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Stability Analysis of Switched Time-Delay Systems

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Abstract— **This paper addresses the asymptotic stability of switched time delay systems with heterogenous time invariant time delays. Piecewise Lyapunov-Razumikhin functions are introduced for the switching candidate systems to investigate the stability in the presence of infinite number of switchings. We provide sufficient conditions in terms of the minimum dwell time to guarantee asymptotic stability under the assumptions that each switching candidate is delay-independently or delay-dependently stable. Conservatism analysis is also provided by comparing with the dwell time conditions for switched delay free systems.**

I. Introduction

Switching control offers a new look into the design of complex control systems (e.g. nonlinear systems, parameter varying systems and uncertain systems) [1], [8], [9], [19], [17], [21], [28]. Unlike the conventional adaptive control techniques that rely on continuous tuning, the switching control method updates the controller parameters in a discrete fashion based on the switching logic. The resulting closed-loop systems have hybrid behaviors (e.g. continuous dynamics, discrete time dynamics and jump phenomena, etc.). One of the most challenging issues in the area of hybrid systems is the stability analysis in the presence of control switching. We refer to [9] for a general review on switching control methods.

In particular, we are interested in the stability analysis of switched time delay systems. In fact, time delay systems are ubiquitous in chemical processes, aerodynamics, and communication networks [3], [14]. To further complicate the situation, the time delays are usually time varying and uncertain [24], [25]. It has been shown that robust $\mathcal{H}^\infty$ controllers can be designed for such infinite dimensional plants, where robustness can be guaranteed within some uncertainty bounds [4]. In order to incorporate larger operating range or better robustness, controller switching can be introduced, which results in switched closed-loop systems with time delays. For delay free systems, stability analysis and design methodology have been investigated recently in the framework of hybrid dynamical systems [1], [2], [8], [11], [19], [21], [26]. In particular, [21] provided sufficient conditions on the stability of the switching control systems based on Filippov solutions to discontinuous differential equations and Lyapunov functionals; [19] proposed a dwell-time based switching control, where a sufficiently large dwell-time can guarantee the system stability. A more flexible result was obtained in [10], where the average dwell-time was introduced for switching control. In [26] the results of [10] were extended to LPV systems. LaSalle's invariance principle was extended to a class of switched linear systems for stability analysis [8]. Despite the variety and significance of the many results on hybrid system stability, stability of switched time delay systems hasn't been adequately addressed due to the general difficulty of infinite dimensional systems [7].

Two important approaches in the stability analysis of time delay systems are (1) Lyapunov-Krasovskii method, and (2) Lyapunov-Razumikhin method [6], [20]. Various sufficient conditions with respect to the stability of time delay systems have been given using Riccati-type inequalities or LMIs [3], [12], [14], [24]. In the meanwhile, stability analysis in the presence of switching has been discussed in some recent works [16], [18], [22]. In [18] stability and stabilizability were discussed for discrete time switched time delay systems; [16] considered similar stability problem in continuous time domain. Note that [18] and [16] are *trajectory dependent* results without taking admissible switching signals into considerations.

The main contribution of this paper is a collection of results on the *trajectory independent* stability of continuous time switched time delay systems using piecewise Lyapunov-Razumikhin functions. The dwell time of the switching signals is constructively given, which guarantees asymptotic stability for the delay independent case and the delay dependent case, respectively. Note that the asymptotic stability of finite dimensional linear systems indicates exponential stability, whereas this is not the case for infinite dimensional systems, [7], [15]. This poses the key challenge in the analysis of switched time delay systems.

The paper is organized as follows. The problem is defined in Section II. In Section III, the main results on the stability of switched time delay systems are presented in terms of the dwell time of the switching signals. Conservatism analysis is provided by comparing with the dwell time conditions for switching delay free systems in Section IV, followed by concluding remarks in Section V.

II. Problem Definition

For convenience, we would like to employ the following notation. The general Retarded Functional Differential Equations (RFDE) with time delay r can be described as

$$\dot{x}(t) = f(t, x_t) \tag{1}$$

with initial condition $\phi(\cdot) \in C([-r, 0], \mathbb{R}^n)$, where x_t denotes the state defined by $x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$.

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We use $\|\cdot\|$ to denote the Euclidean norm of a vector in $\mathbb{R}^n$, and $|f|_{[t-r,t]}$ for the ∞-norm of f, i.e.

$$|f|_{[t-r,t]} := \sup_{t-r\le\theta\le t} \|f(\theta)\|,$$

where f is an element of the Banach space $C([t-r,t],\mathbb{R}^n)$.

Consider the following switched time delay systems:

$$\Sigma_t : \begin{cases} \dot{x}(t) &= A_{q(t)}x(t) + \bar{A}_{q(t)}x(t-\tau_{q(t)}), \quad t\ge 0 \\ x_0(\theta) &= \phi(\theta), \quad \forall\theta\in[-\tau_{max},0] \end{cases} \tag{2}$$

where $x(t)\in\mathbb{R}^n$ and $q(t)$ is a piecewise switching signal taking values on the set $\mathcal{F} := \{1,2,...,l\}$, i.e. $q(t)=k_j$, $k_j\in\mathcal{F}$, for $\forall t\in[t_j,t_{j+1})$, where t_j, $j\in\mathbb{Z}^+\cup\{0\}$, is the j^{th} switching time instant. It is clear that the trajectory of Σ_t in any arbitrary switching interval $t\in[t_j,t_{j+1})$ obeys:

$$\Sigma_{k_j} : \begin{cases} \dot{x}(t) &= A_{k_j}x(t) + \bar{A}_{k_j}x(t-\tau_{k_j}), t\in[t_j,\ t_{j+1}) \\ x_{t_j}(\theta) &= \phi_j(\theta), \quad \forall\theta\in[-\tau_{k_j},0], \end{cases} \tag{3}$$

where $\phi_j(\theta)$ is defined as:

$$\phi_j(\theta) = \begin{cases} x(t_j+\theta) & -\tau_{k_j}\le\theta<0 \\ \lim_{h\to 0^-} x(t_j+h), & \theta=0 \end{cases} \tag{4}$$

We introduce the triplet $\Sigma_i := (A_i,\bar{A}_i,\tau_i)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}\times\mathbb{R}^+$ to describe the i^{th} candidate system of (2). Thus for $\forall t\ge 0$, we have $\Sigma_t\in\mathcal{A} := \{\Sigma_i : i\in\mathcal{F}\}$, where $\mathcal{A}$ is the family of candidate systems of (2). In (2), $\phi(\cdot) : [-\tau_{max},0]\to\mathbb{R}^n$ is a continuous and bounded vector-valued function, where $\tau_{max} = \max_{i\in\mathcal{F}}\{\tau_i\}$ is the maximal time delay of the candidate systems in $\mathcal{A}$.

Similar to [8], we say that the switched time-delay system Σ_t described by (2) is *stable* if there exists a function $\bar{\alpha}$ of class $\mathcal{K}$ [1] such that

$$\|x(t)\| \le \bar{\alpha}(|x|_{[t_0-\tau_{max},t_0]}), \quad \forall t\ge t_0\ge 0, \tag{5}$$

along the trajectory of (2). Furthermore, Σ_t is *asymptotically stable* when Σ_t is stable and $\lim_{t\to+\infty} x(t) = 0$.

Lemma 2.1: ([3], [14]) Suppose for a given triplet $\Sigma_i\in\mathcal{A}$, $i\in\mathcal{F}$, there exists symmetric and positive-definite $P_i\in\mathbb{R}^{n\times n}$, such that the following LMI with respect to P_i is satisfied for some $p_i>1$ and $\alpha_i>0$:

$$\begin{bmatrix} P_iA_i + A_i^TP_i + p_i\alpha_iP_i & P_i\bar{A}_i \\ \bar{A}_i^TP_i & -\alpha_iP_i \end{bmatrix} < 0. \tag{6}$$

Then Σ_i is asymptotically stable independent of delay.

If all candidate systems of (2), $\Sigma_i\in\mathcal{A}$, are delay-independently asymptotically stable satisfying (6), we denote $\mathcal{A}$ by $\tilde{\mathcal{A}}$.

Lemma 2.2: ([3], [14]) Suppose for a given triplet $\Sigma_i\in\mathcal{A}$, $i\in\mathcal{F}$, there exists symmetric and positive-definite $P_i\in\mathbb{R}^{n\times n}$, and a scalar $p_i>1$, such that

$$\begin{bmatrix} \tau_i^{-1}\Omega_i & P_i\bar{A}_iM_i \\ M_i^T\bar{A}_i^TP_i & -R_i \end{bmatrix} < 0 \tag{7}$$

[1]A continuous function $\bar{\alpha}(\cdot) : \mathbb{R}^+\to\mathbb{R}^+$ is a class $\mathcal{K}$ function if it is strictly increasing and $\bar{\alpha}(0)=0$.

where

$$\begin{aligned} \Omega_i &= (A_i+\bar{A}_i)^TP_i + P_i(A_i+\bar{A}_i) + \tau_ip_i(\alpha_i+\beta_i)P_i, \\ M_i &= [A_i\ \bar{A}_i], \\ R_i &= diag(\alpha_iP_i, \beta_iP_i), \end{aligned}$$

and $\alpha_i>0$, $\beta_i>0$ are scalars. Then Σ_i is asymptotically stable dependent of delay.

Similarly we denote $\mathcal{A}$ by $\tilde{\mathcal{A}}_d$ if all candidate systems of (2) are delay-dependently asymptotically stable satisfying (7).

In what follows, we will establish sufficient conditions to guarantee stability of switched system (2) for the delay independent case and the delay dependent case. Therefore, we will assume that $\mathcal{A}=\tilde{\mathcal{A}}$ and $\mathcal{A}=\tilde{\mathcal{A}}_d$ respectively in the corresponding sections in this paper. An important method in stability analysis of switched systems is based on the construction of the common Lyapunov function (CLF), which allows for arbitrary switching. However, this method is too conservative from the perspective of controller design because it is usually difficult to find the CLF for all the candidate systems, particularly for time delay systems whose stability criteria are only sufficient in most of the circumstances. A recent paper [29] explored the CLF method for switched time delays systems with three very strong assumptions: (i) each candidate system has the same time delay τ; (ii) each candidate is assumed to be delay independently stable; (iii) The A-matrix is always symmetric and the $\bar{A}$-matrix is always in the form of δI. In the present paper, we consider an alternative method using piecewise Lyapunov-Razumikhin functions for a general class of systems (2) and obtain stability conditions in terms of the dwell time of the switching signal. This method can be used for the case with delay independent criterion (6) and the case with delay dependent criterion (7).

III. Main Results on Dwell Time Based Switching

For a given positive constant τ_D, the switching signal set based on the dwell time τ_D is denoted by $S[\tau_D]$, where for any switching signal $q(t)\in S[\tau_D]$, the distance between any consecutive discontinuities of $q(t)$, $t_{j+1}-t_j$, $j\in\mathbb{Z}^+\cup\{0\}$, is larger than τ_D [10], [19]. Sufficient condition on the minimum dwell time to guarantee the stable switching will be given using piecewise Lyapunov-Razumikhin functions. Note that the dwell time based switching is trajectory-independent [8].

Before presenting the main result of this paper, we recall the following lemma [7] for general Retarded Functional Differential Equations (1).

Lemma 3.1: [7] Suppose $u,v,w,p : \mathbb{R}^+\to\mathbb{R}^+$ are continuous,nondecreasing functions, $u(0)=v(0)=0$, $u(s)$, $v(s)$, $w(s)$, $p(s)$ positive for $s>0$, $p(s)>s$, and $v(s)$ strictly increasing. If there is a continuous function $V:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ such that

$$u(\|x(t)\|) \le V(t,x) \le v(\|x(t)\|), \quad t\in\mathbb{R},\ x\in\mathbb{R}^n, \tag{8}$$

and

$$\dot{V}(t,x(t)) \le -w(\|x(t)\|), \tag{9}$$

if

$$V(t+\theta, x(t+\theta)) < p(V(t, x(t))) \quad \forall \theta \in [-r, 0], \quad (10)$$

then the solution $x = 0$ of the RFDE is uniformly asymptotically stable.

A particular case of (1) is a linear time delay system Σ_i, $i \in \mathcal{F}$, where we can construct the corresponding Lyapunov-Razumikhin function in the quadratic form

$$V_i(t, x) = x^T(t) P_i x(t), \quad P_i = P_i^T > 0. \quad (11)$$

Apparently V_i can be bounded by

$$u_i(\|x(t)\|) \le V_i(t, x) \le v_i(\|x(t)\|), \ \forall x \in \mathbb{R}^n, \quad (12)$$

where

$$u_i(s) := \kappa_i s^2, \quad v_i(s) := \bar{\kappa}_i s^2, \quad (13)$$

in which $\kappa_i := \sigma_{min}[P_i] > 0$ denotes the smallest singular value of P_i and $\bar{\kappa}_i := \sigma_{max}[P_i] > 0$ the largest singular value of P_i.

Proposition 3.2: For each time delay systems Σ_i with Lyapunov-Razumikhin function defined by (11) assume (9) and (10) are satisfied for some $w_i(s)$. Then we have

$$|x|_{[t_m-\tau_i, t_m]} \le \left(\frac{\bar{\kappa}_i}{\kappa_i}\right)^{1/2} |x|_{[t_n-\tau_i, t_n]}, \forall t_m \ge t_n \ge 0. \ (14)$$

Proof. Define

$$\bar{V}_i(t, x) := \sup_{-\tau_i \le \theta \le 0} V_i(t+\theta, x(t+\theta)) \quad (15)$$

for $t \ge 0$, we have

$$\kappa_i(|x|_{[t-\tau_i, t]})^2 \le \bar{V}_i(t, x) \le \bar{\kappa}_i(|x|_{[t-\tau_i, t]})^2, \quad t \ge 0 \quad (16)$$

The definition of $\bar{V}_i(t, x)$ implies $\exists \theta_0 \in [-\tau_i, 0]$, such that $\bar{V}_i(t, x) = V(t+\theta_0, x(t+\theta_0))$. Introduce the upper right-hand derivative of $\bar{V}_i(t, x)$ as

$$\dot{\bar{V}}_i^+ = \limsup_{h \to 0^+} \frac{1}{h}[\bar{V}_i(t+h, x(t+h)) - \bar{V}_i(t, x(t))],$$

we have

(i). If $\theta_0 = 0$, i.e. $V_i(t+\theta, x(t+\theta)) \le V_i(t, x(t)) < p(V_i(t, x(t)))$, we have $\dot{V}_i(t, x) < 0$ by (9). Therefore $\dot{\bar{V}}_i^+ \le 0$.

(ii). If $-\tau_i < \theta_0 < 0$, we have $\bar{V}_i(t+h, x(t+h)) = \bar{V}_i(t, x)$ for $h > 0$ sufficiently small, which results in $\dot{\bar{V}}_i^+ = 0$.

(iii). If $\theta_0 = -\tau_i$, the continuity of $V_i(t, x)$ implies $\dot{\bar{V}}_i^+ \le 0$.

The above analysis shows that

$$\bar{V}_i(t_m) \le \bar{V}_i(t_n), \quad \forall t_m \ge t_n \ge 0. \quad (17)$$

Recall (16), we have

$$\kappa_i(|x|_{[t_m-\tau_i, t_m]})^2 \le \bar{V}_i(t_m) \le \bar{V}_i(t_n) \le \bar{\kappa}_i(|x|_{[t_n-\tau_i, t_n]})^2, \quad (18)$$

for any $t_m \ge t_n \ge 0$. This implies (14) and proves the result. ■

Suppose all of the conditions of Lemma 3.1 are satisfied for general RFDE (1), we also have the following result.

Lemma 3.3: [7] Suppose $|\phi|_{[t_0-r, t_0]} \le \bar{\delta}_1, \bar{\delta}_1 > 0$, and $\bar{\delta}_2 > 0$ such that $v(\bar{\delta}_1) = u(\bar{\delta}_2)$. For all η satisfying $0 < \eta \le \bar{\delta}_2$, we have

$$V(t, x) \le u(\eta), \quad \forall\, t \ge t_0 + T. \quad (19)$$

Here

$$T = \frac{N v(\bar{\delta}_1)}{\gamma} \quad (20)$$

is defined by $\gamma = \inf_{v^{-1}(u(\eta)) \le s \le \bar{\delta}_2} w(s)$ and $N = \lceil (v(\bar{\delta}_1) - u(\eta))/a \rceil$, where $\lceil \cdot \rceil$ is the ceiling integer function and $a > 0$ satisfies $p(s) - s > a$ for $u(\eta) \le s \le v(\bar{\delta}_1)$.

A. The Case with Delay Independent Criterion

Consider the switched time delay systems Σ_t defined by (2) and assume each candidate system Σ_i, $i \in \mathcal{F}$ delay-independently asymptotically stable satisfying (6) (i.e. $\mathcal{A} = \tilde{\mathcal{A}}$). A sufficient condition on the minimum dwell time to guarantee the asymptotic stability can be derived using multiple piecewise Lyapunov-Razumikhin functions. In order to state the main result we make some preliminary definitions.

For the switched delay systems (2), first assume $\tau_D > \tau_{max}$. Consider an arbitrary switching interval $[t_j, t_{j+1})$ of the piecewise switching signal $q(t) \in S[\tau_D]$, where $q(t) = k_j, k_j \in \mathcal{F}$ for $\forall t \in [t_j, t_{j+1})$ and t_j is the j^{th} switching time instant for $j \in \mathbb{Z}^+ \cup \{0\}$ and $t_0 = 0$. The state variable $x_j(t)$ defined on this interval obeys (3). For the convenience of using "sup", we define $x_j(t_{j+1}) = \lim_{h \to 0^-} x_j(t_{j+1}+h) = x_{j+1}(t_{j+1})$ based on the fact that $x(t)$ is continuous for $t \ge 0$. Therefore $x_j(t)$ is now defined on a compact set $[t_j, t_{j+1}]$. Recall (4), the initial condition $\phi_j(t)$ of Σ_{k_j} is $\phi_j(t) = x(t) = x_{j-1}(t), t \in [t_j - \tau_{k_j}, t_j]$ for $j \in \mathbb{Z}^+$, which is true because $\tau_D > \tau_{max}$.

Construct the Lyapunov-Razumikhin function

$$V_{k_j}(x_j, t) = x_j^T(t) P_{k_j} x_j(t), \quad t \in [t_j, t_{j+1}] \quad (21)$$

for (3), then we have

$$\kappa_{k_j}\|x_j(t)\|^2 \le V_{k_j}(t, x_j) \le \bar{\kappa}_{k_j}\|x_j(t)\|^2, \ \forall x_j \in \mathbb{R}^n. \ (22)$$

A straightforward calculation gives the time derivative of $V_{k_j}(t, x_j(t))$ along the trajectory of (3)

$$\begin{aligned} \dot{V}_{k_j}(t, x_j) &= x_j^T(A_{k_j}^T P_{k_j} + P_{k_j} A_{k_j}) x_j \\ &\quad + 2x_j^T(t) P_{k_j} \bar{A}_{k_j} x_j(t - \tau_{k_j}), \end{aligned} \quad (23)$$

where

$$\begin{aligned} & 2x_j^T(t) P_{k_j} \bar{A}_{k_j} x_j(t-\tau_{k_j}) \\ \le\ & \alpha_{k_j} x_j^T(t-\tau_{k_j}) P_{k_j} x_j(t-\tau_{k_j}) \\ & + \alpha_{k_j}^{-1} x_j^T(t) P_{k_j} \bar{A}_{k_j} P_{k_j}^{-1} \bar{A}_{k_j}^T P_{k_j} x_j(t), \quad \forall \alpha_{k_j} > 0. \end{aligned}$$

Applying Razumikhin condition with $p(s) = p_{k_j} s$, $p_{k_j} > 1$, we obtain

$$x_j^T(t-\tau_{k_j}) P_{k_j} x_j(t-\tau_{k_j}) \le p_{k_j} x_j^T(t) P_{k_j} x_j(t) \quad (24)$$

for

$$V_{k_j}(t+\theta, x_j(t+\theta)) < p_{k_j} V_{k_j}(t, x_j(t)) \quad \forall \theta \in [-\tau_{k_j}, 0].$$

Let

$$S_{k_j} := -(A_{k_j}^T P_{k_j} + P_{k_j} A_{k_j} + p_{k_j}\alpha_{k_j} P_{k_j} + \alpha_{k_j}^{-1} P_{k_j}\bar{A}_{k_j} P_{k_j}^{-1}\bar{A}_{k_j}^T P_{k_j}) \quad (25)$$

we have

$$\dot{V}_{k_j}(t, x_j) \leq -x_j^T(t) S_{k_j} x_j(t). \quad (26)$$

Because $\Sigma_t \in \tilde{\mathcal{A}}$, we have $S_{k_j} > 0$ from Lemma 2.1. Furthermore we can select $w(s) = w_{k_j} s^2$ in Lemma 3.1, such that (9) is satisfied, where $w_{k_j} := \sigma_{min}[S_{k_j}] > 0$.

Define

$$\lambda := \max_{i\in\mathcal{F}} \frac{\bar{\kappa}_i}{\kappa_i}, \quad (27)$$

and

$$\mu := \max_{i\in\mathcal{F}} \frac{\bar{\kappa}_i}{w_i}. \quad (28)$$

Now we are ready to state the main result.

Theorem 3.4: Let the dwell time be defined by $\tau_D := T^* + \tau_{max}$, where

$$T^* := \lambda\mu\lfloor \frac{\lambda - 1}{\bar{p} - 1} + 1\rfloor, \quad (29)$$

with $\bar{p} := \min_{i\in\mathcal{F}}\{p_i\} > 1$, and $\lfloor\cdot\rfloor$ being the floor integer function. Then the system (2) with $\Sigma_t \in \tilde{\mathcal{A}}$ is asymptotically stable for any switching rule $q(t) \in S[\tau_D]$.

Proof. First we claim that for all $\tau > \tau_D$, there exist $0 < \beta < 1$ and $0 < \alpha < 1$, such that $\tau \geq \bar{T} + \tau_{max}$, where

$$\bar{T} := \frac{\lambda\mu}{\alpha^2}\lceil \frac{\lambda - \alpha^2}{\alpha^2\beta(\bar{p} - 1)}\rceil. \quad (30)$$

For a given τ, to find such α and β define $\tilde{T} + \tau_{max} := \tau > \tau_D = T^* + \tau_{max}$, and consider two cases below.

1) If $\lfloor(\lambda - 1)/(\bar{p} - 1)\rfloor =: k < (\lambda - 1)/(\bar{p} - 1) < k + 1$, then can find $\Delta_1 > 0$ and $\Delta_2 > 0$ small enough, such that

$$\lceil \frac{\lambda - \alpha_1^2}{\alpha_1^2\beta(\bar{p} - 1)}\rceil = \lceil \frac{\lambda - 1}{\bar{p} - 1}\rceil = k + 1 = \lfloor \frac{\lambda - 1}{\bar{p} - 1} + 1\rfloor$$

with $\alpha_1 = (1 + \Delta_1)^{-\frac{1}{2}} < 1$ and $\beta = (1 + \Delta_2)^{-\frac{1}{2}} < 1$. Let $\tilde{T} = T^* + \epsilon$, $\epsilon > 0$. It is easy to check that

$$\frac{\lambda\mu}{\alpha_2^2}\lceil \frac{\lambda - \alpha_1^2}{\alpha_1^2\beta(\bar{p} - 1)}\rceil = \frac{\lambda\mu}{\alpha_2^2}(k + 1) \leq (k + 1)\lambda\mu + \epsilon = \tilde{T}, \quad (31)$$

where $0 < \alpha_2 = (1 + \Delta_3)^{-\frac{1}{2}} < 1$ with $0 < \Delta_3 \leq \frac{\epsilon}{(k+1)\lambda\mu}$. Now choosing $0 < \alpha = \max\{\alpha_1, \alpha_2\} < 1$, we have $\bar{T} \leq \tilde{T}$, which is straightforward from (30) and (31).

2) If $(\lambda - 1)/(\bar{p} - 1) = k > 0$ is an integer. We can similarly find $0 < \alpha_1 < 1$ and $0 < \beta < 1$ such that

$$\lceil \frac{\lambda - \alpha_1^2}{\alpha_1^2\beta(\bar{p} - 1)}\rceil = \lceil \frac{\lambda - 1}{\bar{p} - 1} + 1\rceil = k + 1 = \lfloor \frac{\lambda - 1}{\bar{p} - 1} + 1\rfloor$$

In the same fashion as 1), we can constructively have $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\bar{T} \leq \tilde{T}$.

This proves the first claim.

The second claim we make is that $\|x_j(t)\| \leq \alpha\delta_j$ for any $t \geq t_j + \bar{T}, t \in [t_j, t_{j+1}]$, where we assume $|\phi_j(t)|_{[t_j - \tau_{k_j}, t_j]} \leq \delta_j$. To show this fact, we can choose $\bar{\delta}_1 = \delta_j$, $\bar{\delta}_2 = \bar{\delta}_1\sqrt{\bar{\kappa}_{k_j}/\kappa_{k_j}} \geq \bar{\delta}_1$, and select $\eta = \alpha\bar{\delta}_1$ in Lemma 3.3. It is straightforward that $0 < \eta < \bar{\delta}_1 \leq \bar{\delta}_2$. Recall (19) and (20), we have

$$V_{k_j}(t, x_j) \leq \kappa_{k_j}\eta^2, \quad \text{for} \quad t \geq t_j + T, \quad (32)$$

where

$$T = \frac{Nv(\bar{\delta}_1)}{\gamma}\frac{\lceil(v(\bar{\delta}_1) - u(\eta))/a\rceil v(\bar{\delta}_1)}{\inf_{v^{-1}(u(\eta))\leq s\leq\bar{\delta}_2} w(s)} = \frac{\bar{\kappa}_{k_j}^2\lceil(v(\bar{\delta}_1) - u(\eta))/a\rceil}{\alpha^2 w_{k_j}\kappa_{k_j}} \quad (33)$$

Combining (22) and (32) yields

$$\|x_j(t)\| \leq \alpha\delta_j, \quad \text{for} \quad t \geq t_j + T. \quad (34)$$

Now choosing $a = \beta(p_{k_j} - 1)\kappa_{k_j}\eta^2$, we have

$$T = \frac{\bar{\kappa}_{k_j}^2\lceil\frac{\frac{\bar{\kappa}_{k_j}}{\kappa_{k_j}} - \alpha^2}{\alpha^2\beta(p_{k_j} - 1)}\rceil}{\alpha^2 w_{k_j}\kappa_{k_j}} \leq \bar{T} \quad (35)$$

Therefore from (34) and (35) we have

$$|x_j|_{[t_j + \bar{T}, t_{j+1}]} \leq \alpha\delta_j, \quad (36)$$

as claimed.

Now recall that $t_{j+1} - t_j > \tau_D$. Therefore $t_{j+1} - t_j \geq \bar{T} + \tau_{max} \geq \bar{T} + \tau_{k_{j+1}}$. Also notice that $\phi_{j+1}(t) = x_j(t), t \in [t_{j+1} - \tau_{k_{j+1}}, t_{j+1}]$. We have

$$|\phi_{j+1}|_{[t_{j+1} - \tau_{k_{j+1}}, t_{j+1}]} = |x_j|_{[t_{j+1} - \tau_{k_{j+1}}, t_{j+1}]} \leq |x_j|_{[t_j + \bar{T}, t_{j+1}]} \leq \alpha\delta_j := \delta_{j+1} \quad (37)$$

and δ_0 is defined as $\delta_0 := |\phi|_{[-\tau_{max}, 0]} \geq |\phi|_{[-\tau_{k_0}, 0]}$. Therefore we obtain a convergent sequence $\{\delta_i\}, i = 0, 1, 2, \ldots$, where $\delta_i = \alpha^i\delta_0$.

Meanwhile, (14) implies

$$|x_j|_{[t - \tau_{k_j}, t]} \leq \sqrt{\frac{\bar{\kappa}_{k_j}}{\kappa_{k_j}}}|x_j|_{[t_j - \tau_{k_j}, t_j]}, \quad \forall t \in [t_j, t_{j+1}]. \quad (38)$$

Hence

$$\sup_{t\in[t_j, t_{j+1}]}\|x_j(t)\| \leq \sup_{t\in[t_j, t_{j+1}]}|x_j|_{[t - \tau_{k_j}, t]} \leq \sqrt{\lambda}|x_j|_{[t_j - \tau_{k_j}, t_j]} \leq \sqrt{\lambda}\delta_j = \alpha^j\sqrt{\lambda}\delta_0, \quad (39)$$

which implies the asymptotic stability of the switched time delay system Σ_t with the switching signal $q(t) \in S_{[\tau_D]}$. ■

B. The Case with Delay Dependent Criterion

In a similar fashion, we can investigate the stability of the switched time delay system Σ_t of (2) under the assumption that $\Sigma_t \in \bar{\mathcal{A}}_d$. Hence each candidate system Σ_i, $i \in \mathcal{F}$ is delay-dependently asymptotically stable satisfying (7). We assume $\tau_D^d > 2\tau_{max}$ in this scenario. Similar to the proof of Theorem 3.4, we consider an arbitrary switching interval $[t_j, t_{j+1})$ of the piecewise switching signal $q(t) \in S[\tau_D^d]$, where the state variable $x_j(t)$ defined on this interval obeys (3). The first order model transformation [7] of (3) results in

$$\dot{x}_j(t) = (A_{k_j} + \bar{A}_{k_j})x_j(t) - \bar{A}_{k_j}\int_{-\tau_{k_j}}^{0}[A_{k_j}x_j(t+\theta) + \bar{A}_{k_j}x(t+\theta-\tau_{k_j})]d\theta \quad (40)$$

where the initial condition $\psi_j(t)$ is defined as $\psi_j(t) = x_{j-1}(t)$, $t \in [t_j - 2\tau_{k_j}, t_j]$ for $j \in \mathbb{Z}^+$, and $\psi_0(t)$ defined by

$$\psi_0(t) = \begin{cases} \phi(t), & t \in [-\tau_{max}, 0] \\ \phi(-\tau_{max}), & t \in [-2\tau_{max}, -\tau_{max}) \end{cases}$$

By using the Lyapunov-Razumikhin function (21), we obtain the time derivative of $V_{k_j}(t, x_j(t))$ along the trajectory of (40)

$$\dot{V}_{k_j}(t, x_j) = x_j^T(t)[P_{k_j}(A_{k_j} + \bar{A}_{k_j}) + (A_{k_j} + \bar{A}_{k_j})^T P_{k_j}]x_j(t) - \int_{-\tau_{k_j}}^{0}[2x_j^T(t)P_{k_j}\bar{A}_{k_j}(A_{k_j}x_j(t+\theta) + \bar{A}_{k_j}x_j(t+\theta-\tau_{k_j})]d\theta.$$

Assume $V_{k_j}(t+\theta, x_j(t+\theta)) < p(V_{k_j}(t, x_j(t)))$ for $\forall\theta \in [-2\tau_{k_j}, 0]$, where $p(s) = p_{k_j}s$, $p_{k_j} > 1$, we have [3], [14]

$$\dot{V}_{k_j}(t, x_j) \le -x_j^T(t)S_{k_j}^d x_j(t), \quad (41)$$

where

$$\begin{aligned} S_{k_j}^d := &- \{P_{k_j}(A_{k_j} + \bar{A}_{k_j}) + (A_{k_j} + \bar{A}_{k_j})^T P_{k_j} \\ &+ \tau_{k_j}[\alpha_{k_j}^{-1}P_{k_j}\bar{A}_{k_j}A_{k_j}P_{k_j}^{-1}\bar{A}_{k_j}^T A_{k_j}^T P_{k_j} \\ &+ \beta_i^{-1}P_{k_j}(\bar{A}_{k_j})^2 P_{k_j}^{-1}(\bar{A}_{k_j}^T)^2 P_{k_j} \\ &+ p_{k_j}(\alpha_{k_j} + \beta_{k_j})P_{k_j}]\}. \end{aligned} \quad (42)$$

Because $\Sigma_t \in \tilde{\mathcal{A}}_d$, we have $S_{k_j}^d > 0$ from Lemma 2.2. Therefore we can select $w(s) = w_{k_j}^d s^2$ in Lemma 3.1, such that (9) holds, where $w_{k_j}^d := \sigma_{min}[S_{k_j}^d] > 0$.

Theorem 3.5: Let the dwell time be $\tau_D^d := T_d^* + 2\tau_{max}$, where

$$T_d^* := \lambda\mu_d\lfloor\frac{\lambda - 1}{\bar{p} - 1} + 1\rfloor, \quad (43)$$

with

$$\mu_d := \max_{i\in\mathcal{F}}\frac{\bar{\kappa}_i}{w_i^d} \quad (44)$$

and the other parameters are the same as those defined in Theorem 3.4. Then, the system (2) with $\Sigma_t \in \tilde{\mathcal{A}}_d$ is asymptotically stable for any switching rule $q(t) \in S[\tau_D^d]$.

Proof. We can apply similar arguments used in the proof of Theorem 3.4 to obtain the following inequality:

$$\sup_{t\in[t_j, t_{j+1}]}\|x_j(t)\| \le \sqrt{\lambda}\delta_j^d, \quad (45)$$

where $|\psi_j(t)|_{[t_j - 2\tau_{k_j}, t_j]} \le \delta_j^d$, and $\delta_{j+1}^d = \alpha\delta_j^d$. Note that δ_0^d can be selected as

$$\delta_0^d := |\psi|_{[-2\tau_{max}, 0]} = |\phi|_{[-\tau_{max}, 0]} = \delta_0.$$

It is clear that $|\psi|_{[-2\tau_{k_0}, 0]} \le \delta_0^d$, which further implies $\delta_j^d = \delta_j$, $j \in \mathbb{Z}^+ \cup \{0\}$. The upper bound of the state variable $x(t)$ of the switched time delay systems Σ_t is bounded by a decreasing sequence $\{\delta_i\}, i = 0, 1, 2, \ldots$ converging to zero, which implies the asymptotic stability and proves this theorem. ■

The dwell time based stability analysis proposed in this paper is general in the sense that it can be used for other stability results based on Razumikhin theorems as long as the correspondingly Lyapunov functions are in quadratic forms. Particularly, Theorem 3.5 can be extended easily to the case where Σ_t has time-varying time delays and parameter uncertainties, which has important applications such as TCP (Transmission Control Protocol) congestion control of computer networks [13], [25].

IV. Conservatism Analysis

The dwell time based stability results had been obtained for switched linear systems free of delays [10], [19]. It is interesting to compare the conservatism of the results presented in this paper with those for delay free systems.

In fact, one extreme case of the switched system Σ_t is $\tau_i = 0$ and $\bar{A}_i = 0$ for $i \in \mathcal{A}$, which corresponds to the delay free scenario. For each candidate system $\dot{x} = A_i x$, a sufficient and necessary condition to guarantee asymptotic stability is $\exists P_i = P_i^T > 0$, such that $Q_i := -(A_i^T P_i + P_i A_i) > 0$. Correspondingly a dwell time based stability for such switched delay free system is $q(t) \in S_{[\tilde{\tau}_D]}$, where

$$\tilde{\tau}_D = \tilde{\mu}\ln\lambda, \quad (46)$$

where λ is defined by (27) and

$$\tilde{\mu} := \max_{i\in\mathcal{F}}\frac{\bar{\kappa}_i}{\tilde{w}_i}, \quad (47)$$

where $\tilde{w}_i := \sigma_{min}[Q_i] > 0$.

On the other hand in our case, for $\tau_i = 0$ and $\bar{A}_i = 0$, we observe that

$$\lim_{\alpha_i\to 0^+} S_i = \lim_{\alpha_i,\beta_i\to 0^+} S_i^d = Q_i, \quad i \in \mathcal{F} \quad (48)$$

from (25) and (42), which indicates $\mu = \mu_d = \tilde{\mu}$ by (28), (44), and (47). Accordingly we can select $p_i > 1$, $i \in \mathcal{F}$ sufficiently large such that $\lfloor\frac{\lambda-1}{\bar{p}-1} + 1\rfloor = 1$ in (29) and (43), and obtain

$$\tau_D = T^* = \lambda\mu = \lambda\mu_d = T_d^* = \tau_D^d. \quad (49)$$

Therefore

$$\tau_D = \tau_D^d = \lambda\tilde{\mu} > \tilde{\mu}\ln\lambda = \tilde{\tau}_D. \quad (50)$$

The dwell times derived for switched time delay systems are proportional to λ, as opposite to the logarithm of λ for switched delay free systems. This gap is due to the fact that asymptotic stability for linear delay free systems implies exponential stability. However, for time delay systems, the sufficient stability conditions based on Lyapunov-Razumikhin theorem do not guarantee exponential stability. As a matter of fact, the exponential estimates for time delay systems require additional assumptions besides asymptotic stability [15].

It is noticeable that stability conditions for switched time delay systems are also considered in [22], [23], where the authors give a sufficient condition to guarantee *uniform* stability (see Theorem 6.1 of [22] for the notation and details): $\Gamma e^{L(\Lambda+h)} \leq 1$. Apparently, this condition does not hold for the switched system (2) because in our case $\Gamma = 1$, and hence

$$\Gamma e^{L(\Lambda+h)} = e^{L(\Lambda+h)} > 1, \ \ \forall \ \ \Lambda > 0, \ L > 0, \ h > 0.$$

The reader is referred to the journal version of this paper, [27], for numerical examples where the calculated dwell times for switched delay systems are also compared to that of delay free systems.

V. Concluding Remarks

We provided stability analysis for switched linear systems with time delays, where each candidate system is assumed to be delay-independently or delay-dependently asymptotically stable. We showed the existence of a dwell time of the switching signal, such that the switched time delay system is asymptotically stable independent of the trajectory. The dwell time values for both scenarios are constructively given. The results are compared with the dwell time conditions for switched delay free systems. Optimization of the minimum dwell times we have derived, in terms of the free parameters appearing in the LMI conditions, is an interesting open problem.

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