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ON DIAGONALIZATION BY DYNAMIC OUTPUT FEEDBACK

A. BÜLENT ÖZGÜLER, VASFI ELDEM

The purpose of this paper is to draw attention to a *causality degree-dominance* property in diagonalization problems by dynamic output feedback and constant precompensator. Even in the well-investigated special case of square transfer matrices, the property of degree-dominance yields new insight into the structure of diagonalizable transfer matrices.

1. INTRODUCTION

Transfer matrix diagonalization (decoupling) problems have been receiving a renewed attention for the last few years. The reason is the desire to apply the powerful tools developed in the last decades for dealing with such difficult problems [1] as well as a new stimulant introduced by the successful resolution of the much considered Morgan's problem [2, 3].

It has been noted that the introduction of a *dynamic* precompensator brings an extra degree of freedom into decoupling problems and that very general versions of the problem can quite successfully be solved [4]. However, the interest in the classical dynamic output feedback problem with *constant* precompensator still persists due to its simplicity of implementation and because these harder problems gave way to the development of many interesting structural system theory concepts [5, 6].

The problems considered in this paper are the following:

(P): Given a strictly proper $p \times m$ transfer matrix Z determine the conditions under which a constant $m \times p$ matrix L (*precompensator*) and a proper $p \times p$ rational matrix Z_c (*feedback compensator*) exist such that

(1) $$\hat{Z} := ZL(I + Z_c ZL)^{-1}$$

is diagonal and nonsingular.

(GP): Given Z as in (P), determine the conditions for the existence of an $m \times p$

constant L and $m \times p$ proper rational Z_c such that

(2) $$\hat{Z} := Z(I + Z_c Z)^{-1} L$$

is diagonal and nonsingular.

The usual assumption "Z is strictly proper" assures that the inverses in (1) and (2) exist and are proper. It can actually be discarded at the cost of a more refined analysis.

The main result of this section indicates that all the transfer matrices Z for which (P) or (GP) is solvable share a certain "causality degree dominance" property. This observation allows us to reduce both of these feedback problems to open-loop, constant precompensation problems. These problems are in turn expected to be easier to tackle; an expectation which has been fulfilled by the resolution of (P) in [7].

In the special case of square Z, i.e., in the case $p = m$, the causality degree dominance property yields a new solvability condition to the problems as an alternative to the existing condition in terms of Z^{-1} of [8] and [9]. The new condition allows one to more clearly see into the structure of diagonalizable transfer matrices.

Note that if the pair (Z_c, L) is a solution to (P), then (LZ_c, L) is a solution to (GP). Thus, the solvability of (P) implies the solvability of (GP). On the other hand, it can be shown that the reverse implication does not hold. A counterexample to (GP) implies (P) is

$$Z = \begin{bmatrix} z^{-2} & z^{-6} & z^{-3} \\ z^{-5} & z^{-4} & z^{-3} \end{bmatrix}.$$

Clearly, the two problems are equivalent in the case of a square transfer matrix. Due to the analogy between the restricted state feedback problem, general state feedback problem and (P), (GP), respectively, one may call (P) the "restricted" and (GP) the "general" dynamic output feedback problem of diagonalization.

A short description of the terminology and notation used in the paper is as follows.

We define the (causality) degree of a rational function of z with real coefficients [i.e., an element of the field of real rational functions $\mathscr{R}(z)$] to be the difference between the degrees of its numerator and denominator polynomials. Thus, if a is in $\mathscr{R}(z)$, then $a = p/q$ for (coprime) polynomials p and q in $\mathscr{R}(z)$ and $\deg a := \deg p - \deg q$. The degree of 0 is assigned to be $-\infty$ for convenience. An element a in $\mathscr{R}(z)$ is called *proper* iff $\deg a \leqq 0$ and *strictly proper* iff $\deg a < 0$. The set of proper elements of $\mathscr{R}(z)$ forms the ring of transfer functions, denoted by $\mathscr{R}_{\text{pr}}(z)$. A degree 0 element (unit) of this ring is said to be *biproper*.

If A is in $\mathscr{R}(z)^{p \times m}$, then $\deg A := \max\{\deg A_{ij}\}$; $i = 1, \ldots, p$ and $j = 1, \ldots, m$. A rational matrix is *proper* iff $\deg A \leqq 0$ and *strictly proper* iff $\deg A < 0$. If A is a rational matrix, then it can uniquely be written as $A = A^- + A^+$, where A^- is a proper matrix and A^+ is a strictly polynomial matrix. A $p \times m$ rational matrix is called *left biproper* iff it is proper and admits a proper right inverse, i.e., there exists a proper $m \times p$ matrix B such that $AB = I$; it is called *right biproper* iff its transpose A' is left biproper. If a square rational matrix is left (or right) biproper, then it is called *biproper*. If A_0 denotes the coefficient of z^0 in the Laurent series expansion in z^{-1} of A, then a proper A is well known to be left biproper iff rank $A_0 = p$, or equivalently, A_0 is left invertible. We now define the concept of row properness for a rational matrix. This is actually equivalent to the notion of the rows of a matrix being *properly independent* for transfer matrices, see [10]. Let A be in $\mathscr{R}(z)^{p \times m}$ with A_i denoting its ith row; a $1 \times m$ matrix. Let $\mu_i := \deg A_i$ and define the *row degree matrix* D of A to be $D := \operatorname{diag}\{z^{\mu_i}\}$ which is a diagonal nonsingular matrix. Let $B := D^{-1}A$. The rational matrix A is called *row proper* iff B is left biproper. The zero coefficient matrix B_0 of the proper matrix B is called the *highest row coefficient matrix of* A and is denoted by A_{h}. We note that if A is a polynomial matrix, then our concept of row properness is precisely that of [6]. It is easy to see that any rational A (row proper or not) can uniquely be represented as $A = F(A_{\mathrm{h}} + Y)$ for some strictly proper rational matrix Y. Given a square rational matrix A, it can uniquely be decomposed into its *diagonal* and *off-diagonal parts* as $A = A_{\mathrm{d}} + A_{\mathrm{off}}$, where the ijth entry of A_{d} is equal to the corresponding entry of A if $i = j$ and is equal to 0 for $i \neq j$.

2. DIAGONALIZATION PROBLEM (P)

We first prove the following preliminary result.

Lemma 1. The problem (P) is solvable if and only if rank $Z = p$, there exists an $m \times p$ constant matrix L such that ZL is nonsingular, and for all $i, j = 1, \ldots, p$ with $i \neq j$ the inequality

(3) $$\deg (ZL)_{ij} \leqq \deg (ZL)_{ii} + \deg (ZL)_{jj}$$

holds.

Proof. [Only if] By nonsingularity of $\hat{Z}$ in (1), ZL is also nonsingular implying the necessity of rank $Z = p$. Equality (1) further implies that

$$\hat{Z}^{-1} Z L \hat{Z}^{-1} = \hat{Z}^{-1} + Z_{\mathrm{c}}(I - \hat{Z} Z_{\mathrm{c}})^{-1} ,$$

where $I - \hat{Z} Z_{\mathrm{c}}$ is biproper by strict properness of $\hat{Z}$. Then,

$$\hat{Z}^{-1} (ZL)_{\mathrm{off}} \hat{Z}^{-1} = [Z_{\mathrm{c}}(I - \hat{Z} Z_{\mathrm{c}})^{-1}]_{\mathrm{off}} ,$$

yielding $\deg (ZL)_{ij} \leqq \deg \hat{Z}_{ii} + \deg \hat{Z}_{jj}$; $i \neq j$, as the right hand side is proper. Note, however, that $\hat{Z}^{-1}(ZL)_{\mathrm{d}} - I$ is strictly proper by (1), or that $\hat{Z}^{-1}(ZL)_{\mathrm{d}}$ is biproper. Thus,

$$\deg (ZL)_{ii} = \deg \hat{Z}_{ii} ; \quad i = 1, \ldots, p$$

yielding (3). [If] Let rank $Z = p$ and suppose (3) holds for some L such that ZL is nonsingular. We first note that (3) implies that $(ZL)_{\mathrm{d}}$ is nonsingular; since, otherwise, a whole row or column of ZL would be zero. Let us prove that $Z_{\mathrm{c}} :=$

$:= -[(ZL)^{-1}]_{\text{off}}$ is a solution to (P). Let $E := (ZL)_{\text{d}}$ to simplify the notation and note by (3) that $E^{-1}(ZL)_{\text{off}}E^{-1}$ is proper. Hence, $I + E^{-1}(ZL)_{\text{off}}$ is biproper by strict properness of E and all off-diagonal entries of

$$(ZL)^{-1} = E^{-1} - [I + E^{-1}(ZL)_{\text{off}}]^{-1} E^{-1}(ZL)_{\text{off}} E^{-1}$$

are proper. This establishes that Z_c is proper. Further, $\hat{Z} := [(ZL)^{-1} + Z_c]^{-1} = \{[(ZL)^{-1}]_{\text{d}}\}^{-1}$ is diagonal. Therefore, Z_c is a solution to (P). □

Definition. Let us call a nonsingular $p \times p$ rational matrix R *degree-dominant* iff

(4) $$\deg R_{ij} \leqq \deg R_{ii} + \deg R_{jj}\,; \quad i, j = 1, \ldots, p\,; \quad i \neq j\,.$$

Among the following matrices Z^1 is degree-dominant whereas Z^2 and Z^3 are not degree-dominant:

$$Z^1 = \begin{bmatrix} z & z^{-1} \\ z & z \end{bmatrix}, \quad Z^2 = \begin{bmatrix} z^{-2} & z^{-3} \\ z^{-3} & z^{-2} \end{bmatrix}, \quad Z^3 = \begin{bmatrix} z & z^2 \\ z^{-2} & 0 \end{bmatrix}.$$

It is clear that a strictly proper and degree-dominant R is row proper as the degree inequality above implies that $\deg R_{ij} < \deg R_{ii}$ for all $i \neq j$ since $\deg R_{jj} < 0$.

Lemma 1 yields the following main result of this section. The second part is a trivial consequence of the first; it is stated separately as it yields a readily checkable sufficient condition for the solvability of problem (P).

Theorem 1. (i) The problem (P) is solvable if and only if there exists an $m \times m$ constant matrix K such that a $p \times p$ minor of ZK is nonsingular and degree-dominant. (ii) If any of the $p \times p$ minors of Z is nonsingular and degree-dominant, then (P) has a solution.

Proof. Let (P) be solvable for Z so that, by Lemma 1, there exists an $m \times p$ constant L such that ZL is nonsingular and degree-dominant. It follows that L is of full column rank and hence for some constant $m \times m$ matrix K, $L = K[I : 0]'$. Consequently, the $p \times p$ minor of ZK of column indexes $j = 1, \ldots, p$ is degree-dominant. Conversely, if for some K, a $p \times p$ minor of ZK is nonsingular and degree-dominant, let L be the $m \times p$ matrix of 0's and 1's picking that minor, i.e., such that ZKL is equal to that minor. Since ZKL is nonsingular and degree-dominant, by Lemma 1 it follows that (P) is solvable. Statement (ii) follows trivially from (i) on setting $K = I$. □

The merit of the result of Theorem 1 is that it reduces the solvability of (P) to an open loop problem, namely, the existence of a constant $m \times p$ matrix L such that ZL is degree-dominant. Theorem 1 is the starting point of Eldem and Özgüler [7] where a complete solution to (P) has been obtained.

In the case of square Z, Theorem 1 yields a new solvability condition for (P) which sheds more light into the structure of diagonalizable transfer matrices than the already existing condition (see (ii) below) in terms of Z^{-1} of [8] and [9].

Corollary 1. Let Z be a strictly proper $p \times p$ transfer matrix. The following are equivalent:
(i) (P) is solvable for Z.
(ii) Z is nonsingular and $(Z_h Z^{-1})_{ij}$ is proper for all $i \neq j$.
(iii) Z_h is nonsingular and ZZ_h^{-1} is degree-dominant.

Proof. Note that, A^+ is diagonal iff A_{ij} is proper for all $i \neq j$. The equivalence of (i) and (ii), then, follows by [8]. Let (P) be solvable for Z so that, by Theorem 1, there exists a constant $p \times p$ matrix L such that ZL is nonsingular and degree-dominant. It follows in particular that ZL is row proper. This implies by nonsingularity of L that Z_h is nonsingular. We now show that $L = Z_h^{-1} C_d$ for some constant diagonal nonsingular C_d. In fact, if ZL is degree-dominant, then

$$X := (D^{-1}ZLD^{-1})_{\text{off}} = D^{-1}(ZL)_{\text{off}}\, D^{-1}$$

is proper since $\deg (ZL)_{ii} = i\text{-}th$ *row degree of* Z. Using the representation $Z = D(Z_h + Y)$ of Z, where Y is strictly proper, we can write $(Z_h L)_{\text{off}} + (YL)_{\text{off}} = XD$. Here, $XD - (YL)_{\text{off}}$ is strictly proper and $(Z_h L)_{\text{off}}$ is constant. Hence, $(Z_h L)_{\text{off}} = 0$. By nonsingularity of $Z_h L$, we then have $Z_h L = C_d$ for some constant diagonal nonsingular C_d as claimed above. It is easy to see that ZZ_h^{-1} is degree-dominant iff $ZZ_h^{-1}C_d$ is degree-dominant for any constant diagonal nonsingular C_d. Thus, (i) implies (iii). The converse is a straightforward consequence of Theorem 1.

□

Remark. The role of degree-dominance in (P) is best observed by considering the restricted problem: (P) *with* $L = I$. By Corollary 1, this problem is solvable iff Z is degree-dominant.

3. GENERAL PROBLEM (GP)

The second problem (GP) requires a closer attention and the reduction of the problem to an open loop one is not so straightforward. However, if the plant transfer matrix Z is row proper, then a version of the concept of degree-dominance is easily seen to be central to the solvability of (GP).

Theorem 2. Let Z be row proper with row degrees $\{\mu_i;\ i = 1, \ldots, p\}$. Then (GP) is solvable if and only if there exists a constant matrix L such that ZL is nonsingular and

(5) $$\deg (ZL)_{ij} \leqq \mu_i + \deg (ZL)_{jj}\,,$$

for all $i, j = 1, \ldots, p$ and $i \neq j$.

Proof. Let us write $Z = DZ_b$ for a diagonal proper D and a left biproper Z_b. Note that $\mu_i = \deg D_{ii}$ for $i = 1, \ldots, p$. If (GP) is solvable, then by (2) $\hat{Z} + ZZ_c\hat{Z} = ZL$ implying that $\deg (ZL)_{ii} = \deg [(ZL)_d]_{ii} = \deg (\hat{Z})_{ii}$ for all $i = 1, \ldots, p$ and that

$$D^{-1}(ZL)_{\text{off}}\hat{Z}^{-1} = (Z_b Z_c)_{\text{off}}$$

with the right hand side proper. Thus, (5) should hold. Conversely, if (5) holds, then $D^{-1}(ZL)_{\text{off}}(ZL)_{\text{d}}^{-1} =: Y$ is proper. Letting $Z_c := Y_b Y$, where Y_b is a proper right inverse of Z_b, it is easy to verify that (2) holds with $\hat{Z} = (ZL)_d$. □

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Dr. A. Bülent Özgüler, Electrical and Electronics Eng., Bilkent University, P. O. Box 8, Maltepe 06572, Ankara. Turkey.

Dr. Vasfi Eldem, Research Institute for Basic Sciences, Tübitak, P. O. Box 74, Gebze 41470, Kocaeli. Turkey.