, \rho_{BC}, \rho_{CD}$ is beyond the scope of this paper. Known rigorous results in this case are mostly sporadic, see [20] and references therein. For the fermionic version one can find further information in [12, 11].

Section 2 contains a brief account of the classical marginal problem and its connection with Bell's inequalities.

Section 3 starts with a survey of some recent results that laid the ground of the quantum marginal problem, followed up by a solution of the problem based on geometric invariant theory.

The last section 4 deals with one point reduced density matrix of a system of N fermions. It includes a solution of general N -representability problem for one point reduced density matrix, as well as explicit inequalities for systems of rank ≤ 8 . A representation theoretical interpretation of N -representability plays crucial role in the calculations.

The results of this section imply some inequalities between spectra of Riemann and Ricci curvatures, see Remark 4.2.5. Recall that in general relativity Ricci curvature is governed by the energy-momentum tensor, i.e. by physical content of the space, while Riemann curvature is responsible for its geometry and topology. The above constraints impose some bounds on influence of matter on geometry.

2. Classical marginal problem

2.1. Marginal distributions

Let's start with the *classical marginal problem* (MP) which asks for existence of a "body" in \mathbb{R}^n with given projections onto some coordinate subspaces $\mathbb{R}^I \subset \mathbb{R}^n, I \subset \{1, 2, \dots, n\}$, i.e. existence of a *probability density* $p(x) = p(x_1, x_2, \dots, x_n)$ with given *marginal distributions*

$$p_I(x_I) = \int_{\mathbb{R}^J} p(x) dx_J, J = \{1, 2, \dots, n\} \setminus I.$$

The discrete version of the classical MP amounts to calculation of an image of a multidimensional simplex, say $\Delta = \{p_{ijk} \geq 0 \mid \sum p_{ijk} = 1\}$, under a linear map like

$$\begin{aligned} \pi : \quad \mathbb{R}^{\ell mn} &\rightarrow \mathbb{R}^{\ell m} \oplus \mathbb{R}^{mn} \oplus \mathbb{R}^{n\ell}, \\ p_{ijk} &\mapsto (p_{ij}, p_{jk}, p_{ki}), \\ p_{ij} &= \sum_k p_{ijk}, \quad p_{jk} = \sum_i p_{ijk}, \quad p_{ki} = \sum_j p_{ijk}. \end{aligned}$$

The image $\pi(\Delta)$ is a convex hull of the projections of vertices of Δ . So the classical MP amounts to the calculation of facets of a convex hull. In high dimensions this might be a computational nightmare [25, 15].

2.2. Classical realism

Let $X : \mathcal{H}_A \rightarrow \mathcal{H}_A$ be an observable of quantum system A . Actual measurement of X produces a random quantity x with values in $\text{Spec}(X)$. The density $p(x)$ is implicitly determined by the expectations

$$\langle f(x) \rangle = \langle \psi | f(X) | \psi \rangle$$

for all functions f on spectrum $\text{Spec}(X)$. For *commuting* observables $X_i, i \in I$ the random variables $x_I = \{x_i, i \in I\}$ have *joint distribution* $p_I(x_I)$ defined by the similar equation

$$\langle f(x_I) \rangle = \langle \psi | f(X_I) | \psi \rangle, \quad \forall f. \quad (2.1)$$

Classical realism postulates existence of a hidden joint distribution of *all variables* x_i . This amounts to compatibility of the marginal distributions (2.1) for *commuting* sets of observables X_I . Hence *Bell inequalities*, designed to test classical realism, stem from the classical marginal problem.

2.2.1 Example. Observations of disjoint components of composite system $\mathcal{H}_A \otimes \mathcal{H}_B$ always commute. For two qubits with two measurements per site their compatibility is given by 16 inequalities obtained from the *Clauser-Horne-Shimony-Holt* inequality [9]

$$\langle a_1 b_1 \rangle + \langle a_2 b_1 \rangle + \langle a_2 b_2 \rangle - \langle a_1 b_2 \rangle + 2 \geq 0$$

by spin flips $a_i \mapsto \pm a_j$ and permutation of the components $A \leftrightarrow B$. Here $\langle a_i b_j \rangle$ is expectation of the product of spin projections onto directions i, j at sites A, B .

2.2.2 Example. For three qubits with two measurements per site the marginal constraints amount to 53856 independent inequalities [26]. This example may help to disabuse us from overoptimistic expectations for the *quantum marginal problem* to be discussed below.

2.2.3 Example. Univariant marginal distributions $p_i(x_i)$ are always compatible, e.g. we can consider x_i as *independent* random variables. However under additional constraints, say for a “body” of constant density, even univariant marginal problem becomes nontrivial. For its discrete version the *Gale-Ryser theorem* [16] tells that partitions λ and μ are margins of a rectangular 0/1 matrix iff the majorization inequality $\lambda \prec \mu^t$ holds. Here, the marginal values arranged in decreasing order are treated as *Young diagrams*

$$\lambda = (5, 4, 2, 1) = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \\ \hline \square & \square & & & \\ \hline \square & & & & \\ \hline \end{array} \quad \lambda^t = (4, 3, 2, 2, 1) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array},$$

μ^t stands for *transposed* diagram, and the *majorization order* $\lambda \prec \nu$ is defined by inequalities

$$\begin{array}{rcl} \lambda_1 & \leq & \nu_1 \\ \lambda_1 + \lambda_2 & \leq & \nu_1 + \nu_2 \\ \lambda_1 + \lambda_2 + \lambda_3 & \leq & \nu_1 + \nu_2 + \nu_3 \\ \dots\dots\dots & \dots & \dots\dots\dots \end{array}$$

3. Quantum marginal problem

3.1. Reduced states

Density matrix ρ_{AB} of a composite system AB can be written as a linear combination of separable states

$$\rho_{AB} = \sum_{\alpha} a_{\alpha} \rho_A^{\alpha} \otimes \rho_B^{\alpha}, \tag{3.1}$$

where $\rho_A^{\alpha}, \rho_B^{\alpha}$ are mixed states of the components A, B respectively, and the coefficients a_{α} are *not* necessarily positive. Its *reduced matrices* or *marginal states* may be defined by equations

$$\begin{aligned} \rho_A &= \sum_{\alpha} a_{\alpha} \text{Tr}(\rho_B^{\alpha}) \rho_A^{\alpha} := \text{Tr}_B(\rho_{AB}), \\ \rho_B &= \sum_{\alpha} a_{\alpha} \text{Tr}(\rho_A^{\alpha}) \rho_B^{\alpha} := \text{Tr}_A(\rho_{AB}). \end{aligned}$$

The reduced states ρ_A, ρ_B are independent of the decomposition (3.1) and can be characterized intrinsically by the following property

$$\langle X_A \rangle_{\rho_{AB}} = \text{Tr}(\rho_{AB} X_A) = \text{Tr}(\rho_A X_A) = \langle X_A \rangle_{\rho_A}, \tag{3.2}$$

which holds for all observables X_A of component A . In other words ρ_A is a “visible” state of subsystem A . This justifies the chosen terminology.

3.1.1 *Example.* Let's identify pure state of two component system

$$\psi = \sum_{ij} \psi_{ij} \alpha_i \otimes \beta_j \in \mathcal{H}_A \otimes \mathcal{H}_B$$

with its matrix $[\psi_{ij}]$ in orthonormal bases α_i, β_j of $\mathcal{H}_A, \mathcal{H}_B$. Then the reduced states of ψ in respective bases are given by matrices

$$\rho_A = \psi^\dagger \psi, \quad \rho_B = \psi \psi^\dagger, \quad (3.3)$$

which have identical non negative spectra

$$\text{Spec} \rho_A = \text{Spec} \rho_B = \lambda \quad (3.4)$$

except extra zeros if $\dim \mathcal{H}_A \neq \dim \mathcal{H}_B$. The isospectrality implies the so-called *Schmidt decomposition*

$$\psi = \sum_i \sqrt{\lambda_i} \psi_i^A \otimes \psi_i^B, \quad (3.5)$$

where ψ_i^A, ψ_i^B are eigenvectors of ρ_A, ρ_B with the same eigenvalue λ_i .

Thus the reduced states of a two component system are strongly correlated. Similar correlations for multicomponent systems are at the heart of the quantum marginal problem discussed below.

3.2. Statement of the problem

The quantum analogue of the classical marginal distribution is the *reduced state* ρ_A of the composite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Accordingly, the most general *quantum marginal problem* (QMP) asks about existence of mixed state ρ_I of composite system

$$\mathcal{H}_I = \bigotimes_{i \in I} \mathcal{H}_i$$

with given reduced states ρ_J for some $J \subset I$ (cf. with classical settings of section 2). Additional constraints on state ρ_I may be relevant. Here we consider only two variations:

- *Pure quantum marginal problem*

dealing with marginals of a pure state $\rho_I = |\psi\rangle\langle\psi|$, and more general

- *Mixed quantum marginal problem*

corresponding to a state with given spectrum $\lambda_I = \text{Spec} \rho_I$.

Both versions are nontrivial even for univariant margins (cf. Example 2.2.3). In this case reduced states ρ_i can be diagonalized by local unitary transformations and their compatibility depends only on the spectra $\lambda_i = \text{Spec} \rho_i$. Note that mixed QMP say for two component system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ is formally equivalent to the pure one for three component system $\mathcal{H}_{AB} \otimes \mathcal{H}_A \otimes \mathcal{H}_B$.

The pure quantum marginal problem has no classical analogue, since the projection of a point is a point. For a two component system $\mathcal{H}_A \otimes \mathcal{H}_B$ marginal constraints amount to isospectrality: $\text{Spec} \rho_A = \text{Spec} \rho_B$, see Example 3.1.1. For a three component system the problem can be stated in plain language as follows.

Problem 3.2.1. *Let $\psi = [\psi_{\alpha\beta\gamma}]$ be complex cubic matrix and ρ_A, ρ_B, ρ_C be the Gram matrices formed by Hermitian dot products of parallel slices of ψ . The question is what are relations between spectra of matrices ρ_A, ρ_B, ρ_C ?*

Unfortunately methods of this paper can't be applied directly to overlapping marginals like $\rho_{AB}, \rho_{BC}, \rho_{CA}$.

3.3. Some known results

Here are some recent results that laid the ground of the quantum marginal problem. They all stem from quantum information theory in a couple of years.

Theorem (Higuchi-Sudbery-Szulc [18]). *For an array of qubits $\bigotimes_{i=1}^n \mathcal{H}_i$, $\dim \mathcal{H}_i = 2$, all constraints on the margins ρ_i of a pure state are given by the polygonal inequalities*

$$\lambda_i \leq \sum_{j(\neq i)} \lambda_j$$

for λ_i the minimal eigenvalue of ρ_i .

This characterization was discovered independently by Sergey Bravyi who also managed to crack the mixed two qubit problem.

Theorem (Bravyi [6]). *For two qubits $\mathcal{H}_A \otimes \mathcal{H}_B$ the solution of the mixed QMP is given by inequalities*

$$\begin{aligned} \min(\lambda_A, \lambda_B) &\geq \lambda_3^{AB} + \lambda_4^{AB}, \\ \lambda_A + \lambda_B &\geq \lambda_2^{AB} + \lambda_3^{AB} + 2\lambda_4^{AB}, \\ |\lambda_A - \lambda_B| &\leq \min(\lambda_1^{AB} - \lambda_3^{AB}, \lambda_2^{AB} - \lambda_4^{AB}), \end{aligned}$$

where λ_A, λ_B are minimal eigenvalues of ρ_A, ρ_B and $\lambda_1^{AB} \geq \lambda_2^{AB} \geq \lambda_3^{AB} \geq \lambda_4^{AB}$ is spectrum of ρ_{AB} .

Finally for three qutrits the problem was solved by Matthias Franz using rather advanced mathematical technology and help of a computer. An elementary solution was found independently by Astashi Higuchi.

Theorem (Franz [14], Higuchi [19]). *All constraints on margins of a pure state of three qutrit system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ are given by the following inequalities*

$$\begin{aligned} \lambda_2^a + \lambda_1^a &\leq \lambda_2^b + \lambda_1^b + \lambda_2^c + \lambda_1^c, \\ \lambda_3^a + \lambda_1^a &\leq \lambda_2^b + \lambda_1^b + \lambda_3^c + \lambda_1^c, \\ \lambda_3^a + \lambda_2^a &\leq \lambda_2^b + \lambda_1^b + \lambda_3^c + \lambda_2^c, \\ 2\lambda_2^a + \lambda_1^a &\leq 2\lambda_2^b + \lambda_1^b + 2\lambda_2^c + \lambda_1^c, \\ 2\lambda_1^a + \lambda_2^a &\leq 2\lambda_2^b + \lambda_1^b + 2\lambda_1^c + \lambda_2^c, \\ 2\lambda_2^a + \lambda_3^a &\leq 2\lambda_2^b + \lambda_1^b + 2\lambda_2^c + \lambda_3^c, \\ 2\lambda_2^a + \lambda_3^a &\leq 2\lambda_1^b + \lambda_2^b + 2\lambda_3^c + \lambda_2^c, \end{aligned}$$

where a, b, c is a permutation of A, B, C , and the marginal spectra are arranged in increasing order $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

Note that in contrast to the classical marginal problem, linearity of the quantum marginal constraints is a surprising nontrivial fact.

3.4. Main theorem

A general solution of the quantum marginal problem, based on geometric invariant theory, has been found recently [20]. We state the result for two component systems. Its extension to multicomponent case is straightforward.

Theorem 3.4.1. For the two component system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ of format $m \times n$ all constraints on spectra $\lambda^{AB} = \text{Spec } \rho_{AB}$, $\lambda^A = \text{Spec } \rho_A$, $\lambda^B = \text{Spec } \rho_B$ arranged in decreasing order are given by linear inequalities

$$\sum_i a_i \lambda_{u(i)}^A + \sum_j b_j \lambda_{v(j)}^B \leq \sum_k (a+b)_k^\downarrow \lambda_{w(k)}^{AB}, \quad (3.6)$$

where $a : a_1 \geq a_2 \geq \dots \geq a_m$, $b : b_1 \geq b_2 \geq \dots \geq b_n$, $\sum a_i = \sum b_j = 0$ are “test spectra”, the spectrum $(a+b)^\downarrow$ consists of numbers $a_i + b_j$ arranged in decreasing order, and $u \in S_m$, $v \in S_n$, $w \in S_{mn}$ are permutations, subject to a topological condition $c_{uv}^w(a, b) \neq 0$ that will be explained later.

3.4.2 Remark. The coefficient $c_{uv}^w(a, b)$ depends only on the order in which quantities $a_i + b_j$ appear in the spectrum $(a+b)^\downarrow$. The order changes when a pair (a, b) crosses hyperplane

$$H_{ij|kl} : a_i + b_j = a_k + b_l.$$

The hyperplanes cut the set of all pairs (a, b) into finite number of pieces called *cubicles*. For each cubicle one have to check inequality (3.6) only for its *extremal edges*. Hence the marginal constraints amount to a *finite system* of inequalities, but the total number of extremal edges increases rapidly. Here are some sample data for arrays of qubits.

# qubits	2	3	4	5	6
# edges	2	4	12	125	11344

Unfortunately, for most systems the marginal constraints are too numerous to be reproduced here. Therefore we only give a summarizing table of the number of independent marginal inequalities, which shows how complicate the answer may be.

System	Rank	#Inequalities
2×2	2	7
$2 \times 2 \times 2$	3	40
2×3	3	41
2×4	4	234
3×3	4	387
$2 \times 2 \times 3$	4	442
$2 \times 2 \times 2 \times 2$	4	805

3.5. Hidden geometry and topology

Here we explain the meaning of the coefficient $c_{uv}^w(a, b)$ in the statement of the theorem and show how it can be calculated. Let's start with the set of all Hermitian operators $X_A : \mathcal{H}_A \rightarrow \mathcal{H}_A$ with given spectrum $\text{Spec}(X_A) = a$ and call it *flag variety*

$$\mathcal{F}_a(\mathcal{H}_A) := \{X_A \mid \text{Spec}(X_A) = a\}.$$

For two flag varieties $\mathcal{F}_a(\mathcal{H}_A)$ and $\mathcal{F}_b(\mathcal{H}_B)$ define the map

$$\begin{aligned} \varphi_{ab} : \mathcal{F}_a(\mathcal{H}_A) \times \mathcal{F}_b(\mathcal{H}_B) &\longrightarrow \mathcal{F}_{a+b}(\mathcal{H}_A \otimes \mathcal{H}_B), \\ X_A \times X_B &\longmapsto X_A \otimes 1 + 1 \otimes X_B. \end{aligned}$$

The coefficients $c_{uv}^w(a, b)$ come from the induced morphism of cohomology

$$\varphi_{ab}^* : H^*(\mathcal{F}_{a+b}(\mathcal{H}_{AB})) \rightarrow H^*(\mathcal{F}_a(\mathcal{H}_A)) \otimes H^*(\mathcal{F}_b(\mathcal{H}_B))$$

written in the basis of *Schubert cocycles* σ^w

$$\varphi_{ab}^* : \sigma^w \mapsto \sum_{u,v} c_{uv}^w(a,b) \sigma^u \otimes \sigma^v.$$

We'll give below an algorithm for their calculation. For this we need a description of the cohomology of flag varieties due to Bernstein-Gelfand-Gelfand [4]. Specifically, for a *simple* spectrum a eigenspaces of the operator $X_A \in \mathcal{F}_a(\mathcal{H}_A)$ of given eigenvalue a_i form a line bundle \mathcal{L}_i^A on the flag variety $\mathcal{F}_a(\mathcal{H}_A)$. Their Chern classes $x_i^A = c_1(\mathcal{L}_i^A)$ generate the cohomology ring $H^*(\mathcal{F}_a(\mathcal{H}_A))$ and in this setting the morphism φ_{ab}^* admits a simple description:

$$\varphi_{ab}^* : x_k^{AB} \mapsto x_i^A + x_j^B$$

for $(a+b)_k^\downarrow = a_i + b_j$. In terms of the *canonical generators* $x_i = c_1(\mathcal{L}_i)$ the Schubert cocycle σ^w is given by the so-called *Schubert polynomial* [22]

$$S_w(x_1, x_2, \dots) = \partial_{w^{-1}w_0} (x_1^{n-1} x_2^{n-2} \dots x_{n-1}),$$

where $w \in S_n$ is a permutation of $1, 2, \dots, n$, $w_0 = (n, n-1, \dots, 2, 1)$, and the operator $\partial_w = \partial_{i_1} \partial_{i_2} \dots \partial_{i_\ell}$ is defined via decomposition $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$ into product of transpositions $s_i = (i, i+1)$, $\ell = \ell(w)$ is the number of inversion in w called its *length*. Finally, $\partial_i = \partial_{s_i}$ is divided difference operator

$$\partial_i f = \frac{f(\dots, x_i, x_{i+1}, \dots) - f(\dots, x_{i+1}, x_i, \dots)}{x_i - x_{i+1}}.$$

This leads to the *computational formula*

$$c_{uv}^w(a,b) = \partial_u^A \partial_v^B S_w(x^{AB}) \Big|_{x_k^{AB} = x_i^A + x_j^B} \quad (3.7)$$

where $\ell(w) = \ell(u) + \ell(v)$, so that the right hand side is a scalar, and operators ∂_u^A and ∂_v^B acts on the variables x^A and x^B respectively. These variables emerge from substitution $x_k^{AB} \mapsto x_i^A + x_j^B$ in Schubert polynomial $S_w(x^{AB})$, and the indices i, j, k come from the equation $(a+b)_k^\downarrow = a_i + b_j$. The formula can be easily implemented into a computer program. Recall that in order to get a finite system of inequalities one have also to find all the extremal edges and use them as the test spectra (a, b) .

3.5.1 Example. Note that for identical permutations u, v, w the coefficient $c_{uv}^w(a, b)$ is equal to 1. Hence the inequality

$$\sum_i a_i \lambda_i^A + \sum_j b_j \lambda_j^B \leq \sum_k (a+b)_k \lambda_k^{AB}$$

holds for all test spectra (a, b) . This amounts to a finite system of *basic inequalities* [17]

$$\begin{aligned} \lambda_1^A + \lambda_2^A + \dots + \lambda_k^A &\leq \lambda_1^{AB} + \lambda_2^{AB} + \dots + \lambda_{kn}^{AB}, & k \leq m = \dim \mathcal{H}_A, \\ \lambda_1^B + \lambda_2^B + \dots + \lambda_\ell^B &\leq \lambda_1^{AB} + \lambda_2^{AB} + \dots + \lambda_{m\ell}^{AB}, & \ell \leq n = \dim \mathcal{H}_B. \end{aligned}$$

The calculations needed in Theorem 3.4.1 can be essentially reduced using the following result, which appears in [20] as a conjecture. The proof, based on Belkale arguments [2], will be published elsewhere.

Theorem 3.5.2. *In the setting of Theorem 3.4.1 all marginal constraints are given by inequalities (3.6) with $c_{uv}^w(a, b) = 1$.*

We use it in the next section to figure out structure of the marginal constraints in an array of qubits.

3.6. Array of qubits

Let ρ be a mixed state of n qubit system $\mathcal{H}^{\otimes n}$, $\dim \mathcal{H} = 2$, and $\rho^{(i)}$ be the reduced state of i -th component. Multicomponent version of Theorem 3.4.1 tells that all constraints on spectra $\lambda = \text{Spec } \rho$ and $\lambda^{(i)} = \text{Spec } \rho^{(i)}$ are given by inequalities

$$\sum_i (-1)^{u_i} a_i (\lambda_1^{(i)} - \lambda_2^{(i)}) \leq \sum_{\pm} (\pm a_1 \pm a_2 \pm \dots \pm a_n)_k \downarrow \lambda_{w(k)} \quad (3.8)$$

for all “test spectra” $\pm a_i$, and all permutations $u_i \in S_2$, $w \in S_{2^n}$ subject to condition $c_{u_1 u_2 \dots u_n}^w(a) \neq 0$.

The quantity $c_{u_1 u_2 \dots u_n}^w(a)$ is equal to the coefficient at $x_1^{u_1} x_2^{u_2} \dots x_n^{u_n}$ in the specialization of Schubert polynomial

$$S_w(z_1, z_2, \dots, z_{2^n})|_{z_k = \pm x_1 \pm x_2 \pm \dots \pm x_n}, \quad (3.9)$$

where the signs are taken from k -th term of the spectrum $(\pm a_1 \pm a_2 \pm \dots \pm a_n)_k^\downarrow$. Here we use isomorphism $S_2 \simeq \mathbb{Z}_2$ to identify $u_i \in S_2$ with binary variable $u_i = 0, 1$.

Theorem 3.5.2 implies that all marginal constraints are given by inequalities (3.8) with *odd* coefficient $c_{u_1 u_2 \dots u_n}^w(a)$. Reduction of the specialization (3.9) modulo two amounts to multinomial

$$S_w(1, 1, \dots, 1)(x_1 + x_2 + \dots + x_n)^{\ell(w)} \quad (3.10)$$

which contains a multiplicity free term $x_1^{u_1} x_2^{u_2} \dots x_n^{u_n}$ only for $\ell(w) = 0$ or 1 . This leaves us with two possibilities:

- w and u_i are identical permutations. This give us the basic inequality

$$\sum_i a_i (\lambda_1^{(i)} - \lambda_2^{(i)}) \leq \sum_{\pm} (\pm a_1 \pm a_2 \pm \dots \pm a_n)_k \downarrow \lambda_k \quad (3.11)$$

- $w = (k, k+1)$ is a transposition and all u_i except one are identical permutations.

The Schubert polynomial for a transposition is well known $S_{(k, k+1)}(z) = z_1 + z_2 + \dots + z_k$. Hence for even k the coefficient $S_w(1, 1, \dots, 1)$ in (3.10) is even. This bound us to transpositions $w = (k, k+1)$ with odd k . As result we get

Theorem 3.6.1. *For an array of qubits all marginal constraints can be obtained from the basic inequality (3.11) by transposition $\lambda_k \leftrightarrow \lambda_{k+1}$, $k = \text{odd}$, in RHS combined with sign change $a_i \mapsto -a_i$ of a term in LHS.*

To get a finite system of inequalities one has only to find the extremal edges. For large n this may be a challenge, see Remark 3.4.2, but conceptually the theorem reduces QMP for array of qubits to finding facets of a convex polytope given by an explicite system of linear inequalities.

3.6.2 Example. For 3-qubits the theorem returns the following list of marginal inequalities grouped by their extremal edges. The first inequality in each group is the basic one. The transposed eigenvalues in modified inequalities are typeset in bold face. Below we expect the differences $\Delta_i = \lambda_1^{(i)} - \lambda_2^{(i)}$ to be arranged in increasing order $\Delta_1 \leq \Delta_2 \leq \Delta_3$.

$$\begin{aligned} \Delta_3 &\leq \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8. \\ \Delta_2 + \Delta_3 &\leq 2\lambda_1 + 2\lambda_2 - 2\lambda_7 - 2\lambda_8. \\ \Delta_1 + \Delta_2 + \Delta_3 &\leq 3\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - 3\lambda_8, \\ -\Delta_1 + \Delta_2 + \Delta_3 &\leq 3\lambda_2 + \lambda_1 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - 3\lambda_8, \\ -\Delta_1 + \Delta_2 + \Delta_3 &\leq 3\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 - \lambda_8 - 3\lambda_7. \\ \Delta_1 + \Delta_2 + 2\Delta_3 &\leq 4\lambda_1 + 2\lambda_2 + 2\lambda_3 - 2\lambda_6 - 2\lambda_7 - 4\lambda_8, \\ -\Delta_1 + \Delta_2 + 2\Delta_3 &\leq 4\lambda_2 + 2\lambda_1 + 2\lambda_3 - 2\lambda_6 - 2\lambda_7 - 4\lambda_8, \\ -\Delta_1 + \Delta_2 + 2\Delta_3 &\leq 4\lambda_1 + 2\lambda_2 + 2\lambda_4 - 2\lambda_6 - 2\lambda_7 - 4\lambda_8, \\ -\Delta_1 + \Delta_2 + 2\Delta_3 &\leq 4\lambda_1 + 2\lambda_2 + 2\lambda_3 - 2\lambda_5 - 2\lambda_7 - 4\lambda_8, \\ -\Delta_1 + \Delta_2 + 2\Delta_3 &\leq 4\lambda_1 + 2\lambda_2 + 2\lambda_3 - 2\lambda_6 - 2\lambda_8 - 4\lambda_7. \end{aligned}$$

4. N-representability problem

4.1. Physical background

The quantum marginal problem may be complicated by additional constraints on state ψ . For example, the Pauli principle implies that state space of N identical particles shrinks to symmetric tensors $S^N\mathcal{H} \subset \mathcal{H}^{\otimes N}$ for bosons and to skew symmetric tensors $\wedge^N\mathcal{H}$ for fermions. For such systems reduced density matrices (RDM) appear in the second quantization formalism in the form

$$\begin{aligned}\rho^{(1)} &= \langle \psi | a_i^\dagger a_j | \psi \rangle = 1 \text{ particle RDM,} \\ \rho^{(2)} &= \langle \psi | a_i^\dagger a_j^\dagger a_k a_l | \psi \rangle = 2 \text{ particle RDM, etc.}\end{aligned}$$

Their physical importance stems from the observation that, say for *fermionic system*, like a multi-electron atom or molecule, with pairwise interaction

$$H = \sum_i^N H_i + \sum_{i<j} H_{ij}$$

the energy of state ψ depends only on the 2-point RDM

$$E = \binom{N}{2} \text{Tr} (H^{(2)} \rho^{(2)}),$$

where $H^{(2)} = \frac{1}{N-1}[H_1 + H_2] + H_{12}$ is a reduced two particle Hamiltonian. This allows, for example, to express the energy of the ground state E_0 via 2-point RDM

$$E_0 = \binom{N}{2} \min_{\rho^{(2)}=\text{RDM}} \text{Tr}(H^{(2)} \rho^{(2)}).$$

The problem however is that it is not obvious what conditions the RDM itself should satisfy. This is what the quantum marginal problem is about. In this settings it was known from early sixties as *N-representability problem* [13, 10]. Later the problem was regarded as one of ten most prominent research challenges in quantum chemistry [27]. Its solution allows to calculate nearly all properties of matter which are of interest to chemists and physicists. For current state of affairs and more history see [12, 11].

4.2. One point N-representability

Here we outline a solution of the problem for one point reduced states. Following chemists we treat them as electron density and accordingly use the normalization $\text{Tr} \rho^{(1)} = N$ while keeping $\text{Tr} \rho = 1$. There are few cases where complete solution of one point *N-representability* was known prior 2005:

- Pauli principle: $0 \leq \lambda_i \leq 1$, $\lambda = \text{Spec} \rho^{(1)}$. This condition provides a criterion for *mixed N-representability* [10].
- Criterion for pure *N-representability* for two particles $\wedge^2\mathcal{H}_r$ or two holes $\wedge^{r-2}\mathcal{H}_r$ is given by even degeneration of all eigenvalues of $\rho^{(1)}$, except 0 (resp. 1) for odd rank $r = \dim \mathcal{H}_r$ [10, 3].
- For system of three fermions of rank six $\wedge^3\mathcal{H}_6$ all constraints on one point reduced matrix of a pure state are given by the following (in)equalities

$$\lambda_1 + \lambda_6 = \lambda_2 + \lambda_5 = \lambda_3 + \lambda_4 = 1, \quad \lambda_4 \leq \lambda_5 + \lambda_6, \quad (4.1)$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \geq \lambda_6$ is spectrum of $\rho^{(1)}$.

The last result belongs to Borland and Dennis [5] who commented it as follows:

We have no apology for consideration of such a special case. The general N -representability problem is so difficult and yet so fundamental for many branches of science that each concrete result is useful in shedding light on the nature of general solution.

For more then 30 years passed after this theorem no other solution of N -representability problem has been found. Borland and Dennis derived their criterion from an extensive computer experiment, and later proved it with help provided by M.B. Ruskai and R.L. Kingsley. They also conjectured solutions for systems $\wedge^3\mathcal{H}_7, \wedge^4\mathcal{H}_7, \wedge^4\mathcal{H}_8$, e.g. for $\wedge^3\mathcal{H}_7$ one point pure representability is given by 4 inequalities

$$\begin{aligned} \lambda_1 + \lambda_6 + \lambda_7 &\geq 1, & \lambda_2 + \lambda_5 + \lambda_7 &\geq 1, \\ \lambda_3 + \lambda_4 + \lambda_7 &\geq 1, & \lambda_3 + \lambda_5 + \lambda_6 &\geq 1, \end{aligned} \quad (4.2)$$

but they failed to prove them. The conjectures turn out to be true and covered by the following general result.

Theorem 4.2.1. *For mixed state ρ of an n -fermion system $\wedge^n\mathcal{H}_r$ of rank $r = \dim \mathcal{H}_r$ all constraints on spectra $\nu = \text{Spec } \rho$ and $\lambda = \text{Spec } \rho^{(1)}$ are given by inequalities*

$$\sum_i a_i \lambda_{\nu(i)} \leq \sum_j (\wedge^n a)_j \nu_{w(j)} \quad (4.3)$$

for all “test spectra” $a : a_1 \geq a_2 \geq \dots \geq a_r, \sum a_i = 0$. Here $\wedge^n a = \{a_{i_1} + a_{i_2} + \dots + a_{i_n}\}^\downarrow$ consists of all sums $a_{i_1} + a_{i_2} + \dots + a_{i_n}, i_1 < i_2 < \dots < i_n$ arranged in decreasing order, and $v \in S_r, w \in S_{\binom{n}{r}}$ are permutations subject to a topological condition $c_w^v(a) \neq 0$ to be explained below.

4.2.2 Remark. Recall that the spectra λ and ν are arranged in decreasing order and normalized to trace n and 1 respectively. Similarly to Theorem 3.4.1 the coefficients $c_w^v(a)$ are defined via flag variety $\mathcal{F}_a(\mathcal{H}_r) := \{X : \mathcal{H}_r \rightarrow \mathcal{H}_r \mid \text{Spec}(X) = a\}$ and morphism

$$\begin{aligned} \varphi_a : \mathcal{F}_a(\mathcal{H}_r) &\rightarrow \mathcal{F}_{\wedge^n a}(\wedge^n \mathcal{H}_r) \\ X &\mapsto X^{(n)} \end{aligned}$$

where operator $X^{(n)} : \wedge^n \mathcal{H}_r \rightarrow \wedge^n \mathcal{H}_r$ acts as differential

$$X^{(n)} : \alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_n \mapsto \sum_i \alpha_1 \wedge \alpha_2 \wedge \dots \wedge X\alpha_i \wedge \dots \wedge \alpha_n.$$

The coefficients $c_w^v(a)$ come from the induced morphism of cohomology

$$\varphi_a^* : H^*(\mathcal{F}_{\wedge^n a}(\wedge^n \mathcal{H})) \rightarrow H^*(\mathcal{F}_a(\mathcal{H}))$$

written in the basis of *Schubert cocycles* σ_w

$$\varphi_a^* : \sigma_w \mapsto \sum_v c_w^v(a) \sigma_v.$$

They can be calculated by equation

$$c_w^v(a) = \partial_v S_w(z) \Big|_{z_k = x_{i_1} + x_{i_2} + \dots + x_{i_n}},$$

where the indices come from k -th term $a_{i_1} + a_{i_2} + \dots + a_{i_n}$ of the spectrum $\wedge^n a$, and the operator ∂_v acts on variables x_i , cf. section 3.5.

4.2.3 *Example.* For system $\wedge^2\mathcal{H}_4$ the marginal constraints on $\nu = \text{Spec } \rho$ and $\lambda = \text{Spec } \rho^{(1)}$ are given by inequalities

$$\begin{aligned}
2\lambda_1 &\leq \nu_1 + \nu_2 + \nu_3 \\
2\lambda_4 &\geq \nu_4 + \nu_5 + \nu_6 \\
2(\lambda_1 - \lambda_4) &\leq \nu_1 + \nu_2 - \nu_5 - \nu_6 \\
\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 &\leq \nu_1 - \nu_6 \\
\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 &\leq \min(\nu_1 - \nu_5, \nu_2 - \nu_6) \\
|\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4| &\leq \min(\nu_1 - \nu_4, \nu_2 - \nu_5, \nu_3 - \nu_6) \\
2 \max(\lambda_1 - \lambda_3, \lambda_2 - \lambda_4) &\leq \min(\nu_1 + \nu_3 - \nu_5 - \nu_6, \nu_1 + \nu_2 - \nu_4 - \nu_6) \\
2 \max(\lambda_1 - \lambda_2, \lambda_3 - \lambda_4) &\leq \min(\nu_1 + \nu_3 - \nu_4 - \nu_6, \nu_2 + \nu_3 - \nu_5 - \nu_6, \nu_1 + \nu_2 - \nu_4 - \nu_5).
\end{aligned} \tag{4.4}$$

For reasons that will become apparent in remark 4.2.5, here we keep the standard normalization $\text{Tr } \rho = \text{Tr } \rho^{(1)}$.

4.2.4 *Example.* Similar compatibility conditions for system $\wedge^2\mathcal{H}_5$ contain 522 independent inequalities which are too numerous to be reproduced here. They can be obtained from the cite <http://www.fen.bilkent.edu.tr/~murata/FermIneq5x2.pdf>.

4.2.5 *Remark.* As we've yet mentioned in the Introduction Ricci curvature operator $\text{Ric} : T \rightarrow T$ is the contraction of Riemann curvature $R : \wedge^2 T \rightarrow \wedge^2 T$. Hence inequalities (4.4) impose constraints on spectra of Riemann and Ricci curvatures of a Riemann four-manifold.

Recall that in general relativity Ricci curvature is governed by energy-momentum tensor, i.e. by physical content of the space, while Riemann curvature is responsible for its geometry and topology. The above constraints impose some bounds on the influence of matter on geometry.

4.3. Pure N -representability in dimension ≤ 8

Here I'll give an account of joint work with Murat Altunbulak [1]. The details will be published elsewhere.

Formally, the solution of pure marginal problem can be deduced from inequalities (4.3) of Theorem 4.2.1 by putting $\nu_i = 0$ for $i \neq 1$. However for a system like $\wedge^4\mathcal{H}_8$ we are confronted with an immense symmetric group of degree $\binom{8}{4} = 70$. Besides, listing the extremal edges for a system of this size is all but impossible. A way out of this is provided by a representation-theoretical interpretation of N -representability.

Let's start with decomposition of a symmetric power of $\wedge^n\mathcal{H}$, called *plethysm*, into irreducible components

$$S^m(\wedge^n\mathcal{H}) = \sum_{\lambda} m_{\lambda} \mathcal{H}_{\lambda} \tag{4.5}$$

of the unitary group $U(\mathcal{H})$. The components \mathcal{H}_{λ} , entering into the decomposition with some multiplicities $m_{\lambda} \geq 0$, are parameterized by *Young diagrams*

$$\lambda : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$$

of size $|\lambda| = \sum_i \lambda_i = n \cdot m$ that fit into $r \times m$ rectangular, $r = \dim \mathcal{H}$. It is instructive to treat the diagrams as *spectra*. We are interested in asymptotic of these spectra as $m \rightarrow \infty$ and therefor normalize them to a fixed size $\tilde{\lambda} = \lambda/m$, $\text{Tr } \tilde{\lambda} = n$.

Theorem 4.3.1. *Every $\tilde{\lambda}$ obtained from irreducible component $\mathcal{H}_{\lambda} \subset S^d(\wedge^n\mathcal{H})$ is spectrum of reduced matrix $\rho^{(1)}$ of a pure state $\psi \in \wedge^n\mathcal{H}$. Moreover every one point reduced spectrum is a convex combination of such $\tilde{\lambda}$ with bounded $m \leq M$.*

A similar result holds in standard settings of the quantum marginal problem [14, 7, 20, 8]. Note that the representation $S^m(\wedge^{r-n}\mathcal{H})$ is dual to $S^m(\wedge^n\mathcal{H})$ and hence

$$S^m(\wedge^{r-n}\mathcal{H}) = \sum_{\lambda} m_{\lambda} \mathcal{H}_{\lambda^*}, \quad (4.6)$$

where λ^* is complement of the diagram $\lambda \subset r \times m$ to the rectangle $r \times m$, and the multiplicity m_{λ} is the same as in (4.5). Thus we arrived at the following *particle-hole duality*.

Corollary 4.3.2. *Marginal constraints on spectrum of one point reduced matrix of a pure state for system $\wedge^{r-n}\mathcal{H}_r$ can be obtained from that of the system $\wedge^n\mathcal{H}_r$ by substitution $\lambda_i \mapsto 1 - \lambda_{r+1-i}$.*

4.3.3 Example. There are few cases where decomposition (4.5) is explicitly known, for example

$$S^m(\wedge^2\mathcal{H}_r) = \sum_{|\lambda|=2m, \lambda=\text{even}} \mathcal{H}_{\lambda},$$

where the sum is extended over diagrams $\lambda \subset r \times m$ with even multiplicity of every nonzero row [21, 23]. Together with theorem 4.3.1 and the particle-hole duality this implies Coleman's criteria of pure N-representability for systems of two particles $\wedge^2\mathcal{H}_r$ and two holes $\wedge^{r-2}\mathcal{H}_r$ mentioned at the beginning of section 4.2.

4.3.4 Example. Borland-Dennis equations (4.1) mean that every component $\mathcal{H}_{\lambda} \subset S^m(\wedge^3\mathcal{H}_6)$ is selfdual $\lambda = \lambda^*$. It seems mathematicians missed this fact, which holds only for this specific system. Observe that wedge product ensure selfduality of $\wedge^3\mathcal{H}_6$ and hence of the plethysm $S^m(\wedge^3\mathcal{H}_6)$. However apparently there is no simple way to extend this to every component $\mathcal{H}_{\lambda} \subset S^m(\wedge^3\mathcal{H}_6)$.

Theorem 4.3.1 for any fixed M gives an inner approximation to the set of all possible reduced spectra, while any set of inequalities (4.3) of theorem 4.2.1 amounts to its outer approximation. This suggests the following approach to pure N -representability problem, which combines both theorems.

- Find all irreducible components $\mathcal{H}_{\lambda} \subset S^m(\wedge^n\mathcal{H})$ for $m \leq M$ starting with $M = 1$.
- Calculate convex hull of the corresponding reduced spectra $\tilde{\lambda}$.
- Check whether or not all inequality defining facets of the convex hull fit into the form (4.3) of Theorem 4.2.1.
- If they do then all inequalities are found. Otherwise increase $M \mapsto M + 1$.

4.3.5 Remark. The success of this approach depends on the degrees of generators of the module of covariants of the system $\wedge^n\mathcal{H}_r$. Generically the degrees are expected to be huge as well as the whole number of the resulting inequalities. However for systems of rank $r \leq 8$ and for $r = 9$, $n \neq 4, 5$ the module of covariants is free [28] and the degrees of the generators should be reasonably small.

Indeed an inexpensive PC, assisted with some dirty tricks, managed to resolve N -representability problem for rank $r \leq 8$. Recall that for two fermions or two holes the answer is known, see section 4.2 and example 4.3.3. Together with the particle-hole duality this bounds us to the range $3 \leq n \leq r/2$. The corresponding constraints are listed below. They are grouped by the extremal edges and use the chemical normalization $\sum_i \lambda_i = n$ for the reduced spectrum.

- $\wedge^3\mathcal{H}_6$.

$$\lambda_1 + \lambda_6 = \lambda_2 + \lambda_5 = \lambda_3 + \lambda_4 = 1, \quad \lambda_4 \leq \lambda_5 + \lambda_6$$

- $\wedge^3 \mathcal{H}_7$.

$$\begin{aligned}
-4\lambda_1 + 3\lambda_2 + 3\lambda_3 + 3\lambda_4 + 3\lambda_5 - 4\lambda_6 - 4\lambda_7 &\leq 2 \\
3\lambda_1 - 4\lambda_2 + 3\lambda_3 + 3\lambda_4 - 4\lambda_5 + 3\lambda_6 - 4\lambda_7 &\leq 2 \\
3\lambda_1 + 3\lambda_2 - 4\lambda_3 - 4\lambda_4 - 3\lambda_5 + 3\lambda_6 - 4\lambda_7 &\leq 2 \\
3\lambda_1 + 3\lambda_2 - 4\lambda_3 + 3\lambda_4 - 4\lambda_5 - 4\lambda_6 + 3\lambda_7 &\leq 2
\end{aligned}$$

- $\wedge^3 \mathcal{H}_8$.

$$3\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 + 3\lambda_8 \leq 1$$

$$\begin{aligned}
-\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 &\leq 1 \\
\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 + \lambda_5 + \lambda_6 - \lambda_7 - \lambda_8 &\leq 1 \\
\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 + \lambda_7 - \lambda_8 &\leq 1 \\
\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 + \lambda_6 - \lambda_7 - \lambda_8 &\leq 1
\end{aligned}$$

$$\begin{aligned}
2\lambda_1 + \lambda_2 - 2\lambda_3 - \lambda_4 - \lambda_6 + \lambda_8 &\leq 1 \\
2\lambda_1 - \lambda_2 - \lambda_4 + \lambda_6 - 2\lambda_7 + \lambda_8 &\leq 1 \\
\lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_6 - \lambda_7 + \lambda_8 &\leq 1 \\
\lambda_1 + 2\lambda_2 - 2\lambda_3 - \lambda_5 - \lambda_6 + \lambda_8 &\leq 1 \\
2\lambda_1 - \lambda_2 + \lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_8 &\leq 1
\end{aligned}$$

$$\begin{aligned}
5\lambda_1 + 5\lambda_2 - 7\lambda_3 - 3\lambda_4 - 3\lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 &\leq 3 \\
5\lambda_1 - 3\lambda_2 - 3\lambda_3 + \lambda_4 + \lambda_5 + 5\lambda_6 - 7\lambda_7 + \lambda_8 &\leq 3
\end{aligned}$$

$$\begin{aligned}
5\lambda_1 + \lambda_2 - 3\lambda_3 + \lambda_4 - 3\lambda_5 + \lambda_6 - 3\lambda_7 + \lambda_8 &\leq 3 \\
\lambda_1 + \lambda_2 + \lambda_3 + 5\lambda_4 - 3\lambda_5 - 3\lambda_6 - 3\lambda_7 + \lambda_8 &\leq 3 \\
\lambda_1 + 5\lambda_2 - 3\lambda_3 + \lambda_4 + \lambda_5 - 3\lambda_6 - 3\lambda_7 + \lambda_8 &\leq 3
\end{aligned}$$

$$\begin{aligned}
9\lambda_1 + \lambda_2 - 7\lambda_3 - 7\lambda_4 - 7\lambda_5 + \lambda_6 + \lambda_7 + 9\lambda_8 &\leq 3 \\
9\lambda_1 - 7\lambda_2 - 7\lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 - 7\lambda_7 + 9\lambda_8 &\leq 3
\end{aligned}$$

$$\begin{aligned}
7\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 + 7\lambda_6 - 9\lambda_7 - \lambda_8 &\leq 5 \\
7\lambda_1 - \lambda_2 - \lambda_3 + 7\lambda_4 - 9\lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 &\leq 5 \\
7\lambda_1 + 7\lambda_2 - 9\lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 &\leq 5 \\
-\lambda_1 - \lambda_2 + 7\lambda_3 + 7\lambda_4 - \lambda_5 - \lambda_6 - 9\lambda_7 - \lambda_8 &\leq 5 \\
-\lambda_1 + 7\lambda_2 - \lambda_3 + 7\lambda_4 - \lambda_5 - 9\lambda_6 - \lambda_7 - \lambda_8 &\leq 5 \\
-\lambda_1 + 7\lambda_2 - \lambda_3 - \lambda_4 + 7\lambda_5 - \lambda_6 - 9\lambda_7 - \lambda_8 &\leq 5
\end{aligned}$$

$$\begin{aligned}
-3\lambda_1 + 5\lambda_2 + 5\lambda_3 + 13\lambda_4 - 11\lambda_5 - 3\lambda_6 - 11\lambda_7 + 5\lambda_8 &\leq 7 \\
5\lambda_1 + 13\lambda_2 - 11\lambda_3 + 5\lambda_4 - 11\lambda_5 - 3\lambda_6 - 3\lambda_7 + 5\lambda_8 &\leq 7 \\
5\lambda_1 - 3\lambda_2 + 5\lambda_3 + 13\lambda_4 - 11\lambda_5 - 11\lambda_6 - 3\lambda_7 + 5\lambda_8 &\leq 7 \\
5\lambda_1 + 13\lambda_2 - 11\lambda_3 - 3\lambda_4 + 5\lambda_5 - 11\lambda_6 - 3\lambda_7 + 5\lambda_8 &\leq 7
\end{aligned}$$

$$\begin{aligned}
19\lambda_1 + 11\lambda_2 - 21\lambda_3 - 13\lambda_4 - 5\lambda_5 - 5\lambda_6 + 3\lambda_7 + 11\lambda_8 &\leq 9 \\
19\lambda_1 - 13\lambda_2 - 5\lambda_3 - 5\lambda_4 + 3\lambda_5 + 11\lambda_6 - 21\lambda_7 + 11\lambda_8 &\leq 9 \\
11\lambda_1 + 19\lambda_2 - 21\lambda_3 - 5\lambda_4 - 13\lambda_5 - 5\lambda_6 + 3\lambda_7 + 11\lambda_8 &\leq 9 \\
-5\lambda_1 + 3\lambda_2 + 11\lambda_3 + 19\lambda_4 - 21\lambda_5 - 13\lambda_6 - 5\lambda_7 + 11\lambda_8 &\leq 9
\end{aligned}$$

• $\wedge^4\mathcal{H}_8$.

$$\begin{aligned}
5\lambda_1 + \lambda_2 + \lambda_3 - 3\lambda_4 + \lambda_5 - 3\lambda_6 - 3\lambda_7 + \lambda_8 &\leq 4 \\
\lambda_1 + \lambda_2 + 5\lambda_3 - 3\lambda_4 + \lambda_5 + \lambda_6 - 3\lambda_7 - 3\lambda_8 &\leq 4 \\
\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 5\lambda_5 - 3\lambda_6 - 3\lambda_7 - 3\lambda_8 &\leq 4 \\
\lambda_1 + 5\lambda_2 + \lambda_3 - 3\lambda_4 + \lambda_5 - 3\lambda_6 + \lambda_7 - 3\lambda_8 &\leq 4 \\
5\lambda_1 - 3\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 - 3\lambda_7 - 3\lambda_8 &\leq 4 \\
5\lambda_1 + \lambda_2 + \lambda_3 - 3\lambda_4 - 3\lambda_5 + \lambda_6 + \lambda_7 - 3\lambda_8 &\leq 4 \\
5\lambda_1 + \lambda_2 - 3\lambda_3 + \lambda_4 + \lambda_5 - 3\lambda_6 + \lambda_7 - 3\lambda_8 &\leq 4 \\
\\
-\lambda_1 + 3\lambda_2 + 3\lambda_3 - \lambda_4 + 3\lambda_5 - \lambda_6 - \lambda_7 - 5\lambda_8 &\leq 4 \\
3\lambda_1 + 3\lambda_2 - \lambda_3 - \lambda_4 + 3\lambda_5 - 5\lambda_6 - \lambda_7 - \lambda_8 &\leq 4 \\
3\lambda_1 + 3\lambda_2 + 3\lambda_3 - 5\lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 &\leq 4 \\
3\lambda_1 - \lambda_2 + 3\lambda_3 - \lambda_4 + 3\lambda_5 - \lambda_6 - 5\lambda_7 - \lambda_8 &\leq 4 \\
3\lambda_1 + 3\lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 + 3\lambda_7 - 5\lambda_8 &\leq 4 \\
3\lambda_1 - \lambda_2 - \lambda_3 + 3\lambda_4 + 3\lambda_5 - \lambda_6 - \lambda_7 - 5\lambda_8 &\leq 4 \\
3\lambda_1 - \lambda_2 + 3\lambda_3 - \lambda_4 - \lambda_5 + 3\lambda_6 - \lambda_7 - 5\lambda_8 &\leq 4
\end{aligned}$$

4.3.6 Remark. The marginal inequalities are independent and written in the form (4.3) of theorem 4.2.1. Using the normalization equation $\text{Tr } \rho = n$ they can be transformed in many different ways. For example, the above constraints for system $\wedge^3\mathcal{H}_7$ are equivalent to inequalities (4.2). The inequalities for $\wedge^4\mathcal{H}_8$ can be recast into a nice form found experimentally by Borland and Dennis [5]

$$|x_1| + |x_2| + |x_3| + |x_4| + |x_5| + |x_6| + |x_7| \leq 4, \quad (4.7)$$

where

$$\begin{aligned}
x_1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 \\
x_2 &= \lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 - \lambda_3 - \lambda_4 - \lambda_7 - \lambda_8 \\
x_3 &= \lambda_1 + \lambda_3 + \lambda_5 + \lambda_7 - \lambda_2 - \lambda_4 - \lambda_6 - \lambda_8 \\
x_4 &= \lambda_1 + \lambda_4 + \lambda_6 + \lambda_7 - \lambda_2 - \lambda_3 - \lambda_5 - \lambda_8 \\
x_5 &= \lambda_2 + \lambda_3 + \lambda_6 + \lambda_7 - \lambda_1 - \lambda_4 - \lambda_5 - \lambda_8 \\
x_6 &= \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 - \lambda_1 - \lambda_3 - \lambda_6 - \lambda_8 \\
x_7 &= \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 - \lambda_1 - \lambda_2 - \lambda_7 - \lambda_8
\end{aligned}$$

Borland and Dennis numerical data were inconclusive for the system $\wedge^3\mathcal{H}_8$ described by 31 inequalities. One may wonder whether they can be written in a compact form like (4.7).

5. Conclusion

A recent progress drastically improves our understanding of relations between state of a composite quantum system and reduced states of the components. This is especially true for an array of qubits where the constraints are given by an *explicit* system of linear inequalities.

A longstanding problem of one point N -representability has been resolved. Explicit criteria of N -representability found for systems of rank ≤ 8 after more than 30 years of stagnation.

New connections of the quantum marginal problem with flag varieties, representations of the symmetric group, and Riemann geometry are established.

On the other hand the quantum marginal problem with overlapping margins is still obscure and intractable, as well as two-point N -representability. Even for the theoretically resolved problems computational difficulties may be formidable.

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