# Schlesinger transformations of Painlevé II-V

U. Muğan
*Department of Mathematics, Bilkent University, 06533 Bilkent, Ankara, Turkey*

A. S. Fokas
*Department of Mathematics and Computer Science, Clarkson University, Potsdam, New York 13699*

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The explicit form of the Schlesinger transformations for the second, third, fourth, and fifth Painlevé equations is given.

## I. INTRODUCTION

A powerful method for studying the initial value problem for certain nonlinear ODE's was introduced in Refs. 1 and 2. This method, which is the extension of the inverse spectral method to ODE's, is called the inverse monodromic (or isomonodromic) method. It can be thought of as a nonlinear analogous of the Laplace's method.

The six Painlevé transcendents, PI–PVI, are the most well-known nonlinear ODE's that can be studied using the inverse monodromy method. A rigorous investigation of PII–PV using this method has been recently carried out in Refs. 3 and 4. In particular, in these papers, it is shown that certain Riemann–Hilbert problems, occurring in the process of implementing the inverse monodromy method, can be rigorously investigated. This implies that the Cauchy problems of PII–PV admit, in general, global meromorphic in $t$ solutions. Furthermore, for special relations among the monodromy data, and for certain restrictions of the constant parameters appearing in PII–PV, these solutions have no poles. This provides the motivation for studying how the solutions of a Painlevé equation depend on their associated constant parameters.

Here, we present a systematic investigation of the Schlesinger transformations associated with PII–PV. These transformations imply the relations among the solutions of a given Painlevé equation when its parameters are shifted by an integer.

Let $y(t)$ be a solution of a Painlevé equation corresponding to the parameter $\theta$ (for PII, $y_{tt}=2y^3+ty+\theta$). This equation is associated with the monodromy problem $Y_z=AY$, where $z$ plays the role of the spectral parameter. The implementation of the isomonodromy method necessitates the investigation of the analytic properties of $Y(z)$. It turns out that there exists a sectionally meromorphic function $Y(z)$, with certain jumps across the certain contours of the complex $z$ plane; these jumps are specified by the so-called monodromy data, denoted by MD. We denote by $y'$ and by $Y'$, $y$ and $Y$ when $\theta\to\theta'$. It turns out that it is possible to find an appropriate transformation of $\theta$ (namely, $\theta'=\theta+n$ or $\theta'=\theta+n/2$, $n\in\mathbf{Z}$) such that the MD are invariant. Then $Y'(z)=R(z)Y(z)$, and the Schlesinger transformation matrix $R(z)$, can be found in *closed form*, by solving a certain simple Riemann–Hilbert problem [since the MD of $Y$ and $Y'$ are the same, $R(z)$ has very simple jumps in the complex $z$ plane].

## II. THE SECOND PAINLEVÉ EQUATION

The second Painlevé equation,

$$\frac{d^2y}{dt^2}=2y^3+ty+\alpha, \tag{2.1}$$

can be obtained as the compatibility condition of the following linear system of equations:

$$Y_z(z)=A(z)Y(z), \tag{2.2a}$$

$$Y_t(z)=B(z)Y(z), \tag{2.2b}$$

where

$$A(z)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}z^2+\begin{pmatrix}0&u\\-\frac{2v}{u}&0\end{pmatrix}z+\begin{pmatrix}v+\frac{t}{2}&-uy\\-\frac{2}{u}(\theta+yv)&-\left(v+\frac{t}{2}\right)\end{pmatrix},$$

$$B(z)=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}z+\frac{1}{2}\begin{pmatrix}0&u\\-2\frac{v}{u}&0\end{pmatrix}. \tag{2.3}$$

The equation $Y_{zt}=Y_{tz}$ implies

$$\frac{dv}{dt}=-2yv-\theta,\quad \frac{du}{dt}=-uy,\quad \frac{dy}{dt}=v+y^2+\frac{t}{2}. \tag{2.4}$$

Thus $y$ satisfies the second Painlevé equation (2.1), with the parameter

  

$$\alpha=\tfrac{1}{2}-\theta. \tag{2.5}$$

### A. Solution about $z=\infty$

The formal solution $\widetilde{Y}_\infty(z)=(\widetilde{Y}_\infty^{(1)}(z),\ \widetilde{Y}_\infty^{(2)}(z))$ of equation in (2.2a) in the neighborhood of the irregular singular point $z=\infty$ has the expansion

$$\widetilde{Y}_\infty^{(1)}(z)=\left(\frac{1}{z}\right)^{\theta}e^{q(z)}\left\{\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}-K\\ \dfrac{v}{u}\end{pmatrix}\frac{1}{z}+\cdots\right\}$$

$$=\left(\frac{1}{z}\right)^{\theta}e^{q(z)}\widehat{Y}_\infty^{(1)}(z),$$

$$\widetilde{Y}_\infty^{(2)}(z)=\left(\frac{1}{z}\right)^{-\theta}e^{-q(z)}\left\{\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}-\dfrac{u}{2}\\ K\end{pmatrix}\frac{1}{z}+\cdots\right\}$$

$$=\left(\frac{1}{z}\right)^{-\theta}e^{-q(z)}\widehat{Y}_\infty^{(2)}(z), \tag{2.6}$$

where

$$K=\frac{1}{2}v^2+\left(y+\frac{t}{2}\right)v+\theta y,\quad q(z)=\frac{z^3}{3}+\frac{t}{2}z.$$

Let $Y_k(z)$, $k=1,...,6$ be solutions of (2.2), such that $\det Y_k(z)=1$ and $Y_k(z)\sim\widetilde{Y}_\infty(z)$ as $|z|\to\infty$ in the sector $S_k$, where the sectors $S_k$ are given by

$$S_1:-\frac{\pi}{6}\leqslant\arg z<\frac{\pi}{6},\quad S_2:\frac{\pi}{6}\leqslant\arg z<\frac{\pi}{2},\quad S_3:\frac{\pi}{2}\leqslant\arg z<\frac{5\pi}{6},$$

$$S_4:\frac{5\pi}{6}\leqslant\arg z<\frac{7\pi}{6},\quad S_5:\frac{7\pi}{6}\leqslant\arg z<\frac{3\pi}{2},$$

$$S_6:\frac{3\pi}{2}\leqslant\arg z<\frac{11\pi}{2}. \tag{2.7}$$

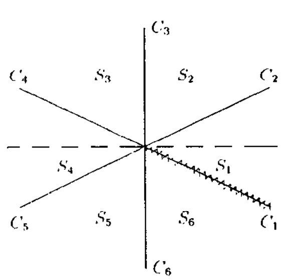


Diagram 1.

The solutions $Y_k(z)$ are related by the Stokes matrices $G_k$,

$$Y_{j+1}(z)=Y_j(z)G_j,\quad j=1,...,5,$$

$$Y_1(z)=Y_6(ze^{2i\pi})G_6e^{2i\pi\theta\sigma_3}, \tag{2.8}$$

where

$$G_1=\begin{pmatrix}1&0\\a&1\end{pmatrix},\quad G_2=\begin{pmatrix}1&b\\0&1\end{pmatrix},\quad G_3=\begin{pmatrix}1&0\\c&1\end{pmatrix},$$

$$G_4=\begin{pmatrix}1&d\\0&1\end{pmatrix},\quad G_5=\begin{pmatrix}1&0\\e&1\end{pmatrix},\quad G_6=\begin{pmatrix}1&f\\0&1\end{pmatrix},$$

$$\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{2.9}$$

and $a$, $b$, $c$, $d$, $e$, $f$ are constants with respect to $z$. The monodromy data, $MD=\{a,b,c,d,e,f\}$, satisfy the consistency condition

$$\prod_{j=1}^{6}G_je^{2i\pi\theta\sigma_3}=I. \tag{2.10}$$

### B. Schlesinger transformations

Let $Y'(z)$ correspond to $\theta'$. We consider the transformation

$$Y'(z)=R(z)Y(z), \tag{2.11}$$

and we demand that $Y'$ has the same monodromy data as $Y$. Since Eq. (2.10) is invariant if $\theta$ is shifted by an integer, we let $\theta'=\theta+n$, $n\in\mathbb{Z}$. Let $R(z)=R_k(z)$ when $z$ in $S_k$; then the definition of the Stokes matrices (2.8) implies that the transformation matrix $R(z)$ satisfies the RH problem along the contour $C_k$, $k=1,...,6$, indicated in Diagram 1:

$$R_{j+1}(z)=R_j(z)\ \text{on}\ C_{j+1},\quad j=1,...,5,$$

$$R_1(z)=R_6(ze^{2i\pi})\ \text{on}\ C_1, \tag{2.12}$$

with the boundary condition

$$R_k(z)\sim\widehat{Y}'_\infty(z)(1/z)^{n\sigma_3}\widehat{Y}_\infty^{-1}(z),\ \text{as}\ z\to\infty,\ z\ \text{in}\ S_k. \tag{2.13}$$

Equation (2.12) implies that the transformation matrix $R(z)$ is analytic everywhere in $z$ plane and can be determined explicitly by using the boundary conditions (2.13). It is enough to consider the particular cases $\theta'=\theta+1$ and $\theta'=\theta-1$. Solving the above RH problem for these two cases we find

$\theta' = \theta + 1$:

$$R_{(1)}(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} 0 & -\frac{u}{v} \\ \frac{v}{u} & -\frac{\theta}{u} - y \end{pmatrix},$$

$$\theta' = \theta - 1: \quad R_{(2)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} y & \frac{u}{2} \\ -\frac{2}{u} & 0 \end{pmatrix}. \tag{2.14}$$

Successive applications of the transformation matrices $R_{(i)}(z)$, $i=1,2$, map $\theta$ to $\theta' = \theta + n$, $n \in \mathbb{Z}$. If, $y', u', v', \theta' = \theta + 1$ are the transformed quantities of $y$, $u$, $v$, $\theta$ under the transformation given by $R_{(1)}(z)$, i.e.,

$$Y'(z;t,y',u',v',\theta') = R_{(1)}(z;t,y,u,v,\theta)\, Y(z;t,y,u,v,\theta), \tag{2.15}$$

and if $y'', u'', v'', \theta'' = \theta' - 1$ are the transformed quantities of $y'$, $u'$, $v'$, $\theta'$ under the transformation given by $R_{(2)}(z)$, i.e.,

$$Y''(z,t;y'',u'',v'',\theta'') = R_{(2)}(z,t;y',u',v',\theta')\, Y(z,t;y',u',v',\theta'), \tag{2.16}$$

then

$$R_{(2)}(z,t;y'(y,u,v,\theta),\ldots) R_{(1)}(z;t,y,u,v,\theta) = I. \tag{2.17}$$

Also,

$$R_{(1)}(z;t,y'(y,u,v,\theta),\ldots) R_{(1)}(z;t,y,u,v,\theta) = R_{(3)}(z),$$

$$R_{(2)}(z;t,y'(y,u,v,\theta),\ldots) R_{(2)}(z;t,y,u,v,\theta) = R_{(4)}(z), \tag{2.18}$$

where $R_{(3)}(z)$ and $R_{(4)}(z)$ shift the exponent $\theta \to \theta' = \theta + 2$ and $\theta \to \theta' = \theta - 2$, respectively.

The linear equation (2.2a) under the Schlesinger transformation given by Eq. (2.11) is transformed as follows:

$$Y_z'(z) = A'(z) Y',$$

$$A'(z) = [R(z)A(z) + R_z(z)] R^{-1}(z). \tag{2.19}$$

For the particular case of $R_{(2)}$, the quantities $y$, $u$, $v$, $\theta$ are transformed by

$$\theta' = \theta - 1, \quad y' = -y - \frac{\theta'}{2y^2 + v + t},$$

$$u' = (u/2)v', \quad v' = -v - 2y^2 - t. \tag{2.20}$$

That is, if $y(t)$ solves the second Painlevé equation with a parameter $\alpha = \frac{1}{2} - \theta$, then $y'(t)$ solves the second Painlevé equation with parameter $\alpha' = \alpha + 1$. From Eq. (2.20), the well-known Bäcklund transformation for the second Painlevé equation can be obtained:

$$y' = -y + \frac{2\alpha + 1}{2y^2 + 2y_t + t}, \quad \alpha' = \alpha + 1. \tag{2.21}$$

## III. THE THIRD PAINLEVÉ EQUATION

The third Painlevé equation,

$$\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \tag{3.1}$$

is the compatibility condition of the linear systems of equations

$$Y_z(z) = A(z) Y(z), \tag{3.2a}$$

$$Y_t(z) = B(z) Y(z), \tag{3.2b}$$

where

$$A(z) = \frac{t}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} \frac{-\theta_\infty}{2} & u \\ v & \frac{\theta_\infty}{2} \end{pmatrix}\frac{1}{z} + \begin{pmatrix} s - \frac{t}{2} & -ws \\ \frac{1}{w}(s-t) & -\left(s - \frac{t}{2}\right) \end{pmatrix}\frac{1}{z^2},$$

$$B(z) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z + \frac{1}{t}\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} - \frac{1}{t}\begin{pmatrix} s - \frac{t}{2} & -ws \\ \frac{1}{w}(s-t) & -\left(s - \frac{t}{2}\right) \end{pmatrix}\frac{1}{z}. \tag{3.3}$$

The equation $Y_{zt} = Y_{tz}$ implies

$$\frac{du}{dt} = \frac{\theta_\infty}{t}u - 2ws, \quad \frac{dv}{dt} = -\frac{\theta_\infty}{t}v + \frac{2}{w}(t-s),$$

$$t\left(\frac{ds}{dt}\right) = -4ys^2 + (4yt - 2\theta_\infty + 1)s + (\theta_0 + \theta_\infty)t, \tag{3.4}$$

$$t\frac{dw}{dt} = w\left[\frac{t}{s}(\theta_0 + \theta_\infty) - 2ty + \theta_\infty\right],$$

$$\frac{dy}{dt} = 4sy^2 - 2ty^2 + (2\theta_\infty - 1)y + 2t,$$

where $y = -u/sw$ and

$$\frac{\theta_0}{2} = -\frac{s-t}{wt}\left(u - \frac{\theta_\infty}{2}w\right) + \frac{s}{t}\left(wv + \frac{\theta_\infty}{2}\right). \tag{3.5}$$

Thus $y$ satisfies the third Painlevé equation (3.1) with the parameters

$$\alpha = 4\theta_0, \quad \beta = 4(1-\theta_\infty), \quad \gamma = 4, \quad \delta = -4. \tag{3.6}$$

### A. Solution about $z = \infty$

The two linearly independent formal solution $\widetilde{Y}^{(\infty)}(z) = (\widetilde{Y}^{(\infty)}_{(1)}(z), \widetilde{Y}^{(\infty)}_{(2)}(z))$ of Eq. (3.2a) in the neighborhood of the irregular singular point $z = \infty$ has the form

$$\widetilde{Y}^{(\infty)}_{(1)}(z) = \left(\frac{1}{z}\right)^{\theta_\infty/2} e^{zt/2}\left\{\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\frac{t}{2} - s - \frac{uv}{t}\\ \frac{v}{t}\end{pmatrix}\frac{1}{z} + \begin{pmatrix}\cdots\\ K^{(\infty)}_{12}\end{pmatrix}\frac{1}{z^2} + \cdots\right\} = \left(\frac{1}{z}\right)^{\theta_\infty/2} e^{zt/2}\widehat{Y}^{(\infty)}_{(1)}(z),$$

$$\widetilde{Y}^{(\infty)}_{(2)}(z) = \left(\frac{1}{z}\right)^{-\theta_\infty/2} e^{-zt/2}\left\{\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}-\frac{u}{t}\\ -\left(\frac{t}{2} - s - \frac{uv}{t}\right)\end{pmatrix}\frac{1}{z} + \begin{pmatrix}K^{(\infty)}_{21}\\ \cdots\end{pmatrix}\frac{1}{z^2}\cdots\right\} = \left(\frac{1}{z}\right)^{-\theta_\infty/2} e^{-zt/2}\widehat{Y}^{(\infty)}_{(2)}(z), \tag{3.7}$$

where

$$K^{(\infty)}_{12} = \frac{1}{t}\left[\frac{v}{t}(1+\theta_\infty) + v\left(\frac{t}{2} - s - \frac{uv}{t}\right) + \frac{1}{w}(s-t)\right],$$

$$K^{(\infty)}_{21} = \frac{1}{t}\left[u(1-\theta_\infty) + ws - u\left(s - \frac{t}{2} + \frac{uv}{t}\right)\right]. \tag{3.8}$$

Let $Y^{(\infty)}_j(z)$, $j = 1,2$, be the solution of Eq. (3.2), such that $\det Y^{(\infty)}_j(z) = 1$ and

$$Y^{(\infty)}_j(z) \sim \widetilde{Y}^{(\infty)}(z) \text{ as } |z| \to \infty, \text{ in } S^{(\infty)}_j, \quad j = 1,2, \tag{3.9}$$

where the sectors $S^{(\infty)}_j$ are indicated in Diagram 2 and given by

$$S^{(\infty)}_1: -\frac{\pi}{2} \leqslant \arg z < \frac{\pi}{2}, \quad S^{(\infty)}_2: \frac{\pi}{2} \leqslant \arg z < \frac{3\pi}{3}, \quad |z| \geqslant z_0, \tag{3.10}$$

where $z_0$ is a constant and $0 < z_0 < \infty$. The solutions $Y^{(\infty)}_j(z)$ are related by the Stokes matrices $G^{(\infty)}_j$. For $t > 0$ this relation is given by

$$Y^{(\infty)}_2(z) = Y^{(\infty)}_1(z)G^{(\infty)}_1,$$

$$Y^{(\infty)}_1(z) = Y^{(\infty)}_2(ze^{2i\pi})G^{(\infty)}_2 e^{i\pi\theta_\infty\sigma_3}, \tag{3.11}$$

where

$$G^{(\infty)}_1 = \begin{pmatrix}1 & 0\\ a_\infty & 1\end{pmatrix}, \quad G^{(\infty)}_2 = \begin{pmatrix}1 & b_\infty\\ 0 & 1\end{pmatrix}. \tag{3.12}$$

For $t < 0$ the Stokes matrices are the transpose of those for the case $t > 0$.

### B. Solution about $z = 0$

The formal solution $\widetilde{Y}^{(0)}(z) = (\widetilde{Y}^{(0)}_{(1)}(z), \widetilde{Y}^{(0)}_{(2)}(z))$ of (3.2) in the neighborhood of the irregular singular point $z = 0$ has the form

$$\widetilde{Y}^{(0)}_{(1)}(z)=z^{\theta_0/2}e^{t/2z}\left\{\begin{pmatrix} wk \\ k \end{pmatrix}+\begin{pmatrix} -wk\left[s\left(1-\frac{t}{2s}\right)+\frac{\bar{u}\bar{v}}{t}\right]-l\frac{\bar{v}}{t} \\ -k\left[s\left(1-\frac{t}{2s}\right)+\frac{\bar{u}\bar{v}}{t}\right]-l\frac{\bar{v}}{t}\left(\frac{s-t}{ws}\right) \end{pmatrix}z+\cdots\right\}=z^{\theta_0/2}e^{t/2z}\widehat{Y}^{(0)}_{(1)}(z),$$

$$\widetilde{Y}^{(0)}_{(2)}(z)=z^{-\theta_0/2}e^{-t/2z}\left\{\begin{pmatrix} l \\ l\frac{s-t}{ws} \end{pmatrix}+\begin{pmatrix} k\frac{w\bar{u}}{t}+l\left[s\left(1-\frac{t}{2s}\right)+\frac{\bar{u}\bar{v}}{t}\right] \\ k\frac{\bar{u}}{t}+l\frac{s-t}{ws}\left[s\left(1-\frac{t}{2s}\right)+\frac{\bar{u}\bar{v}}{t}\right] \end{pmatrix}z+\cdots\right\}$$

$$=z^{-\theta_0/2}e^{-t/2z}\widehat{Y}^{(0)}_{(2)}(z), \tag{3.13}$$

where

$$\bar{u}=l^2\left[\left(\frac{s-t}{ws}\right)^2u-v-\frac{\theta_\infty}{ws}(s-t)\right],$$

$$\bar{v}=k^2(\theta_\infty w-u+w^2v), \tag{3.14}$$

and $k$, $l$ satisfy the following conditions:

$$\ln l=\int^t\frac{1}{t'}\left\{u\left(\frac{s-t'}{ws}\right)-\frac{sw}{t'}\left[u\left(\frac{s-t'}{ws}\right)^2-v-\frac{\theta_\infty}{ws}(s-t')\right]\right\}dt',$$

$$\ln k=\int^t\frac{1}{t'}\left[wv-\frac{1}{t'}\left(\frac{s-t}{w}\right)(w\theta_\infty-u-w^2v)\right]dt',$$

$$kl=-s/t. \tag{3.15}$$

Let $Y^{(0)}_j(z)$, $j=1,2$, be a solution of (3.2) such that $\det Y^{(0)}_j(z)=1$ and $Y^{(0)}_j\sim\widetilde{Y}^{(0)}(z)$ as $z\to 0$ in $S^{(0)}_j$, where the sectors $S^{(0)}_j$ are given by

$$S^{(0)}_1:-\frac{\pi}{2}\leqslant\arg z<\frac{\pi}{2},\quad S^{(0)}_2:\frac{\pi}{2}\leqslant\arg z<\frac{3\pi}{2},\quad |z|<z_0. \tag{3.16}$$

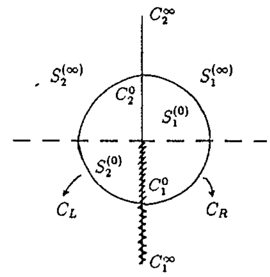


Diagram 2.

The solutions $Y^{(0)}_j(z)$ are related by the Stokes matrices $G^{(0)}_j$; this relation is given by

$$Y^{(0)}_2(z)=Y^{(0)}_1(z)G^{(0)}_1,$$

$$Y^{(0)}_1(z)=Y^{(0)}_2(ze^{2i\pi})G^{(0)}_2e^{-i\pi\theta_0\sigma_3}, \tag{3.17}$$

where

$$G^{(0)}_1=\begin{pmatrix}1 & 0\\ a_0 & 1\end{pmatrix},\quad G^{(0)}_2=\begin{pmatrix}1 & b_0\\ 0 & 1\end{pmatrix}. \tag{3.18}$$

The solutions $Y^{(0)}_1(z)$ and $Y^{(\infty)}_1(z)$ are related by the connection matrix $E$:

$$Y^{(\infty)}_1(z)=Y^{(0)}_1(z)E,\quad E=\begin{pmatrix}\mu & \nu\\ \zeta & \eta\end{pmatrix},\quad \det E=1, \tag{3.19}$$

$$Y^{(\infty)}_2(z)=Y^{(0)}_2(z)\times\begin{cases}[G^{(0)}_1]^{-1}EG^{(\infty)}_1, & \text{if } \operatorname{Im} z>0,\\ G^{(0)}_2e^{-i\pi\theta_0\sigma_3}E[G^{(\infty)}_2e^{i\pi\theta_\infty\sigma_3}]^{-1}, & \text{if } \operatorname{Im} z<0.\end{cases} \tag{3.20}$$

The monodromy data $MD=\{a_0,b_0,a_\infty,b_\infty,\mu,\nu,\zeta,\eta\}$ satisfy the consistency condition

$$G^{(\infty)}_1G^{(\infty)}_2e^{i\pi\theta_\infty\sigma_3}=E^{-1}G^{(0)}_1G^{(0)}_2e^{-i\pi\theta_0\sigma_3}E. \tag{3.21}$$

In particular,

$$2\cos\pi\theta_\infty+a_\infty b_\infty e^{-i\pi\theta_\infty}=2\cos\pi\theta_0+a_0b_0e^{i\pi\theta_0}. \tag{3.22}$$

### C. Schlesinger transformations

Since the consistency condition of the monodromy data (3.21) [or (3.22)] is invariant if $\theta_0$ and $\theta_\infty$ are shifted by integers, we let $\theta_0'=\theta_0+n$, $\theta_\infty'=\theta_\infty+m$, $n$, $m\in\mathbb{Z}$. If $Y'$ corresponds to $\theta_0'$ and $\theta_\infty'$, we let $Y'(z)=R(z)Y(z)$, where

$$R(z)=R_j^{(\infty,0)}(z),\quad \text{when } z \text{ in } S_j^{(\infty,0)},\quad j=1,2. \tag{3.23}$$

Then the definition of the Stokes matrices (3.11) and (3.17), and of the connection matrix (3.19) and (3.20), imply that the transformation matrix $R(z)$ satisfies the RH problem

$$\begin{aligned}
&R_2^{(\infty)}(z)=R_1^{(\infty)}(z) \text{ on } C_2^\infty,\\
&R_1^{(\infty)}(z)=(-1)^m R_2^{(\infty)}(ze^{2i\pi}) \text{ on } C_1^\infty,\\
&R_2^{(0)}(z)=R_1^{(0)}(z) \text{ on } C_2^0,\\
&R_1^{(0)}(z)=(-1)^n R_2^{(0)}(ze^{2i\pi}) \text{ on } C_1^0,
\end{aligned}\tag{3.24}$$

$$\begin{aligned}
&R_1^{(\infty)}(z)=R_1^{(0)}(z) \text{ on } C_R,\\
&R_2^{(\infty)}(z)=R_2^{(0)}(z)\begin{cases} I \text{ on } C_L, & \operatorname{Im} z>0,\\ (-1)^{m+n}I \text{ on } C_L, & \operatorname{Im} z<0,\end{cases}
\end{aligned}\tag{3.25}$$

where the contours $C_L$, $C_R$, $C_j^0$, $C_j^\infty$, $j=1,2$, are indicated in Diagram 2. The continuity of the RH problem along the contour $C_L$ implies that $n+m=2k$, $k\in\mathbb{Z}$. Hence,

$$(\theta_0{}',\theta_\infty{}')=(\theta_0+m-n,\theta_\infty+m+n),\quad m,n\in\mathbb{Z}. \tag{3.26}$$

It is enough to consider the following four cases:

$$(1){:}\begin{cases}\theta_0{}'=\theta_0-1\\ \theta_\infty{}'=\theta_\infty+1\end{cases}\quad (2){:}\begin{cases}\theta_0{}'=\theta_0+1\\ \theta_\infty{}'=\theta_\infty+1,\end{cases}$$

$$(3){:}\begin{cases}\theta_0{}'=\theta_0-1\\ \theta_\infty{}'=\theta_\infty-1,\end{cases}\quad (4){:}\begin{cases}\theta_0{}'=\theta_0+1\\ \theta_\infty{}'=\theta_\infty-1.\end{cases} \tag{3.27}$$

For all four cases the RH problem (3.24) and (3.25) can be written as

$$\begin{aligned}
&R^+(z)=R^-(z) \text{ on } C_2^0+C_2^\infty,\\
&R^+(z)=-R^-(ze^{2i\pi}) \text{ on } C_1^0+C_1^\infty,
\end{aligned}\tag{3.28}$$

with the boundary conditions

$$R(z)\sim[\hat{Y}^{(\infty)}(z)]'\left(\frac{1}{z}\right)^{\frac{1}{2}(m+n)\sigma_3}[\hat{Y}^{(\infty)}(z)]^{-1}$$

as $|z|\to\infty$,

$$R(z)\sim[\hat{Y}^{(0)}(z)]'\left(\frac{1}{z}\right)^{\frac{1}{2}(m-n)\sigma_3}[\hat{Y}^{(0)}(z)]^{-1}$$

as $z\to0$, (3.29)

where $R^+(z)$ and $R^-(z)$ are sectionally analytic functions in sectors $S_1^{(0)}\cup S_1^{(\infty)}$ and $S_2^{(0)}\cup S_2^{(\infty)}$, respectively. The solution of the RH problem (3.28) is given as

$$R(z)=z^{-1/2}\hat{R}(z), \tag{3.30}$$

where $\hat{R}(z)$ is bounded at $z=0$. The explicit form of the transformation matrices $R(z)$ are obtained from Eq. (3.29) and are

$$R_{(1)}(z)=\begin{pmatrix}0&0\\0&1\end{pmatrix}z^{1/2}+\begin{pmatrix}1 & -\dfrac{ws}{s-t}\\ -\dfrac{v}{t} & \dfrac{v}{t}\dfrac{ws}{s-t}\end{pmatrix}z^{-1/2}, \tag{3.31}$$

$$R_{(2)}(z)=\begin{pmatrix}0&0\\0&1\end{pmatrix}z^{1/2}+\begin{pmatrix}1 & -w\\ -\dfrac{v}{t} & \dfrac{wv}{t}\end{pmatrix}z^{-1/2}, \tag{3.32}$$

$$R_{(3)}(z)=\begin{pmatrix}1&0\\0&0\end{pmatrix}z^{1/2}+\begin{pmatrix}-\dfrac{u}{t}\dfrac{s-t}{ws} & \dfrac{u}{t}\\ -\dfrac{s-t}{ws} & 1\end{pmatrix}z^{-1/2}, \tag{3.33}$$

$$R_{(4)}(z)=\begin{pmatrix}1&0\\0&0\end{pmatrix}z^{1/2}+\begin{pmatrix}-\dfrac{u}{tw} & \dfrac{u}{t}\\ -\dfrac{1}{w} & 1\end{pmatrix}z^{-1/2}. \tag{3.34}$$

The transformation matrices $R_{(i)}(z)$, $i=1,\dots,4$, generate all possible transformation matrices. If

$$\begin{aligned}&Y'(z,t;y',u',v',\bar{u}',\bar{v}',w',s',\theta_\infty{}',\theta_0{}')\\ &\quad=R_{(k)}(z,t;y,\dots,\theta_0)\,Y(z,t;y,\dots,\theta_0)\end{aligned}\tag{3.35}$$

and

$$\begin{aligned}&Y''(z,t;y'',u'',v'',\bar{u}'',\bar{v}'',w'',s'',\theta_\infty{}'',\theta_0{}'')\\ &\quad=R_{(l)}(z,t;y',\dots,\theta_0{}')\,Y'(z,t;y',\dots,\theta_0{}'),\end{aligned}\tag{3.36}$$

then

$$R_{(k)}(z,t;y'(y,u,\dots,\theta_0),\dots)R_{(l)}(z,t;y,\dots,\theta_0)=I, \tag{3.37}$$

for $k,l=2,3$ and $k,l=1,4$.

## IV. THE FOURTH PAINLEVÉ EQUATION

The fourth Painlevé equation,

$$\frac{d^2y}{dt^2}=\frac{1}{2y}\left(\frac{dy}{dt}\right)^2+\frac{3}{2}y^3+4ty^2+2(t^2-\alpha)y+\frac{\beta}{y}, \tag{4.1}$$

is the compatibility condition of the linear problem

$$Y_z(z)=A(z)Y(z), \tag{4.2a}$$

$$Y_t(z)=B(z)Y(z), \tag{4.2b}$$

where

$$A(z)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}z+\begin{pmatrix} t & u\\ \frac{2}{u}(v-\theta_0-\theta_\infty) & -t\end{pmatrix}+\begin{pmatrix}\theta_0-v & -\frac{uy}{2}\\ 2\frac{v}{uy}(v-2\theta_0) & -(\theta_0-v)\end{pmatrix}\frac{1}{z},$$

$$B(z)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}z+\begin{pmatrix} 0 & u\\ \frac{2}{u}(v-\theta_0-\theta_\infty) & 0\end{pmatrix}. \tag{4.3}$$

The equation $Y_{zt}=Y_{tz}$ implies that

$$\frac{du}{dt}=-u(y+2t),$$

$$\frac{dv}{dt}=-\frac{2}{y}v^2+\left(\frac{4\theta_0}{y}-y\right)v+(\theta_0+\theta_\infty)y,$$

$$\left(\frac{dy}{dt}\right)=-4v+y^2+2ty+4\theta_0. \tag{4.4}$$

Thus $y$ satisfies the fourth Painlevé equation (4.1) with the parameters

$$\alpha=2\theta_\infty-1,\quad \beta=-8\theta_0^2. \tag{4.5}$$

### A. Solution about $z=\infty$

The formal solution of the Eq. (4.2a) in the neighborhood of the irregular singular point $z=\infty$, $\widetilde{Y}_\infty(z)=(\widetilde{Y}_\infty^{(1)}(z),\ \widetilde{Y}_\infty^{(2)}(z))$ has the form

$$\widetilde{Y}_\infty^{(1)}(z)=\left(\frac{1}{z}\right)^{\theta_\infty}e^{q(z)}\left\{\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}-K\\ \frac{1}{u}(v-\theta_0-\theta_\infty)\end{pmatrix}\frac{1}{z}+\cdots\right\}=\left(\frac{1}{z}\right)^{\theta_\infty}e^{q(z)}\widehat{Y}_\infty^{(1)}(z),$$

$$\widetilde{Y}_\infty^{(2)}(z)=\left(\frac{1}{z}\right)^{-\theta_\infty}e^{-q(z)}\left\{\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}-\frac{u}{2}\\ K\end{pmatrix}\frac{1}{z}+\cdots\right\}$$

$$=\left(\frac{1}{z}\right)^{-\theta_\infty}e^{-q(z)}\widehat{Y}_\infty^{(2)}(z), \tag{4.6}$$

where

$$K=\frac{v}{y}(v-2\theta_0)-(v-\theta_0-\theta_\infty)\left(t+\frac{y}{2}\right),$$

$$q(z)=\frac{z^2}{2}+tz.$$

Let $Y_j(z)$ be the solution of (4.2) defined by $Y_j(z)\sim Y_\infty(z)$ as $|z|\to\infty$, $z$ in the sector $S_j$, $j=1,2,3,4$, where the sectors $S_j$ are given by

$$S_1:-\frac{\pi}{4}\leqslant\arg z<\frac{\pi}{4},\quad S_2:\frac{\pi}{4}\leqslant\arg z<\frac{3\pi}{4},$$

$$S_3:\frac{3\pi}{4}\leqslant\arg z<\frac{5\pi}{4},\quad S_4:\frac{5\pi}{4}\leqslant\arg z<\frac{7\pi}{4}. \tag{4.7}$$

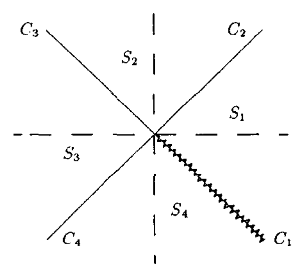


Diagram 3.

The solutions $Y_j(z)$ are related by the Stokes matrices $G_j$ via

$$Y_{k+1}(z)=Y_k(z)G_k,\quad k=1,2,3,$$

$$Y_1(z)=Y_4(ze^{2i\pi})G_4e^{2i\pi\theta_\infty\sigma_3}, \tag{4.8}$$

$$G_1=\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix},\quad G_2=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix},$$

$$G_3=\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix},\quad G_4=\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}. \tag{4.9}$$

### B. Solution about $z=0$

The solution $Y_0(z)=(Y_0^{(1)}(z),\ Y_0^{(2)}(z))$ of Eqs. (4.2) in the neighborhood of the regular singular point $z=0$ for $\theta_0\neq n_2$, $n\in\mathbf{Z}$ has the form

$$Y_0^{(1)}(z)=z^{\theta_0}e^{-\sigma(t)}\left\{\begin{pmatrix}1\\ -\frac{2v}{uy}\end{pmatrix}+\frac{1}{2\theta_0+1}\begin{pmatrix}H_{(1)}^{(1)}\\ \cdots\end{pmatrix}z+\cdots\right\}$$

$$=z^{\theta_0}\hat{Y}_0^{(1)}(z),$$

$$Y_0^{(2)}(z)=z^{-\theta_0}e^{\sigma(t)}\left(1-\frac{v}{2\theta_0}\right)\left\{\begin{pmatrix}\frac{uy}{2(2\theta_0-v)}\\ 1\end{pmatrix}+\frac{1}{2\theta_0-1}\begin{pmatrix}H_{(1)}^{(2)}\\ \cdots\end{pmatrix}z+\cdots\right\}$$

$$=z^{-\theta_0}\hat{Y}_0^{(2)}(z), \tag{4.10}$$

where

$$H_{(1)}^{(1)}=-y(v-\theta_0-\theta_\infty)-vt+(1+2\theta_0-v)(t+2vy),$$

$$H_{(1)}^{(2)}=\frac{uy}{2}\left[t+\frac{2(1-v)}{y}-\frac{y}{2\theta_0-v}\times\left(v-\theta_0-\theta_\infty-\frac{t(1-v)}{y}\right)\right], \tag{4.11}$$

$$\sigma(t)=\int^t\frac{2v}{y}\,dt'.$$

The monodromy matrix about $z=0$ is given as

$$Y_0(ze^{2i\pi})=Y_0(z)e^{2i\pi\theta_0\sigma_3}, \tag{4.12}$$

and the relation between $Y_1(z)$ and $Y_0(z)$ is given by connection matrix $E$,

$$Y_1(z)=Y_0(z)E,\quad E=\begin{pmatrix}\mu & \nu\\ \zeta & \eta\end{pmatrix},\quad \det E=1. \tag{4.13}$$

The monodromy data $MD=\{a,b,c,d,\mu,\nu,\zeta,\eta\}$ satisfy the consistency condition

$$\prod_{j=1}^{4}G_je^{2i\pi\theta_\infty\sigma_3}=E^{-1}e^{-2i\pi\theta_0\sigma_3}E; \tag{4.14}$$

in particular,

$$(1+bc)e^{2i\pi\theta_\infty}+[ad+(1+cd)(1+ab)]e^{-2i\pi\theta_\infty}=2\cos 2\pi\theta_0. \tag{4.15}$$

### C. Schlesinger transformations

The consistency condition of the monodromy data (4.14) is invariant if $\theta_0$ and $\theta_\infty$ are shifted as

$$\text{a:}\begin{cases}\theta_0{}'=\theta_0+n\\ \theta_\infty{}'=\theta_\infty+m'\end{cases}\quad \text{b:}\begin{cases}\theta_0{}'=\theta_0+\frac{2n+1}{2}\\ \theta_\infty{}'=\theta_\infty+\frac{2m+1}{2}\end{cases}. \tag{4.16}$$

If $Y'$ corresponds to $\theta_0'$ and $\theta_\infty'$, we let $Y'(z)=R(z)Y(z)$, where $R(z)=R_j(z)$ when $z$ in $S_j$, $j=1,2,3,4$. Then Eq. (4.8) implies a RH problem for $R(z)$:

$$\text{a:}\begin{cases}R_{k+1}(z)=R_k(z) & \text{on } C_{k+1},\ k=1,2,3\\ R_1(z)=R_4(ze^{2i\pi}) & \text{on } C_1,\end{cases}$$

$$\text{b:}\begin{cases}R_{k+1}(z)=R_k(z) & \text{on } C_{k+1},\ k=1,2,3\\ R_1(z)=-R_4(ze^{2i\pi}) & \text{on } C_1,\end{cases} \tag{4.17}$$

with the boundary conditions

$$\text{a:}\begin{cases}R(z)\sim\hat{Y}_0'(z)z^{n\sigma_3}\hat{Y}_0^{-1}(z) & \text{as } z\to 0,\\ R(z)\sim\hat{Y}_\infty'(z)\left(\frac{1}{z}\right)^{m\sigma_3}\hat{Y}_\infty'(z) & \text{as } |z|\to\infty,\end{cases}$$

$$\text{b:}\begin{cases}R(z)\sim\hat{Y}_0'(z)z^{((2n+1)/2)\sigma_3}\hat{Y}_0^{-1}(z) & \text{as } z\to 0\\ R(z)\sim\hat{Y}_\infty'(z)\left(\frac{1}{z}\right)^{((2m+1)/2)\sigma_3}\hat{Y}_\infty'(z) & \text{as } |z|\to\infty,\end{cases} \tag{4.18}$$

where the contours $C_j$, $j=1,2,3,4$, are indicated in Diagram 3.

For the case a, there exists a function $R(z)$ which is analytic everywhere and

$$R(z)\equiv R_1(z)=R_2(Z)=R_3(z)=R_4(z). \tag{4.19}$$

The boundary conditions (4.18) specify $R(z)$ as a rational function of $z$. For the case b, there exists a function $R(z)$ which is analytic everywhere except along the contour $C_1$. The solution of the RH problem (4.17b) is given as

$$R(z)=z^{-1/2}\hat{R}(z), \tag{4.20}$$

where $\hat{R}(z)$ is bounded at $z=0$. By using the boundary conditions for $R(z)$, the function $\hat{R}(z)$ can be determined.

All possible Schlesinger transformations admitted by the linear problem (4.2) can be generated by the following transformation matrices $R_j(z)$, $j=1,2,3,4$,

$$\begin{cases}\theta_0'=\theta_0-\frac{1}{2}\\ \theta_\infty'=\theta_\infty+\frac{1}{2}\end{cases},\quad R_{(1)}(z)=\begin{pmatrix}0&0\\0&1\end{pmatrix}z^{1/2}+\begin{pmatrix}1&\dfrac{uy}{2(v-2\theta_0)}\\ -\dfrac{v-\theta_0-\theta_\infty}{u}&-\dfrac{y(v-\theta_0-\theta_\infty)}{2(v-2\theta_0)}\end{pmatrix}z^{-1/2}, \tag{4.21}$$

$$\begin{cases}\theta_0'=\theta_0+\frac{1}{2}\\ \theta_\infty'=\theta_\infty-\frac{1}{2}\end{cases},\quad R_{(2)}(z)=\begin{pmatrix}1&0\\0&0\end{pmatrix}z^{1/2}+\begin{pmatrix}\dfrac{v}{y}&\dfrac{u}{2}\\ \dfrac{2v}{uy}&1\end{pmatrix}z^{-1/2}, \tag{4.22}$$

$$\begin{cases}\theta_0'=\theta_0+\frac{1}{2}\\ \theta_\infty'=\theta_\infty+\frac{1}{2}\end{cases},\quad R_{(3)}(z)=\begin{pmatrix}0&0\\0&1\end{pmatrix}z^{1/2}+\begin{pmatrix}1&\dfrac{uy}{2v}\\ -\dfrac{v-\theta_0-\theta_\infty}{u}&-\dfrac{y(v-\theta_0-\theta_\infty)}{2v}\end{pmatrix}z^{-1/2}, \tag{4.23}$$

$$\begin{cases}\theta_0'=\theta_0-\frac{1}{2}\\ \theta_\infty'=\theta_\infty-\frac{1}{2}\end{cases},\quad R_{(4)}(z)=\begin{pmatrix}1&0\\0&0\end{pmatrix}z^{1/2}+\begin{pmatrix}\dfrac{v-2\theta_0}{y}&\dfrac{u}{2}\\ \dfrac{2}{uy}(v-2\theta_0)&1\end{pmatrix}z^{-1/2}. \tag{4.24}$$

If $y',u',v',\theta_0'=\theta_0-\frac{1}{2}$, $\theta_\infty'=\theta_\infty+\frac{1}{2}$ are the transformed quantities of $y,u,v,\theta_0,\theta_\infty$ under the transformation given by $R_{(1)}$, i.e.,

$$Y'(z,t;y',u',v',\theta_0',\theta_\infty')=R_{(1)}(z,t;y,u,v,\theta_0,\theta_\infty)\,Y(z,t;y,u,v,\theta_0,\theta_\infty), \tag{4.25}$$

and if $y'',u'',v'',\theta_0''=\theta_0'+\frac{1}{2}$, $\theta_\infty''=\theta_\infty'-\frac{1}{2}$ are the transformed quantities of $y'$, $u'$, $v'$, $\theta_0'$, $\theta_\infty'$ under the transformation given by $R_{(2)}(z)$, i.e.,

$$Y''(z,t;y'',u'',v'',\theta_0'',\theta_\infty'')=R_{(2)}(z,t;y',u',v',\theta_0',\theta_\infty')\,Y(z,t;y',u',v',\theta_0',\theta_\infty'), \tag{4.26}$$

then

$$R_{(2)}(z,t;y'(y,u,\ldots),\ldots)R_{(1)}(z,t;y,\ldots)=I. \tag{4.27}$$

Similarly,

$$R_{(3)}(z,t;y'(y,u,\ldots),\ldots)R_{(4)}(z,t;y,\ldots)=I. \tag{4.28}$$

Also,

$$R_{(1)}(z,t;y'(y,u,\ldots),\ldots)R_{(3)}(z,t;y,\ldots)=R_{(5)},$$

$$R_{(2)}(z,t;y'(y,u,\ldots),\ldots)R_{(4)}(z,t;y,\ldots)=R_{(6)}, \tag{4.29}$$

where $R_{(5)}$ and $R_{(6)}$ are

$$R_{(5)}(z)=\begin{pmatrix}0&0\\0&1\end{pmatrix}z+\begin{pmatrix}0&\dfrac{u}{v-\theta_0-\theta_\infty}\\ -\dfrac{v-\theta_0-\theta_\infty}{u}&-\dfrac{v(v-2\theta_0)}{y(v-\theta_0-\theta_\infty)}+t\end{pmatrix}$$

$$R_{(6)}(z)=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{2\theta_0+1}{N}\begin{pmatrix}-1&-\dfrac{uy}{2v}\\ \dfrac{2v}{uy}&1\end{pmatrix}z^{-1}$$

$$N=2\left[t+\frac{v}{y}+\frac{y}{2v}(v-\theta_0-\theta_\infty)\right], \tag{4.30}$$

and shift the exponents $\theta_0'=\theta_0$, $\theta_\infty'=\theta_\infty+1$ and $\theta_0'=\theta_0+1$, $\theta_\infty'=\theta_\infty$, respectively. Hence, the successive application of the Schlesinger transformations defined by the transformation matrices $R_{(j)}$, $j=1,2,3,4$, maps $\theta_0$, $\theta_\infty$ to $\theta_0'=\theta_0+n/2$, $\theta_\infty'=\theta_\infty+m/2$, $n,m\in\mathbb{Z}$.

## V. THE FIFTH PAINLEVÉ EQUATION

The fifth Painlevé equation,

$$\frac{d^2y}{dt^2}=\left(\frac{1}{2y}+\frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2-\frac{1}{t}\frac{dy}{dt}+\frac{(y-1)^2}{t^2}\left(\alpha y+\frac{\beta}{y}\right)+\frac{\gamma y}{t}+\frac{\delta y(y+1)}{y-1}, \tag{5.1}$$

is the compatibility condition of

$$Y_z=A(z)Y, \tag{5.2a}$$

$$Y_t=B(z)Y, \tag{5.2b}$$

where

$$A(z)=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\begin{pmatrix}v+\frac{\theta_0}{2} & -u(v+\theta_0)\\ \frac{v}{u} & -\left(v+\frac{\theta_0}{2}\right)\end{pmatrix}\frac{1}{z}+\begin{pmatrix}-w & uy\left(w-\frac{\theta_1}{2}\right)\\ -\frac{1}{uy}\left(w+\frac{\theta_1}{2}\right) & w\end{pmatrix}\frac{1}{z-1},$$

$$B(z)=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}z+\frac{1}{t}\begin{pmatrix}0 & u\left[v+\theta_0-y\left(w-\frac{\theta_1}{2}\right)\right]\\ \frac{1}{u}\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right] & 0\end{pmatrix},\quad w=v+\tfrac{1}{2}(\theta_0+\theta_\infty). \tag{5.3}$$

The equation $Y_{zt}=Y_{tz}$ implies

$$t\frac{dy}{dt}=ty-2v(y-1)^2-\frac{1}{2}(y-1)[(\theta_0-\theta_1+\theta_\infty)y-(3\theta_0+\theta_1+\theta_\infty)],$$

$$t\frac{dv}{dt}=yv\left(w-\frac{\theta_1}{2}\right)-\frac{1}{y}(v+\theta_0)\left(w+\frac{\theta_1}{2}\right),$$

$$t\frac{du}{dt}=u\left[-2t-\theta_0+y\left(w-\frac{\theta_1}{2}\right)+\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right]. \tag{5.4}$$

Thus $y$ satisfies the fifth Painlevé equation (5.1) with the parameters

$$\alpha=\frac{1}{2}\left(\frac{\theta_0-\theta_1+\theta_\infty}{2}\right)^2,\quad \beta=-\frac{1}{2}\left(\frac{\theta_0-\theta_1-\theta_\infty}{2}\right)^2,$$

$$\gamma=1-\theta_0-\theta_1,\quad \delta=-\tfrac{1}{2}. \tag{5.5}$$

### A. Solution about $z=0$

The solution $Y_{(0)}(z)=(Y_{(0)}^{(1)}(z),\ Y_{(0)}^{(2)}(z))$ of Eq. (5.2) in the neighborhood of the regular singular point $z=0$, for $|z|<1$ and for $\theta_0\neq$integer, has the form

$$Y_{(0)}^{(1)}(z)=z^{\theta_0/2}e^{-\sigma_0}\frac{v}{u\theta_0}\left\{\begin{pmatrix}\frac{u}{v}(v+\theta_0)\\ 1\end{pmatrix}+\begin{pmatrix}\cdots\\ K_{(0)}^{(2)}\end{pmatrix}z+\cdots\right\}=z^{\theta_0/2}\hat{Y}_{(0)}^{(1)}(z),$$

$$Y_{(0)}^{(2)}(z)=z^{-\theta_0/2}e^{-\sigma_0}\left\{\begin{pmatrix}u\\ 1\end{pmatrix}+\begin{pmatrix}\cdots\\ L_{(0)}^{(2)}\end{pmatrix}z+\cdots\right\}=z^{-\theta_0/2}\hat{Y}_{(0)}^{(2)}(z), \tag{5.6}$$

where

$$K_{(0)}^{(2)}=\frac{v-1}{1+\theta_0}\left[\left(\frac{t}{2}+w\right)\left(1+\frac{v+\theta_0}{v-1}\right)-\frac{v+\theta_0}{vy}\times\left(w+\frac{\theta_1}{2}\right)+\frac{vy}{1-v}\left(w-\frac{\theta_1}{2}\right)\right],$$

$$L_{(0)}^{(2)}=\frac{u(1+v)}{1-\theta_0}\left[\left(\frac{t}{2}+w\right)\left(1+\frac{v+\theta_0}{v+1}\right)-\frac{v+\theta_0}{y(1+v)}\times\left(w+\frac{\theta_1}{2}\right)-y\left(w-\frac{\theta_1}{2}\right)\right],$$

$$\sigma_0=\int^t\left\{\frac{1}{t'}\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right]-\frac{1}{2}\right\}dt'. \qquad (5.7)$$

The monodromy matrix about the regular singular point $z=0$ is defined as

$$Y_{(0)}(ze^{2i\pi})=Y_{(0)}(z)e^{i\pi\theta_0\sigma_3}, \quad |z|<1. \qquad (5.8)$$

### B. Solution about $z=1$

The solution $Y_{(1)}(z)=(Y_{(1)}^{(1)}(z),\ Y_{(1)}^{(2)}(z))$ of (5.2) in the neighborhood of the regular singular point $z=1$, for $|z-1|<1$ and $\theta_1\neq$integer, has the form

$$Y_{(1)}^{(1)}(z)=(z-1)^{\theta_1/2}e^{-\sigma_1}\left(\frac{\theta_1-2w}{2\theta_1}\right)\times\left\{\begin{pmatrix}1\\ \frac{1}{uy}\frac{w+\theta_1/2}{w-\theta_1/2}\end{pmatrix}+\begin{pmatrix}\cdots\\ K_{(1)}^{(2)}\end{pmatrix}(z-1)+\cdots\right\}=(z-1)^{\theta_1/2}\hat{Y}_{(1)}^{(1)}(z),$$

$$Y_{(1)}^{(2)}(z)=(z-1)^{-\theta_1/2}e^{-\sigma_1}\left\{\begin{pmatrix}uy\\ 1\end{pmatrix}+\begin{pmatrix}\cdots\\ L_{(1)}^{(2)}\end{pmatrix}(z-1)+\cdots\right\}=(z-1)^{-\theta_1/2}\hat{Y}_{(1)}^{(2)}(z), \qquad (5.9)$$

where

$$K_{(1)}^{(2)}=\frac{1}{1+\theta_1}\left[\frac{1}{uy^2}\frac{w+\theta_1/2}{w-\theta_1/2}(v+\theta_0)-\frac{1}{uy}\left(\frac{t}{2}+v+\frac{\theta_0}{2}\right)\times\left(w+\frac{\theta_1}{2}\right)\left(\frac{1+2w}{w-\theta_1/2}\right)+\frac{v}{u}\left(1+w+\frac{\theta_1}{2}\right)\right],$$

$$L_{(1)}^{(2)}=\frac{1+w-\theta_1/2}{1-\theta_1}\left[\frac{w+\theta_1/2}{y(1+w-\theta_1/2)}(v+\theta_0)+vy\left(\frac{t}{2}+v+\frac{\theta_0}{2}\right)\left(\frac{1+2w}{1+w-\theta_1/2}\right)\right],$$

$$\sigma_1(t)=\int^t\left\{\frac{y}{t'}\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right]-\frac{1}{2}\right\}dt'. \qquad (5.10)$$

The monodromy matrix about the point $z=1$ is defined as

$$Y_{(1)}(ze^{2i\pi})=Y_{(1)}(z)e^{i\pi\theta_1\sigma_3}, \quad |z-1|<1. \qquad (5.11)$$

### C. Solution about $z=\infty$

The formal solution $\widetilde{Y}_\infty(z)=(\widetilde{Y}_\infty^{(1)}(z),\ \widetilde{Y}_\infty^{(2)}(z))$ of Eq. (5.2a) in the neighborhood of the irregular singular point $z=\infty$ has the form

$$\widetilde{Y}_\infty^{(1)}(z)=\left(\frac{1}{z}\right)^{\theta_\infty/2}e^{q(z)}\left\{\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}-K_\infty\\ \frac{1}{ut}\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right]\end{pmatrix}\frac{1}{z}+\cdots\right\}=\left(\frac{1}{z}\right)^{\theta_\infty/2}e^{q(z)}\hat{Y}_\infty^{(1)}(z),$$

$$\widetilde{Y}_\infty^{(2)}(z)=\left(\frac{1}{z}\right)^{-\theta_\infty/2}e^{-q(z)}\left\{\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}\frac{u}{t}\left[v+\theta_0-y\left(w-\frac{\theta_1}{2}\right)\right]\\ K_\infty\end{pmatrix}\frac{1}{z}+\cdots\right\}=\left(\frac{1}{z}\right)^{-\theta_\infty/2}e^{-q(z)}\hat{Y}_\infty^{(2)}(z), \qquad (5.12)$$

where

$$K_\infty=-\frac{1}{t}\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right]\left[v+\theta_0+y\left(w-\frac{\theta_1}{2}\right)\right]-w,$$

$$q(z)=(zt/2).$$

Let $Y_{(j)}(z)$, $j=1,2$, be the solution of (5.2), such that $\det Y_j(z)=1$ and $Y_j(z)\sim \widetilde{Y}_\infty(z)$, as $|z|\to\infty$ in the sectors $S_j$, where the sectors $S_j$ are given by

$$S_1: -\frac{\pi}{2}\leqslant \arg z<\frac{\pi}{2},\quad S_2: \frac{\pi}{2}\leqslant \arg z<\frac{3\pi}{2}. \tag{5.13}$$

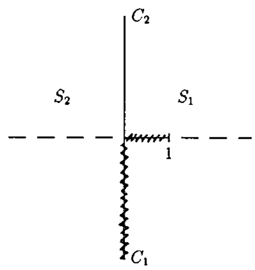


Diagram 4.

The solutions $Y_j(z)$ are related by the Stokes matrices $G_j$ and the relation is

$$Y_2(z)=Y_1(z)G_1,\quad Y_1(z)=Y_2(ze^{2i\pi})G_2e^{i\pi\theta_\infty\sigma_3}, \tag{5.14}$$

$$G_1=\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix},\quad G_2=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}. \tag{5.15}$$

The relation between the solutions $Y_{(1)}(z)$ and $Y_{(0)}(z)$ are given by the connection matrices $E_0$, $E_1$, respectively,

$$Y_1(z)=Y_{(0)}(z)E_0,\quad E_0=\begin{pmatrix}\mu_0 & \nu_0\\ \zeta_0 & \eta_0\end{pmatrix},\quad \det E=1,$$

$$Y_1(z)=Y_{(1)}(z)E_1,\quad E_1=\begin{pmatrix}\mu_1 & \nu_1\\ \zeta_1 & \eta_1\end{pmatrix},\quad \det E_1=1. \tag{5.16}$$

Let $Y_1^+(z)$ and $Y_1^-(z)$ be the limit values of $Y_1(z)$, as $z$ approaches to contour $C_3$ (see Diagram 5) from above and from below, respectively. Then they are related as

$$Y_1^+(z)=Y_1^-(z)E_1^{-1}e^{i\pi\theta_1\sigma_3}E_1\quad \text{for } 0<z<1,$$

$$Y_1^+(z)=Y_1^+(z)\quad \text{for } z>1. \tag{5.17}$$

The monodromy data $MD=\{a,b,\mu_0,\nu_0,\zeta_0,\eta_0,\mu_1,\nu_1,\zeta_1,\eta_1\}$ satisfy the consistency condition

$$G_1G_2e^{i\pi\theta_\infty\sigma_3}=E_0^{-1}e^{-i\pi\theta_0\sigma_3}E_0E_1^{-1}e^{-i\pi\theta_1\sigma_3}E_1. \tag{5.18}$$

## D. Schlesinger transformations

The monodromy data or equivalently the consistency condition of the monodromy data (5.18) is invariant under the transformation

$$\text{a:}\begin{cases}\theta_0'=\theta_0+n\\ \theta_1'=\theta_1\\ \theta_\infty'=\theta_\infty+m\end{cases},\quad \text{b:}\begin{cases}\theta_0'=\theta_0\\ \theta_1'=\theta_1+n\\ \theta_\infty'=\theta_\infty+m\end{cases},$$

$$\text{c:}\begin{cases}\theta_0'=\theta_0+n\\ \theta_1'=\theta_1+m,\\ \theta_\infty'=\theta_\infty\end{cases} \tag{5.19}$$

where $n$ and $m$ are either even or odd integers. It is enough to consider the cases $n,m=\pm 1$. Let

$$R(z)=R_1^+(z)\quad \text{when } z \text{ in } S_1^+,$$

$$R(z)=R_1^-(z)\quad \text{when } z \text{ in } S_1^-,$$

$$R(z)=R_2(z)\quad \text{when } z \text{ in } S_2, \tag{5.20}$$

where the sectors $S_1^\pm$ are

$$S_1^+: 0\leqslant \arg z<\frac{\pi}{2},\qquad S_1^-: -\frac{\pi}{2}\leqslant \arg z<0. \tag{5.21}$$

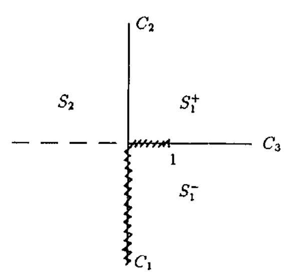


Diagram 5.

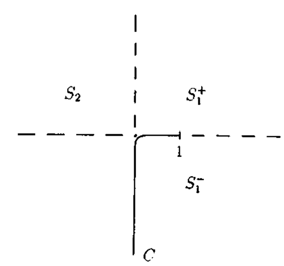


Diagram 6.

If $Y'$ corresponds to $\theta_0'$, $\theta_1'$, and $\theta_\infty'$, we let

$$[Y_1^\pm(z)]'=R_1^\pm(z)Y_1^\pm(z)\quad \text{when } z \text{ in } S_1^\pm,$$

$$Y_2'(z)=R_2(z)Y_2(z)\quad \text{when } z \text{ in } S_2. \tag{5.22}$$

The definition of the Stokes matrices (5.14) and Eq. (5.17) imply that the transformation matrix $R(z)$ satisfies the following RH problem along the contours $C_k$, $k=1,2,3$, indicated in Diagram 5:

$$\text{a:}\begin{cases} R_2(z) = R_1^+(z) & \text{on } C_2, \\ R_1^+(z) = R_1^-(z) & \text{on } C_3, \\ R_1^-(z) = -R_2(ze^{2i\pi}) & \text{on } C_1, \end{cases} \tag{5.23}$$

$$\text{b:}\begin{cases} R_2(z) = R_1^+(z) \quad \text{on } C_2 \\ R_1^+(z) = R_1^-(z) \begin{cases} -I, & 0<z<1 \quad \text{on } C_3 \\ I, & z<1 \quad \text{on } C_3 \end{cases} \\ R_1^-(z) = R_2(ze^{2i\pi}) \quad \text{on } C_1, \end{cases} \tag{5.24}$$

$$\text{c:}\begin{cases} R_2(z) = R_1^+(z) \quad \text{on } C_2 \\ R_1^+(z) = R_1^-(z) \begin{cases} -I, & 0<z<1 \quad \text{on } C_3 \\ I, & z<1 \quad \text{on } C_3 \end{cases} \\ R_1^-(z) = R_2(ze^{2i\pi}) \quad \text{on } C_1, \end{cases} \tag{5.25}$$

with the following boundary conditions:

$$\text{a:}\begin{cases} R_1^+(z) \sim Y'_{(0)}(z) z^{\pm\sigma_3} Y_{(0)}^{-1}(z) & \text{as } z\to 0, \ z \text{ in } S_1^+, \\ R_1^+(z) \sim Y'_{(1)}(z) Y_{(1)}^{-1}(z) & \text{as } z\to 1, \ z \text{ in } S_1^+, \\ R_1^+(z) \sim Y'_{(\infty)}(z)(1/z)^{\pm\sigma_3} Y_{(\infty)}^{-1}(z) & \text{as } |z|\to\infty, \ z \text{ in } S_1^+, \end{cases} \tag{5.26}$$

$$\text{b:}\begin{cases} R_1^+(z) \sim Y'_{(0)}(z) Y_{(0)}^{-1}(z) & \text{as } z\to 0, \ z \text{ in } S_1^+, \\ R_1^+(z) \sim Y'_{(1)}(z)(z-1)^{\pm\sigma_3} Y_{(1)}^{-1}(z) & \text{as } z\to 1, \ z \text{ in } S_1^+, \\ R_1^+(z) \sim Y'_{(\infty)}(z)(1/z)^{\pm\sigma_3} Y_{(\infty)}^{-1}(z) & \text{as } |z|\to\infty, \ z \text{ in } S_1^+, \end{cases} \tag{5.27}$$

$$\text{c:}\begin{cases} R_1^+(z) \sim Y'_{(0)}(z) z^{\pm\sigma_3} Y_{(0)}^{-1}(z) & \text{as } z\to 0, \ z \text{ in } S_1^+, \\ R_1^+(z) \sim Y^{+}_{(1)}(z)(z - z^{\pm\sigma_3} Y_{(0)}^{-1}(z) 1)^{\pm\sigma_3} Y_{(1)}^{-1}(z) & \text{as } z\to 1, \ z \text{ in } S_1^+, \\ R_1^+(z) \sim Y'_{(\infty)}(z) Y_{(\infty)}^{-1}(z) & \text{as } |z|\to\infty, \ z \text{ in } S_1^+ . \end{cases} \tag{5.28}$$

In the case a, there exists a function $R_a(z)$ which is analytic everywhere except along the contour $C_1$ on which it satisfies the jump condition,

$$R_a^+(z) = -R_a^-(z). \tag{5.29}$$

The solution of the above RH problem is given as

$$R_a(z) = z^{-1/2}\hat{R}_a(z), \tag{5.30}$$

where $\hat{R}_a(z)$ is bounded at $z=0$. For the case b, the RH problem (5.24) implies that there exists a function $R_b(z)$ which is analytic everywhere except along the contour $C$ indicated in Diagram 6 and the jump is given by

$$R_b^+(z) = -R_b^-(z). \tag{5.31}$$

The solution of the RH problem is

$$R_b(z) = (z-1)^{-1/2}\hat{R}_b(z), \tag{5.32}$$

where $\hat{R}_b$ is bounded at $z=1$. For the case c, Eq. (5.25) yields the following RH problem along the contour $C_3$ for $0<z<1$,

$$R_c^+(z) = -R_c^-(z), \tag{5.33}$$

and its solution is given as

$$R_c(z) = z^{-1/2}(z-1)^{1/2}\hat{R}_c(z), \tag{5.34}$$

$\hat{R}_c(z)$ is bounded at $z=0$ and $z=1$.

It is enough to determine the transformation matrix $R(z)$ for $n,m=\pm 1$. The explicit form of $R(z)$ can be listed as follows:

$$\begin{cases}\theta_0'=\theta_0+1,\\ \theta_1'=\theta_1,\\ \theta_\infty'=\theta_\infty+1,\end{cases}\quad R_{(1)}(z)=\begin{pmatrix}0&0\\0&1\end{pmatrix}z^{1/2}+\begin{pmatrix}1 & -\frac{u}{v}(v+\theta_0)\\ -\frac{1}{tu}\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right] & \frac{1}{tv}(v+\theta_0)\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right]\end{pmatrix}z^{-1/2},\tag{5.35}$$

$$\begin{cases}\theta_0'=\theta_0-1,\\ \theta_1'=\theta_1,\\ \theta_\infty'=\theta_\infty-1,\end{cases}\quad R_{(2)}(z)=\begin{pmatrix}1&0\\0&1\end{pmatrix}z^{1/2}+\begin{pmatrix}\frac{1}{t}\left[v+\theta_0-\frac{1}{y}\left(w-\frac{\theta_1}{2}\right)\right] & -\frac{u}{t}\left[v+\theta_0-y\left(w-\frac{\theta_1}{2}\right)\right]\\ -\frac{1}{u} & 1\end{pmatrix}z^{-1/2},\tag{5.36}$$

$$\begin{cases}\theta_0'=\theta_0+1,\\ \theta_1'=\theta_1,\\ \theta_\infty'=\theta_\infty-1,\end{cases}\quad R_{(3)}(z)=\begin{pmatrix}1&0\\0&0\end{pmatrix}z^{1/2}+\begin{pmatrix}\frac{1}{t}\left[v+\theta_0-y\left(w-\frac{\theta_1}{2}\right)\right]\frac{v}{v+\theta_0} & -\frac{u}{t}\left[v+\theta_0-y\left(w-\frac{\theta_1}{2}\right)\right]\\ -\frac{v}{u(v+\theta_0)} & 1\end{pmatrix}z^{-1/2},\tag{5.37}$$

$$\begin{cases}\theta_0'=\theta_0-1,\\ \theta_1'=\theta_1,\\ \theta_\infty'=\theta_\infty+1,\end{cases}\quad R_{(4)}(z)=\begin{pmatrix}0&0\\0&1\end{pmatrix}z^{1/2}+\begin{pmatrix}1 & -u\\ -\frac{1}{tu}\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right] & \frac{1}{t}\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right]\end{pmatrix}z^{-1/2},\tag{5.38}$$

$$\begin{cases}\theta_0'=\theta_0,\\ \theta_1'=\theta_1+1,\\ \theta_\infty'=\theta_\infty+1,\end{cases}\quad R_{(5)}(z)=\begin{pmatrix}0&0\\0&1\end{pmatrix}(z-1)^{1/2}+\begin{pmatrix}1 & -\frac{uy}{w_1}\\ -\frac{1}{tu}\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right] & \frac{y}{t}\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right]\frac{1}{w_1}\end{pmatrix}(z-1)^{-1/2},\tag{5.39}$$

$$\begin{cases}\theta_0'=\theta_0,\\ \theta_1'=\theta_1-1,\\ \theta_\infty'=\theta_\infty-1,\end{cases}\quad R_{(6)}=\begin{pmatrix}1&0\\0&0\end{pmatrix}(z-1)^{1/2}+\begin{pmatrix}\frac{1}{ty}\left[v+\theta_0-y\left(w-\frac{\theta_1}{2}\right)\right] & -\frac{u}{t}\left[v+\theta_0-y\left(w-\frac{\theta_0}{2}\right)\right]\\ -\frac{1}{uy} & 1\end{pmatrix}(z-1)^{-1/2},\tag{5.40}$$

$$\begin{cases}\theta_0'=\theta_0,\\ \theta_1'=\theta_1+1,\\ \theta_\infty'=\theta_\infty-1,\end{cases}\quad R_{(7)}(z)=\begin{pmatrix}1&0\\0&0\end{pmatrix}(z-1)^{1/2}+\begin{pmatrix}\frac{w_1}{ty}\left[v+\theta_0-y\left(w-\frac{\theta_1}{2}\right)\right] & -\frac{u}{t}\left[v+\theta_0-y\left(w-\frac{\theta_1}{2}\right)\right]\\ -\frac{1}{uy}w_1 & 1\end{pmatrix}(z-1)^{-1/2},\tag{5.41}$$

$$\begin{cases}\theta_0'=\theta_0,\\ \theta_1'=\theta_1-1,\\ \theta_\infty'=\theta_\infty+1,\end{cases}\quad R_{(8)}(z)=\begin{pmatrix}0&0\\0&1\end{pmatrix}(z-1)^{1/2}+\begin{pmatrix}1 & -uy\\ -\frac{1}{tu}\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right] & \frac{y}{t}\left[v-\frac{1}{y}\left(w+\frac{\theta_1}{2}\right)\right]\end{pmatrix}(z-1)^{-1/2},\tag{5.42}$$

where

$$w_1=\frac{w+\theta_1/2}{w-\theta_1/2}.\tag{5.43}$$

The transformation matrices $R_{(j)}(z)$, $j=1,2,\ldots,8$, are sufficient to obtain the transformation matrix $R(z)$ which shifts the exponents $\theta_0,\theta_1,\theta_\infty$ to $\theta_0'$, $\theta_1'$, $\theta_\infty'$ with any integer differences. If

$$Y'(z,t;y',u',v',\theta_0',\theta_1',\theta_\infty')=R_{(j)}(z,t;y,\ldots,\theta_\infty)Y(z,t;y,\ldots,\theta_\infty),\tag{5.44}$$

and

$$Y''(z,t;y'',u'',v'',\theta_0'',\theta_1',\theta_\infty')$$

$$= R_{(k)}(z,t;y',...,\theta_\infty')\,Y(z,t;y',...,\theta_\infty'). \tag{5.45}$$

Then

$$R_{(k)}(z,t;y'(y,u,...,\theta_\infty),...)R_{(j)}(z,t;y,...,\theta_\infty) = I,$$

$$\text{for } k=j+1, \quad j=1,3,5,7. \tag{5.46}$$

Also, $R_{(1)}(z)R_{(7)}(z) = R_{(9)}(z)$ shifts the exponents as $\theta_0' = \theta_0 + 1$, $\theta_1' = \theta_1 + 1$, $\theta_\infty' = \theta_\infty$, and $R_{(2)}(z)R_{(8)}(z) = R_{(10)}(z)$ shifts the exponents as $\theta_0' = \theta_0 - 1$, $\theta_1' = \theta_1 - 1$, $\theta_\infty = \theta_\infty$. The explicit form of $R_{(9)}$ and $R_{(10)}$ are

$$\begin{cases} \theta_0' = \theta_0 + 1, \\ \theta_1' = \theta_1 + 1, \\ \theta_\infty' = \theta_\infty, \end{cases} \quad R_{(9)}(z) = z^{1/2}(z-1)^{-1/2}\left[I + \frac{1}{g_{21}f_{11} - g_{12}f_{21}} \begin{pmatrix} g_{21}f_{11} & -g_{11}f_{11} \\ g_{21}f_{21} & -g_{11}f_{21} \end{pmatrix} \frac{1}{z}\right], \tag{5.47}$$

$$\begin{cases} \theta_0' = \theta_0 - 1, \\ \theta_1' = \theta_1 - 1, \\ \theta_\infty' = \theta_\infty, \end{cases} \quad R_{(10)}(z) = z^{-1/2}(z-1)^{1/2}\left[I + \frac{1}{g_{22}f_{12} - g_{12}f_{22}} \begin{pmatrix} -g_{12}f_{22} & g_{12}f_{12} \\ -g_{22}f_{22} & g_{22}f_{12} \end{pmatrix} \frac{1}{z-1}\right], \tag{5.48}$$

where

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{\theta_0}(v+\theta_0)e^{-\sigma_0(t)} & ue^{\sigma_0(t)} \\ \frac{v}{u\theta_0}e^{-\sigma_0(t)} & e^{\sigma_0(t)} \end{pmatrix},$$

$$F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} -\frac{2w-\theta_1}{2\theta_1}e^{-\sigma_1(t)} & uye^{\sigma_1(t)} \\ -\frac{1}{uyw_1}\left(\frac{2w-\theta_1}{2\theta_1}\right)e^{-\sigma_1(t)} & e^{\sigma_1(t)} \end{pmatrix}, \tag{5.49}$$

where $\sigma_0$, $\sigma_1$, $w$, and $w_1$ are given in (5.7), (5.10), (5.3), and (5.43), respectively.

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