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Theory and Methodology

Sequencing jobs on a single machine with a common due date and stochastic processing times

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Abstract: This paper presents a procedure for sequencing jobs on a single machine with jobs having a common due date and stochastic processing times. The performance measure to be optimized is the expected incompletion cost. Job processing times are normally distributed random variables, and the variances of the processing times are proportional to their means. The optimal sequences are shown to have a W- or V-shape. Based on this property computationally attractive solution methods are presented.

Keywords: Stochastic scheduling, single machine

1. Introduction

In this paper we consider the problem of sequencing N jobs on a single machine with jobs having a common due data and stochastic processing times. The common due date can also be viewed as the cycle time. Consequently, a job, if not completed within the due date, incurs a fixed incompletion cost corresponding to the amount required for its completion at some other facility or the penalty to be paid for it being late. The incompletion cost is different for different jobs. The objective is to sequence jobs so that the expected incompletion cost or the sum of the weighted incompletion probabilities is minimized. This problem is like a single-machine sequencing problem with a nonlinear loss function, however, the loss function here is defined as the expected incompletion cost. When the incompletion costs of the jobs are all equal to unity, then the criterion considered reduces to that of minimizing the expected number of tardy jobs.

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Several studies have been reported in the literature for the single-machine problem with various loss functions. One of the earlier attempts for solving the problem was made by McNaughton [5], who described a procedure for finding the optimal schedule to the single-machine problem with linear loss functions and deterministic job processing times. Panwalker et al. [6] have studied the deterministic single-machine problem with linear earliness and tardiness penalties and common due date. They have shown that the optimal sequence is v-shaped (lpt–spt ordering). Lawler [4] extended McNaughton's [5] study for nonlinear loss functions by using a dynamic programming (dp) approach. Lawler [4] also presented some linear programming formulations for the multiple-machine case with nonlinear loss functions and deterministic processing times. Schild and Fredman [9] developed criteria for quadratic loss functions to determine the relative order in which two jobs should appear in the optimal sequence. For general loss functions, the number of computations required by this algorithm grows exponentially with increase in N. Townsend [10] developed a branch-and-bound solution to the single-machine problem with quadratic loss function of job flowtimes. The procedure is not practical for large problems, and an approximate solution which requires generation of $\frac{1}{2}N(N+1)$ nodes is recommended. Bagga and Kalra [1] further suggested a node elimination procedure for Townsend's [10] algorithm. Gupta and Sen [3] curtailed the enumeration tree of Townsend's [10] algorithm at the branching stage by recognizing certain conditions which give a priori precedence relations among some of the jobs in the optimal sequence. Regarding the consideration of stochastic processing times of jobs, one of the earlier attempts was made by Banarjee [2] for a single-machine problem. Lately, considerable research has been reported in the area of stochastic scheduling. For a review, the reader is referred to papers by Pinedo and Schrage [7] and Weiss [11]. Pinedo [8] gives the optimal static and dynamic policies for the single machine problem with exponential processing times and common due date, which is a random variable with an arbitrary distribution, for the criterion of minimizing expected weighted number of tardy jobs.

Our main result shows that if the job processing times are normally distributed with certain assumptions about job variance and incompletion cost satisfied, then the optimal sequence, minimizing the total expected incompletion cost on a single machine, must be w- or v-shaped. This property substantially reduces the number of sequences that must be considered and serves as a basis for a branch and bound solution, like Townsend's [10] for the case of quadratic loss functions.

In the sequel, we first present some notation and the assumptions used in the paper. Section 3 contains our main results. The procedure to generate the promising sequences for the optimal solution is presented in Section 4, followed by computational experience. Finally, a heuristic, using a truncated version of the branching tree, is discussed briefly.

2. Notation and Assumptions

Consider a single facility with N jobs waiting. Assume that the facility is free at the moment, and we wish to decide the sequence in which the jobs should be processed on that facility. The performance measure to be optimized is the expected incompletion cost. Let

$C_i(s)$ = completion time of job i in sequence $s \in S$, for $i = 1, \dots, N$, where S is the set of all permutations of the N jobs;
d = common due date for all jobs;
IC_i = incompletion cost of job i, for $i = 1, \dots, N$.

The performance measure can be expressed as

$$\min_{s \in S} \sum_{i=1}^{N} \mathrm{IC}_i \Pr[C_i(s) > d].$$

We assume that job duration times are distributed normally with known means and variances. This could be the case for example when each job consists of a large number of elementary tasks with stochastic

Table 1
Lower bounds on the expected performance times of the jobs

ε	$\mu_i \geqslant$	Lower bound on μ_i for all i				
		$a = 0.1$	$a = 0.2$	$a = 0.3$	$a = 0.5$	$a = 1.0$
0.20	0.706a	0.071	0.141	0.212	0.353	0.706
0.10	1.638a	0.164	0.328	0.491	0.819	1.638
0.05	2.706a	0.271	0.541	0.812	1.353	2.706
0.03	3.346a	0.353	0.707	1.060	1.767	3.534
0.01	5.406a	0.541	1.081	1.622	2.037	5.406

processing times. In order to ensure that job processing times are nonnegative, the processing time distributions are truncated at zero. In addition, job processing time variances are expected to be proportional to their means as, for example, a job with a large expected processing time contains a large number of elementary tasks, consequently resulting in a large variance for the job. We also assume that the incompletion cost of a job is proportional to its complexity, i.e. to its mean processing time. Accordingly, let $\sigma_i^2 = a\,\mu_i$ and $\mathrm{ic}_i = r\,\mu_i$ for $i = 1, \ldots, N$, where a and r are constants and μ_i and σ_i^2 are the mean and variance of the processing time of job i, respectively. Such a relationship between σ_i^2 and μ_i is not uncommon. The Poisson and binomial distributions, for instance, follow this property and are approximated by the normal distribution for certain parameter values.

The truncation of the job performance time distributions at zero can be made if the probability that a normally distributed random variable can take negative values is small enough. Next, we develop some conditions under which this is true. To that end, let E represent the area to the left of zero under a normal distribution with mean μ_i and variance σ_i^2. Let ε be a small quantity greater than zero. If $\Phi(\cdot)$ is the cumulative normal distribution function, then the desired condition is as follows:

$$E = \Phi(-\mu_i/\sigma_i) \leqslant \varepsilon \quad \text{for } i = 1, \ldots, N.$$

The above condition reduces to the following expression:

$$\mu_i \geqslant a\left[\Phi^{-1}(\varepsilon)\right]^2 \quad \text{for } i = 1, \ldots, N.$$

In other words, the truncation of the normal distribution can be ignored if the expected performance times of the jobs are larger than the above value determined as a function of ε and a. Table 1 depicts the lower bounds on the expected performance times for different ε and a values. Note that for practical job performance times, $a < 1.0$, because it is highly improbable to have the variance of a job performance time to be greater than its expected value.

3. Main Results

Consider the incompletion probability function

$$p(x) = 1 - \Phi\left[(d - x)/\sqrt{ax}\right].$$

Note that $p(x = d) = 0.5$ since $\Phi[0.0] = 0.5$, and $p(x)$ approaches one as x goes to infinity. First, we prove an important property of the incompletion probability function, which will be used in the remainder of the paper.

Theorem 1. *The incompletion probability function, $p(x)$ is monotonically increasing and convex over the interval $0 \leqslant x \leqslant d'$ and monotonically increasing and concave for $x \geqslant d'$, for some $d' < d$.*

Proof. The incompletion probability function, $p(x)$, can be represented as follows:

$$p(x) = 1 - \Phi\left(\frac{d-x}{\sqrt{ax}}\right) = \Phi\left(\frac{x-d}{\sqrt{ax}}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{b(x)} e^{-z^2/2}\,dz,$$

where $b(x) = (x-d)/\sqrt{ax}$. Let $f(z) = e^{-z^2/2}$.

$$\frac{dp(x)}{dx} = \frac{1}{\sqrt{2\pi}} f(b) \frac{db}{dx} = \frac{1}{\sqrt{2\pi}} e^{-b^2/2} \frac{(x+d)}{2\sqrt{a}\,x^{3/2}}.$$

Hence, $dp(x)/dx > 0$, for $x > 0$. Next consider $d^2p(x)/dx^2$.

$$\frac{d^2p(x)}{dx^2} = \frac{1}{\sqrt{2\pi}}\left[f'(b)\left(\frac{db}{dx}\right)^2 + f(b)\frac{d^2b}{dx^2}\right], \tag{1}$$

where

$$f'(b) = -b\,e^{-b^2/2} = -bf(b). \tag{2}$$

Substituting equation (2) into equation (1) yields

$$\frac{d^2p(x)}{dx^2} = \frac{f(b)}{\sqrt{2\pi}}\left[\frac{d^2b}{dx^2} - b\left(\frac{db}{dx}\right)^2\right]. \tag{3}$$

Moreover,

$$\left(\frac{db}{dx}\right)^2 = \frac{(x+d)^2}{4ax^3}, \tag{4}$$

and

$$\frac{d^2b}{dx^2} = -\frac{x+3d}{4\sqrt{a}\,x^{5/2}}. \tag{5}$$

Therefore, substituting equations (4) and (5) into equation (3) yields

$$\frac{d^2p(x)}{dx^2} = -\frac{e^{-b^2/2}}{\sqrt{2\pi}}\left[\frac{ax(x+3d) + (x-d)(x+d)^2}{4a^{3/2}x^{7/2}}\right].$$

Let

$$\Delta(x) = ax(x+3d) + (x-d)(x+d)^2.$$

Note that the signs of $d^2p(x)/dx^2$ and $\Delta(x)$ are opposite of each other. To determine the nature of $\Delta(x)$, consider

$$d\Delta(x)/dx = 3x^2 + 2x(a+d) + 3ad - d^2.$$

As x approaches zero, $d\Delta(x)/dx \geqslant 0$ if $a > d/3$, and $d\Delta(x)/dx \leqslant 0.0$ if $a \leqslant d/3$. On the other hand, for $x \geqslant d$, $d\Delta(x)/dx \geqslant 0.0$. Thus, for the case $a \leqslant d/3$, the slope of $\Delta(x)$ changes sign as x moves from d to zero. For this case, let $0.0 \leqslant y^* \leqslant d$ be such that $d\Delta(y^*)/dx = 0.0$. Since

$$d^2\Delta(y^*)/dx^2 = 6y^* + 2(a+d) \geqslant 0.0,$$

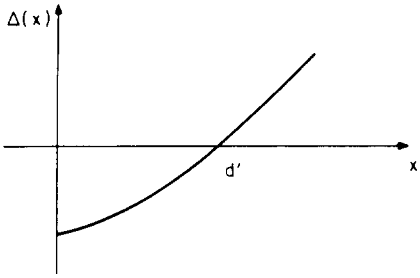

Figure 1. $\Delta(x)$ for $a > d/3$

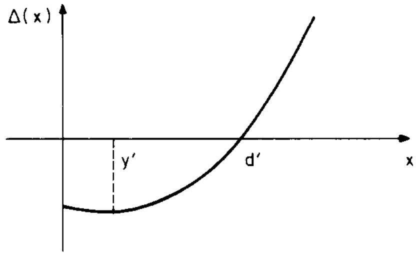

Figure 2. $\Delta(x)$ for $a < d/3$

then y^* is a local minimum of $\Delta(x)$. In addition, for $x = 0.0$, $\Delta(x) < 0$ and for $x \geqslant d$, $\Delta(x) > 0$; therefore, it follows that $\Delta(x) \leqslant 0$ over the interval $0 \leqslant x \leqslant d'$, for some $d' \leqslant d$, and $\Delta(x) \geqslant 0$ for $x \geqslant d'$. $\Delta(x)$ is depicted in Figure 1 for the case where $a \geqslant d/3$, and in Figure 2 for the case where $a \leqslant d/3$. Consequently, $\mathrm{d}^2p(x)/\mathrm{d}x^2 \geqslant 0.0$ for $0 \leqslant x \leqslant d'$, and $\mathrm{d}^2p(x)/\mathrm{d}x^2 \leqslant 0.0$ for $x \geqslant d'$. This proves that the incompletion probability function, $p(x)$ is monotonically increasing and convex over the interval $0 \leqslant x \leqslant d'$, and monotonically increasing and concave for $x \geqslant d'$, where $d' < d$. This completes the proof. □

The derivation of d' gets quite complicated. However, when the values of d and a are known, its computation is straightforward. Note that d' is the root of $\Delta(x)$ which is cubic in X and hence its root can be determined using the standard expression. To that end, let

$$h = -(a+d)^2/3 + 3ad - d^2,$$

and

$$q = 2\left[\frac{a+d}{3}\right]^3 - \frac{(a+d)(3ad-d^2)}{3} - d^3.$$

If $V = [h/3]^3 + [q/2]^2$, then

$$d' = \left[\frac{q}{2} + \sqrt{V}\right]^{1/3} + \left[-\frac{q}{2} - \sqrt{V}\right]^{1/3} - \frac{(a+d)}{3}.$$

We computed d' values for different d and a values. Table 2 depicts d' values for the values of d in the range from 1.0 to 20.0, and for a values of 0.2, 0.5 and 1.0. The ratios of d' and d are also depicted in the table. As it is seen, for $d \geqslant 10$, d' gets quite close to d. In addition, the ratio of d' and d is inversely proportional to the value of a; in fact, as a approaches zero, the difference between d and d' goes to zero. Thus, the value of $a = 1.0$ results in the smallest d'/d values; while those for $a = 0.2$ result in the largest values; by assumption $a \leqslant 1.0$. Hence, based on the above analysis, the difference between d' and d for practical problem parameters with $d > 10$ and $a < 0.5$ can be ignored; the error involved will be negligible.

Consider an arbitrary sequence R in which a pair of adjacent jobs, i and j, with j following i, exists such that $\mathrm{IC}_i \geqslant \mathrm{IC}_j$. In the sequence R', the jobs i and j are interchanged. Let Z represent all the jobs preceding job i and y represents the set of jobs following j in R. Let μ_Z and σ_Z^2 be the sum of means and variances of the jobs in Z, respectively, and let cost(R) and cost(R') be the total expected incompletion costs of the two sequences.

Theorem 2.
(i) *If* $\mu_Z + \mu_i + \mu_j \leqslant d'$ *and* $\mathrm{IC}_i \geqslant \mathrm{IC}_j$ *then* $\mathrm{cost}(R) \leqslant \mathrm{cost}(R')$.
(ii) *If* $\mu_Z \geqslant d'$ *and* $\mathrm{IC}_i \geqslant \mathrm{IC}_j$ *then* $\mathrm{cost}(R) \geqslant \mathrm{cost}(R')$.
(iii) *If* $\mu_Z \leqslant d' < \mu_Z + \mu_j$ *and* $\mathrm{IC}_i \geqslant \mathrm{IC}_j$ *then* $\mathrm{cost}(R) \geqslant \mathrm{cost}(R')$.

Proof. Let

$$p_i = 1 - \Phi\left[\{d - (\mu_Z + \mu_i)\}/\sqrt{\sigma_Z^2 + \sigma_i^2}\right]$$

Table 2
Variation in the ratio of d' and d for different a and d values

a	d	d'	d'/d
0.2	1.0	0.815	0.815
	2.0	1.806	0.903
	3.0	2.804	0.935
	4.0	3.802	0.951
	5.0	4.802	0.960
	10.0	9.801	0.980
	15.0	14.801	0.987
	20.0	19.800	0.990
0.5	1.0	0.585	0.585
	2.0	1.536	0.768
	3.0	2.524	0.841
	4.0	3.517	0.879
	5.0	4.513	0.903
	10.0	9.507	0.951
	15.0	14.505	0.967
	20.0	19.503	0.975
1.0	1.0	0.352	0.352
	2.0	1.167	0.584
	3.0	2.103	0.701
	4.0	3.074	0.769
	5.0	4.057	0.811
	10.0	9.027	0.903
	15.0	14.018	0.925
	20.0	19.013	0.951

be the incompletion probability of job i in sequence R;

$$p_j = 1 - \Phi\left[\left\{d - (\mu_Z + \mu_j)\right\} / \sqrt{\sigma_Z^2 + \sigma_j^2}\right]$$

be the incompletion probability of job j in sequence R';

$$p_Z = 1 - \Phi\left[(d - \mu_Z)/\sigma_Z^2\right]$$

be the incompletion probability of the job preceding job i in sequence R or job j in sequence R';

$$p = 1 - \Phi\left[\left\{d - (\mu_Z + \mu_i + \mu_j)\right\} / \sqrt{\sigma_Z^2 + \sigma_i^2 + \sigma_j^2}\right]$$

be the incompletion probability of job j in sequence R or that of job i in sequence R'. From the definition of the expected incompletion cost function we get that

$$\text{cost}(R) \leqslant \text{cost}(R') \quad \text{iff } p_i \text{IC}_i + p\text{IC}_j \leqslant p_j \text{IC}_j + p\text{IC}_i. \tag{6}$$

After substituting for IC_i and IC_j and rearranging, this is equivalent to

$$(p - p_j)/\mu_i \leqslant (p - p_i)/\mu_j. \tag{7}$$

In order to prove the theorem we use simple geometric arguments instead of a much more complicated algebraic derivation of the results. For case (i) note that the left side of the above inequality is tan α in Figure 3 while the right side is tan β and tan $\alpha \leqslant$ tan β follows directly from the fact that p(x) is monotone increasing and convex when $\mu_Z + \mu_i + \mu_j \leqslant d'$.

For case (ii) we use Figure 4. Here tan $\alpha = (p - p_j)/\mu_i$ and tan $\beta = (p - p_i)/\mu_j$ again and tan $\alpha \geqslant$ tan β follows from the fact that $p(x)$ is monotone increasing and concave when $\mu_Z \geqslant d'$. Finally for case (iii)

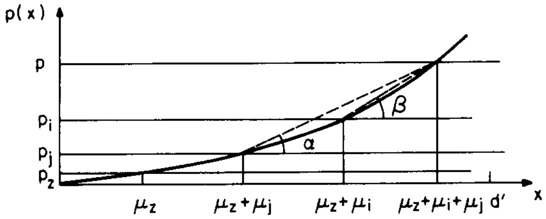

Figure 3. Incompletion probability function for the case when $x < d'$

consider Figure 5. Here again $\tan\alpha = (p - p_j)/\mu_i$ and $\tan\beta = (p - p_i)/\mu_j$ and $\tan\alpha \geqslant \tan\beta$ is a consequence of the fact that $p(x)$ is monotone increasing and concave for $x \geqslant \mu_Z + \mu_j \geqslant d'$. This completes the proof. □

Theorem 2 implies that if $s = (s(1), s(2), \ldots, s(N))$ is a sequence of the N jobs in which the kth job, $s(k)$, has the property that

$$\sum_{i=1}^{k-1} \mu_{s(i)} \leqslant d' \quad \text{and} \quad \sum_{i=1}^{k} \mu_{s(i)} > d',$$

then the sequence s is dominated by a sequence s' (i.e. the expected incompletion cost of s' is less than or equal to the expected incompletion cost of s), where s' has the same jobs in the first $k-1$ positions as s ordered in a nonascending order of their incompletion costs, and s' also has the same jobs in the last $N-k$ positions as s, ordered in a nondescending order of their incompletion costs. The job in position k can be *any* job, which means that it is possible to have the following four cases for the shape of the optimal sequence $t = (t(1), t(2)), \ldots, t(N))$, where k always denotes the position for which

$$\sum_{i=1}^{k-1} \mu_{t(i)} \leqslant d' \quad \text{and} \quad \sum_{i=1}^{k} \mu_{t(i)} > d':$$

(a) $\mu_{t(1)} \geqslant \mu_{t(2)} \geqslant \cdots \geqslant \mu_{t(k-1)} \leqslant \mu_{t(k)} \geqslant \mu_{t(k+1)} \leqslant \mu_{t(k+2)} \leqslant \cdots \leqslant \mu_{t(N)}$.

(b) $\mu_{t(1)} \geqslant \mu_{t(2)} \geqslant \cdots \geqslant \mu_{t(k-1)} \leqslant \mu_{t(k)} < \mu_{t(k+1)} \leqslant \mu_{t(k+2)} \leqslant \cdots \leqslant \mu_{t(N)}$.

(c) $\mu_{t(1)} \geqslant \mu_{t(2)} \geqslant \cdots \geqslant \mu_{t(k-1)} > \mu_{t(k)} \geqslant \mu_{t(k+1)} \leqslant \mu_{t(k+2)} \leqslant \cdots \leqslant \mu_{t(N)}$.

(d) $\mu_{t(1)} \geqslant \mu_{t(2)} \geqslant \cdots \geqslant \mu_{t(k-1)} > \mu_{t(k)} < \mu_{t(k+1)} \leqslant \mu_{t(k+2)} \leqslant \cdots \leqslant \mu_{t(N)}$.

Thus we have the following result:

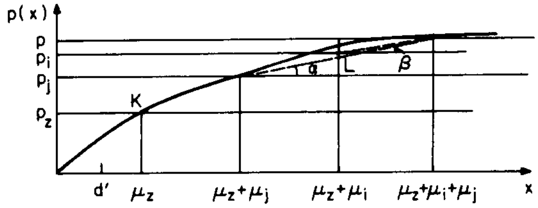

Figure 4. Incompletion probability function for the case when $x > d'$

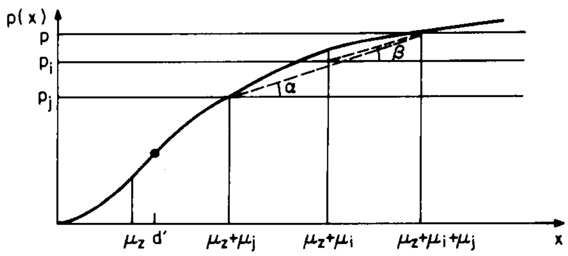

Figure 5. Incompletion probability function for case (iii) of Theorem 2

Corollary 3. *The optimal sequence is always W-shaped (case* (a)) *or V-shaped (cases* (b), (c), (d)), *considering the shape of the function* $\mu_{t(i)}$.

The above result helps to tremendously cut down the number of sequences that need to be considered. Such sequences will hereafter be called 'promising sequences'. In the next section, a procedure to generate the promising sequences is described, followed by a numerical example.

4. Procedure to generate promising sequences

The proposed procedure to generate the promising sequences is a special enumeration tree whose nodes represent arrangements of jobs in the sequence. These nodes are pruned, further branched from or are evaluated depending upon the outcome of a fathoming step based on the results developed above.

Step 1 (*Initialization step*). Order the jobs in the nonascending order of their incompletion costs. Let the first job in the sequence be numbered 1, the second job as 2, and so on.

Step 2 (*Check of the trivial case*). If the sum of the expected processing times of all the jobs is less than or equal to d', then the order obtained in the initialization step is optimal.

Step 3 (*Branching step*). A node with 'g' number of jobs is branched into $N-g$ nodes depending on which one of the $N-g$ unsequenced jobs comes next in the partial solution corresponding to the branch.

Step 4 (*Fathoming step*). Suppose the current node considered corresponds to the partial sequence $s=(s(1), s(2), \ldots, s(m))$.

(i) If $\sum_{i=1}^{m}\mu_{s(i)} \leqslant d'$ and $\mathrm{IC}_{s(m-1)} < \mathrm{IC}_{s(m)}$, then prune this node because it violates Theorem 2(i).

(ii) If $\sum_{i=1}^{m}\mu_{s(i)} \geqslant d'$ then following Theorem 2(ii), complete s by arranging the remaining $N-m$ jobs in nondescending order of their incompletion costs. This is a promising sequence and is therefore evaluated for its cost.

(iii) If

$$\sum_{i=1}^{m}\mu_{s(i)} \leqslant d'$$

and

$$\sum_{i=1}^{m}\mu_{s(i)} + \min_{j\in Y}\mu_j > d',$$

where Y denotes the remaining jobs (not in s), then following Theorem 2(iii), complete the sequence by arranging the jobs in Y in nondescending order of their incompletion costs. This is a promising sequence and is therefore evaluated for its cost.

Step 5. Go back to Step 3.

Next, we illustrate this procedure on an example problem. This example problem consists of 6 jobs. The other relevant data are shown in Table 3 with $d'=10$.

Table 3
Parameters of the example problem

Job (i)	Mean (μ_i)	Incompletion cost (IC_i)
1	10.0	5.0
2	8.0	4.0
3	6.0	3.0
4	4.0	2.0
5	2.0	1.0
6	1.0	0.5

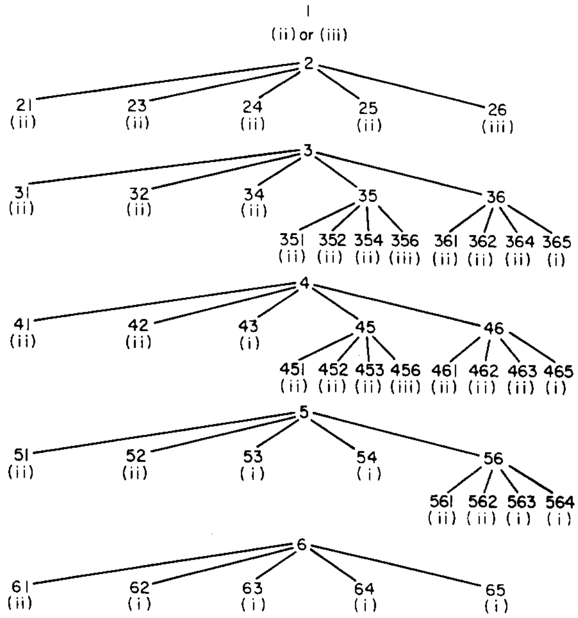

Figure 6. Enumeration tree of the example problem. Legend:

(i) Means that the partial sequence satisfies condition (i) of Step 4, thus it can be pruned.

(ii) The partial sequence satisfies conditions (ii) of Step 4, thus it can be completed by sequencing the remaining jobs in nondescending order and this sequence should be evaluated.

(iii) The partial sequence satisfies conditions (iii) of Step 4, thus it can be completed by sequencing the remaining jobs in nondescending order and this sequence should be evaluated.

The enumeration tree is shown in Figure 6. The nodes that are pruned without evaluation are labelled "(i)" and those that are obtained by completing the sequence and evaluating it are labelled "(ii)" and/or "(iii)". Note that the status of a node is indicated by the condition number of Step 4 of the generating procedure. For example, node {21} generates the sequence {2–1–6–5–4–3} which is a promising sequence (it satisfies condition (ii)); so it is evaluated and labelled "(ii)". On the other hand, node {43} violates condition (i) and the node is pruned without evaluation. Note that the tree has only 41 leaves, substantially less than $6! = 720$.

5. Computational experience

Although the number of sequences generated is cut down tremendously by the fathoming step of the procedure, it still reaches quite a large value for problems with $N > 20$. The situation worsens if d' is in the neighborhood of $\frac{1}{2}\Sigma_{i=1}^{N}\mu_i$. To further investigate the performance of the algorithm, the ratio of the best solution obtained by exploring the first 100 nodes generated by the procedure to the optimal solution was computed. In the experimentation, three sets of problems with 10, 15 and 20 jobs were created, each set containing 10 problems. d' was computed as $d' = b\Sigma_{i=1}^{N}\mu_i$, and for each set three different values of b, namely, 0.25, 0.5 and 0.75 were used. Thus, a total of 90 problems were created and solved. In the test problems, $\mu_i \sim \mathrm{U}[0;\ 20]$ with $\sigma_i^2 = \mathrm{RAN}_1 \cdot \mu_i$ and $\mathrm{IC}_i = \mathrm{RAN}_2 \cdot \mu_i$ where $\mathrm{RAN}_1 \sim \mathrm{N}[0.3;\ 0.067]$ and $\mathrm{RAN}_2 \sim \mathrm{N}[0.05;\ 0.01]$. The maximum, minimum and average ratio values for the problems solved are summarized in Table 4. If the ratio value is 1.00, then the solution obtained at the end of the 100-th node is either optimal or very close to the optimal value. As it is seen from the table, the procedure always generated a solution that is within 2.5% of the optimal solution during the evaluation of the first 100 nodes, thus it is quite robust as a heuristic. To put this in perspective, Table 5 depicts the maximum, minimum and average number of nodes evaluated in order to obtain the optimal solution for different

Table 4
Ratios of the values of the solutions obtained at the end of the 100-th promising sequence to that of the optimal solution

No. of jobs	No. of problems	$d' = b \times \sum_{i=1}^{N} \mu_i$		
		Ratio		
		Average	Minimum	Maximum
$b = 0.25$				
10	10	1.001	1.000	1.006
15	10	1.002	1.000	1.005
20	10	1.005	1.001	1.017
$b = 0.50$				
10	10	1.001	1.000	1.007
15	10	1.005	1.001	1.015
20	10	1.016	1.005	1.025
$b = 0.75$				
10	10	1.002	1.000	1.006
15	10	1.006	1.002	1.013
20	10	1.006	1.002	1.012

Table 5
Number of promising sequences generated to obtain the optimal solutions of the example problems

No. of jobs	No. of problems	$d' = b \times \sum_{i=1}^{N} \mu_i$		
		Number of promising sequences		
		Average	Minimum	Maximum
$b = 0.25$				
10	10	323	229	424
15	10	7824	9050	5772
20	10	203 590	158 009	260 845
$b = 0.50$				
10	10	881	641	957
15	10	34 712	31 479	36 819
20	10	1 392 685	1 228 380	1 484 019
$b = 0.75$				
10	10	379	326	455
15	10	10 154	8213	11 026
20	10	188 515	112 238	247 308

problems. It should be noted that, to generate the first 100 nodes, it requires negligible computation time as compared to the large computation time required to obtain the optimal solution. Hence, the proposed procedure was very effective as a heuristic and generated almost optimal solutions very fast.

6. Conclusions

For the problem of sequencing jobs on a single processor with a common due date and normally distributed processing times, we have developed some conditions to order jobs so as to minimize the expected incompletion cost. These conditions are implemented in a tree search procedure and they help in cutting down the number of sequences generated tremendously. For large problems, an approximate solution procedure has generated almost optimal sequences.

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