

# THE METHOD OF ANALYTICAL REGULARIZATION IN THE ELECTROMAGNETIC WAVE SCATTERING BY THIN DISKS

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## Abstract

We consider the problem of diffraction of an arbitrary electromagnetic wave by a thin disk made from different materials and located in free space. Here we imply a zero-thickness perfectly electrically conducting (PEC) disk, and also thin electrically resistive (ER) and dielectric disks whose thickness is much smaller than the disk radius and the free space wavelength, and also much smaller than the skin-layer depth in the ER disk case. The method used for the modeling is based on the integral equation (IE) technique and analytical regularization. Starting with Maxwell's equations, boundary conditions and the radiation condition at infinity we obtain a set of coupled dual IEs (DIEs) for the unknowns and then reduce this set of equations to the coupled IEs of the Fredholm second kind. To verify our results we calculate the far field characteristics in the case of the PEC disk with the incident field being the field of horizontal electrical dipole located on the disk axis.

## 1 Introduction

The problem of electromagnetic field scattering by a thin disk has been interested in since long ago. This is explained by many different applications of this canonical shape. Besides of traditional applications in the printed disk antennas with PEC or ER disks, thin dielectric disk is used as a simplified model of the tree leaf [1]. Still besides, thin few-micron-radius disks are met as resonators of semiconductor lasers with ultralow thresholds [2]. Many methods of computational electromagnetics have been used in their analysis, starting from empirical methods and finishing with the method of moments (MoM) and finite difference time domain method (FDTD). However, the use of these methods leads to difficulties such as prohibitively large matrices and impossibility of estimation of the error of calculations. In contrast, by using the method of IEs with analytical regularization it is possible to escape these difficulties.

## 2 Problem statement

We consider the problem of diffraction of a given time-harmonic electromagnetic field by a zero-thickness dielectric disk. The geometry of this problem is illustrated in Figure 1.

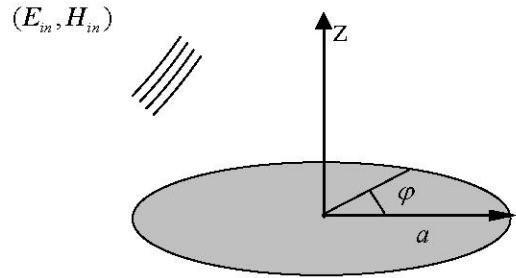


Figure 1. Disk geometry

The disk of the radius  $a$  is located in free space at the plane ( $z=0$ ). We introduce the cylindrical coordinates  $(r, \varphi, z)$  with the origin on the axis of symmetry of the disk and denote total field as a sum of the fields scattered by the disk and the incident one:

$$\mathbf{E} = \mathbf{E}_{sc} + \mathbf{E}_{in}, \quad \mathbf{H} = \mathbf{H}_{sc} + \mathbf{H}_{in} \quad (1)$$

The components of the scattered field must satisfy the set of homogeneous Maxwell equations outside the disk, the 3-D radiation condition, the condition of local integrability of power (edge condition), and the generalized boundary conditions (see [3]) on the disk surface, at ( $z=0, r < a$ ):

$$\begin{aligned} \frac{1}{2} [\mathbf{E}_{tg}^+ + \mathbf{E}_{tg}^-] &= Z_0 R \cdot \vec{n} \times [\mathbf{H}_{tg}^+ - \mathbf{H}_{tg}^-], \\ \frac{Z_0}{2} [\mathbf{H}_{tg}^+ + \mathbf{H}_{tg}^-] &= -S \cdot \vec{n} \times [\mathbf{E}_{tg}^+ - \mathbf{E}_{tg}^-]. \end{aligned} \quad (2)$$

Here,  $Z_0$  is the free-space impedance, and  $R$  and  $S$  are the electric and magnetic resistivities. For a thin dielectric disk they are given by

$$R = \frac{iZ}{2} \cot \left( \frac{\sqrt{\epsilon_r \mu_r} k \tau}{2} \right), \quad |\epsilon_r \mu_r| \gg 1, \quad \tau \ll \lambda_0. \quad (3)$$

$$S = \frac{i}{2Z} \cot \left( \frac{\sqrt{\epsilon_r \mu_r} k \tau}{2} \right), \quad |\epsilon_r \mu_r| \gg 1, \quad \tau \ll \lambda_0. \quad (4)$$

Here,  $Z$  is the relative impedance of the disk material,  $k = \omega/c$  is the wavenumber,  $\epsilon_r$  is the relative permittivity,  $\mu_r$  is the relative conductivity,  $\lambda_0$  is the wavelength in free space, and  $\tau$  is the thickness of the disk.

For a thin electrically resistive (ER) disk they are given by

$$R = \frac{1}{Z_0 \sigma \tau}, \quad \sigma / \omega \gg 1, \quad S = \infty. \quad (5)$$

For a zero-thickness PEC disk they are given by

$$R = 0, \quad S = \infty \quad (6)$$

On the rest part of the plane ( $z=0$ ) the components of the field are continuous.

### 3 Basic equations

We express the scattered field components normal to the disk in terms of the azimuth-angle Fourier series and the radial inverse integral Hankel transform:

$$E_{sc,z}^{\pm} = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_0^{\infty} e^{\pm i\gamma(\kappa)\zeta} e_{m,z}^{\pm}(\kappa) J_{|m|}(\kappa\rho) \kappa d\kappa, \quad (7)$$

$$Z_0 H_{sc,z}^{\pm} = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_0^{\infty} e^{\pm i\gamma(\kappa)\zeta} h_{m,z}^{\pm}(\kappa) J_{|m|}(\kappa\rho) \kappa d\kappa, \quad (8)$$

where  $e_{m,z}^{\pm}(\kappa)$  and  $h_{m,z}^{\pm}(\kappa)$  are the images of the normal to the disk field components in the spectral domain. They are the unknown functions to be found. Then the tangential to the disk components can be written as follows:

$$\begin{pmatrix} E_{sc,r}^{\pm} \\ -iE_{sc,\varphi}^{\pm} \end{pmatrix} = \sum_{m=-\infty}^{\infty} e^{im\varphi} \cdot \int_0^{\infty} e^{\pm i\gamma(\kappa)\zeta} \bar{H}_m(\kappa\rho) \begin{pmatrix} \pm i\gamma(\kappa) e_{m,z}^{\pm}(\kappa) \\ -\kappa h_{m,z}^{\pm}(\kappa) \end{pmatrix} d\kappa \quad (9)$$

$$\begin{pmatrix} Z_0 H_{sc,r}^{\pm} \\ -iZ_0 H_{sc,\varphi}^{\pm} \end{pmatrix} = \sum_{m=-\infty}^{\infty} e^{im\varphi} \cdot \int_0^{\infty} e^{\pm i\gamma(\kappa)\zeta} \bar{H}_m(\kappa\rho) \begin{pmatrix} \pm i\gamma(\kappa) h_{m,z}^{\pm}(\kappa) \\ -\kappa e_{m,z}^{\pm}(\kappa) \end{pmatrix} d\kappa \quad (10)$$

Here

$$\bar{H}_m(\kappa\rho) = \begin{pmatrix} J'_{|m|}(\kappa\rho) & mJ_{|m|}(\kappa\rho)/(\kappa\rho) \\ mJ_{|m|}(\kappa\rho)/(\kappa\rho) & J'_{|m|}(\kappa\rho) \end{pmatrix} \quad (11)$$

is the kernel matrix function of the vector Hankel transform,

$$\rho = r/a, \quad \zeta = z/a \quad \text{and} \quad \gamma(\kappa) = \sqrt{(ka)^2 - \kappa^2}.$$

On substituting (9) and (10) to (2) and continuity conditions outside the disk, we, in general case, obtain the set of coupled DIEs for the four unknown functions in the spectral domain:

$$\begin{cases} \int_0^{\infty} \bar{H}_m(\kappa\rho) \begin{pmatrix} i\gamma(\kappa) \left( u_m^-(\kappa) + e^{i\gamma(\kappa)\zeta_0} u_m^{\zeta_0,-}(\kappa) \right) + 2R i \kappa a u_m^-(\kappa) \\ -\kappa a \left( v_m^+(\kappa) + e^{i\gamma(\kappa)\zeta_0} v_m^{\zeta_0,+}(\kappa) \right) - 2R \gamma(\kappa) v_m^+(\kappa) \end{pmatrix} d\kappa = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \rho < 1, \\ \int_0^{\infty} \bar{H}_m(\kappa\rho) \begin{pmatrix} i\kappa a u_m^-(\kappa) \\ -\gamma(\kappa) v_m^+(\kappa) \end{pmatrix} d\kappa = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \rho > 1, \end{cases} \quad (12)$$

$$\begin{cases} \int_0^{\infty} \bar{H}_m(\kappa\rho) \begin{pmatrix} i\gamma(\kappa) \left( v_m^-(\kappa) + e^{i\gamma(\kappa)\zeta_0} v_m^{\zeta_0,-}(\kappa) \right) + 2S i \kappa a v_m^-(\kappa) \\ \kappa a \left( u_m^+(\kappa) + e^{i\gamma(\kappa)\zeta_0} u_m^{\zeta_0,+}(\kappa) \right) + 2S \gamma(\kappa) u_m^+(\kappa) \end{pmatrix} d\kappa = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \rho < 1, \\ \int_0^{\infty} \bar{H}_m(\kappa\rho) \begin{pmatrix} i\kappa a v_m^-(\kappa) \\ \gamma(\kappa) u_m^+(\kappa) \end{pmatrix} d\kappa = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \rho > 1, \end{cases} \quad (13)$$

Here  $u_m^{\pm}(\kappa)$  и  $v_m^{\pm}(\kappa)$  are linear combination of original unknown image function:

$$\begin{aligned} u_m^{\pm}(\kappa) &= \left( e_{m,z}^+(\kappa) \pm e_{m,z}^-(\kappa) \right) / 2 \\ v_m^{\pm}(\kappa) &= \left( h_{m,z}^+(\kappa) \pm h_{m,z}^-(\kappa) \right) / 2 \end{aligned} \quad (14)$$

The obtained set of DIEs can be written in the following abstract form:

$$\begin{aligned} \int_0^{\infty} \bar{H}_m(\kappa\rho) \cdot \bar{S}_1(ka, \kappa, \rho) \cdot X(\kappa) d\kappa \\ = \int_0^{\infty} \bar{H}_m(\kappa\rho) \cdot \bar{S}_2(ka, \kappa, \rho) \cdot X_0(\kappa) d\kappa \end{aligned} \quad (15)$$

where  $\bar{S}_{1,2}(ka, \kappa, \rho)$  are singular diagonal matrix-functions of two variables with the singular point at infinity ( $\kappa = \infty$ ),  $X(\kappa)$  is the unknown vector-function,  $X_0(\kappa)$  is the given vector-function. Note that the singularity of  $\bar{S}_2(ka, \kappa, \rho)$  leads to the limitation of  $X_0(\kappa)$  in terms of the allowed class

of functions. To solve (15), we use the method of analytical regularization [4]. It consists of the following two parts: First, we split the function  $\bar{S}_1(ka, \kappa, \rho)$  into the singular and regular parts:

$$\bar{S}_1(ka, \kappa, \rho) = \bar{\Sigma}_1(ka, \kappa, \rho) + \bar{P}_1(ka, \kappa, \rho) \quad (16)$$

Then (15) can be rewritten as follows:

$$\int_0^\infty \bar{H}_m(\kappa\rho) \cdot \bar{\Sigma}_1(ka, \kappa, \rho) \cdot X(\kappa) d\kappa = F(\rho, X(\dots)), \quad (17)$$

where the right-hand part is

$$F(\rho, X(\dots)) = \int_0^\infty \bar{H}_m(\kappa\rho) \cdot \left( -\bar{P}_1(ka, \kappa, \rho) \cdot X(\kappa) + \bar{S}_2(ka, \kappa, \rho) \cdot X_0(\kappa) \right) d\kappa \quad (18)$$

The solution of ‘‘canonical’’ IE is sought in the following form:

$$X(\lambda) = \int_0^\infty \bar{K}_m(\lambda, \rho) \cdot F(\rho, X(\dots)) d\rho \quad (19)$$

Now we substitute expression (18) into (19) and reduce the coupled DIEs (15) to the following Fredholm IE of the second kind:

$$\begin{aligned} x_m(\lambda) - \int_0^\infty iw(\kappa)x_m(\kappa)K_m^{(1)}(\kappa, \lambda)d\kappa + ika2mA_mI_m^{(1)}\lambda^{-1/2}(\lambda+1)^{-1}J_{m+3/2}(\lambda) + \\ + i2mB_mI_m^{(2)}\lambda^{-1/2}(\lambda+1)^{-1}J_{m-1/2}(\lambda) = \int_0^\infty i\kappa^{-1}\gamma(\kappa)(\kappa+1)^2 e^{i\gamma(\kappa)|\zeta_0|} x_m^0(\kappa)K_m^{(1)}(\kappa, \lambda)d\kappa \\ y_m(\lambda) - \int_0^\infty w(\kappa)y_m(\kappa)\lambda\gamma^{-1}(\lambda)K_m^{(-1)}(\kappa, \lambda)d\kappa + 2mA_mI_m^{(3)}\lambda^{1/2}\gamma^{-1}(\lambda)(\lambda+1)^{-1}J_{m+1/2}(\lambda) + \\ + 2mkaB_mI_m^{(4)}\lambda^{1/2}(\lambda+1)^{-1}J_{m-3/2}(\lambda) = -\int_0^\infty i(\kappa+1)^2 e^{i\gamma(\kappa)|\zeta_0|} y_m^0(\kappa)\lambda\gamma^{-1}(\lambda)K_m^{(-1)}(\kappa, \lambda)d\kappa \\ - \int_0^\infty iw(\kappa)x_m(\kappa)G_m^{(1)}(\kappa)d\kappa + ika2mA_mI_m^{(1)} - i2mB_mI_m^{(2)} = \int_0^\infty i\kappa^{-1}\gamma(\kappa)(\kappa+1)^2 e^{i\gamma(\kappa)|\zeta_0|} x_m^0(\kappa)G_m^{(1)}(\kappa)d\kappa \\ - \int_0^\infty w(\kappa)y_m(\kappa)G_m^{(-1)}(\kappa)d\kappa + 2mA_mI_m^{(3)} - ka2mB_mI_m^{(4)} = -\int_0^\infty i(\kappa+1)^2 e^{i\gamma(\kappa)|\zeta_0|} y_m^0(\kappa)G_m^{(-1)}(\kappa)d\kappa \end{aligned} \quad (22)$$

Here  $x_m(\lambda)$ ,  $y_m(\lambda)$  are the unknown functions,  $A_m$ ,  $B_m$  are the unknown constants of integration,  $K_m^{(\pm 1)}(\kappa, \lambda)$ ,  $G_m^{(\pm 1)}(\kappa)$  are the kernel functions,  $w(\kappa)$  is the weight function,  $x_m^0(\lambda)$

$$\begin{aligned} X(\lambda) + \int_0^\infty \bar{K}_m(\lambda, \rho) \cdot \int_0^\infty \bar{H}_m(\kappa\rho) \cdot \bar{P}_1(ka, \kappa, \rho) \cdot X(\kappa) d\kappa d\rho \\ = \int_0^\infty \bar{K}_m(\lambda, \rho) \cdot \int_0^\infty \bar{H}_m(\kappa\rho) \cdot \bar{S}_2(ka, \kappa, \rho) \cdot X_0(\kappa) d\kappa d\rho \end{aligned} \quad (20)$$

Extraction of the singular part of integral operators in (12), (13) leads to two different-type DIEs in the case of PEC ( $R=0$  and  $S=\infty$ ) and dielectric or ER disks. Finding the solutions in both cases with ( $m \neq 0$ ) is based on decoupling the DIEs by the integration in  $\rho$  and introduction of four (in general case) constants of integration. This procedure leads to the DIEs, which can be cast to the following short form:

$$\begin{cases} \int_0^\infty \left\{ \kappa^\delta u_m(\kappa) - f(\kappa) \right\} J_m(\kappa\rho) d\kappa = 0 & (\rho < 1), \\ \int_0^\infty u_m(\kappa) J_m(\kappa\rho) d\kappa = 0 & (\rho > 1), \end{cases} \quad (21)$$

Such DIEs can be solved analytically by using the Abel integral transform and inverse Hankel integral transform in the case of  $\delta = \pm 1$  or just inverse Hankel integral transform in the case of  $\delta = 0$ . Additional equations for finding the constants of integration follow from the conditions of local integrability of power near the disk edge.

As an example, we show the final integral equation of the Fredholm second kind ( $m \neq 0$ ) in case when the diffraction by the PEC disk is considered.

and  $y_m^0(\lambda)$  are the given functions determined by the incident field, and  $I_m^{(1,2,3,4)}$  are known coefficients.

## 4 Numerical results

The Fredholm second kind nature of the derived equations guarantees the existence of solutions of (22) and the convergence of numerical algorithm based on any reasonable discretization scheme. For example, it can be a simple projection scheme with the basis functions taken as the so-called step functions:

$$\delta_n^s(\lambda) = \begin{cases} 1 & \lambda \in [s(n-1), sn) \\ 0 & \lambda \notin [s(n-1), sn) \end{cases} \quad (23)$$

To verify the obtained equations we calculate the main characteristics in the case of the PEC disk with the incident field being the field of a horizontal electrical dipole located on the disk axis and oriented along the line ( $\varphi = 0$ ). They are the far-field radiation patterns and the total radiated power. Besides, we calculate truncation and discretization errors in the computation of the total radiated power.

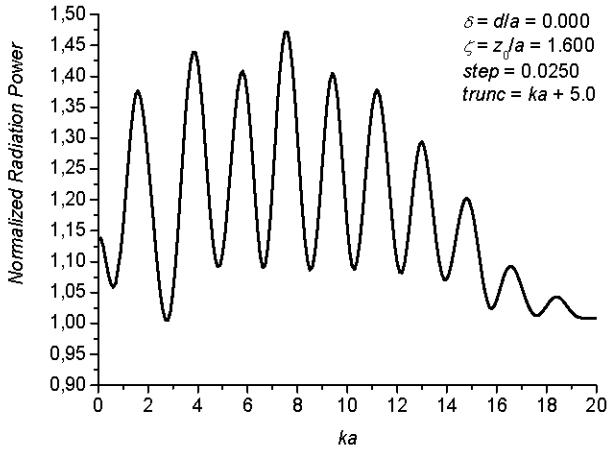


Figure 2a. Normalized radiation power vs. the normalized disk radius.

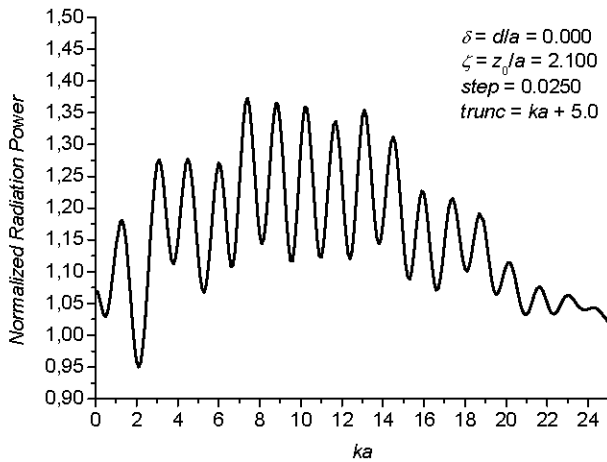


Figure 2b. Normalized radiation power vs. the normalized disk radius.

Figures 2a-2b show the normalized radiation power vs. the frequency normalized by the disk radius. Here  $\zeta$  is the normalized distance from the dipole to the disk center,  $trunc$  is the truncation value of the domain of integration in (22) that is adapted to the frequency as  $ka + 5$ . Figures 3a-3b show the relative computation errors given by:

$$Err_{curr} = \left| \frac{P_{curr} - P_{prev}}{P_{prev}} \right| \quad (24)$$

where  $P_{prev}$  is the normalized total power at the previous computation point,  $P_{curr}$  is the normalized total power at the current point. Figure 4 shows the dependence of  $P$  vs.  $\zeta$ , which is the normalized distance between disk and dipole. Figure 5 shows the far field radiation patterns for some characteristic values of parameters.

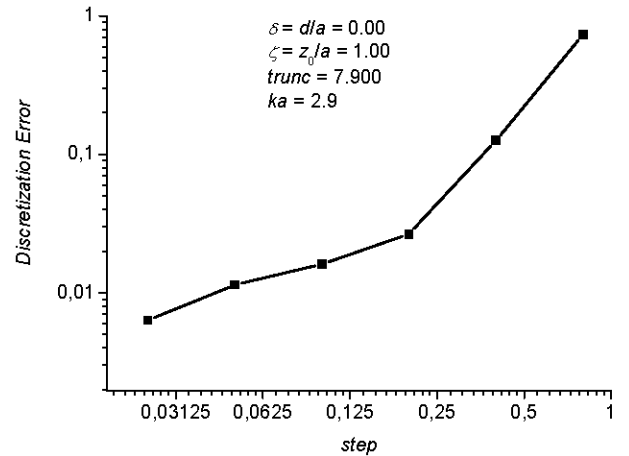


Figure 3a. Discretization error vs. the grid step.

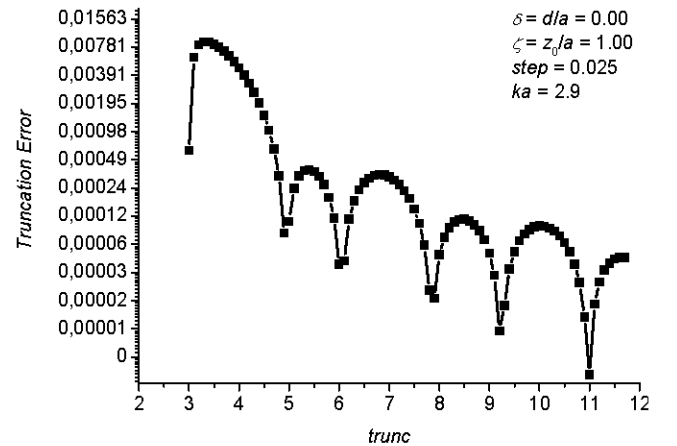


Figure 3b. The computation error vs. the value of truncation of the integration domain.

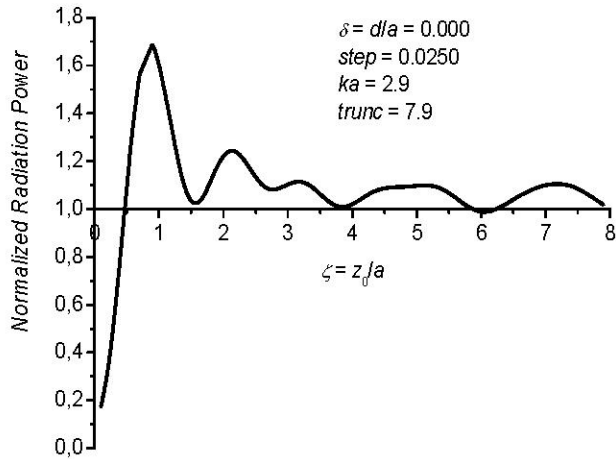


Figure 4. Normalized radiation power vs. the distance from the disk to the dipole.

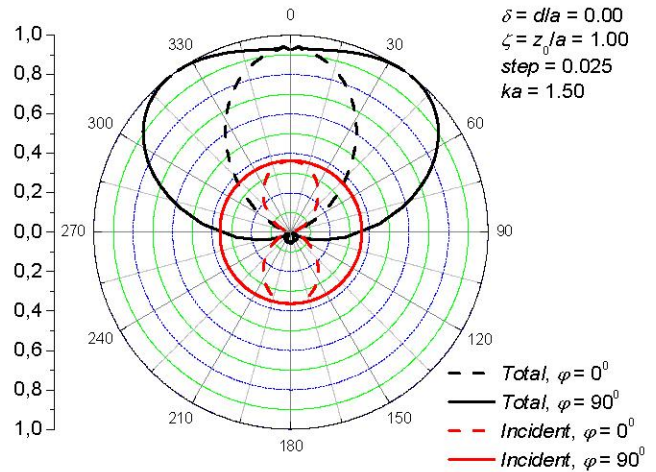


Figure 5b. Far field radiation pattern  $ka=1.5$ .

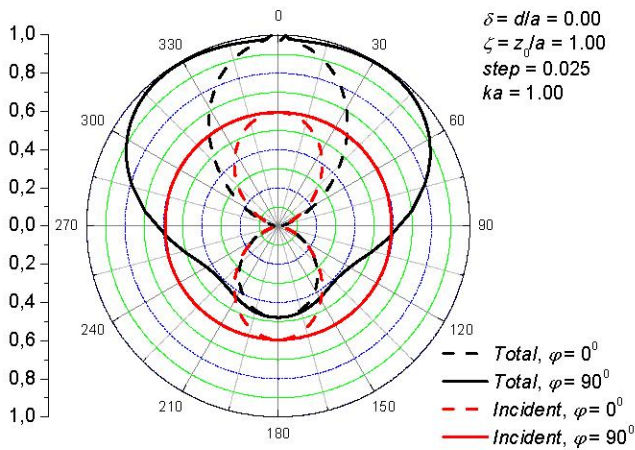


Figure 5a. Far field radiation pattern for  $ka=1$ .

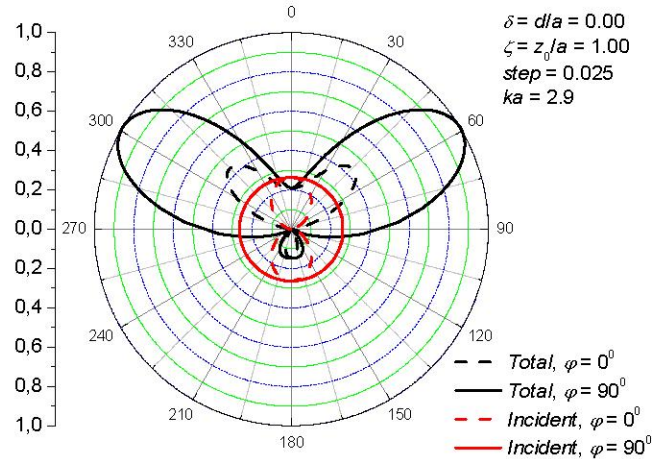


Figure 5c. Far field radiation pattern  $ka=2.9$ .

## 5 Acknowledgements

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