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On the number of conjugacy classes of maximal subgroups in a finite soluble group

By

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Abstract. We show that for many formations $\mathfrak{F}$, there exists an integer $n = \overline{m}(\mathfrak{F})$ such that every finite soluble group G *not* belonging to the class $\mathfrak{F}$ has at most n conjugacy classes of maximal subgroups belonging to the class $\mathfrak{F}$. If $\mathfrak{F}$ is a local formation with formation function f, we bound $\overline{m}(\mathfrak{F})$ in terms of the $\overline{m}(f(p))$ $(p \in \mathbb{P})$. In particular, we show that $\overline{m}(\mathfrak{N}^k) = k+1$ for every nonnegative integer k, where $\mathfrak{N}^k$ is the class of all finite groups of Fitting length $\leqq k$.

1. Introduction. For every nonnegative integer n, let $\mathfrak{N}^n$ denote the class of all finite groups having a series of length n with nilpotent factors. Then the *Fitting length* of a finite soluble group G is the least integer n such that $G \in \mathfrak{N}^n$. Now assume that the finite soluble group G has Fitting length n and let M be a maximal subgroup of G. In [1], Doerk shows that M has Fitting length $n-2$, $n-1$, or n, and that G has at most $n-2$ conjugacy classes of maximal subgroups whose Fitting length is $n-2$. While it is easy to see that a group of Fitting length $n > 0$ can have any number of maximal normal subgroups of Fitting length n (see Example 2.5 below), we show that G cannot have more than n conjugacy classes of maximal subgroups whose Fitting length is at most $n-1$; moreover, for every integer k with $2 \leqq k \leqq n$, we construct groups G of Fitting length n such that G has k conjugacy classes of maximal subgroups of Fitting length $n-1$ and $n-k$ conjugacy classes of maximal subgroups of Fitting length $n-2$; see Example 4.3 below.

More generally, let $\mathfrak{X}$ be a class of groups, and denote with $\overline{m}(\mathfrak{X})$ the least upper bound (possibly ∞) on the number of conjugacy classes of maximal $\mathfrak{X}$-subgroups of a finite soluble group G not belonging to the class $\mathfrak{X}$, and put $\overline{m}(\mathfrak{X}) = -\infty$ if no such group G exists, that is, if $\mathfrak{X} = \mathfrak{S}$, the class of all finite soluble groups. Here a subgroup M of a group G is a maximal $\mathfrak{X}$-subgroup of G if M is a maximal subgroup of G and belongs to the class $\mathfrak{X}$.

Observe that we have $\overline{m}(\mathfrak{X}) \leqq k$ if every finite soluble group with more than k conjugacy classes of maximal $\mathfrak{X}$-subgroups belongs to $\mathfrak{X}$.

For instance, by a theorem of P. Hall, we have $\overline{m}(\mathfrak{S}_\pi) = 1$ for every set of primes $\pi \neq \mathbb{P}$, where $\mathfrak{S}_\pi$ is the class of all finite soluble π-groups and $\mathbb{P}$ denotes the set of all primes. Note that in this case, or more generally, in the case when the class $\mathfrak{X}$ is a formation, it does not make sense to bound the number of conjugacy classes of maximal $\mathfrak{X}$-subgroups of an $\mathfrak{X}$-group (see Example 2.5).

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Moreover, in order to show that a finite group of Fitting length n has at most n conjugacy classes of maximal subgroups of Fitting length $\leqq n-1$, and that this bound is attained, we have to show that $\overline{m}(\mathfrak{N}^{n-1}) = n$.

In the case when $\mathfrak{X}$ is a formation, we obtain the following results, which are proved in Proposition 3.1, Theorem 3.2 and Theorem 4.1 below. Recall that if $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of groups, then $\mathfrak{X}\mathfrak{Y}$ denotes the class of all groups G which possess a normal subgroup $N \in \mathfrak{X}$ such that $G/N \in \mathfrak{Y}$. Moreover, $\mathfrak{N}$ denotes the class of all finite nilpotent groups.

Theorem. *Let $\mathfrak{F}$ be a formation. Then the following statements hold.*

(a) $\overline{m}(\mathfrak{S}_\pi\mathfrak{F}) \leqq \overline{m}(\mathfrak{F}) + 1$ *for every set of primes π, with equality if $\mathfrak{S}_{\pi'}\mathfrak{F} = \mathfrak{F}$.*

(b) $\overline{m}(\mathfrak{N}\mathfrak{F}) = \overline{m}(\mathfrak{F}) + 1$.

(c) *If $\mathfrak{F}$ is a local formation of characteristic $\pi \neq \emptyset$ defined by a formation function F satisfying $\mathfrak{S}_p F(p) = F(p)$ for every prime $p \in \pi$, and $\mathfrak{F} \neq \mathfrak{S}_\pi$, then*

$$\overline{m}(\mathfrak{F}) \leqq \sup_{p\in\pi} \overline{m}(F(p)) + 1;$$

moreover if $F(p) \subseteqq \mathfrak{F}$ for every prime $p \in \pi$, then $\overline{m}(F(p)) \leqq \overline{m}(\mathfrak{F})$ for every prime $p \in \pi$, and if $\mathfrak{F} \neq \mathfrak{S}_\pi$, then also

$$\overline{m}(\bigcap_{p\in\pi} F(p)) \leqq \mathfrak{F} - 1.$$

(d) *If f is any formation function for the local formation $\mathfrak{F}$, then*

$$\overline{m}(\mathfrak{F}) \leqq \sup_{p\in\mathbb{P}} \overline{m}(f(p)) + 2\,.$$

Note that every local formation $\mathfrak{F}$ can be defined by a formation function F satisfying $\mathfrak{S}_p F(p) = F(p) \subseteqq \mathfrak{F}$ for every prime p; see [2, IV, 3.7].

Since $\overline{m}(\mathfrak{N}^0) = 1$ and $\mathfrak{N}^n = \mathfrak{N}(\mathfrak{N}^{n-1})$, it follows from statement (b) above that $\overline{m}(\mathfrak{N}^n) = n + 1$ for every nonnegative integer n, thus proving the statement at the beginning of the introduction.

Observe also that there exist local formations $\mathfrak{F}$ for which it is not possible to find a formation function f satisfying $f(p) \subseteqq \mathfrak{F}$ and $\overline{m}(f(p)) \leqq \overline{m}(\mathfrak{F}) - 1$; see Example 3.5.

However, $\overline{m}(\mathfrak{F})$ need not always be finite: Example 2.4 shows that $\overline{m}(\mathfrak{A}) = \overline{m}(\mathfrak{A}^2) = \overline{m}(\mathfrak{U}) = \overline{m}(\mathfrak{N}\mathfrak{A}) = \infty$, where $\mathfrak{A}$ and $\mathfrak{U}$ denote the classes of finite abelian and finite supersoluble groups, respectively.

Our notation is standard and follows [2]. A definition of local formations can also be found at the beginning of Section 3.

2. Preliminary results and examples. The following simple observation will prove useful for many induction arguments.

Lemma 2.1. *Let $\mathfrak{X}$ be a class of groups which is closed with respect to factor groups and n a positive integer. If the group G has n (nonconjugate) maximal subgroups belonging to $\mathfrak{X}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{X}$, or G/N has at least n (nonconjugate) maximal subgroups belonging to $\mathfrak{X}$.*

Proof. Let $M_1, \ldots, M_n \in \mathfrak{X}$ be (nonconjugate) maximal subgroups of G. Then either $M_i N = G$ for some i and so $G/N \cong M_i/M_i \cap N \in \mathfrak{X}$, or the $M_i/N \in \mathfrak{X}$ are n (nonconjugate) maximal subgroups of G/N. $\square$

Recall that a class $\mathfrak{H}$ of finite soluble groups is a *Schunck class* if $G \in \mathfrak{H}$ if and only if $G/\mathrm{Core}_G(M) \in \mathfrak{H}$ for every maximal subgroup M of G (note that $1 \in \mathfrak{H}$). The *boundary* $b(\mathfrak{H})$ of a Schunck class $\mathfrak{H}$ consists of all finite soluble groups whose proper epimorphic images are $\mathfrak{H}$-groups but G does not belong to $\mathfrak{H}$. Lemma 2.1 can, for instance, be used to determine $\overline{m}(\mathfrak{H})$ for Schunck classes $\mathfrak{H}$; in particular $\overline{m}(\mathfrak{H})$ is finite if the Schunck class has a finite boundary.

Proposition 2.2. *Let $\mathfrak{V}$ be a class of finite soluble groups which is closed with respect to factor groups and let $\mathfrak{H}$ be a Schunck class. If every $\mathfrak{V}$-group in the boundary of $\mathfrak{H}$ has at most n conjugacy classes of maximal subgroups belonging to the Schunck class $\mathfrak{H}$, then every $\mathfrak{V}$-group which does not belong to $\mathfrak{H}$ has at most n conjugacy classes of maximal subgroups belonging to the Schunck class $\mathfrak{H}$.*

Proof. Assume that $G \in \mathfrak{V}$ has $n+1$ nonconjugate maximal subgroups $M_1, \ldots, M_{n+1}$. By induction on the order of G, we have $G/N \in \mathfrak{H}$ for every nontrivial normal subgroup N of G. Therefore $G \in \mathfrak{H}$ or G is primitive. But in the latter case, we have $G \in b(\mathfrak{H})$, a contradiction. □

Note that if $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of groups with $\mathfrak{X} \subseteqq \mathfrak{Y}$, then we need not have $\overline{m}(\mathfrak{X}) \leqq \overline{m}(\mathfrak{Y})$. In fact, while $\overline{m}(\mathfrak{S}_\pi) = 1$ if $\pi \neq \mathbb{P}$, Example 4.3 below shows that $\overline{m}(\mathfrak{N} \cap \mathfrak{S}_\pi) = 2$ if $|\pi| \geqq 2$. However, $\overline{m}$ is well-behaved with respect to intersections.

Lemma 2.3. *Let I be a set and assume that $\mathfrak{X}_i$ is a group class for every $i \in I$. Then*

$$\overline{m}\big(\bigcap_{i \in I} \mathfrak{X}_i\big) \leqq \sup_{i \in I} \overline{m}(\mathfrak{X}_i).$$

Proof. Assume that $n = \sup_{i \in I} \overline{m}(\mathfrak{X}_i)$ is finite and let G be a finite soluble group having $n+1$ nonconjugate maximal subgroups $M_1, \ldots, M_{n+1} \in \mathfrak{X}$, where $\mathfrak{X}$ denotes the intersection of all $\mathfrak{X}_i$ $(i \in I)$. Then $M_i \in \mathfrak{X}_i$ for every $i \in I$, and hence $G \in \mathfrak{X}_i$. It follows that $G \in \mathfrak{X}$ and $\overline{m}(\mathfrak{X}) \leqq n$, as required. □

The following example shows that $\overline{m}(\mathfrak{A}) = \overline{m}(\mathfrak{A}^2) = \overline{m}(\mathfrak{N}\mathfrak{A}) = \overline{m}(\mathfrak{U}) = \infty$.

Example 2.4. Let p be a prime such that $p \geqq n-1$ and let P be an extraspecial p-group of order p^3. Then P has at least $p+1 \geqq n$ maximal (normal) subgroups P_i whose order is p^2. So the P_i are abelian and $P = P_i P_j$ for all $1 \leqq i \neq j \leqq p+1$, but P is nonabelian.

Let q be a prime, $q \neq p$ and put $F = GF(q)$. By [2, B, 10.7], P has a faithful irreducible FP-module N, so if $G = N \rtimes P$ denotes the semidirect product of P with N, then $A_i = NP_i \in \mathfrak{N}\mathfrak{A}$ for every i and $G = A_i A_j$ for all $i \neq j$. But since N is the Fitting subgroup of G and P is nonabelian, we have $G \notin \mathfrak{N}\mathfrak{A}$.

Now suppose that p^2 divides $q-1$ (by Dirichlet's remainder theorem, for every prime p, there exists such a prime q). By Maschke's theorem, N is a completely reducible FP_i-module for every i. Moreover, since F has a primitive p^2th root of unity, a simple FP_i-module has F-dimension 1 by [2, B, 9.2]. This shows that $A_i = P_i N$ is supersoluble for every i. But $G \notin \mathfrak{U} \subseteqq \mathfrak{N}\mathfrak{A}$. Observe also that the A_i have derived length 2 but G has derived length 3.

It is also easy to show that an $\mathfrak{F}$-group can have any number of conjugacy classes of maximal subgroups in $\mathfrak{F}$, where $\mathfrak{F}$ is any formation.

Example 2.5. Let $\mathfrak{F}$ be a nonempty formation, $G \in \mathfrak{F}$ and n an integer. Since $\mathfrak{F}$ is closed with respect to factor groups, it contains a cyclic group C of order p for some prime p. Moreover, $\mathfrak{F}$ is closed with respect to direct products, and so the direct product $D = G \times C \times \ldots \times C$, where C occurs n times, belongs to $\mathfrak{F}$. Clearly, every maximal subgroup of D containing G belongs to $\mathfrak{F}$, and so D has at least n maximal $\mathfrak{F}$-subgroups.

Next, we show that the classes $\mathfrak{S}_\pi$, where $\pi \neq \mathbb{P}$, are the only examples of local formations $\mathfrak{F}$ with $\overline{m}(\mathfrak{F}) = 1$. Recall that the characteristic of a class of groups $\mathfrak{X}$ is the set of all primes p such that $\mathfrak{X}$ contains a group of order p.

Theorem 2.6. *Let $\mathfrak{F} \subseteq \mathfrak{S}$ be a class of groups satisfying $\overline{m}(\mathfrak{F}) = 1$. If $\mathfrak{F}$ is a formation or closed with respect to normal subgroups, then $\mathfrak{S}_\pi \subseteq \mathfrak{F}$, where π denotes the characteristic of $\mathfrak{F}$. In particular, if $\mathfrak{F}$ is a Fitting class, a saturated formation or a formation which is closed with respect to normal subgroups, then $\overline{m}(\mathfrak{F}) = 1$ if and only if $\mathfrak{F} = \mathfrak{S}_\pi$ for some set of primes $\pi \neq \mathbb{P}$.*

Proof. Assume that $\mathfrak{S}_\pi$ is not contained in $\mathfrak{F}$, and among the groups of minimal exponent in $\mathfrak{S}_\pi \setminus \mathfrak{F}$ choose a group G of minimal order. Then $G \neq 1$ and every proper subgroup of G belongs to $\mathfrak{F}$. Since $\overline{m}(\mathfrak{F}) = 1$, the group G has a unique maximal subgroup M, and since G is soluble, M is normal in G and G/M is cyclic of prime order $p \in \pi$. Now $M = \Phi(G)$ and so G is a p-group, hence is cyclic of order p^n for some positive integer n.

Since $p \in \pi$, the class $\mathfrak{F}$ contains a cyclic group C of order p, hence we may assume that $M \neq 1$. Now G can be embedded into the wreath product $W = M \wr C$; see e.g. [2, A, 18.9]. Let B denote the base group of W, then the exponent of B, being the direct product of p copies of $M \in \mathfrak{F}$, is p^{n-1}, and so B belongs to $\mathfrak{F}$. Moreover, let $N = [B, C]CB^p$, then N/B^p has exponent p by [2, A, 18.10], and so the exponent of N is at most p^{n-1}. Now B and N are maximal $\mathfrak{F}$-subgroups of W and so $W \in \mathfrak{F}$. Since W is a p-group, G is subnormal in W, hence is an $\mathfrak{F}$-group if $\mathfrak{F}$ is closed with respect to normal subgroups. If $\mathfrak{F}$ is a formation, $\mathfrak{F} \cap \mathfrak{S}_p$ is subgroup-closed by [2, IV, 1.16], and so $G \in \mathfrak{F}$.

The second part of the theorem follows from the fact that if $\mathfrak{F}$ is a Fitting class or a saturated formation, then $\mathfrak{F}$ is contained in $\mathfrak{S}_\pi$; see [2, IV, 4.3] and [2, IX, 1.7], respectively. The same is obviously true if $\mathfrak{F} \subseteq \mathfrak{S}$ is closed with respect to factor groups and normal subgroups. □

On the other hand, there are examples of formations $\mathfrak{F}$ of finite soluble groups which satisfy $\overline{m}(\mathfrak{F}) = 1$ but which are not saturated.

Proposition 2.7. *Let $\sigma \subseteq \pi \subseteq \mathbb{P}$ and define $\mathfrak{F}_\pi^\sigma$ to be the class of all groups $G \in \mathfrak{S}_\pi$ such that every central principal factor of G is a σ-group. Then*

(a) *$\mathfrak{F}_\pi^\sigma$ is a formation; moreover, it is closed with respect to extensions and products of normal subgroups.*

(b) *$\mathfrak{F}_\pi^\sigma$ is saturated if and only if $\sigma = \pi$, that is, if and only if $\mathfrak{F}_\pi^\sigma = \mathfrak{S}_\pi$.*

(c) *$\overline{m}(\mathfrak{F}_\pi^\sigma) = 1$ if and only if $\sigma \neq \mathbb{P}$.*

Proof. By [2, IV, 1.3], $\mathfrak{F}_\pi^\sigma$ is a formation, and evidently σ is the characteristic of $\mathfrak{F}_\pi^\sigma$. It is easy to see that $\mathfrak{F}_\pi^\sigma$ is closed with respect to extensions and factor groups, and hence with respect to products of normal subgroups.

If $\mathfrak{F}$ is a saturated formation, we have $\mathfrak{F} \subseteq \mathfrak{S}_\sigma$ by [2, IV, 4.3], and so $\sigma = \pi$.

In order to see that $\overline{m}(\mathfrak{F}_\pi^\sigma) \leqq 1$, let $G \in \mathfrak{S}$ and suppose that $M_1, M_2 \in \mathfrak{F}_\pi^\sigma$ are two nonconjugate maximal subgroups of G: then $G = M_1 M_2$ by [2, A, 16.1], and so $G \in \mathfrak{S}_\pi$. Let H/K be a central principal factor of G and assume that H/K is a p-group, where p is a prime. If $G = KM_1$, then $1 \neq H \cap M_1 / K \cap M_1$, and so M_1 has a central p-principal factor. Since $M_1 \in \mathfrak{F}_\pi^\sigma$, it follows that $p \in \sigma$, as required.

Thus we may assume that K is contained in both M_1 and M_2. Since the complements of N/K in G/K are conjugate by [2, A, 15.6], it follows that $H \leqq M_1$ or $H \leqq M_2$, and so M_1 or M_2 has a central p-principal factor. Thus again we have $p \in \sigma$, and hence $G \in \mathfrak{F}_\pi^\sigma$. This proves that $\overline{m}(\mathfrak{F}_\pi^\sigma) \leqq 1$.

If $p \in \mathbb{P} \setminus \sigma$, then the identity subgroup of a group C of order p is a maximal $\mathfrak{F}_\pi^\sigma$-subgroup of C, proving that $\overline{m}(\mathfrak{F}_\pi^\sigma) = 1$ if $\sigma \neq \mathbb{P}$. Conversely, if $\sigma = \mathbb{P}$, then we have $\mathfrak{F}_\pi^\sigma = \mathfrak{S}$, and so $\overline{m}(\mathfrak{F}_\pi^\sigma) = -\infty$. $\square$

3. Local formations. Let π be a set of primes and f a function assigning to every $p \in \pi$ a nonempty formation $f(p)$. Then f is a *formation function*, and the formation

$$LF(f) = \mathfrak{S}_\pi \cap \bigcap_{p \in \pi} \mathfrak{S}_{p'} \mathfrak{S}_p f(p)$$

is the *local formation* defined by f; note that π equals the characteristic of $\mathfrak{F}$. A formation $\mathfrak{F}$ is *local* if $\mathfrak{F} = LF(f)$ for some formation function f. A formation function f for the local formation $\mathfrak{F}$ of characteristic π is *full* if $f(p) = \mathfrak{S}_p f(p)$ for every $p \in \pi$; it is *integrated* if $f(p) \subseteq \mathfrak{F}$ for every $p \in \pi$. Note that by [2, IV, 3.7], every local formation of finite soluble groups possesses exactly one formation function which is both full and integrated. Note that Example 3.5 shows that in the following proposition, we have $\overline{m}(\mathfrak{S}_\pi \mathfrak{F}) = \overline{m}(\mathfrak{F}) + 1$ if $\mathfrak{F}$ satisfies $\mathfrak{F} = \mathfrak{S}_{\pi'} \mathfrak{F}$.

Proposition 3.1. *Let $\mathfrak{F}$ be a formation of finite soluble groups. Then $\overline{m}(\mathfrak{S}_\pi \mathfrak{F}) \leqq \overline{m}(\mathfrak{F}) + 1$.*

Proof. Let $n = \overline{m}(\mathfrak{F})$ and suppose that the finite soluble group G has $n+2$ nonconjugate maximal subgroups $M_1, \ldots, M_{n+2} \in \mathfrak{S}_\pi \mathfrak{F}$. We have to show that $G \in \mathfrak{S}_\pi \mathfrak{F}$.

Clearly, we may exclude the case $|G| = 1$. By Lemma 2.1 and induction on the order of G, it follows that $G/N \in \mathfrak{S}_\pi \mathfrak{F}$ for every nontrivial normal subgroup N of G. Thus we may assume that G has a unique minimal normal subgroup N of prime exponent $p \notin \pi$. It follows that $F(G) = O_p(G)$ is a p-group. Since N has at most one conjugacy class of complements in G by [2, A 15.2], we may assume that $N \leqq M_i$ for $i = 1, \ldots, n+1$. We show that $M_i \in \mathfrak{F}$ for all $i \leqq n+1$, for then $G \in \mathfrak{F} \subseteq \mathfrak{S}_\pi \mathfrak{F}$, as required.

If $O_p(G) \leqq M_i$, it follows that $O_\pi(M_i) \leqq C_G(O_p(G))$ which is contained in $O_p(G) = F(G)$ by [2, A 10.6], proving that $O_\pi(M_i) = 1$ and $M_i \in \mathfrak{F}$. Otherwise $G = O_p(G) M_i$, and since N is properly contained in $F(G) = O_p(G)$, we have $\Phi(G) \neq 1$ and consequently $N \leqq \Phi(G)$. Let $q \in \pi$, then $O_q(G/N) \leqq F(G/N) = F(G)/N$ which is a p-group. This shows that $O_\pi(G/N) = 1$ and so $G/N \in \mathfrak{F}$. Thus $M_i / M_i \cap O_p(G) \cong G/O_p(G)$ is contained in $\mathfrak{F}$. On the other hand, also $M_i / O_\pi(M_i)$ belongs to $\mathfrak{F}$ and since $O_p(G) \cap O_\pi(M_i) = 1$, we have $M_i \in \mathfrak{F}$. $\square$

The upper bounds on $\overline{m}(\mathfrak{F})$ for local formations $\mathfrak{F}$ now follow directly from Proposition 3.1.

Theorem 3.2. *Let $\mathfrak{F}$ be a saturated formation of finite soluble groups with formation function f. Let $\emptyset \neq \pi$ be the support of f and let $n = \sup_{p\in\pi} \overline{m}(f(p))$. If f is full, then $\overline{m}(\mathfrak{F}) \leqq n+1$, and in any case, $\overline{m}(\mathfrak{F}) \leqq n+2$.*

Proof. Suppose that f is full. Then $\mathfrak{F} = \mathfrak{S}_\pi \cap \bigcap_{p\in\pi} \mathfrak{S}_{p'}f(p)$. Since $\overline{m}(\mathfrak{S}_{p'}f(p)) \leqq \overline{m}(f(p))+1$ for every $p \in \pi$ by Proposition 3.1 and obviously $\overline{m}(\mathfrak{S}_\pi) = 1$, it follows from Lemma 2.3 that $\overline{m}(\mathfrak{F}) \leqq \sup_{p\in\pi} \overline{m}(f(p))$.

Now let f be any formation function for $\mathfrak{F}$. Then F, defined by $F(p) = \mathfrak{S}_p F(p)$ for all $p \in \pi$, is a full formation function for $\mathfrak{F}$, and for every $p \in \pi$, we have $\overline{m}(F(p)) \leqq \overline{m}(f(p))+1$ by Proposition 3.1. Thus the second statement follows from the first. □

On the other hand, one might ask whether, for every saturated formation $\mathfrak{F}$, there exists a formation function f for $\mathfrak{F}$ such that the values of $\overline{m}(f(p))$ are related to $\overline{m}(\mathfrak{F})$. Indeed, this is true for the (unique) full and integrated local definition of $\mathfrak{F}$.

Proposition 3.3. *Suppose that $\mathfrak{F}$ is a saturated formation with full and integrated formation function F. Then $\overline{m}(F(p)) \leqq \overline{m}(\mathfrak{F})$ for every prime p.*

Proof. Let $n = \overline{m}(\mathfrak{F})$ and p a prime. If $F(p)$ is the empty class, there is nothing to prove, so assume that $F(p)$ is a nonempty formation. Suppose that the finite soluble group G has n nonconjugate maximal subgroups $M_1, M_2, \ldots, M_{n+1} \in F(p)$ and let W be the regular wreath product of a cyclic group of order p with G. If N denotes the base group of W, then $M_1N, \ldots, M_{n+1}N$ are maximal subgroups of W belonging to the class $\mathfrak{S}_p F(p) = F(p) \subseteqq \mathfrak{F}$. Since $n+1 > \overline{m}(\mathfrak{F})$, we have $W \in \mathfrak{F}$. Now $\mathfrak{F}$ is contained in $\mathfrak{S}_{p'}\mathfrak{S}_p F(p) = \mathfrak{S}_{p'}F(p)$, and since $O_{p'}(W) = 1$ by construction, it follows that $W \in F(p)$ and so also $G \cong W/N \in F(p)$, as required. □

Sometimes, the following result can be used to show that the bound in Theorem 3.2 is, indeed, best-possible.

Proposition 3.4. *Suppose that $\mathfrak{F}$ is a saturated formation of characteristic $\pi \neq \emptyset$, $\mathfrak{F} \neq \mathfrak{S}_\pi$, and F a full and integrated formation function for $\mathfrak{F}$. Then*

$$\overline{m}(\bigcap_{p\in\pi} F(p)) \leqq \overline{m}(\mathfrak{F}) - 1.$$

Proof. Let $n = \overline{m}(\mathfrak{F})$, then, in view of Theorem 2.6, we may assume that $n \geqq 2$ and thus that $|\pi| \geqq 2$. Let G be a finite soluble group with n nonconjugate maximal subgroups $M_1, M_2, \ldots, M_n$ belonging to $F(p)$ for every $p \in \pi$. By induction on the order of G, we may assume that G has a unique minimal normal subgroup N, which is an elementary abelian p-group for some prime p, such that $G/N \in F(q)$ for every prime $q \in \pi$. Since $G = M_1M_2$ by [2, A, 16.1], it follows that G is a π-group and $p \in \pi$. Thus G belongs to the class $\mathfrak{S}_p F(p) = F(p)$; in particular G belongs to $\mathfrak{F}$.

Now let $q \in \pi$ with $q \neq p$ and let $F = GF(q)$ be the prime field of characteristic q, then by [2, B, 10.7], there exists a faithful irreducible FG-module K. Put $H = G \ltimes K$, then $M_1K, \ldots, M_nK$ are maximal $F(q)$-subgroups of H. Thus $M_1K, \ldots, M_nK$ and G are nonconjugate maximal $\mathfrak{F}$-subgroups of H, and H is an $\mathfrak{F}$-group. Since $O_{q'}(H) = 1$, we have

$H \in F(q)$ and so also $G \in F(q)$, as required. $\square$

The following examples show that the bounds obtained above are best-possible.

Example 3.5. Let $\mathfrak{F} \neq \emptyset$ be a formation and $\emptyset \neq \pi \subsetneqq \mathbb{P}$. Then the formation function F defined by

$$F(p) = \begin{cases} \mathfrak{S}_\pi \mathfrak{F} & \text{if } p \in \pi \\ \mathfrak{S}_{\pi'} \mathfrak{S}_\pi \mathfrak{F} & \text{if } p \notin \pi \end{cases}$$

for every $p \in \mathbb{P}$, is a full and integrated local definition of the formation $\mathfrak{S}_{\pi'} \mathfrak{S}_\pi \mathfrak{F}$.

Now by Proposition 3.4, we have $\overline{m}(\mathfrak{S}_\pi \mathfrak{F}) \leqq \overline{m}(\mathfrak{S}_{\pi'} \mathfrak{S}_\pi \mathfrak{F}) - 1$ and so by Proposition 3.1, $\overline{m}(\mathfrak{S}_{\pi'} \mathfrak{S}_\pi \mathfrak{F}) = \overline{m}(\mathfrak{S}_\pi \mathfrak{F}) + 1$. In particular, the bounds in Proposition 3.1 and Proposition 3.4 are both attained.

For every prime p, let $f(p)$ be the formation consisting of the factor groups of the groups $G/O_p(G)$, where $G \in F(p)$. Assume that $|\pi| \geqq 2$ and $|\pi'| \geqq 2$ and choose $p, q \in \pi$ with $p \neq q$. Let C be a cyclic group of order q and $G \in F(p)$, then $W = C \wr G$, the regular wreath product of C and G, belongs to $F(p)$. Since every element of G operates nontrivially on the base group of W, it follows that $O_{q'}(W) = 1$; in particular $O_p(W) = 1$. This shows that $f(p) = F(p)$ for every $p \in \pi$. A similar argument shows that also $f(p) = F(p)$ for every $p \in \pi'$.

If g is any integrated formation function for $\mathfrak{S}_{\pi'} \mathfrak{S}_\pi \mathfrak{F}$, then we have $f(p) \subseteqq g(p) \subseteqq F(p)$ for every prime p by [2, IV, 3.7, 3.8 and 3.10], and so it follows that F is the unique full and integrated formation function for $\mathfrak{S}_{\pi'} \mathfrak{S}_\pi \mathfrak{F}$. Thus we have $\overline{m}(g(p)) = \overline{m}(\mathfrak{S}_{\pi'} \mathfrak{S}_\pi \mathfrak{F})$ for every $p \notin \pi$. This shows that the bounds in Theorem 3.2 and Proposition 3.3 are attained.

4. The nilpotent-by-$\mathfrak{F}$ case.

Theorem 4.1. *Let $\mathfrak{F}$ be a formation. Then $\overline{m}(\mathfrak{N}\mathfrak{F}) = \overline{m}(\mathfrak{F}) + 1$.*

Proof. Let $n = \overline{m}(\mathfrak{F})$. In order to prove that $\overline{m}(\mathfrak{N}\mathfrak{F}) \leqq n + 1$, we may clearly assume that n is finite. Let $M_1, \ldots, M_{n+2} \in \mathfrak{N}\mathfrak{F}$ be nonconjugate maximal subgroups of the finite soluble group G.

Since $\mathfrak{N}\mathfrak{F}$ is a local formation, by induction on the group order, we may assume that G is primitive and nonnilpotent: by [2, A 15.6], G has a unique minimal normal subgroup $N = C_G(N) = F(G)$, and N has a unique conjugacy class of complements. Thus we may assume that $N \leqq M_i$ for every $i \leqq n + 1$. We will show that $M_1/N, \ldots, M_{n+1}/N \in \mathfrak{F}$, for then $G/N \in \mathfrak{F}$ and $G \in \mathfrak{N}\mathfrak{F}$, as required. Therefore let $i \in \{1, \ldots, n + 1\}$.

Since $M_i \in \mathfrak{N}\mathfrak{F}$, we clearly have $M_i/F(M_i) \in \mathfrak{F}$. Now $F(M_i)$ contains N, and since N is self-centralising, $F(M_i)$ must be a p-group, where p is the exponent of N.

Let $L/N = F(G/N)$. Since G is soluble, L/N is nontrivial, and it follows that $M_i L = G$. Since G/L belongs to $\mathfrak{F}$ by induction hypothesis, we also have $M_i/M_i \cap L \in \mathfrak{F}$. Since L/N is a p'-group, we have $L \cap F(M_i) = N$, and so $M_i/N \in \mathfrak{F}$, as required.

To prove the other inequality let $f(p) = \mathfrak{S}_p \mathfrak{F}$ for every prime p, then f is a full and integrated local definition for the saturated formation $\mathfrak{N}\mathfrak{F}$. Since

$$\mathfrak{F} = \bigcap_{p \in \mathbb{P}} \mathfrak{S}_p \mathfrak{F},$$

it follows from Proposition 3.4 that $\overline{m}(\mathfrak{F}) \leqq \overline{m}(\mathfrak{N}\mathfrak{F}) - 1$; hence we have $\overline{m}(\mathfrak{N}\mathfrak{F}) = \overline{m}(\mathfrak{F}) + 1$. $\square$

Let $\mathfrak{F}$ be the class of all groups of order 1, then clearly $\overline{m}(\mathfrak{F}) = 1$, so that by an easy induction argument and the previous result, we have $\overline{m}(\mathfrak{N}^n) = n + 1$ for every positive integer n; thus we obtain:

Corollary 4.2. *Let G be a finite group with Fitting length $n \geqq 1$. Then G has at most n conjugacy classes of maximal subgroups whose Fitting length is at most $n - 1$.*

Our final example shows that if n and k are positive integers with $2 \leqq k \leqq n$, then there exists a finite group of Fitting length n G having k conjugacy classes of maximal subgroups of Fitting length $n - 1$ and $n - k$ conjugacy classes of maximal subgroups of Fitting length $n - 2$.

Example 4.3. Let n be a positive integer and $p_1, \ldots, p_n$ be primes such that $p_i \neq p_{i+1}$ for all $1 \leqq i \leqq n - 1$. Let $G_0 = 1$ and for each $i \geqq 0$, let G_{i+1} be the semidirect product of G_i with a faithful irreducible $GF(p_{i+1})G_i$-module $(1 \leqq i \leqq n - 1)$. Then $G = G_n$ has a unique principal series $G = N_0 \triangleright N_1 \triangleright \ldots \triangleright N_n = 1$, where N_{i-1}/N_i is an elementary abelian p_i-group and $G/N_i \cong G_i$. For each $i = 1, \ldots, n + 1$, let M_i be a maximal subgroup that avoids N_{i-1}/N_i and covers all other principal factors. If $2 \leqq i \leqq n - 1$ and $p_{i-1} = p_{i+1}$, it follows that M_i has Fitting length $n - 2$. Otherwise, we have $p_{i-1} \neq p_{i+1}$,and M_i has Fitting length $n - 1$.

Also, a construction as in Example 2.5 may be used to provide similar examples with any number of conjugacy classes of maximal subgroups having Fitting length n.

References

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