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A Locally Optimal Seasonal Unit-Root Test

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This article proposes a locally best invariant test of the null hypothesis of seasonal stationarity against the alternative of seasonal unit roots at all or individual seasonal frequencies. An asymptotic distribution theory is derived and the finite-sample properties of the test are examined in a Monte Carlo simulation. My test is also compared with the Canova and Hansen test. The proposed test is superior to the Canova and Hansen test in terms of both size and power.

KEY WORDS: Locally best invariant test; Maximum likelihood estimation; Monte Carlo; Seasonal differencing filter; Seasonal stationarity.

Seasonality in economic time series can be viewed as either deterministic or stochastic. Fitting dummy variables is the usual way of handling seasonality in a deterministic way. To differentiate among various patterns of seasonality, statistical tests are introduced in the literature.

Hylleberg, Engle, Granger, and Yoo (HEGY) (1990) developed tests of the null hypothesis of a unit root at one or more seasonal frequencies against the alternative of stationary seasonality. Their test is an extension of the unit-root test of Dickey and Fuller (1979) from the zero frequency to the seasonal frequencies. This test has low power in finite samples near unit roots, so it is difficult to reject the false unit-root hypothesis at a single or a set of seasonal frequencies (Canova and Hansen 1995).

In the tests that are developed by Canova and Hansen (CH) (1995), stationary seasonality forms the null hypothesis. The alternative hypothesis is nonstationarity due to seasonal unit roots. They generalized the unit-root test of Kwiatkowski, Phillips, Schmidt, and Shin (KPSS) (1992) from the zero frequency to the seasonal frequencies. Their test statistics are Lagrange multiplier (LM) tests that are modified to include serially correlated and heteroscedastic processes. Only least squares techniques are needed in their LM-type test, and autocorrelation is handled by using a nonparametric adjustment. Tam and Reinsel (1995) also contributed to this literature by developing tests for moving average (MA) seasonal unit roots. Their test is mainly the extension of the Saikkonen and Luukkonen (1993) unit-root test to seasonal frequencies.

In this article, I propose a test procedure in which the null hypothesis of stationary seasonality is tested against the alternative of seasonal nonstationarity. I generalize the unit-root test of Leybourne and McCabe (1994) from zero frequency to the seasonal frequency. To test the null hypothesis, I propose a locally best invariant test that is derived from the framework of King and Hillier (1985). The test statistics depend on the residuals, which are calculated via maximum likelihood and then using least squares. The large-sample distribution under the null is the generalized von Mises distribution that does not depend on the nuisance parameters.

There are two major differences between my test and the CH test for seasonal stability. In my test, autocorrelation is taken into account in a parametric way, but the CH test uses a nonparametric correction. Second, my test statistic is consistent to the order N, whereas the CH test is of order N/z, where z is the lag truncation parameter. The nonparametric adjustment of autocorrelation suggested by Canova and Hansen (1995) fails to give good finite-sample performance when a large autoregressive (AR) component is present in the data. This problem is due to the significant truncation errors in the finite samples.

The aim of this article is to overcome this problem by introducing a test statistic that accounts for autocorrelation parametrically. A Monte Carlo exercise has been conducted to examine and compare the finite-sample properties of the proposed test with those of the CH test. It is shown that the proposed test has better size and power properties than the CH test in an AR type of autocorrelation.

Section 1 introduces the regression model. Section 2 presents the structural and reduced form of the model, and a locally best invariant test statistic is also derived for testing unit roots at the seasonal frequencies. The rest of the section develops the asymptotic distribution for this test statistic. In Section 3, a Monte Carlo exercise is conducted, and the size and power properties of the proposed test are compared to those of the CH test. Section 4 concludes the article. Proofs of the theorems are discussed in the Appendix. A GAUSS program for calculating test statistics is available on request from me.

1. THE MODEL

A linear time series model with stationary seasonality is considered:

$$\Phi(L)y_t = \mu + S_t + e_t, \qquad t = 1, 2, \ldots, N. \tag{1}$$

In the preceding equation, $\Phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$ is a pth-order AR polynomial in the lag operator L with roots outside the unit circle, y_t is real valued, and S_t is a real-valued deterministic seasonal process of period s, in which s is a positive even integer and error term e_t is distributed as iid $(0, \sigma_e^2)$. The number of observations is N. If there are T years of data, $N = Ts$.

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In this article, because I am interested in unit roots at the seasonal frequencies, I require that y_t not have a unit root at the zero frequency because it is not very difficult to transform a series with nonstationarity at the zero frequency to a stationary series; various applications of this test will be possible. The autocorrelation in the series is accounted for by including the lagged terms in y_t.

A trigonometric representation is presented for the deterministic seasonal pattern S:

$$S_t = \sum_{j=1}^{q} f'_{jt}\gamma_j, \tag{2}$$

where $q = s/2$ ($s = 4$ for quarterly data and $s = 12$ for monthly data) and, for $j < q, f'_{jt} = [\cos((j/q)\pi t), \sin((j/q)\pi t)]$, when $j = q, f_{qt} = \cos(\pi t)$. Expressing the right side of (2) in a vector,

$$S_t = f'_t\gamma, \tag{3}$$

where

$$\gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_q \end{pmatrix}, \quad f_t = \begin{pmatrix} f_{1t} \\ \vdots \\ f_{qt} \end{pmatrix}. \tag{4}$$

These γ and f_t vectors have $(s-1)$ elements. Substituting (3) into (1), the regression equation is

$$\Phi(L)y_t = \mu + f'_t\gamma + e_t, \qquad t = 1, 2, \ldots, N. \tag{5}$$

This representation allows seasonality to be a cylical process. At the seasonal frequency $j\pi/q$, the cylical processes are elements of f_t. Moreover, f_t is a zero-mean process whenever N is a multiple of s. The coefficients γ_j represent the effect of each cycle on the deterministic seasonal component S_t. This cylical formulation of seasonality is common in the time series literature (Hannan 1970, p. 174; Harvey 1989, p. 42).

2. THE TEST FOR SEASONAL UNIT ROOTS

2.1 The Structural Model and the Reduced Form

To test whether seasonal patterns are stable or not, I need to present a specific alternative hypothesis. One form of the alternative hypothesis is to allow a unit root in γ_t. This idea was suggested by Hannan (1970) and used by Canova and Hansen (1995) and Leybourne and McCabe (1994). The structural model is

$$\Phi(L)y_t = \mu + f'_t\gamma_t + e_t, \qquad t = 1, 2, \ldots, N, \tag{6}$$

and

$$\gamma_t = \gamma_{t-1} + u_t, \tag{7}$$

γ_0 fixed. It is assumed that u_t is iid mean 0, independent of e_t and f_t and its covariance matrix are

$$Eu_tu'_t = \sigma_u^2 G, \tag{8}$$

where G is an $(s-1)\times(s-1)$ matrix and σ_u^2 is a scalar. The unit roots at different seasonal frequencies are determined by G matrix. It can be easily seen from (6) and (7) that, whenever $\sigma_u^2 \neq 0$, then there will be seasonal unit roots.

The structural model (6)–(7) is second-order equivalent in moments to the reduced-form model:

$$\Phi(L)S(L)y_t = \mu' + \Theta(L)\zeta_t, \tag{9}$$

where ζ_t is distributed $(0, \sigma_\zeta^2), S(L) = \sum_{j=0}^{s-1} L^j$ is a seasonal filter, and $\mu' = s\mu$.

This last term $\Theta(L)$ is an MA$(s-1)$ polynomial. The derivation of (9) can be obtained from me on demand. For the importance of this representation, the reader can consult Leybourne and McCabe (1994). This reduced form is very similar to the model used by Tam and Reinsel (1995). Their model did not, however, result in a test that can differentiate between unit roots at various seasonal frequencies and the zero frequency.

2.2 The Hypothesis Test

I wish to test whether the seasonal patterns are stable or not. In other words I need to develop a hypothesis so that I can determine whether a given series has stationary seasonality or not. One such hypothesis is a stationary AR(p) process, against the alternative of nonstationary seasonality. This can be easily formulated in terms of the structural model (6)–(7) as $H_0 : \rho = 0$ against $H_1 : \rho > 0$, where $\rho = \sigma_u^2/\sigma_e^2$. I shall examine the local departures from the null hypothesis in the following section.

Another task that I face is developing a test statistic. From the structural model (6)–(7) and using the framework of King and Hillier (1985), the locally best invariant test statistic for $H_0 : \rho = 0$ is

$$D = \hat{\sigma}_e^{-2}N^{-2}\sum_{t=1}^{N}\hat{F}'_tG\hat{F}_t, \tag{10}$$

where $\hat{F}_t = \sum_{i=1}^{t} f_i\hat{e}_i, \hat{\sigma}_e^2 = \hat{e}'\hat{e}/N$ is a consistent estimator of σ_e^2, and $\hat{e}$ is an $N\times 1$ vector of residuals $\hat{e}_t$.

The residuals $\hat{e}_t$ are obtained via the following procedure: First find the maximum likelihood estimates of (ϕ) from the fitted model,

$$y_t^* = \mu' + \sum_{l=1}^{p}\phi_l y_{t-l}^* + \Theta(L)\zeta_t, \tag{11}$$

where $y_t^* = S(L)y_t$. Then construct the series

$$\bar{y}_t = y_t - \sum_{l=1}^{p}\phi_l^* y_{t-l}, \tag{12}$$

where ϕ_l^* are the maximum likelihood estimates of ϕ_l obtained from (11). Then regress $\bar{y}_t$ on an intercept and seasonal dummies to obtain $\hat{e}_t$. Even though I do not assume the normality of e_t, this is necessary for the "optimality" of the tests.

As pointed out by Saikonnen and Luukonnen (1993) and Leybourne and McCabe (1994), one wants to estimate ϕ_l consistently both under the null and the alternative hypothesis in the reduced form, so I use maximum likelihood estimation rather than ordinary least squares. Even though maximum likelihood estimation might have drawbacks, they are found to have no effect on the finite-sample properties of my test in the simulations that I have conducted in the former version of the article (Ansley and Newbold 1980; Galbraith and Zinde-Walsh 1994). An alternative way of estimation is the instrumental-variable technique. But this approach did not give good results in this case.

An important point is the structure of G matrix. Different specifications of the alternative hypotheses depend on the structure of G. When the alternative hypothesis is unit roots at all seasonal frequencies, then G must be nonsingular and γ_t must be time varying. If the alternative hypothesis is unit roots at specific seasonal frequencies, then G must be block diagonal with nonzero elements in only selected blocks and a subset of γ_t is time varying.

2.3 The Asymptotic Distribution

If the alternative hypothesis is seasonal nonstationarity, then I should have a joint unit-root test at all seasonal frequencies. It was first suggested by Nyblom (1989) in the likelihood context and later applied to econometric models by Hansen (1990, 1992) that, when $G/\sigma_e^2 = (\Omega^f)^{-1}$, then the asymptotic distribution of the test statistic is easy to evaluate. In the preceding discussion, Ω^f is the long-run covariance matrix of $f_i e_i$ (see Canova and Hansen 1995).

Because e_t is serially uncorrelated and homoscedastic, I can use the consistent estimator

$$\hat{\Omega}^f = \hat{\sigma}_e^2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots \end{bmatrix}. \tag{13}$$

To study the large-sample distribution of D, I use the following notation: $\xrightarrow{d}$ denotes convergence in distribution, W_m denotes a vector standard Brownian bridge of dimension m, and $VM(m)$ is a random variable obtained by the following operation:

$$VM(m) = \int_0^1 W_m(r)' W_m(r)\, dr.$$

As was suggested by Canova and Hansen (1995), $VM(m)$ will be referred to as the generalized von Mises distribution with m df. Critical values were given in table 1 of Canova and Hansen (1995). The main theorem of this section is proved in the Appendix.

Theorem 1. In (1), if $\Phi(L)$ is a finite AR polynomial in the lag operator with roots outside the unit circle and if e_t is iid, $Ee_t = 0$, and $Ee_t^2 = \sigma_e^2 < \infty$, then, under H_0,

$$D \xrightarrow{d} VM(s-1).$$

Following Section 2.2 and using theorem 3 of Canova and Hansen (1995), I have the individual test statistics

$$D_{j\pi/q} = \frac{2}{\hat{\sigma}_e^2 N^2} \sum_{t=1}^{N} \hat{F}_{jt}' \hat{F}_{jt}, \qquad j < q, \tag{14}$$

and

$$D_{\pi} = \frac{1}{\hat{\sigma}_e^2 N^2} \sum_{t=1}^{N} \hat{F}_{qt}^2, \qquad j = q. \tag{15}$$

The individual test statistics can be calculated as a by-product of the joint test. Their asymptotic distribution is given in Theorem 2.

Theorem 2. Under the conditions in Theorem 1, (1) for $j < q, D_{j\pi/q} \xrightarrow{d} VM(2)$, and (2) $j = q, D_{\pi} \xrightarrow{d} VM(1)$.

The individual tests supply us with more information about the nature of the seasonal process when there is seasonal nonstationarity in the joint test. Nonstationarity can be caused by the unit roots at the individual seasonal frequencies.

The consistency of the joint and individual tests under H_1 can be obtained via the method described by Leybourne and McCabe (1994). My test is a generalization of the Leybourne and McCabe test at zero frequency to the seasonal frequency, whereas the CH test is a generalization of the KPSS test. Both tests have the same limiting distribution. If I analyze the advantages of the D test, first I should begin by comparing my test with the CH test. The CH test accounts for autocorrelation in a nonparametric fashion, but in finite-samples this can cause problems if the data structure contains higher-order terms in the AR polynomial. The nonparametric adjustment then is not able to capture the serial correlation in data. My test focuses on this problem. Autocorrelation is allowed by introducing lagged terms in y_t. This parametric correction is the main advantage of the test and, with a significant AR component in the data, this results in better finite-sample performance. According to Leybourne and McCabe (1994), the test statistic is consistent at a rate $O_p(N)$ under H_1, but in the KPSS test this rate is $O_p(N/z)$ (z is the bandwidth parameter in KPSS and CH tests). These rates also apply to my test statistic and the CH test statistic under H_1. Therefore, I expect my test's power to be better.

3. MONTE CARLO STUDY

To examine the size and power properties of the proposed test statistics, a Monte Carlo exercise is conducted. Two quarterly models are considered. The first model is

$$\Phi(L) y_t = \mu + \sum_{j=1}^{2} f_{jt}' \gamma_{jt} + e_t, \qquad e_t \sim N(0,1), \tag{16}$$

and

$$\gamma_t = \delta \gamma_{t-1} + u_t, \qquad u_t, \sim N(0, \sigma_u^2 G), \tag{17}$$

where $\gamma_0 = [1,1,1]$, $\gamma_t = (\gamma_{1t}, \gamma_{2t})'$, and $0 < \delta \leq 1$. γ_{2t} is a 2×1 vector, and G is a 3×3 matrix, $\sigma_e^2 = 1$. $\Phi(L)y_t$ is an

AR(p) process. The second model is given by

$$y_t = \mu + \sum_{j=1}^{2} f'_{jt}\gamma_{jt} + \tau(L)e_t, \qquad e_t, \sim N(0,1), \quad (18)$$

and

$$\gamma_t = \delta\gamma_{t-1} + u_t, \qquad u_t, \sim N(0, \sigma_u^2 G), \quad (19)$$

where $\tau(L) = 1 + \tau_1 L + \tau_2 L^2 + \cdots + \tau_\ell L^\ell$. The model (18)–(19) ensures a fair comparison between the D test and the CH test because my test captures an AR(p) type of autocorrelation. In calculating the size of the tests, I explore the more empirically relevant case of $0 < \delta < 1$ and $\sigma_u^2 \neq 0$. For both models, three different data-generating processes (DGP's) are used under the alternative hypothesis—

$$\text{DGP1}: G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (20)$$

$$\text{DGP2}: G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (21)$$

and

$$\text{DGP3}: G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (22)$$

Under DGP 1, there is a unit root at the π frequency as long as $\sigma_u^2 \neq 0$. The D_π test is designed for this specification. There is a pair of complex conjugate roots at the $\pi/2$ frequency under DGP2 when $\sigma_u^2 \neq 0$. Similarly, when $\sigma_u^2 = 0$ under DGP 3 there are no unit roots, but there are unit roots at all seasonal frequencies if $\sigma_u^2 \neq 0$. One important fact to note is that the covariance structure implied by G is different from the one that is used to construct the D and $D_{\pi/2}$ test statistics.

In the simulations, the order of the AR polynomial p and the order of the MA polynomial ℓ are 1 and 2. Both the AR parameters of (16)–(17) and the MA parameters of (18)–(19) are chosen carefully to understand the effect of autocorrelation on the test statistics that are proposed in this article. I set $\delta = .8$. The signal-to-noise ratio $\rho = \sigma_u^2/\sigma_e^2$ takes the value of .05. I vary the sample size among $N = 50, 100, 200$. I have 1,000 independent realizations for each DGP and parameter configuration. The test statistics are calculated for unit roots at all, π, and $\pi/2$ seasonal frequencies. Robustness issues are explored in Subsection 3.3.

My test's finite-sample properties are compared to those of the CH tests (with and without one lag of the dependent variable included). Hylleberg (1995) showed that the CH test with one lag has low power. Because I analyze various data structures, however, I want to include that test in my study also. The underlying model of the CH test is the same as my model (16)–(17) and (20)–(22), but they assume $p = 0$ or $p = 1$. Because they also assume an α-mixing process for

Table 1. Size and Power Comparison Between the CH and D tests: AR Model

		D			CH_0			CH_1		
DGP	SS	J	π	$\pi/2$	J	π	$\pi/2$	J	π	$\pi/2$
				$y_t + .8y_{t-1} = \mu + f'_t\gamma_t + e_t$						
Size	200	12.9	8.8	13.5	8.9	24.9	.7	6.2	3.7	7.0
Size	100	13.4	11.2	11.7	10.1	29.6	.5	3.4	7.5	7.3
Size	50	14.8	15.0	10.1	13.0	34.6	.7	2.3	3.9	5.1
DGP3	200	99.4	35.3	89.0	95.8	69.2	80.1	91.0	43.3	95.1
DGP3	100	88.5	23.3	77.2	77.9	58.9	54.9	83.1	25.9	83.3
DGP3	50	54.6	30.4	48.5	53.2	45.0	45.1	52.8	17.0	52.6
DGP2	200	97.9	1.5	98.2	90.6	4.3	97.4	94.7	14.0	95.5
DGP2	100	84.3	4.7	86.3	64.4	6.6	85.9	79.2	16.0	80.1
DGP2	50	59.1	12.5	51.4	20.5	9.2	50.4	47.0	8.7	51.9
DGP1	200	48.0	56.0	.9	68.1	71.5	.0	5.3	12.9	2.7
DGP1	100	35.7	45.0	1.6	54.9	57.4	.1	4.6	11.1	3.2
DGP1	50	24.8	31.2	2.4	40.3	40.8	8.7	3.9	12.6	20.0
				$y_t + .8y_{t-2} = \mu + f'_t\gamma_t + e_t$						
Size	200	17.3	16.7	12.0	40.9	.6	55.5	49.9	1.8	63.8
Size	100	21.0	14.3	16.6	47.7	.6	63.7	50.4	1.0	64.1
Size	50	23.6	10.7	20.4	44.7	1.5	61.1	45.7	1.5	58.8
DGP3	200	94.0	83.0	47.5	94.1	64.1	86.3	93.3	61.5	85.5
DGP3	100	81.1	61.9	41.6	78.9	42.1	74.3	77.9	44.2	73.0
DGP3	50	34.5	40.1	19.7	62.0	38.6	61.0	65.1	32.9	59.2
DGP2	200	65.1	.4	72.4	84.9	.1	85.7	84.8	.0	85.9
DGP2	100	51.8	.4	59.1	75.0	.3	74.9	74.7	2.0	75.6
DGP2	50	13.7	.8	14.6	60.4	8.2	62.2	59.4	.2	56.8
DGP1	200	85.3	91.0	2.6	56.4	89.4	7.3	55.9	90.1	6.7
DGP1	100	66.3	73.1	10.5	31.7	67.8	10.7	30.4	68.9	16.5
DGP1	50	29.5	39.9	15.5	19.1	41.0	16.8	24.1	40.9	17.0

NOTE: In both AR parameterizations for the size part, $\gamma_t = .8\gamma_{t-1} + u_t$. For the power part of the program, $\gamma_t = \gamma_{t-1} + u_t$. SS is the sample size. D, CH_0, and CH_1 are the D and the CH tests with no lags and one lag, respectively. J, π, and $\pi/2$ are the tests at all, semiannual, and annual seasonal frequencies, respectively. The DGP column shows the size and the power of the tests (DGP3, DGP2, DGP1).

e_t, their estimates of long-run covariance matrices $\hat{\Omega}_1^f$ and $\hat{\Omega}_0^f$, for $p = 1$ and $p = 0$, respectively, depend on the choice of the kernel and the lag truncation number z. In this study the Bartlett kernel is used and, following Andrews (1991), $z = 3, 4, 6$ is selected for $N = 50, 100, 200$, respectively. One important point about my test is the choice of p, the number of lags in y_t. The D tests are carried out with lag lengths chosen by the Akaike information criterion (AIC) and the Bayesian information criterion (BIC).

The results of the exercise are presented in Tables 1 and 2. The percentage of rejection of the null is given at the 5% significance level. Because the size of the D and CH tests that are calculated in Tables 1 and 2 vary considerably, I calculate the size-adjusted power. In the tables, the power of the tests are size-adjusted power. The critical values for calculating these can be obtained from me on demand.

There are three Monte Carlo studies that compare the relative performance of the tests for seasonal stability. One is by Hylleberg (1995) in which the HEGY tests are contrasted with the CH tests. Ghysels, Lee, and Noh (1994) compared the performance of the HEGY test with the Dickey, Hasza, and Fuller (1984) tests. Canova and Hansen (1995) contrasted the CH tests with HEGY tests, but the DGP is different from the Monte Carlo study of Hylleberg (1995).

3.1 Size and Power of the Test: AR(p) Process

In this section the size and power properties of the D test are compared with the CH test under the model (16)–(17). When analyzing the size of the tests in Table 1, δ is selected to be .8 because this value corresponds to a "near" seasonal unit root. In calculating the power of the tests in Tables 1 and 2, I set $\delta = 1$.

In Table 1, it is easy to see that the size of my tests is slightly above the nominal size of 5% in most of the cases. The CH tests have large size distortions for AR(2) parameterization, however. For example, for $N = 200$ in an AR(2) framework, the size of the joint D test is 17%, whereas the joint CH tests reject the true null in 41–50% of the trials.

The D tests have good power under different alternatives. For $N = 100$, the power of the joint test is 84% when there are seasonal unit roots present at the $\pi/2$ frequency (DGP2). For $N = 200$, in an AR(2) process, the power of the joint test is 85% when there is a seasonal unit root at the π frequency (DGP1).

The CH tests have mixed results under an AR structure. For $N = 100$, in an AR(1) process the joint test has 64–79% power against DGP2. The CH tests with one lag of the dependent variable (CH_1 in Tables 1–2) perform quite poorly in an AR(1) structure. The power is near the nominal size of the tests. Both CH tests also have trouble in an AR(2) structure when only a seasonal unit root at the π frequency is present (DGP1). For $N = 200$, the joint tests have 56% power under DGP1.

Overall, the CH tests do not perform well near seasonal unit roots. They suffer from size distortion. On the other hand, the proposed tests have good size and power. The CH tests performed well in the Monte Carlo study of Canova

Table 2. Comparison of Size and Power: The CH and D tests in an MA Model

		D			CH_0			CH_1		
DGP	SS	J	π	$\pi/2$	J	π	$\pi/2$	J	π	$\pi/2$
				$y_t = \mu + \sum_{j=1}^{2} f_{jt}'\gamma_{jt} + \varepsilon_t + \tau\varepsilon_{t-1}$						
Size	200	9.5	8.8	6.0	8.0	9.3	6.6	8.9	11.6	7.4
Size	100	9.8	9.5	5.4	4.1	6.4	5.5	9.0	14.6	6.8
Size	50	8.4	6.9	7.4	1.7	4.7	4.4	4.7	8.7	6.7
DGP3	200	92.3	51.9	68.5	98.7	83.0	92.9	98.8	78.5	93.6
DGP3	100	73.4	52.4	43.7	91.8	70.3	75.2	91.6	67.0	71.1
DGP3	50	67.2	48.9	30.3	70.4	60.1	42.2	65.0	54.5	39.9
DGP2	200	88.1	.0	93.6	89.1	0.0	93.9	86.2	.0	96.2
DGP2	100	62.9	.0	75.8	69.7	0.0	80.3	64.8	.0	77.7
DGP2	50	24.4	.1	40.5	34.4	0.0	48.2	31.5	.0	45.7
DGP1	200	56.9	61.0	3.2	73.3	82.9	2.2	71.9	81.0	1.6
DGP1	100	43.3	53.5	0.6	65.6	73.5	1.4	61.5	70.2	1.4
DGP1	50	41.8	50.1	3.6	43.0	56.1	4.0	40.3	53.7	3.1
				$y_t = \mu + \sum_{j=1}^{2} f_{jt}'\gamma_{jt} + \varepsilon_t + \tau\varepsilon_{t-2}$						
Size	200	14.1	17.4	9.9	3.0	9.4	1.1	1.6	7.2	1.8
Size	100	14.4	18.0	6.7	2.6	8.6	.6	1.5	9.3	1.0
Size	50	5.6	17.4	0.7	1.6	11.3	.1	1.4	10.8	.2
DGP3	200	97.8	42.2	93.0	99.7	73.4	98.9	99.5	73.1	99.5
DGP3	100	92.2	29.8	90.3	94.2	55.2	90.7	93.9	51.9	94.1
DGP3	50	64.5	13.9	78.0	71.4	34.6	76.0	69.5	29.5	74.7
DGP2	200	91.1	.8	93.9	97.2	2.8	99.3	95.7	2.7	98.8
DGP2	100	78.5	1.6	89.4	88.3	3.7	94.2	87.1	3.4	93.3
DGP2	50	50.3	1.4	77.4	55.6	4.3	75.1	52.6	4.2	72.2
DGP1	200	63.5	72.3	.0	64.8	75.8	.0	64.9	72.9	.0
DGP1	100	45.4	54.1	.0	45.9	54.7	.0	41.8	53.1	.0
DGP1	50	31.5	34.5	.0	24.1	31.6	.1	15.9	27.6	.1

NOTE: τ = .8 in the above parameterizations. For further information on this table, see note to Table 1.

and Hansen (1995) because of the structure of the DGP that they used. Their simulated models are not "near" seasonal unit roots at various frequencies, so it is difficult to determine the size in their study appropriately. In our case the simulated models correspond to an "almost" seasonal nonstationary case.

3.2 Size and Power of the Test: MA(1) Process

In this section the DGP is (18)–(19), which is the case with MA(ℓ) errors. This kind of setup provides a neutral ground for comparing our tests with the CH tests. Two types of MA processes are explored in Table 2. First I use

$$y_t = \mu + \sum_{j=1}^{2} f'_{jt}\gamma_{jt} + e_t + \tau e_{t-1}.$$

Then the following MA(2) process is analyzed:

$$y_t = \mu + \sum_{j=1}^{2} f'_{jt}\gamma_{jt} + e_t + \tau e_{t-2}.$$

I set $\delta = .8, \tau = .8$, and $\sigma_u^2 = .05$. Using the AIC and BIC, the optimal AR lag length p turned out to be 3, 5, and 6, for $N = 50, 100, 200$, respectively.

Table 2 shows that the proposed D tests have good size. The test at the $\pi/2$ frequency performs well even in the small samples. For example, for $N = 50$ in an MA(1) process, the size is 7%. Even though the test at the π frequency performs well in an MA(1) model, however, the size rises above the nominal level and is around 9–21% in an MA(2) setup.

The CH tests also have good size properties. For example, the size of the joint CH test with no lags of the dependent variable (CH_0) is 2–11%. The sizes of both tests do not seem to be affected by the sample size.

Both the D and CH tests have good power under different alternatives. Note, however, that the asymptotic rejection frequency of the D tests is better than that of the CH tests. These results were given by Caner (1996).

3.3 The Robustness Experiments

The results are robust to overfitting of the AR polynomial, correlatedness of u_t and e_t, and overdifferencing of y_t. Specifically, when I tried fitting up to six lags for AR(1) and AR(2) models, there were no significant changes in the power and size of the test. The finite-sample properties of the test were also analyzed by using various σ_u^2's and δ's. The size and the power of the test were not affected by the changes in σ_u^2. Smaller δ and AR coefficients resulted in better size properties for my test. Monte Carlo designs with longer AR polynomials such as 3 and 4 were tried, generating results that were very similar to the case of AR(2) design in Table 1.

4. CONCLUSION

This article proposes a locally best invariant test for detecting the presence of seasonal unit roots in time series models. The null hypothesis of the proposed test is seasonal stationarity, whereas the seasonal unit-root hypothesis forms the alternative. The derived asymptotic distribution is nonstandard and covers serially correlated processes. My test is similar to the CH test for seasonal stability. The main difference between the two arises from handling autocorrelation under the respective null and alternative hypotheses. My test has a parametric correction, but the CH test has a nonparametric adjustment for autocorrelation. According to my simulations the CH test suffers from size distortion in an AR model, whereas the proposed test has good size and power. Moreover, even with different autocorrelation structures and data-generating processes, the proposed tests have good finite-sample properties.

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APPENDIX: DERIVATION OF THEOREMS

Before proving the theorems, I need to prove a lemma and introduce some notation. Let $\Rightarrow$ denote weak convergence on [0, 1] with respect to the uniform metric, let $[\cdot]$ denote integer part, and let $\xrightarrow{p}$ denote convergence in probability.

Lemma 1.

$$\frac{1}{\sqrt{N}}\hat{F}_{[Nr]} = \frac{1}{\sqrt{N}}\sum_{t=1}^{[Nr]} f_t\hat{e}_t \Rightarrow B(r) - rB(1),$$

where $B(r)$ is a three-dimensional Brownian motion with covariance matrix Ω^f.

Proof of Lemma 1. From the structural model under the null hypotheses, I obtain

$$f_t\hat{e}_t = f_t(\mu - \hat{\mu}) + f_t f_t'(\gamma_0 - \hat{\gamma}_0) + f_t\sum_{l=1}^{p}(\phi_l - \phi_l^*)y_{t-l} + f_t e_t. \quad \text{(A.1)}$$

From the first-order conditions, I know that $1/N\sum_{t=1}^{N} f_t\hat{e}_t = 0$, so I have

$$0 = \frac{(\gamma_0 - \hat{\gamma}_0)}{N}\sum_{t=1}^{N} f_t f_t' + \sum_{l=1}^{p}(\phi_l - \phi_l^*)\frac{1}{N}\sum_{t=1}^{N} f_t y_{t-l} + \frac{1}{N}\sum_{t=1}^{N} f_t e_t. \quad \text{(A.2)}$$

Then subtract (A.2) from (A.1) to get

$$f_t\hat{e}_t = f_t(\mu-\hat{\mu}) + (\gamma_0-\hat{\gamma}_0)\left[f_tf_t' - \sum_{t=1}^{N}\frac{f_tf_t'}{N}\right] + \sum_{l=1}^{p}(\phi_l-\phi_l^*)\left[f_ty_{t-l} - \sum_{t=1}^{N}\frac{f_ty_{t-l}}{N}\right] + \left[f_te_t - \frac{1}{N}\sum_{t=1}^{N}f_te_t\right]. \quad (A.3)$$

In this expression each term will be examined in detail. Now, observe that, from the first term on the right side of (A.3),

$$\frac{1}{\sqrt{N}}\sum_{t=1}^{[Nr]}f_t(\mu-\hat{\mu}). \quad (A.4)$$

In (A.4), if $[Nr]$ is a multiple of s, then (A.4) is 0 because f_t is a zero-mean process. If $[Nr]$ is not a multiple of s when $N\to\infty$, (A.4) converges to 0,

$$\sup_{0\le r\le 1}\left|\frac{1}{\sqrt{N}}\sum_{t=1}^{[Nr]}f_t\right| \xrightarrow{p} 0, \quad (A.5)$$

and, from Potscher (1991), $(\mu-\hat{\mu})$ is $o_p(1)$.

The same procedure applies to the second term on the right side of (A.3) as well.

For the third term, under the null hypothesis y_{t-l} is an AR(p) process. Following from the invariance principle for linear processes (Phillips and Solo 1992),

$$\frac{1}{\sqrt{N}}\sum_{t=1}^{[Nr]}\left(f_ty_{t-l} - \frac{1}{N}\sum_{t=1}^{N}f_ty_{t-1}\right)$$

converges weakly and is $O_P(1)$. Then, from Potscher (1991), I know that $(\phi_l-\phi_l^*)$ is $o_p(1)$, so

$$\sum_{l=1}^{p}(\phi_l-\phi_l^*)N^{-1/2}\sum_{t=1}^{[Nr]}\left(f_ty_{t-1} - \frac{1}{N}\sum_{t=1}^{N}f_ty_{t-1}\right) = o_p(1). \quad (A.6)$$

Finally, invoking the functional central limit theorem (Billingsley 1968),

$$\frac{1}{\sqrt{N}}\sum_{t=1}^{[Nr]}\left(f_te_t - \frac{1}{N}\sum_{t=1}^{N}f_te_t\right) = \frac{1}{\sqrt{N}}\sum_{t=1}^{[Nr]}f_te_t - \frac{[Nr]}{N}\frac{1}{\sqrt{N}}\sum_{t=1}^{N}f_te_t \Rightarrow B(r) - rB(1), \quad (A.7)$$

where $B(r)$ is a vector Brownian motion with covariance matrix Ω^f. Combining (A.5), (A.6), and (A.7), I obtain

$$\frac{1}{\sqrt{N}}\sum_{t=1}^{[Nr]}f_t\hat{e}_t = \frac{1}{\sqrt{N}}\sum_{t=1}^{[Nr]}\left(f_te_t - \frac{1}{N}\sum_{t=1}^{N}f_te_t\right) + \frac{1}{\sqrt{N}}\sum_{t=1}^{[Nr]}\sum_{l=1}^{p}(\phi_l-\phi_l^*)\left[f_ty_{t-1} - \sum_{t=1}^{N}\frac{f_ty_{t-l}}{N}\right] + \frac{1}{\sqrt{N}}\sum_{t=1}^{[Nr]}(\gamma_0-\hat{\gamma}_0)\left[f_tf_t' - \sum_{t=1}^{N}\frac{f_tf_t'}{N}\right] + (\mu-\hat{\mu})\sum_{t=1}^{N}\frac{f_t}{\sqrt{N}} \Rightarrow B(r) - rB(1).$$

Proof of Theorem 1. From Lemma 1 and applying the continuous mapping theorem, I obtain

$$D = \frac{1}{N^2}\sum_{t=1}^{N}\hat{F}_t'(\hat{\Omega}^f)^{-1}\hat{F}_t \Rightarrow \int_0^1 W_{s-1}(r)'W_{s-1}(r)\,dr = VM(s-1).$$

Proof of Theorem 2. This theorem is proved in a manner similar to the proof of Theorem 1.

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