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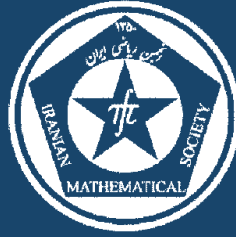Some of the authors of this publication are also working on these related projects:

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STRONGLY CLEAN TRIANGULAR MATRIX RINGS WITH ENDOMORPHISMS

H. CHEN*, H. KOSE AND Y. KURTULMAZ

(Communicated by Davod Khojasteh Salkuyeh)

ABSTRACT. A ring R is strongly clean provided that every element in R is the sum of an idempotent and a unit that commutate. Let $T_n(R, \sigma)$ be the skew triangular matrix ring over a local ring R where σ is an endomorphism of R. We show that $T_2(R, \sigma)$ is strongly clean if and only if for any $a \in 1 + J(R), b \in J(R)$, $l_a - r_{\sigma(b)} : R \to R$ is surjective. Further, $T_3(R, \sigma)$ is strongly clean if $l_a - r_{\sigma(b)}, l_a - r_{\sigma^2(b)}$ and $l_b - r_{\sigma(a)}$ are surjective for any $a \in U(R), b \in J(R)$. The necessary condition for $T_3(R, \sigma)$ to be strongly clean is also obtained.
Keywords: Strongly clean rings, skew triangular matrix rings, local rings.
MSC(2010): Primary: 16D70; Secondary: 16E50.

1. Introduction

We say that an element $a \in R$ is strongly clean provided that there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $eu = ue$. A ring R is strongly clean in case every element in R is strongly clean. Strong cleanness over commutative rings was extensively studied by many authors from very different view points (cf. [1–3] and [5–8]). The problem of deciding the strong cleanness is considerably harder. So far, one considers strong cleanness only over commutative local rings, where a ring R is local provided that R has only a maximal ideal. As is well known, a ring R is local if and only if for any $x \in R$, either x or $1 - x$ is invertible. In [6], Li characterizes when 2×2 matrix ring $M_2(R)$ over a commutative local ring R is strongly clean. The strong cleanness of triangular matrix rings over such a ring is also investigated in [1]. For more discussion of strong cleanness, we refer the reader to [4] and [7].

Let R be a ring, and let σ be an endomorphism of R. Let $T_n(R, \sigma)$ be the set of all upper triangular matrices over the rings R. For any $(a_{ij}), (b_{ij}) \in$

$T_n(R, \sigma)$, we define $(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$, and $(a_{ij})(b_{ij}) = (c_{ij})$ where $c_{ij} = \sum_{k=i}^{n} a_{ik} \sigma^{k-i}(b_{kj})$. Then $T_n(R, \sigma)$ is a ring under the preceding addition and multiplication. Clearly, $T_n(R, \sigma)$ will be $T_n(R)$ only when σ is the identity morphism. Let $a \in R$. $l_a : R \to R$ and $r_a : R \to R$ denote, respectively, the abelian group endomorphisms given by $l_a(r) = ar$ and $r_a(r) = ra$ for all $r \in R$. Thus, $l_a - r_b$ is an abelian group endomorphism such that $(l_a - r_b)(r) = ar - rb$ for any $r \in R$.

The aim of this note is to investigate the strong cleanness over a noncommutative local ring with an endomorphism. We prove that $T_2(R, \sigma)$ is strongly clean if and only if for any $a \in 1 + J(R), b \in J(R), l_a - r_{\sigma(b)} : R \to R$ is surjective. Further, $T_3(R, \sigma)$ is strongly clean if $l_a - r_{\sigma(b)}, l_a - r_{\sigma^2(b)}$ and $l_b - r_{\sigma(a)}$ are surjective for any $a \in U(R), b \in J(R)$. The converse is also true if 1_R can not be the sum of two units. These extend the known results of strong cleanness of matrices over commutative local rings as well.

Throughout, every ring is associative with an identity 1. $J(R)$ and $U(R)$ will denote, respectively, the Jacobson radical and the group of units in the ring R.

2. The rings $T_2(R, \sigma)$

As is well known, the triangular matrix ring $T_2(R)$ over a local ring R is strongly clean if and only if for any $a \in 1 + J(R), b \in J(R), l_a - r_b : R \to R$ is surjective (cf. [5, Theorem 2.2.1]). We extend this result to the skew triangular matrix ring with an endomorphism.

Theorem 2.1. *Let R be a local ring, and let σ be an endomorphism of R. Then the following are equivalent:*

(1) *$T_2(R, \sigma)$ is strongly clean.*
(2) *If $a \in 1 + J(R), b \in J(R)$, then $l_a - r_{\sigma(b)} : R \to R$ is surjective.*

Proof. $(1) \Rightarrow (2)$ Let $a \in 1 + J(R), b \in J(R), v \in R$. Then $A = \begin{pmatrix} a & -v \\ 0 & b \end{pmatrix} \in T_2(R, \sigma)$. By hypothesis, there exists an idempotent $E = \begin{pmatrix} e & x \\ 0 & f \end{pmatrix} \in T_2(R, \sigma)$ such that $A - E \in U(T_2(R, \sigma))$ and $AE = EA$. Since R is local, all idempotents in R are 0 and 1. Thus, we see that $e = 0, f = 1$; otherwise, $A - E \notin U(T_2(R, \sigma))$. So $E = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$. It follows from $AE = EA$ that $v + x\sigma(b) = ax$, and so $ax - v = x\sigma(b)$. Therefore we conclude that $l_a - r_{\sigma(b)} : R \to R$ is surjective.

$(2) \Rightarrow (1)$ Let $A = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in T_2(R, \sigma)$. If $a, b \in U(R)$, then $A \in U(T_2(R, \sigma))$ is strongly clean. If $a, b \in J(R)$, then $A - I_2 \in U(T_2(R, \sigma))$;

hence, $A \in T_2(R, \sigma)$ is strongly clean. Assume that $a \in U(R), b \in J(R)$. If $a - 1 \in U(R)$, then $A - I_2 \in U\big(T_2(R, \sigma)\big)$; hence, $A \in T_2(R, \sigma)$ is strongly clean. If $a - 1 \in J(R)$, by hypothesis, $l_a - r_{\sigma(b)} : R \to R$ is surjective. Thus,

$$ax - x\sigma(b) = -v \text{ for some } x \in R. \text{ Choose } E = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix} \in T_2(R, \sigma). \text{ Then}$$

$E = E^2 \in T_2(R, \sigma)$. In addition, $AE = EA$ and $A - E \in U\big(T_2(R, \sigma)\big)$; hence, $A \in T_2(R, \sigma)$ is strongly clean. Assume that $a \in J(R), b \in U(R)$. If $b - 1 \in U(R)$, then $A - I_2 \in U\big(T_2(R, \sigma)\big)$; hence, $A \in T_2(R, \sigma)$ is strongly clean. If $b - 1 \in J(R)$, by hypothesis, $l_{1-a} - r_{\sigma(1-b)} : R \to R$ is surjective. Thus, $(1 - a)x - x\sigma(1 - b) = v$ for some $x \in R$. As σ is an endomorphism of R, $\sigma(1 - b) = \sigma(1) - \sigma(b) = 1 - \sigma(b)$. Hence, $ax - x\sigma(b) = -v$. Choose

$$E = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \in T_2(R, \sigma). \text{ Then } E = E^2 \in T_2(R, \sigma). \text{ In addition, } AE = EA$$

and $A - E \in U\big(T_2(R, \sigma)\big)$; hence, $A \in T_2(R, \sigma)$ is strongly clean. Therefore we conclude that $A \in T_2(R, \sigma)$ is strongly clean in any case. $\square$

Following Diesl [5], a local ring R is bleached provided that for any $a \in U(R), b \in J(R)$, $l_a - r_b, l_b - r_a$ are both surjective.

Corollary 2.2. *Let R be a local ring, and let σ be an endomorphism of R. If R is bleached, then $T_2(R, \sigma)$ is strongly clean.*

Proof. Let $a \in 1 + J(R), b \in J(R)$. Then $1 - a \in J(R), 1 - b \in 1 + J(R) \subseteq U(R)$. This implies that $\sigma(1 - b) \in U(R)$. By hypothesis, $l_{1-a} - r_{\sigma(1-b)} : R \to R$ is surjective. For any $v \in R$, we can find some $x \in R$ such that $(1 - a)x - x\sigma(1 - b) = -v$. That is, $ax - x\sigma(b) = v$. This implies that $l_a - r_{\sigma(b)} : R \to R$ is surjective. According to Theorem 2.1, $T_2(R, \sigma)$ is strongly clean. $\square$

Corollary 2.3. *Let R be a local ring, and let σ be an endomorphism of R. If $J(R)$ is nil, then $T_2(R, \sigma)$ is strongly clean.*

Proof. Let $a \in U(R), b \in J(R)$. Then we can find some $n \in \mathbb{N}$ such that $b^n = 0$. For any $v \in R$, we choose $x = \big(l_{a^{-1}} + l_{a^{-2}}r_b + \cdots + l_{a^{-n}}r_{b^{n-1}}\big)(v)$. One easily checks that

$$
\begin{aligned}
& \big(l_a - r_b\big)(x) \\
=~& \big(l_a - r_b\big)\big(l_{a^{-1}} + l_{a^{-2}}r_b + \cdots + l_{a^{-n}}r_{b^{n-1}}\big)(v) \\
=~& \big(v + a^{-1}vb + \cdots + a^{-n+1}vb^{n-1}\big) - \big(a^{-1}vb + \cdots + a^{-n}vb^n\big) \\
=~& v.
\end{aligned}
$$

This shows that $l_a - r_b : R \to R$ is surjective. Likewise, $l_b - r_a : R \to R$ is surjective. This implies that R is bleached. In light of Corollary 2.2, $T_2(R, \sigma)$ is strongly clean. $\square$

Example 2.4. *Let $\mathbb{Z}_{p^n} = \mathbb{Z}/p^n\mathbb{Z}(p \text{ is prime}, n \in \mathbb{N})$, and let σ be an endomorphism of $\mathbb{Z}_{p^n}$. Then $T_2(\mathbb{Z}_{p^n}, \sigma)$ is strongly clean. As $\mathbb{Z}_{p^n}$ is a local ring with the Jacobson radical $p\mathbb{Z}_{p^n}$, $J\big(\mathbb{Z}_{p^n}\big)$ is nil, and we are done by Corollary 2.3.*

Example 2.5. *Let* $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$, *let* $R = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4\}$, *and let* $\sigma :$
$R \to R, \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$. *Then* $T_2(R, \sigma)$ *is strongly clean. Obviously,*
σ *is an endomorphism of* R. *It is easy to check that* $J(R) = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in$
$2\mathbb{Z}_4, b \in \mathbb{Z}_4\}$, *and then* $R/J(R) \cong \mathbb{Z}_2$ *is a field. Thus,* R *is a local ring.*
In addition, $(J(R))^4 = 0$, *thus* $J(R)$ *is nil. Therefore we are through from*
Corollary 2.3.

Let σ be an endomorphism of $\mathbb{Z}_{3^n}[x]/(x^2+x+1)$. Analogously, $T_2\big(\mathbb{Z}_{3^n}[x]/(x^2+$ $x+1), \sigma\big)$ is strongly clean.

We say that an element $a \in R$ is very clean provided that for any $x \in R$ there exists an idempotent e such that $ex = xe$ and either $x - e \in U(R)$ or $x + e \in U(R)$. A ring R is very clean in case every element in R is very clean. Every clean ring may be not strongly clean. For instance, $\mathbb{Z}_{(3)} \bigcap \mathbb{Z}_{(5)}$ is a very clean ring, but it is not strongly clean.

Proposition 2.6. *Let* R *be a local ring, and let* σ *be an endomorphism of* R. *Then the following are equivalent:*

    (1) $T_2(R, \sigma)$ *is very clean.*
    (2) $2 \in U(R)$ *or* $T_2(R, \sigma)$ *is strongly clean.*

Proof. (1) $\Rightarrow$ (2) Suppose that $2 \in J(R)$. Let $a \in 1 + J(R), b \in J(R), v \in R$. Then $A = \begin{pmatrix} a & -v \\ 0 & b \end{pmatrix} \in T_2(R, \sigma)$. By hypothesis, there exists an idempotent
$E = \begin{pmatrix} e & x \\ 0 & f \end{pmatrix} \in T_2(R, \sigma)$ such that $A + E$ or $A - E \in U\big(T_2(R, \sigma)\big)$ and $AE = EA$. Since R is local, all idempotents in R are 0 and 1.

If $A - E \in U\big(T_2(R, \sigma)\big)$, then we see that $e = 0, f = 1$; otherwise, $A - E \in U\big(T_2(R, \sigma)\big)$. So $E = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$. It follows from $AE = EA$ that $v + x\sigma(b) = ax$, and so $ax - v = x\sigma(b)$. Therefore we conclude that $l_a - r_{\sigma(b)} : R \to R$ is surjective.

If $A + E \in U\big(T_2(R, \sigma)\big)$, then we see that $f = 1$; otherwise, $A - E \notin U\big(T_2(R, \sigma)\big)$. If $e = 0$, then $E = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$. It follows from $AE = EA$ that $v + x\sigma(b) = ax$, and so $ax - v = x\sigma(b)$. Therefore we conclude that $l_a - r_{\sigma(b)} : R \to R$ is surjective. If $e = 1$, then $2 \in U(R)$, a contradiction. Therefore $T_2(R, \sigma)$ is strongly clean by Theorem 2.1.

(2) $\Rightarrow$ (1) If $T_2(R, \sigma)$ is strongly clean, then $T_2(R, \sigma)$ is very clean. Now we assume that $2 \in U(R)$. Let $A = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in T_2(R, \sigma)$. If $a, b \in U(R)$, then $A \in U\big(T_2(R, \sigma)\big)$ is very clean. If $a, b \in J(R)$, then $A - I_2 \in U\big(T_2(R, \sigma)\big)$;

hence, $A \in T_2(R, \sigma)$ is very clean. Assume that $a \in U(R), b \in J(R)$. If $a - 1 \in U(R)$, then $A - I_2 \in U\big(T_2(R, \sigma)\big)$; hence, $A \in T_2(R, \sigma)$ is very clean.

If $a - 1 \in J(R)$, then $A + I_2 \in U\big(T_2(R, \sigma)\big)$. Hence, A is very clean.

Assume that $a \in J(R), b \in U(R)$. If $b - 1 \in U(R)$, then $A - I_2 \in U\big(T_2(R, \sigma)\big)$; hence, $A \in T_2(R, \sigma)$ is very clean.

If $b - 1 \in J(R)$, then $A + I_2 \in U\big(T_2(R, \sigma)\big)$; hence, $A \in T_2(R, \sigma)$ is very clean. Therefore we conclude that $A \in T_2(R, \sigma)$ is very clean in any case. $\qquad \square$

Example 2.7. *Let* $\mathbb{Z}_{(3)} = \{ \frac{m}{n} \mid m, n \in \mathbb{Z}, 3 \nmid n \}$. *Then* $\mathbb{Z}_{(3)}$ *is a local ring in which* $2 \in U(R)$. *In view of Proposition 2.6,* $T_2(\mathbb{Z}_{(3)}, \sigma)$ *is very clean for any endomorphism* σ *of* R.

3. The rings $T_3(R, \sigma)$

The goal of this section is to investigate strong cleanness of 3×3 skew triangular matrix rings with endomorphisms over a local ring.

Theorem 3.1. *Let* R *be a local ring, and let* σ *be an endomorphism of* R. *If* $l_a - r_{\sigma(b)}, l_a - r_{\sigma^2(b)}$ *and* $l_b - r_{\sigma(a)}$ *are surjective for any* $a \in U(R), b \in J(R)$, *then* $T_3(R, \sigma)$ *is strongly clean.*

Proof. Let $A = (a_{ij}) \in T_3(R, \sigma)$.

Case 1. $a_{11}, a_{22}, a_{33} \in J(R)$. Then $A = I_3 + (A - I_3)$, and so $A - I_3 \in U(T_3(R, \sigma))$. Then $A \in T_3(R, \sigma)$ is strongly clean.

Case 2. $a_{11} \in U(R), a_{22}, a_{33} \in J(R)$. By hypothesis, we can find some $e_{12} \in R$ such that $a_{11}e_{12} - e_{12}\sigma(a_{22}) = -a_{12}$. Further, we can find some $e_{13} \in R$ such that $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = e_{12}\sigma(a_{23}) - a_{13}$. Choose $E = \begin{pmatrix} 0 & e_{12} & e_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in T_3(R, \sigma)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(T_3(R, \sigma))$. In addition,

$$EA = \begin{pmatrix} 0 & e_{12}\sigma(a_{22}) & e_{12}\sigma(a_{23}) + e_{13}\sigma^2(a_{33}) \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

$$AE = \begin{pmatrix} 0 & a_{11}e_{12} + a_{12} & a_{11}e_{13} + a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

and so $EA = AE$. Hence $A \in T_3(R, \sigma)$ is strongly clean.

Case 3. $a_{11} \in J(R), a_{22} \in U(R), a_{33} \in J(R)$. Clearly $\sigma(a_{22}) \in U(R)$. By hypothesis, we can find some $e_{12} \in R$ such that $a_{11}e_{12} - e_{12}\sigma(a_{22}) = a_{12}$. Further, we have some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}\sigma(a_{33}) = -a_{23}$. It follows that $-a_{11}e_{12}\sigma(e_{23}) + a_{12}\sigma(e_{23}) = -e_{12}\sigma(a_{22})\sigma(e_{23}) = e_{12}\sigma(a_{23}) -$

$e_{12}\sigma(e_{23})\sigma^2(a_{33})$. Choose $E = \begin{pmatrix} 1 & e_{12} & -e_{12}\sigma(e_{23}) \\ 0 & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix} \in T_3(R,\sigma)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(T_3(R,\sigma))$. In addition,

$$EA = \begin{pmatrix} a_{11} & a_{12} + e_{12}\sigma(a_{22}) & a_{13} + e_{12}\sigma(a_{23}) - e_{12}\sigma(e_{23})\sigma^2(a_{33}) \\ 0 & 0 & e_{23}\sigma(a_{33}) \\ 0 & 0 & a_{33} \end{pmatrix},$$

$$AE = \begin{pmatrix} a_{11} & a_{11}e_{12} & -a_{11}e_{12}\sigma(e_{23}) + a_{12}\sigma(e_{23}) + a_{13} \\ 0 & 0 & a_{22}e_{23} + a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

and so $EA = AE$. Hence $A \in T_3(R,\sigma)$ is strongly clean.

Case 4. $a_{11}, a_{22} \in J(R), a_{33} \in U(R)$. By hypothesis, we can find some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}\sigma(a_{33}) = a_{23}$. Clearly, $\sigma(a_{33}) \in U(R)$. Thus, we can find some $e_{13} \in R$ such that $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = a_{13} - a_{12}\sigma(e_{23})$. Choose $E = \begin{pmatrix} 1 & 0 & e_{13} \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R,\sigma)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(T_3(R,\sigma))$. In addition,

$$EA = \begin{pmatrix} a_{11} & a_{12} & a_{13} + e_{13}\sigma^2(a_{33}) \\ 0 & a_{22} & a_{23} + e_{23}\sigma(a_{33}) \\ 0 & 0 & 0 \end{pmatrix},$$

$$AE = \begin{pmatrix} a_{11} & a_{12} & a_{11}e_{13} + a_{12}\sigma(e_{23}) \\ 0 & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix},$$

and so $EA = AE$. Hence $A \in T_3(R,\sigma)$ is strongly clean.

Case 5. $a_{11} \in J(R), a_{22}, a_{33} \in U(R)$. By hypothesis, we can find some $e_{12} \in R$ such that $a_{11}e_{12} - e_{12}\sigma(a_{22}) = a_{12}$. Further, we can find some $e_{13} \in R$ such that $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = a_{13} + e_{12}\sigma(e_{23})$. Choose $E = \begin{pmatrix} 1 & e_{12} & e_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R,\sigma)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(T_3(R,\sigma))$. In addition,

$$EA = \begin{pmatrix} a_{11} & a_{12} + e_{12}\sigma(a_{22}) & a_{13} + e_{12}\sigma(a_{23}) + e_{13}\sigma^2(a_{33}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$AE = \begin{pmatrix} a_{11} & a_{11}e_{12} & a_{11}e_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and so $EA = AE$. Hence $A \in T_3(R,\sigma)$ is strongly clean.

Case 6. $a_{11} \in U(R), a_{22} \in J(R), a_{33} \in U(R)$. By hypothesis, we can find some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}\sigma(a_{33}) = a_{23}$. Further, we can find some $e_{12} \in R$ such that $a_{11}e_{12} - e_{12}\sigma(a_{22}) = -a_{12}$. It is easy to verify that

$$e_{12}\sigma(a_{23}) + e_{12}\sigma(e_{23})\sigma^2(a_{33}) = e_{12}\sigma(a_{22}e_{23}) = a_{11}e_{12}\sigma(e_{23}) + a_{12}\sigma(e_{23}).$$

Choose $E = \begin{pmatrix} 0 & e_{12} & e_{12}\sigma(e_{23}) \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix} \in T_3(R,\sigma)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(T_3(R,\sigma))$. In addition,

$$EA = \begin{pmatrix} 0 & e_{12}\sigma(a_{22}) & e_{12}\sigma(a_{23}) + e_{12}\sigma(e_{23})\sigma^2(a_{33}) \\ 0 & a_{22} & a_{23} + e_{23}\sigma(a_{33}) \\ 0 & 0 & 0 \end{pmatrix},$$

$$AE = \begin{pmatrix} 0 & a_{11}e_{12} + a_{12} & a_{11}e_{12}\sigma(e_{23}) + a_{12}\sigma(e_{23}) \\ 0 & a_{22} & a_{22}e_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

and so $EA = AE$. Hence $A \in T_3(R,\sigma)$ is strongly clean.

Case 7. $a_{11}, a_{22} \in U(R), a_{33} \in J(R)$. By hypothesis, we can find some $e_{23} \in R$ such that $a_{22}e_{23} - e_{23}\sigma(a_{33}) = -a_{23}$. Further, we can find some $e_{13} \in R$ such that $a_{11}e_{13} - e_{13}\sigma^2(a_{33}) = -a_{13} - a_{12}\sigma(e_{23})$. Choose $E = \begin{pmatrix} 0 & 0 & e_{13} \\ 0 & 0 & e_{23} \\ 0 & 0 & 1 \end{pmatrix} \in T_3(R,\sigma)$. Then $E = E^2$, and $A = E + (A - E)$, where $A - E \in U(T_3(R,\sigma))$. In addition,

$$EA = \begin{pmatrix} 0 & 0 & e_{13}\sigma^2(a_{33}) \\ 0 & 0 & e_{23}\sigma(a_{33}) \\ 0 & 0 & a_{33} \end{pmatrix},$$

$$AE = \begin{pmatrix} 0 & 0 & a_{11}e_{13} + a_{12}\sigma(e_{23}) + a_{13} \\ 0 & 0 & a_{22}e_{23} + a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

and so $EA = AE$. Hence $A \in T_3(R,\sigma)$ is strongly clean.

Case 8. $a_{11}, a_{22}, a_{33} \in U(R)$. Then $A = 0 + A$, where $A \in U(T_3(R,\sigma))$. Hence $A \in T_3(R,\sigma)$ is strongly clean.

Therefore we conclude that $T_3(R,\sigma)$ is strongly clean.   □

Corollary 3.2. *Let R be a local ring, and let σ be an endomorphism of R. If $J(R)$ is nil, then $T_3(R,\sigma)$ is strongly clean.*

Proof. Let $a \in U(R), b \in J(R)$. Then we can find some $n \in \mathbb{N}$ such that $b^n = 0$; hence, $(\sigma(b))^n = 0$. For any $v \in R$, we choose $x = (l_{a^{-1}} + l_{a^{-2}}r_{\sigma(b)} + \cdots +$

$l_{a^{-n}} r_{(\sigma(b))^{n-1}})(v)$. One easily checks that

$$
\begin{aligned}
& \big(l_a - r_{\sigma(b)}\big)(x) \\
= \ & \big(l_a - r_{\sigma(b)}\big)\big(l_{a^{-1}} + l_{a^{-2}} r_{\sigma(b)} + \cdots + l_{a^{-n}} r_{(\sigma(b))^{n-1}}\big)(v) \\
= \ & \big(v + a^{-1} v \sigma(b) + \cdots + a^{-n+1} v\big(\sigma(b)\big)^{n-1}\big) - \big(a^{-1} v \sigma(b) + \cdots + a^{-n} v\big(\sigma(b)\big)^{n}\big) \\
= \ & v.
\end{aligned}
$$

This shows that $l_a - r_{\sigma(b)} : R \to R$ is surjective. Likewise, $l_a - r_{\sigma^2(b)}, l_b - r_{\sigma(a)} : R \to R$ are surjective. Therefore $T_3(R, \sigma)$ is strongly clean by Theorem 3.1. $\square$

Corollary 3.3. *Let R be a bleached ring, and let σ be an endomorphism of R. If $\sigma\big(J(R)\big) \subseteq J(R)$, then $T_3(R, \sigma)$ is strongly clean.*

Proof. Let $a \in U(R), b \in J(R)$. Since $\sigma\big(J(R)\big) \subseteq J(R)$, we see that $\sigma(b), \sigma^2(b) \in J(R)$. Clearly, $\sigma(a) \in U(R)$. As R is bleached, we get $l_a - r_{\sigma(b)}, l_a - r_{\sigma^2(b)}$ and $l_b - r_{\sigma(a)}$ are surjective . According to Theorem 3.1, the result follows. $\square$

Example 3.4. *Let $R = \big(\mathbb{Z}[x]\big)_{(x)}$ be the localization of $\mathbb{Z}[x]$ at the prime ideal (x). Define $\sigma : R \to R$ given by $\sigma\big(\frac{f(x)}{g(x)}\big) = \frac{f(0)}{g(0)}$. Then $\sigma : R \to R$ is an endomorphism over R. Moreover, R is a bleached local ring and $\sigma\big(J(R)\big) \subseteq J(R)$. In light of Corollary 3.3, $T_3(R, \sigma)$ is strongly clean.*

4. A necessary condition

This section is concerned with the necessary condition on a local ring R under which the skew triangular matrix ring $T_3(R, \sigma)$ is strongly clean.

Theorem 4.1. *Let R be a local ring, and let σ be an endomorphism of R. If $T_3(R, \sigma)$ is strongly clean, then $l_a - r_{\sigma(b)}, l_a - r_{\sigma^2(b)}$ and $l_b - r_{\sigma(a)}$ are surjective for any $a \in 1 + J(R), b \in J(R)$.*

Proof. Let $a \in 1 + J(R), b \in J(R)$. Choose $E = \begin{pmatrix} I_2 & \\ & 0 \end{pmatrix}$. Then $T_2(R, \sigma) \cong ET_3(R, \sigma)E$, and so $T_2(R, \sigma)$ is strongly clean. According to Theorem 2.1, $l_a - r_{\sigma(b)}$ is surjective. As $1 - b \in 1 + J(R)$ and $1 - a \in J(R)$, by the preceding discussion, we see that $l_{1-b} - r_{\sigma(1-a)} : R \to R$ is surjective. For any $v \in R$, we can find some $x \in R$ such that $(1 - b)x - x\sigma(1 - a) = -v$. This implies that $bx - x\sigma(a) = v$. Hence, $l_b - r_{\sigma(a)} : R \to R$ is surjective.

Let $v \in R$ and let $A = \begin{pmatrix} b & 0 & v \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix} \in T_3(R, \sigma)$. Then we can find an idempotent $E = (e_{ij}) \in T_3(R, \sigma)$ such that $A - E \in U\big(T_3(R, \sigma)\big)$ and $EA = AE$. This implies that $e_{11}, e_{22}, e_{33} \in R$ are all idempotents. As $a \in 1 + J(R), b \in J(R)$, we get $e_{11} = 1, e_{22} = 1$ and $e_{33} = 0$; otherwise, $A - E$ is not invertible.

As $E = E^2$, we see that $E = \begin{pmatrix} 1 & 0 & e_{13} \\ 0 & 1 & e_{23} \\ 0 & 0 & 0 \end{pmatrix}$ for some $e_{13}, e_{23} \in R$. Moreover, we see that

$$\begin{pmatrix} b & 0 & be_{13} \\ 0 & b & be_{23} \\ 0 & 0 & 0 \end{pmatrix} = AE = EA = \begin{pmatrix} b & 0 & v + e_{13}\sigma^2(a) \\ 0 & b & e_{23}\sigma(a) \\ 0 & 0 & 0 \end{pmatrix},$$

and so $be_{13} - e_{13}\sigma^2(a) = v$. This means that $l_b - r_{\sigma^2(a)} : R \to R$ is surjective. As $1 - a \in J(R)$ and $1 - b \in 1 + J(R)$, by the preceding discussion, $l_{1-a} - r_{\sigma^2(1-b)} : R \to R$ is surjective. Thus, we can find some $x \in R$ such that $(1 - a)x - x\sigma^2(1 - b) = -v$. This implies that $ax - x\sigma^2(b) = v$, and so $l_a - r_{\sigma^2(b)}$ is surjective, as desired. $\qquad\square$

Corollary 4.2. *Let R be a local ring in which 1_R is not the sum of two units, and let σ be an endomorphism of R. Then the following are equivalent:*

(1) $T_3(R, \sigma)$ *is strongly clean.*
(2) $l_a - r_{\sigma(b)}, l_a - r_{\sigma^2(b)}$ *are surjective for any $a \in 1 + J(R), b \in J(R)$.*

Proof. $(1) \Rightarrow (2)$ is obvious from Theorem 4.1.

$(2) \Rightarrow (1)$ For any $a \in 1 + J(R), b \in J(R)$, as in the proof of Theorem 4.1, we see that $l_b - r_{\sigma(a)} : R \to R$ is surjective. Obviously, $1 + J(R) = U(R)$. For any $u \in U(R)$, we know that either $u - 1 \in J(R)$ or $u - 1 \in U(R)$. Thus $u \in 1 + J(R)$; otherwise, $1 = u + (1 - u)$ is the sum of two units, a contradiction. Therefore $U(R) = 1 + J(R)$. According to Theorem 3.1, $T_3(R, \sigma)$ is strongly clean. $\qquad\square$

Corollary 4.3. *Let R be a local ring in which 1 is not the sum of two units, and let $\sigma = \sigma^2$ be an endomorphism of R. Then the following are equivalent:*

(1) $T_2(R, \sigma)$ *is strongly clean.*
(2) $T_3(R, \sigma)$ *is strongly clean.*
(3) $l_a - r_{\sigma(b)}$ *is surjective for any $a \in 1 + J(R), b \in J(R)$.*

Proof. $(1) \Leftrightarrow (3)$ is proved by Theorem 2.1.

$(2) \Leftrightarrow (3)$ is obvious from Corollary 4.2. $\qquad\square$

Example 4.4. *Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$, and let $G = \{1, g\}$ be the abelian group of order 2. Let $\sigma : \mathbb{Z}_4 G \to \mathbb{Z}_4 G, a + bg \to a + b$ for any $a + bg \in \mathbb{Z}_4 G$. Then $T_2(\mathbb{Z}_4 G, \sigma)$ and $T_3(\mathbb{Z}_4 G, \sigma)$ are strongly clean. Clearly, $\mathbb{Z}_4$ is a local ring with the Jacobson radical $2\mathbb{Z}_4 = \{\overline{0}, \overline{2}\}$. It is easy to verify that $a + bg \in U(\mathbb{Z}_4 G)$ if and only if $a + b \in U(\mathbb{Z}_4)$. Thus, $J(\mathbb{Z}_4 G) = \{a + bg \mid a + b \in J(\mathbb{Z}_4)\}$. If $(a + bg) + (c + dg) = 1$, then $a + c = 1$ and $b + d = 0$, and so $a + b + c + d = 1$. If $a + b$ and $c + d \notin U(\mathbb{Z}_4)$, then $a + b, c + d = 0, 2$. Hence, $a + b + c + d = 0, 2$, a contradiction. Thus, $a + b \in U(\mathbb{Z}_4)$ or $c + d \in U(\mathbb{Z}_4)$. That is, $a + bg \in U(\mathbb{Z}_4 G)$ or $c + dg \in U(\mathbb{Z}_4 G)$. This implies that $\mathbb{Z}_4 G$ is a local ring. If $1_{\mathbb{Z}_4 G} = (a + bg) + (c + dg)$ where*

$a + bg, c + dg \in U\big(\mathbb{Z}_4 G\big)$, then $a + c = 1$ and $b + d = 0$. This yields that $a+c+b+d = 1$. On the other hand, $a+b, c+d = 1, 3$; hence, $a+b+c+d = 0, 2$, a contradiction. This implies that $1_{\mathbb{Z}_4 G}$ is not the sum of two units. Obviously, $\sigma = \sigma^2$. As $\mathbb{Z}_4 G$ is commutative, the preceding condition (3) holds. According to Corollary 4.3, we are done.

We end this note by asking a problem: How to characterize a strongly clean n by n triangular matrix ring $T_n(R, \sigma)$ for $n \geq 4$?

Acknowledgements

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