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RADIATION PHASE OF A DIPOLE FIELD [1]

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In the case of a dipole electromagnetic radiation, the operator of the "radiation phase" is defined. It is shown that this operator has a discrete spectrum with eigenvalues, lying in the segment $[0, 2\pi]$. Some properties of the radiation phase and polarization are discussed.

1. Introduction

Seventy years of investigation of the problem of quantum phase led to the conclusion that there is no unique quantum variable, determining universally the measured phase properties of electromagnetic radiation [1,2]. The operator constructions, describing cosine and sine of the phase, could be different for different schemes of measurement [2]. This fact has accurately been confirmed by a number of recent experiments (see [2–4] and references therein). Thus, it seems to be quite plausible that the quantum phase properties of an electromagnetic radiation are determined by interaction of photons with a macroscopic detecting device.

It is pertinent to ask the following question. Are the quantum phase properties of radiation *completely* determined by such an interaction or the photons have their own *inherent* phase properties which might be measured even if they are modified by interaction with a detecting device?

The universally recognized fact is that the vacuum state of field is degenerated with respect to phase. If a quantum radiation has its inherent phase properties, it means that the degeneration is taken off in the process of generation which is an interaction of the vacuum field with excited states of atoms or molecules. By virtue of this picture proposed in [5], what all one can expect is that the inherent quantum phase properties of radiation are completely determined by a source via the conservation laws, describing the generation process.

Even in this way, it seems to be impossible to determine a unique quantum phase of radiation. As a matter of fact, there are two conservation laws, admitting a nontrivial angular dependence. These are the linear momentum conservation and the angular

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momentum conservation. The former might be related to a *geometrical phase* provided, for example, by measurement of a phase difference between two plane waves with one and the same frequency and polarization, but traveling in different directions. The latter is directly connected with the spin of photons, forming the radiation. Actually, it is the *azimuthal phase of angular momentum* of radiation. Precisely this object seems to be a proper candidate to gain the position of the inherent quantum phase [5].

This so-called *radiation phase* has been introduced in [5] and then examined in our papers [6,7]. It was shown that the "quantum phase information" is coherently transmitted from the atom or molecule to radiation and vice versa [6]. In the Ref.[7], we traced connection between the cosine and sine of the radiation phase operators and generalized Stokes operators had been introduced in [8], and showed a striking difference in the behavior of quantum fluctuations, calculated within our method and within Pegg-Barnett approach [9–11]. Further investigation has led to prediction of a qualitatively new behavior of quantum fluctuations which looks like the quantum "phase bunching" [12].

Our consideration so far have applied to the cosine and sine of the radiation phase operators [5–7,12]. In this paper we report some new results relating to the radiation phase operator *per se* and its eigenstates.

2. Generalized Stokes operators

According to the assumption has made in [5], the radiation phase is connected with the azimuthal phase of spin of photons. Let us remind that the spin of a photon is determined as the minimum value of its angular momentum because the angular momentum of photon cannot be decomposed into the spin and orbital parts [13,14]. The representation of photons with given angular momentum is provided by the multipole expansion of the radiation field. The Cartesian components of the angular momentum are well defined operators in this representation [13,14]. They form a representation of the $SU(2)$ sub-algebra in the Weyl-Heisenberg algebra.

To determine the azimuthal phase operator or corresponding cosine and sine operators, one have to use the polar decomposition of the above mentioned $SU(2)$ sub-algebra. Unfortunately, it is impossible because the enveloping algebra does not contain a uniquely defined scalar [5].

To avoid this difficulty, one can take into account that, within the quantum domain, the polarization of electromagnetic radiation is defined as a given spin state of photons, forming the radiation. Therefore, it seems to be reasonable first to determine the Stokes parameters of a classical multipole field and then to quantize them to find the Stokes operators [6,7]. The components of multipole field are specified by the wave number k, "quantum numbers" j and m, $(-j \leq m \leq j)$, and index λ, describing the type of multipole [15] (the parity, within the quantum picture). There is no loss in generality in choosing fixed j and λ. Actually, these quantum numbers are determined by the selection rules for radiation of a given quantum source. Suppose, for simplicity, that the source under consideration is a localized system of dipole atoms (two-level atoms with the dipole-allowed transitions). On making the further assumption that $j = 1$ and k is fixed, we have dealings with the radiation, which consists of only three modes with different polarization. Precisely, $m = \pm 1$ describes two circularly polarized modes and $m = 0$ specifies the linearly polarized mode which always exists in the dipole radiation [15].

The classical tensor of polarization for such a dipole radiation has the components $\rho_{mm'} = \vec{E}_m \cdot \vec{E}^*_{m'}$. These components can be specified by five real parameters, forming the set of *generalized Stokes parameters* (GSP) [7,8]. Unlike the case of conventional Stokes parameters, we have now dealings with considering in arbitrary space-time points the polarization of the

radiation which can posses any direction of polarization or can be located in any plane in case of elliptical polarization.

The quantum counterpart of ρ determines a representation of the $SU(3)$ sub-algebra in the Weyl-Heisenberg algebra with the following generators [16]

$$\frac{1}{2}(\hat{a}^+_m \hat{a}_{m'} + hc), \qquad \frac{1}{2i}(\hat{a}^+_m \hat{a}_{m'} - hc), \qquad \hat{a}^+_m \hat{a}_m - \hat{a}^+_{m'} \hat{a}_{m'}.$$

Here the operators $\hat{a}_m$ describe the dipole photons with $j = 1$ and given projection m [14]. The generalized Stokes are operators [6]

$$\begin{aligned} \hat{S}_0 &= \sum_m \hat{n}_m, \\ \hat{S}_1 &= (\hat{\mathcal{E}} + \hat{\mathcal{E}}^+)/2, \\ \hat{S}_2 &= (\hat{\mathcal{E}} - \hat{\mathcal{E}}^+)/2i, \\ \hat{S}_3 &= \hat{n}_+ - \hat{n}_-, \\ \hat{S}_4 &= \hat{n}_+ + \hat{n}_- - 2\hat{n}_0, \end{aligned} \tag{1}$$

where

$$\hat{\mathcal{E}} \equiv \hat{a}^+_+ \hat{a}_0 + \hat{a}^+_0 \hat{a}_- + \hat{a}^+_- \hat{a}_+, \qquad [\hat{\mathcal{E}}, \hat{\mathcal{E}}^+] = 0, \tag{2}$$

and $\hat{n}_m \equiv \hat{a}^+_m \hat{a}_m$. Clearly, the operators (1) are some combinations of the above generators of $SU(3)$ sub-algebra.

Let us concentrate on the operators $\hat{S}_{1,2}$ in (1). It was shown in the Refs.[5,8] that, in the case of a single-atom radiation, these operators correspond to the cosine and sine of the azimuthal phase of spin of a photon. The classical counterpart of these operators coincides with two of four conventional Stokes parameters, determined in the circular polarization basis under the assumption that the longitudinal component vanishes. These two conventional Stokes parameters determine the cosine and sine of the classical phase difference between two circularly polarized components.

In view of these facts, it seems to be plausible that the operators $\hat{S}_{1,2}$ in (1) determine some quantum phase properties of the dipole radiation. Being normalized in a special way, they define the cosine and sine of the radiation phase operators [7,12].

As can be seen from the definitions (1), (2), the operators $\hat{S}_{1,2}$ would be just the cosine and sine operators if $\hat{\mathcal{E}}$ were not the normal operator but the unitary one. Nevertheless, if there is a complete orthonormal set of eigenstates of $\hat{\mathcal{E}}$ in the Hilbert space with base vectors $|n_+, n_0, n_-\rangle \equiv \otimes_m |n_m\rangle$, these eigenstates can be used to define the radiation phase operator *per se* in terms of projections. In the next section we look in more detail at the problem of eigenstates and eigenvalues for $\hat{\mathcal{E}}$.

3. Eigenvalues of radiation phase

Since the operator $\hat{\mathcal{E}}$ commutes with the total number of photons $\hat{S}_0$, the eigenstates of $\hat{\mathcal{E}}$ can be specified by the quantum number $n = \sum_m n_m$. Thus, they look like the three-mode number states. Let us use the following notations

$$\hat{\mathcal{E}}|\phi^{(n)}\rangle = \epsilon^{(n)}|\phi^{(n)}\rangle, \qquad \epsilon^{(n)} = |\epsilon^{(n)}|\, e^{-i\phi^{(n)}}. \tag{3}$$

With the assumption that the expansion

$$|\phi^{(n)}\rangle = \sum_{n_+=0}^{n} \sum_{n_0=0}^{n-n_+} \lambda^{(n)}(n_+, n_0)|n_+; n_0; n - n_+ - n_0\rangle \tag{4}$$

exists, the equation (3) can be written as the following recursion relations

$$\epsilon^{(n)}\lambda^{(n)}(n_+, n_0) = \sqrt{n_+(n_0+1)}\lambda^{(n)}(n_+ - 1, n_0 + 1)+ \tag{5}$$

$$\sqrt{n_0(n - n_+ - n_0 + 1)}\lambda^{(n)}(n_+, n_0 - 1) + \sqrt{(n_+ + 1)(n - n_+ - n_0)}\lambda^{(n)}(n_+ + 1, n_0).$$

It yields, at any n, the k-th degree equation for $\epsilon^{(n)}$ where

$$k = \frac{1}{2}(n+2)(n+1).$$

The arguments of these roots determine the eigenvalues of the radiation phase. It is clear that the choice $n = 0$ leads to the solution $\epsilon^{(0)} = 0$ with an arbitrary phase $\phi^{(0)}$. Thus, there is no contradiction with the conventional idea that the phase has the uniform distribution in the vacuum state. The structure of solutions of the equations (5) for a few first n is shown in Fig. 1. As usually, the coefficients $\lambda_k^{(n)}$ are determined by the equations (5) at given $\epsilon_k^{(n)}$ together with the normalization condition. In general case of an arbitrary n, it is possible to prove that the set of roots $\epsilon_k^{(n)}$ is represented by the sites of a right triangular lattice in the complex plane [16]. Moreover, it can be also proven that the functions $|\phi_k^{(n)}\rangle$ form a complete orthonormal system [16]. Therefore, the radiation phase operator under consideration is represented as follows

$$\hat{\Phi} = \sum_{n,k} \phi_k^{(n)} |\phi_k^{(n)}\rangle\langle\phi_k^{(n)}|. \tag{6}$$

Let us stress that the radiation phase operator (6) has a discrete spectrum *at any n*. At the same time, the eigenvalues $\phi_k^{(n)}$ lie between 0 and 2π. Thus, we obtained a well-behaved quantum phase, describing a dipole radiation.

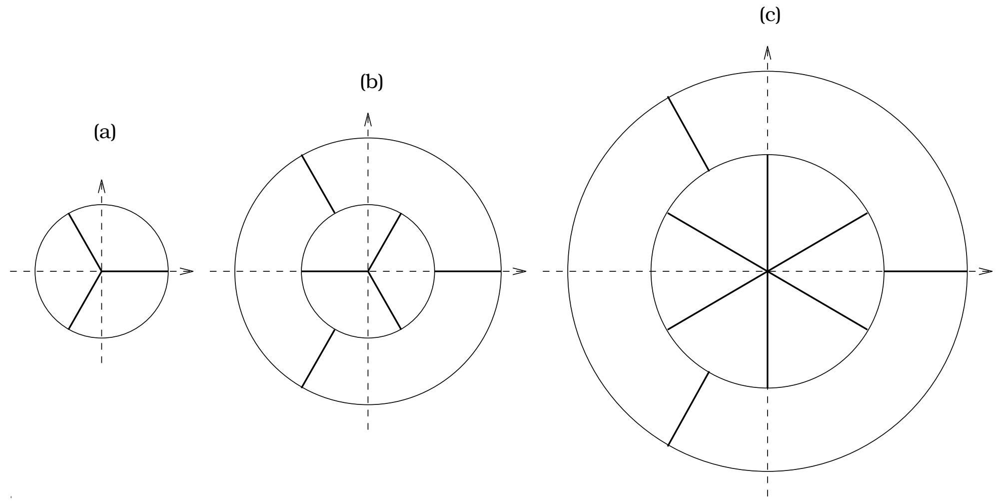

Fig. 1. The eigenvalues of $\hat{\mathcal{E}}$ in the complex plane. (a) corresponds to $n = 1$, $|\epsilon^{(1)}| = 1$, $\phi_k^{(1)} = 2k\pi/3$. (b) corresponds to $n = 1$ with $|\epsilon^{(2)} = 2$, $\phi_k^{(2)} = 2k\pi/3$ and with $|\epsilon^{(2)} = 1$, $\phi_{k'}^{(2)} = (2k' + 1)\pi/3$. (c) corresponds to $n = 3$ with $|\epsilon^{(3)}| = 3$, $\phi_k^{(3)} = 2k\pi/3$ and $|\epsilon^{(3)}| = (3\sqrt{3})^{1/3}$, $\phi_{k'}^{(3)} = \pi(1 + 4k')/6$.

4. Some properties of radiation phase and polarization

By definition, the operators $\hat{\Phi}$, $\hat{\mathcal{E}}$, and $\hat{S}_{1,2}$ should have similar properties. Actually, the latter three of them can be considered as the "weighted" phase operators. All these four operators commute with the total number of photons $\hat{S}_0$. So, all of them can be measured at once. At the same time, they do not commute neither with either $\hat{S}_{3,4}$ nor with any of $\hat{n}_m$. For example

$$[\hat{S}_1, \hat{n}_m] = \frac{1}{2}(-\hat{a}_m^+ \sum_{m' \neq m} \hat{a}_{m'} + hc),$$

which yields the following uncertainty relation

$$V(\hat{S}_1)V(\hat{n}_m) \geq \frac{1}{4}|\langle -\hat{a}_m^+ \sum_{m' \neq m} \hat{a}_{m'} + hc\rangle|^2. \tag{7}$$

In important case of two circularly polarized modes in the coherent states and linearly polarized mode in the vacuum state, corresponding to the radiation field considered in the far zone, the averaging with respect to the state $|\alpha_+\rangle \otimes |0_0\rangle \otimes |\alpha_-\rangle$ gives

$$V(\hat{S}_{1,2})V(\hat{n}_m) = \frac{\bar{n}_m}{2}(\bar{n}_+ + \bar{n}_- \pm \sqrt{\bar{n}_+\bar{n}_-}\cos\Delta_{-+}),$$

where $\bar{n}_m \equiv |\alpha_m|^2$, $m = \pm 1$, and $\Delta_{-+} \equiv \arg\alpha_- - \arg\alpha_+$. As can be seen, the uncertainty of the polarization parameter $s_{1,2} \equiv \langle \hat{S}_{1,2}\rangle$ strongly depends on the phase difference between two coherent modes. In this case, the generalized Stokes parameters $s_{1,2}$ describe the cosine and sine of the phase difference Δ_{-+} respectively. In special case of equal intensities $\bar{n}_+ = \bar{n}_- = \bar{n}$ and $\Delta_{-+} = (2k+1)\pi$ the uncertainty of $\hat{S}_1$ takes the minimum value $\bar{n}/2$ while $V(\hat{S}_2)$ attains the maximum value $3\bar{n}/2$.

The uncertainty relation (7) and that for $\hat{S}_2$ should be taken into account in the operational measurement of polarization within the eight-port detecting scheme [2,17]. To measure $s_{1,2}$, the different inputs of the eight-port scheme should consist of mixture of the linearly polarized component (which could contain a very few photons) with different circularly polarized components such that each output includes all three components [12].

The states (6) can be generated by a localized atomic system. It is clear that a single two-level atom with a dipole transition in a cavity can generate the field in the state $|\phi_k^{(1)}\rangle$ if it is prepared initially in the eigenstate of its own "phase" [6]. A point-like system of two two-level dipole atoms in a cavity can generate the state $|\phi_k^{(2)}\rangle$ if both of them are prepared in corresponding atomic "phase states". At the same time, this two-atom system can be prepared in the state where the total angular momentum is 2. The eigenvalues of the azimuthal phase of this angular momentum form a right pentagon. Such an atomic state cannot generate the photons in one of the eigenstates of the radiation phase. The multi-atom case needs more detailed analysis.

5. Summary

Let us briefly discuss the results. It is shown that the extension of traditional description of polarization in terms of Stokes parameters and corresponding Stokes operators, taking into account the presence of longitudinal component [6,7], leads to definition of a quantum phase operator determined in the whole Hilbert space with a discrete spectrum, lying in the segment $[0, 2\pi]$. In the far zone where one can consider the longitudinal component in the vacuum state, this spectrum determines the possible values of the phase difference between two circularly polarized modes. The physical meaning of this operator in general case needs further discussion.

Definitely, a combination of three coherent states of the form $\otimes_m |\alpha_m\rangle$ is not an eigenstate of the radiation phase. Definition of coherence of the radiation field with all three components also needs further consideration. Investigation of interference in the system of two dipole atoms separated by a distance might be an important step in this direction.

Let us also note that the states (4) can be used to find the radiation phase distribution for different states of the radiation field and then compare them with thous obtained by the appropriate integration of Wigner or Q functions [18–20].

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