RESILIENT PI AND PD CONTROLLERS FOR A CLASS OF UNSTABLE MIMO PLANTS WITH I/O DELAYS [⋆]

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Abstract: Recently (Gündeş et al., 2006) obtained stabilizing PID controllers for a class of MIMO unstable plants with time delays in the input and output channels (I/O delays). Using this approach, for plants with one unstable pole, we investigate resilient PI and PD controllers. Specifically, for PD controllers, optimal derivative action gain is determined to maximize a lower bound of the largest allowable controller gain. For PI controllers, optimal proportional gain is determined to maximize a lower bound of the largest allowable integral action gain. *Copyright © 2006 IFAC*

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1. INTRODUCTION

PID controllers are still very popular in many control applications thanks to their simple structure, (Astrom and Hagglund, 1995; Goodwin et al., 2001). Design of PID controllers for delay systems is still an active research area, see for example the recent book (Silva et al., 2005), and its references. In this paper we consider unstable MIMO plants with time delays. It is clear that, even for delay-free systems, not all unstable plants are stabilizable by a PID controller (strong stabilizability is a necessary condition for stabilization by a PID controller, and there are bounds on the order of strongly stabilizing controllers, (Gündeş et al., 2006; Smith and Sondergeld, 1986; Vidyasagar, 1985)). Moreover, right half plane poles and zeros in the plant transfer matrix, as well as time delays in the input and/or output channels (I/O delays) of the plant, impose additional restrictions on the feedback controllers, see e.g. (Gu et al., 2003; Gümüşsoy and Özbay, 2005; Niculescu, 2001; Stein, 1989; Zeren and Özbay, 2000).

Recently, PID controllers are designed in (Yaniv and Nagurka, 2004) under specified gain margin and sensitivity constraints, and in (Saeki, 2006) under an $\mathcal{H}_\infty$ performance condition. PID controller tuning rules are also discussed in (Kristiansson and Lennartson, 2002; Skogestad, 2003) under different optimality conditions. For SISO unstable systems with delays PID controller tuning has been studied in (Lee et al., 2000; Poulin and Pomerleau, 1999). An extension of predictive control is used in (Fliess et al., 2002) to derive PID controllers for a class of MIMO unstable plants with delays.

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In a recent work (Gündeş et al., 2006) obtained PID controllers from a small gain argument for a class of MIMO unstable plants with delays in the input and output channels (I/O delays). In this paper we use the results of (Gündeş et al., 2006) for plants with one unstable pole, and investigate stabilizing PI and PD controllers with the largest allowable interval for the controller gain. This is an important problem to study, because sensitivity of the closed loop stability to perturbations in the controller coefficients can be minimized this way, and hence resilient PI and PD controllers (see e.g. (Silva et al., 2005) and its references for a discussion of this issue) can be obtained. There are many important practical examples of plants with single unstable pole and time delays, see e.g. (Enns et al., 1992; Lee et al., 2000; Poulin and Pomerleau, 1999; Silva et al., 2005; Stein, 1989) and their references.

Remaining parts of the paper are organized as follows. Preliminary results from (Gündeş et al., 2006) are summarized in Section 2. Main results on PD controller design are given in Section 3, and the results on PI controller are given in Section 4; concluding remarks are made in Section 5.

2. PROBLEM DEFINITION AND PRELIMINARY RESULTS

Consider the linear time invariant (LTI) feedback system shown in Figure 1, where C is the controller to be designed and $G_\Lambda := \Lambda_o G \Lambda_i$ is the plant with r inputs and r outputs. Here G is the delay free part of the system which is assumed to be finite dimensional. Time delays in the input and output channels of the plant are represented by their transfer matrices as $\Lambda_\bullet = \text{diag}\left[e^{-sh_1^\bullet}, \ldots, e^{-sh_r^\bullet}\right]$, where, $h_j^\bullet$ is the j^{th} channel input (when $\bullet = i$) or output (when $\bullet = o$) delay, for $1 \le j \le r$.

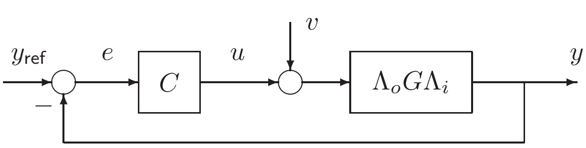

Fig. 1. Unity-Feedback System $Sys(G_\Lambda, C)$ with I/O delays in the plant.

The closed-loop transfer matrix H_{cl} from (y_{ref}, v) to (u, y) is

$$H_{cl} = \begin{bmatrix} C(I+G_\Lambda C)^{-1} & -C(I+G_\Lambda C)^{-1}G_\Lambda \\ G_\Lambda C(I+G_\Lambda C)^{-1} & (I+G_\Lambda C)^{-1}G_\Lambda \end{bmatrix}. \quad (1)$$

In this paper we consider the proper form of PID controllers, (Goodwin et al., 2001),

$$C(s) = C_{pid}(s) = K_p + \frac{K_i}{s} + \frac{K_d\ s}{\tau_d s + 1}\ , \quad (2)$$

where K_p, K_i, K_d are real matrices and $\tau_d > 0$. But we restrict ourselves to PI and PD controllers: $C_{pi} = K_p + \frac{K_i}{s}$ and $C_{pd} = K_p + \frac{K_d\ s}{\tau_d s+1}$ respectively.

Definition. The feedback system $Sys(G_\Lambda, C)$ is stable if all entries of H_{cl} are in $\mathcal{H}_\infty$. We define $\mathcal{S}_{pid}, \mathcal{S}_{pi}, \mathcal{S}_{pd}$ to be the sets of all PID, PI and PD (respectively) controllers stabilizing the feedback system $Sys(G_\Lambda, C)$.

Assumptions.

A1) Finite dimensional part of the plant, G, admits a coprime factorization in the form $G(s) = Y(s)^{-1}X(s) = X(s)Y(s)^{-1}$ where $X \in \mathcal{H}_\infty^{r\times r}$, and $Y(s) = \frac{(s-p)}{(as+1)}I$. Here $p \ge 0$ is the only unstable pole of the plant, and $a > 0$ is arbitrary.

A2) $X(0) = (s-p)G(s)|_{s=0}$ is nonsingular.

Proposition 1. (Gündeş et al., 2006) Consider the plant G_Λ satisfying **A1)** and **A2)**.
i) PD-design: Choose any $\hat{K}_d \in \mathbb{R}^{r\times r}$, $\tau_d > 0$. Define $\widehat{C}_{pd} := X(0)^{-1} + \frac{\hat{K}_d s}{\tau_d s+1}$ and

$$\Phi_\Lambda := s^{-1}\left((s-p)G_\Lambda(s)\widehat{C}_{pd}(s) - I\right)$$

$$\widetilde{\Phi}_\Lambda := s^{-1}\left(\widehat{C}_{pd}(s)(s-p)G_\Lambda(s) - I\right).$$

If $0 \le p < \max\{\|\Phi_\Lambda\|_\infty^{-1}, \|\widetilde{\Phi}_\Lambda\|_\infty^{-1}\}$, then for any positive $\alpha \in \mathbb{R}$ satisfying

$$0 < \ \alpha\ < \max\{\|\Phi_\Lambda\|_\infty^{-1} - p\ ,\ \|\widetilde{\Phi}_\Lambda\|_\infty^{-1} - p\ \}\ , \quad (3)$$

$C_{pd}(s) = (\alpha + p)\widehat{C}_{pd}(s)$ is in $\mathcal{S}_{pd}$.

ii) PID-design: Let C_{pd} be as above, and define $H_{pd} := G_\Lambda(I + C_{pd}G_\Lambda)^{-1}$, $\Upsilon := \frac{H_{pd}(s)H_{pd}(0)^{-1}-I}{s}$, $\widetilde{\Upsilon} := \frac{H_{pd}(0)^{-1}H_{pd}(s)-I}{s}$. Then, for any $\gamma \in \mathbb{R}$ satisfying

$$0 < \gamma < \ \max\{\|\Upsilon\|_\infty^{-1}\ \ ,\ \ \|\widetilde{\Upsilon}\|_\infty^{-1}\}, \quad (4)$$

the PID-controller (5) is in $\mathcal{S}_{pid}$,

$$C_{pid}(s) = C_{pd}(s) + \frac{\gamma\alpha X(0)^{-1}}{s}\ . \quad (5)$$

If (3) and (4) are satisfied for $\hat{K}_d = 0$ then (5) with $\hat{K}_d = 0$ is a PI controller in $\mathcal{S}_{pi}$. □

This result appears in (Gündeş et al., 2006) for systems with possibly uncertain time delays, but for our purposes fixed time delays version stated above is sufficient. Now consider the input delays and output delays separately, with a structural assumption.

Assumption A3.i). $G_\Lambda(s) = G(s)\Lambda_i(s)$, with $G(s) = \frac{1}{s-p}G_0\Lambda_G(s)$ where G_0 is a non-singular constant matrix and $\Lambda_G(s)$ is a stable diagonal matrix with $\Lambda_G(0) = I$, i.e., $\Lambda_G(s) = \text{diag}[g_1(s), \ldots, g_r(s)]$, where $g_1(s), \ldots, g_r(s)$ are stable proper transfer functions with $g_j(0) = 1$, for all $j = 1, \ldots, r$. $\square$

Assumption A3.o). $G_\Lambda(s) = \Lambda_o(s)G(s)$ with $G(s) = \frac{1}{s-p}\Lambda_G(s)G_0$ where G_0 and $\Lambda_G(s)$ are as in **A3.i**. $\square$

Note that with **A3.i** and **A3.o** we have $X(0) = G_0$ and earlier assumptions **A1** and **A2** are satisfied. Moreover, these assumptions result in a diagonal structure in the sensitivity matrices, as demonstrated below. An example for **A3.i** is the transfer matrix of a distillation column with input channel delays, (Friedland, 1986), $G_\Lambda(s) = \frac{1}{s}\, G_0\, \Lambda_G(s)\Lambda_i(s)$, where $G_0 = \begin{bmatrix} 3.04 & -278.2/180 \\ 0.052 & 206.6/180 \end{bmatrix}$, $\Lambda_G(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{180}{(s+6)(s+30)} \end{bmatrix}$.

2.1 PD Control of Systems With Input Delays

Let us now assume that **A3.i** holds, and define $\hat{K}_d = \tilde{K}_d^i X(0)^{-1} = \tilde{K}_d^i G_0^{-1}$. Then, the PD controller of Proposition 1 can be re-written as $C_{pd}(s) = (\alpha + p)\left(I + \tilde{K}_d^i \frac{s}{\tau_d s+1}\right) G_0^{-1}$. Then choosing $\tilde{K}_d^i$ diagonal we have a diagonal input sensitivity matrix $S_i(s) = (I + L_i(s))^{-1}$, where $L_i(s) = \frac{(\alpha+p)}{(s-p)}\left(I + \tilde{K}_d^i \frac{s}{\tau_d s+1}\right)\Lambda_G(s)\Lambda_i(s)$.

Proposition 1 gives a lower bound on the largest controller gain interval: $p < (\alpha + p) < \|\tilde{\Phi}_\Lambda\|_\infty^{-1}$. For the purpose of designing a resilient controller, we would like to maximize the size of the gain interval. This is equivalent to minimizing

$$\mu^i := \|\tilde{\Phi}_\Lambda\|_\infty = \|\frac{\Lambda_{Fi}(s) - I}{s} + \tilde{K}_d^i \frac{\Lambda_{Fi}(s)}{\tau_d s + 1}\|_\infty \quad (6)$$

where $\Lambda_{Fi} := \Lambda_G\Lambda_i$. Therefore in the rest of the paper we will study the problem of minimizing μ^i over the free parameter $\tilde{K}_d^i$. Note that with **A3.i**, $\tilde{\Phi}_\Lambda$ is diagonal whenever $\tilde{K}_d^i := \mathrm{diag}[q_1^i, \ldots, q_r^i]$. Now let $f_j^i(s) := g_j(s)e^{-h_j^i s}$. Then, maximizing the allowable interval for the controller gain $(\alpha+p)$ reduces to the problem of minimizing μ^i over the free parameters $q_1^i, \ldots, q_r^i$, where

$$\mu^i = \max_j \|\frac{f_j^i(s) - 1}{s} + q_j^i\, \frac{f_j^i(s)}{\tau_d s + 1}\|_\infty. \quad (7)$$

2.2 PD Control of Systems With Output Delays

In this section we assume that **A3.o** holds, and define $\hat{K}_d = X(0)^{-1}\tilde{K}_d^o = G_0^{-1}\tilde{K}_d^o$. In this case the PD controller of Proposition 1 is $C_{pd}(s) = (\alpha + p)G_0^{-1}\left(I + \tilde{K}_d^o \frac{s}{\tau_d s+1}\right)$. As before, choosing $\tilde{K}_d^o$ diagonal we have diagonal output sensitivity matrix $S_o(s) = (I + L_o(s))^{-1}$, where $L_o(s) = \frac{(\alpha+p)}{(s-p)}\Lambda_o(s)\Lambda_G(s)\left(I + \tilde{K}_d^o \frac{s}{\tau_d s+1}\right)$.

Proposition 1 gives a lower bound on the largest controller gain interval: $p < (\alpha + p) < \|\Phi_\Lambda\|_\infty^{-1}$. In this case we would like to minimize

$$\mu^o := \|\Phi_\Lambda\|_\infty = \|\frac{\Lambda_{Fo}(s) - I}{s} + \frac{\Lambda_{Fo}(s)\tilde{K}_d^o s}{\tau_d s + 1}\|_\infty (8)$$

where $\Lambda_{Fo} = \Lambda_o\Lambda_G$. As before, we consider $\tilde{K}_d^o = \mathrm{diag}[q_1^o, \ldots, q_r^o]$. Let $f_j^o(s) := g_j(s)e^{-h_j^o s}$. Then the dual problem in the output delay case is to minimize μ^o over the free parameters $q_1^o, \ldots, q_r^o$

$$\mu^o = \max_j \|\frac{f_j^o(s) - 1}{s} + q_j^o\, \frac{f_j^o(s)}{\tau_d s + 1}\|_\infty. \quad (9)$$

2.3 PD Control of Systems With I/O Delays

When we combine input and output delays, the problem at hand cannot be reduced to a set of decoupled scalar optimization problems, unless we introduce "equalizing time delays" in the controller itself. In order to illustrate this point let us examine $\|\tilde{\Phi}_\Lambda\|_\infty = \|\frac{F(s)-I}{s} + \frac{\tilde{K}_d^i\, F(s)}{\tau_d s+1}\|_\infty$, where $F(s) = G_X(0)^{-1}G_X(s)$, and $G_X(s) := (s - p)G_\Lambda(s)$. Even under a structural assumption of the form $G_X = \Lambda_o G_0 \Lambda_G \Lambda_i$, clearly, the function $\tilde{\Phi}_\Lambda$ is not necessarily diagonal, unless $\Lambda_o = I$, or controller has input delays equalizing the time delays in every channel of Λ_o, as illustrated below. Similarly, Φ_Λ is not necessarily diagonal unless $\Lambda_i = I$, or controller has output delays equalizing all the delays in Λ_i. Define $h^o := \max\{h_1^o, \ldots, h_r^o\}$, and $h^i := \max\{h_1^i, \ldots, h_r^i\}$. Now consider the plant $G_\Lambda(s) = \frac{1}{s-p}\Lambda_o(s)G_0\Lambda_G(s)\Lambda_i(s)$ with the controller $C_{pd-eo}(s) = (\alpha+p)\left(I + \frac{\tilde{K}_d^i s}{\tau_d s+1}\right) G_0^{-1}\Lambda_{eo}(s)$ where $\Lambda_{eo}(s) := e^{-h^o s}\Lambda_o^{-1}(s)$. Input channel delay matrix for the controller, Λ_{eo}, is equalizing output delays of the plant. In this case input sensitivity matrix is diagonal as in Section 2.1, and maximizing allowable $(\alpha + p)$ is equivalent to the problem (7) with $f_j^i(s) = g_j(s)e^{-(h_j^i + h^o)s}$.

Similarly, for a plant whose structure is $G(s) = \frac{1}{s-p}\Lambda_o(s)\Lambda_G(s)G_0\Lambda_i(s)$ we can delay the outputs of the controller to equalize the delays in the input channel of the plant: $C_{pd-ei}(s) = (\alpha + p)\Lambda_{ei}(s)G_0^{-1}\left(I + \frac{\tilde{K}_d^o s}{\tau_d s+1}\right)$ where $\Lambda_{ei}(s) := e^{-h^i s}\Lambda_i^{-1}(s)$. In this case Φ_Λ is diagonal and maximizing allowable $(\alpha + p)$ is equivalent to the problem (9) with $f_j^o(s) = g_j(s)e^{-(h_j^o + h^i)s}$.

2.4 PI Control of Systems With Input or Output Delays

Now consider PI controllers with the proportional part $C_p = (\alpha + p)X(0)^{-1}$, where α satisfies (3). The PI controller is then in the form

$$C_{pi}(s) = (\alpha + p)X(0)^{-1} + \frac{\gamma\alpha}{s}X(0)^{-1} \quad (10)$$

where γ satisfies (4). Recall that, under the structural assumption **A3.i**, or **A3.o**, we have $X(0) = G_0$. An interesting problem in this case is to find the largest allowable interval for γ, for a fixed α satisfying (3).

Note that in this case $H_{pd}(s) = H_p(s) = G_\Lambda(I + C_p G_\Lambda)^{-1} = (I + G_\Lambda C_p)^{-1} G_\Lambda$. As in the above discussion on PD controller design we will assume that **A3.i** holds and α is in the interval $0 < \alpha < \|\tilde{\Phi}_\Lambda\|_\infty^{-1} - p$. In this case, since the derivative term is absent, we have $\tilde{\Phi}_\Lambda = \frac{\Lambda(s)-I}{s}$, where $\Lambda = \Lambda_G \Lambda_i$. Then a lower bound for the maximum interval for the allowable "integral action gain" γ is found from (4) where $\widetilde{\Upsilon} = \frac{\alpha\Lambda(s)((s-p)I+(\alpha+p)\Lambda(s))^{-1}-I}{s}$. It is easy to see that in the dual case, under **A3.o** and the added restriction $0 < \alpha < \|\Phi_\Lambda\|_\infty^{-1} - p$, we have $\Upsilon = \frac{\alpha\Lambda(s)((s-p)I+(\alpha+p)\Lambda(s))^{-1}-I}{s}$, where $\Lambda = \Lambda_o \Lambda_G$. Thus, it is interesting to study the upper bound $\gamma_{\max}$ for γ where

$$\gamma_{\max} := \| \frac{\frac{\alpha}{s-p}\Lambda(s)(I + \frac{\alpha+p}{s-p}\Lambda(s))^{-1} - I}{s} \|_\infty^{-1} \quad (11)$$

as a function of α satisfying

$$0 < \alpha < \| \frac{\Lambda(s) - I}{s} \|_\infty^{-1} - p \quad (12)$$

where $\Lambda(s) = \Lambda_G(s)\Lambda_i(s)$ for the input delays case and $\Lambda(s) = \Lambda_o(s)\Lambda_G(s)$ for the output delays case.

3. OPTIMAL DERIVATIVE ACTION GAIN FOR RESILIENT PD CONTROL

Recall from Sections 2.1, 2.2, 2.3 that the optimal designs of the derivative gains (for maximizing a lower bound of the allowable controller gain interval) are determined from a problem which is in the following general form. Given $h > 0$ and a stable transfer function $g(s)$ with $g(0) = 1$, let $f(s) = g(s)e^{-hs}$, and find $q \in \mathbb{R}$ such that μ is minimized, where

$$\mu = \| \frac{f(s) - 1}{s} + q \ \frac{f(s)}{\tau_d s + 1} \|_\infty \ , \quad \tau_d \to 0. \quad (13)$$

We shall denote the optimal solution by q_{opt}. This is a single parameter scalar function $\mathcal{H}_\infty$ norm minimization problem and it can be solved numerically using brute force search. More precisely, such an algorithm would perform the following steps:

0. Choose the candidate values of $q = q_1, \ldots, q_N$, over which the optimization is to be done, and the frequency values $\omega = \omega_1, \ldots, \omega_M$ over which the norm (cost function) is to be computed.

1. For $k = 1, \ldots, N$ and $\ell = 1, \ldots, M$ compute
$$\Psi(q_k, \omega_\ell) := | \frac{f(j\omega_\ell) - 1}{j\omega_\ell} + q_k \ \frac{f(j\omega_\ell)}{j\tau_d\omega_\ell + 1} | \ .$$

2. Define $\mu(q_k) := \max_{\omega_\ell} \Psi(q_k, \omega_\ell)$.

3. Optimal q is $q_{\text{opt}} = \arg\ \min_{q_k} \mu(q_k)$.

As an example, consider the distillation column transfer matrix given in Section 2, where $g_1(s) = 1$ and $g_2(s) = \frac{180}{(s+6)(s+30)}$. Optimal derivative gains are computed in (Gündeş et al., 2006) (see Figure 4 of (Gündeş et al., 2006)) using the numerical procedure given above. However, this procedure is sensitive to the number of grid points chosen for q and ω. So, it would be useful if one could derive a closed form expression for the solution, at least for the simplest case $g(s) = 1$. It turns out that this is possible, and we claim that for $f(s) = e^{-hs}$

$$q_{\text{opt}} = \frac{\sin(2.33)}{2.33} \ h = 0.31 \ h \ . \quad (14)$$

In the rest of this section we discuss how the optimal solution can be computed directly.

Note that (13) is a min-max problem

$$\mu = \min_{q\in\mathbb{R}} \ \max_{\omega\in\mathbb{R}} \ \Psi(q, \omega) \quad (15)$$

where $\Psi(q, \omega) = | \frac{f(j\omega)-1}{j\omega} + q \ \frac{f(j\omega)}{j\tau_d\omega+1} | \ , \ \tau_d \to 0$. Let us now consider the max-min problem where minimization over q is done for each fixed ω. In this case, it is easy to show that optimal q is

$$q_{\text{opt}}(\omega) = -\frac{1}{\omega} \ \frac{\sin(\phi(\omega))}{\rho(\omega)} \quad (16)$$

where $\rho(\omega) = |f(j\omega)|$ is the magnitude and $\phi(\omega) = \angle f(j\omega)$ is the phase of $f(j\omega)$. Inserting (16) into $\Psi(q, \omega)$ we obtain

$$\Psi(q_{\text{opt}}(\omega), \omega) = \left| \frac{\rho(\omega) - \cos(\phi(\omega))}{\omega} \right| =: \eta(\omega). \quad (17)$$

Therefore, solution of the max-min problem is

$$q_o = -\frac{1}{\omega_o} \ \frac{\sin(\phi(\omega_o))}{\rho(\omega_o)} \quad (18)$$

where ω_o is maximizing $\eta(\omega)$. Note that it is very easy to find q_o, we only need to find ω_o numerically. Whereas the algorithm for the min-max problem requires two dimensional search.

Example. Consider $f(s) = e^{-hs}$, $h > 0$. Then $\rho(\omega) = 1$ and $\phi(\omega) = -h\omega$. Hence $\eta(\omega) = \left| \frac{1-\cos(h\omega))}{\omega} \right|$. It is easy to show that the ω value maximizing this function is the solution of

$$\cos(h\omega) + (h\omega)\sin(h\omega) = 1 \ .$$

That gives $h\omega_o = 2.33$ rad., $q_o = 0.31\ h$, and it matches Figure 4 of (Gündeş et al., 2006). □

Now it remains to be shown that q_o given in (18) is equal to the solution q_{opt} of the original problem defined by (15), at least for a large class of functions $f(s)$, including the distillation column example. For this purpose, we need to show that the pair (ω_o, q_o) is a saddle point for the min-max problem (15), i.e. the following inequalities hold

$$\Psi(q_o, \omega) \leq \Psi(q_o, \omega_o) \leq \Psi(q, \omega_o) \quad \forall\, q, \omega \in \mathbb{R}\ . \tag{19}$$

First note that by the definition of $q_{\mathrm{opt}}(\omega)$ we have $\Psi(q_{\mathrm{opt}}(\omega), \omega) \leq \Psi(q, \omega)$ for all $q \in \mathbb{R}$ and $\omega \in \mathbb{R}$. In particular, setting $\omega = \omega_o$ in this inequality we obtain the second part of (19), namely

$$\Psi(q_o, \omega_o) \leq \Psi(q, \omega_o) \quad \forall\, q \in \mathbb{R}\ . \tag{20}$$

For the first inequality of (19) note that, under the assumption $\tau_d = 0$, we have

$$\Psi(q_o, \omega) = |\Psi(q_{\mathrm{opt}}(\omega), \omega) + \Delta_q(\omega)\ f(j\omega)|$$

where $\Delta_q(\omega) = q_o - q_{\mathrm{opt}}(\omega)$.

Claim. The following equality holds:

$$|\Psi(q_o, \omega)|^2 = |\eta(\omega)|^2 + |\Delta_q(\omega)|^2 |\rho(\omega)|^2. \tag{21}$$

Proof. Let us define $R(\omega) + jI(\omega) := \frac{f(j\omega)-1}{j\omega} + q_{\mathrm{opt}}(\omega)\ f(j\omega)$ to be the real and imaginary parts. Similarly, let $R_f(\omega) + jI_f(\omega) := f(j\omega)$ be the real and imaginary parts of f. With these definitions we have $R_f R + I_f I = 0$, which implies (21).

Assumption A4. The function $f(s)$ is such that

$$\Gamma(\omega) := \eta_o^2 - \eta^2(\omega) - |\Delta_q(\omega)|^2 \rho^2(\omega) \geq 0 \quad \forall\ \omega$$

where $\eta(\omega)$ is defined by (17), $\eta_o = \max_\omega \eta(\omega)$, and (16) and (18) define $\Delta_q(\omega) = q_o - q_{\mathrm{opt}}(\omega)$. □

Now with **A4**, (21) and $\eta_o = \Psi(q_o, \omega_o)$, we have

$$\Psi(q_o, \omega) \leq \Psi(q_o, \omega_o) \quad \forall\ \omega \in \mathbb{R}$$

which is the first part of (19). In summary we have the following result.

Proposition 2. Let $f(s) = g(s)e^{-hs}$, with $g \in \mathcal{H}_\infty$, $g(0) = 1$ and $h > 0$, satisfy **A4**. Then, optimal solution of

$$q_{\mathrm{opt}} := \arg \min_{q \in \mathbb{R}} \| \frac{f(s)-1}{s} + q\ \frac{f(s)}{\tau_d s + 1} \|_\infty \quad \tau_d \to 0$$

is given by $q_{\mathrm{opt}} = q_o = -\frac{1}{\omega_o} \frac{\sin(\phi(\omega_o))}{\rho(\omega_o)}$ where ω_o is maximizing $\eta(\omega) := \left| \frac{\rho(\omega) - \cos(\phi(\omega))}{\omega} \right|$. □

Example. Consider the first channel in the distillation column example, where $f(s) = e^{-hs}$, $h > 0$. Figure 2 shows Γ/h versus ω. Since $\Gamma(\omega) \geq 0$ for all ω, **A4** is satisfied, hence the formula $q_{\mathrm{opt}} = 0.31\ h$ is valid. Now for the second channel in the distillation column example, $f(s) = \frac{180}{(s+6)(s+30)} e^{-hs}$, Figure 3 illustrates that **A4** is satisfied. Figure 4 shows q_{opt} and μ versus h for this example. We observe that, as h increases μ increases, which means the allowable interval for the control gain shrinks with increasing h. Note that q_{opt} in Figure 4 is in perfect agreement with Figure 4 of (Gündeş et al., 2006). □

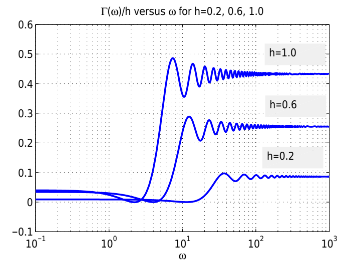

Fig. 2. $\Gamma(\omega)/h$ versus ω for $f(s) = e^{-hs}$.

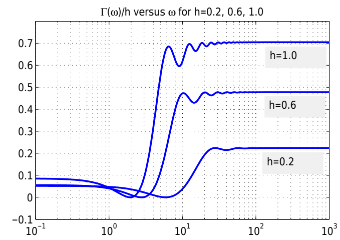

Fig. 3. $\Gamma(\omega)/h$ versus ω for $f(s) = \frac{e^{-hs}180}{(s+6)(s+30)}$.

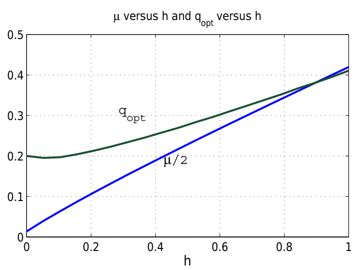

Fig. 4. q_{opt} and μ versus h.

An interesting problem arising in this context is to characterize the class of functions $f(s) = g(s)e^{-hs}$, $g \in \mathcal{H}_\infty$, $g(0) = 1$, $h > 0$, satisfying **A4**. At the moment we do not have a definite answer to this question. But as shown for the distillation column example, **A4** holds for many interesting classes of f. In particular, it holds for all f in the form $f(s) = \frac{e^{-hs}}{1+\tau s}$, and $f(s) = e^{-hs}\frac{1-\tau s}{1+\tau s}$, for all $\tau \geq 0$ and $h > 0$. Unfortunately, there are also many important functions for which it does not hold. For example, $f(s) = e^{-s}\frac{1-s}{1+\tau s}$ satisfies **A4** when $\tau \geq 0.25$; but **A4** is violated when $\tau \leq 0.2$. Similarly, **A4** holds for $f(s) = e^{-s}\frac{1+s}{1+\tau s}$ when $\tau \leq 1.02$, but it is violated when $\tau \geq 1.05$.

4. BOUNDS ON THE INTEGRAL ACTION GAIN IN PI CONTROLLER DESIGN

We now study the bound $\gamma_{\max}$ on the integral action gain γ defined by (11), where $\Lambda(s)$ is a given diagonal matrix in the form $\mathrm{diag}[f_1(s), \ldots, f_r(s)]$ with $f_k(s) = g_k(s)e^{-h_k s}$, $g_k \in \mathcal{H}_\infty$, $g_k(0) = 1$, $h_k > 0$, and α satisfies (12) which is equivalent to $p < \alpha + p < \min_k \| \frac{f_k(s)-1}{s} \|_\infty^{-1}$. With the above definitions we have

$$\gamma_{\max}^{-1} = \max_k \| \frac{\frac{\alpha}{s-p} f_k (1 + \frac{\alpha+p}{s-p} f_k)^{-1} - 1}{s} \|_\infty . \quad (22)$$

Let us define

$$\theta := \max_k \theta_k \quad \text{where} \quad \theta_k := \| \frac{f_k(s) - 1}{s} \|_\infty . \quad (23)$$

Then, a necessary condition for the results stated in Proposition 1 is $0 < \alpha\theta < 1 - p\theta$. After a simple algebra, it can be shown that (22) implies

$$\gamma_\star := \alpha \; \frac{1 - (\alpha + p)\; \theta}{1 + p\; \theta} \leq \gamma_{\max} . \quad (24)$$

The lower bound found in (24) for $\gamma_{\max}$, i.e. $\gamma_\star$, is between 0 and α, and it decreases with increasing θ. Note that θ^{-1} is also an upper bound for the proportional gain $(\alpha + p)$. Therefore, the level of difficulty in controlling the system increases with increasing θ. The other difficulty comes from the $\mathbb{C}_+$ pole of the plant: as p increases $\gamma_\star$ decreases.

Example. Let $f_k(s) = e^{-h_k s}$. Then, $\theta_k = h_k$, and θ is the largest time delay in the system. Now consider $f_1(s) = e^{-h_1 s}$, and $f_2(s) = \frac{180}{(s+6)(s+30)} e^{-h_2 s}$. In this case we have $\theta_1 = h_1$, and $\theta_2 = 0.2 + h_2$. Note that the norm in (23) is attained at $\omega = 0$ for both f_1 and f_2 and the phase of $f_2(j\omega)$ near $\omega \approx 0$ is $-0.2\;\omega$. So, we can see θ_2 as the "effective time delay" in the second channel. Then, $\theta = \max\{h_1, 0.2 + h_2\}$ is the largest effective time delay. □

In the light of (24) an interesting problem to study is to find the optimal α maximizing $\gamma_\star$, subject to $0 < \alpha\theta < 1 - p\theta$. It is easy to see that in this sense the optimal α is

$$\alpha_\star = \frac{1 - p\theta}{2\theta} \quad (25)$$

and the corresponding maximal $\gamma_\star$ is

$$\gamma_{\star,\max} = \frac{\alpha_\star}{2} \; \frac{(1 - p\theta)}{(1 + p\theta)} . \quad (26)$$

Equations (25) and (26) show once again that the difficulty level increases with increasing $p\theta$, where p is the right half plane pole and θ can be seen as the maximal "effective time delay" in the system.

5. CONCLUSIONS

PI and PD controller design problems are studied for unstable MIMO systems with delays in the input or output channels. The results of (Gündeş et al., 2006) are used for plants with single right half plane pole. For PD controller design, optimal derivative action gain is determined for maximizing the interval for the overall controller gain. For PI controller design, optimal proportional gain is calculated for maximizing the interval for the integral action gain. With these results resilient PI and PD controllers can be designed for the class of plants considered. Examples illustrated difficulty of controller design for plants whose products of unstable pole with effective time delay are large.

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