Bose-Einstein condensation in a one-dimensional interacting system due to power-law trapping potentials

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We examine the possibility of Bose-Einstein condensation in one-dimensional interacting Bose gas subjected to confining potentials of the form $V_{\text{ext}}(x) = V_0(|x|/a)^\gamma$, in which $\gamma < 2$, by solving the Gross-Pitaevskii equation within the semiclassical two-fluid model. The condensate fraction, chemical potential, ground state energy, and specific heat of the system are calculated for various values of interaction strengths. Our results show that a significant fraction of the particles is in the lowest energy state for a finite number of particles at low temperature, indicating a phase transition for weakly interacting systems. [S1050-2947(99)00902-6]

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I. INTRODUCTION

The recent observations of Bose-Einstein condensation (BEC) in trapped atomic gases [1–5] have renewed interest in bosonic systems [6,7]. BEC is characterized by a macroscopic occupation of the ground state for $T < T_0$, where T_0 depends on the system parameters. The success of experimental manipulation of externally applied trap potentials brings about the possibility of examining two- or even one-dimensional Bose-Einstein condensates. Since the transition temperature T_0 increases with decreasing system dimension, it was suggested that BEC may be achieved more favorably in low-dimensional systems [8]. The possibility of BEC in one- (1D) and two-dimensional (2D) homogeneous Bose gases is ruled out by the Hohenberg theorem [9]. However, due to spatially varying potentials which break the translational invariance, BEC can occur in low-dimensional inhomogeneous systems. The existence of BEC is shown in a 1D noninteracting Bose gas in the presence of a gravitational field [10], an attractive-δ impurity [11], and power-law trapping potentials [12]. Recently, many authors have discussed the possibility of BEC in 1D trapped Bose gases relevant to the magnetically trapped ultracold alkali-metal atoms [13–18]. Pearson and his co-workers [19] studied the interacting Bose gas in 1D power-law potentials employing the path-integral Monte Carlo (PIMC) method. They have found that a macroscopically large number of atoms occupy the lowest single-particle state in a finite system of hard-core bosons at some critical temperature. It is important to note that the recent BEC experiments are carried out with a finite number of atoms (ranging from several thousands to several 10^6), therefore the thermodynamic limit argument in some theoretical studies [15] does not apply here [8].

The aim of this paper is to study the two-body interaction effects on the BEC in 1D systems under power-law trap potentials. For ideal bosons in harmonic oscillator traps transition to a condensed state is prohibited. It is anticipated that external potentials more confining than the harmonic oscillator type would be possible experimentally. It was also argued [15] that in the thermodynamic limit there can be no BEC phase transition for nonideal bosons in 1D. Since the realistic systems are weakly interacting and contain a finite number of particles, we employ the mean-field theory [20,21] as applied to a two-fluid model. Such an approach has been shown to capture the essential physics in 3D systems [21]. The 2D version [22] is also in qualitative agreement with the results of PIMC simulations on hard-core bosons [23]. In the remaining sections we outline the two-fluid model and present our results for an interacting 1D Bose gas in power-law potentials.

II. THEORY

In this paper we shall investigate the Bose-Einstein condensation phenomenon for 1D interacting Bose gas confined in a power-law potential:

$$V_{\text{ext}}(x) = V_0 \left(\frac{|x|}{a} \right)^\gamma, \tag{1}$$

where V_0 and a are some suitable energy and length parameters defining the external potential, and γ controls the confinement strength. Presumably, they can be experimentally adjusted. Using the semiclassical density of states, the transition temperature T_0 and the fraction of condensed particles N_0/N for the noninteracting system were calculated as [12]

$$k_B T_0 = \left[\frac{N}{\kappa F(\gamma) G(\gamma)} \right]^{2\gamma/(2+\gamma)} \tag{2}$$

and

$$N_0/N = 1 - \left(\frac{T}{T_0} \right)^{1/\gamma + 1/2}, \tag{3}$$

where $\kappa = 2(2m)^{1/2} a / \gamma h V_0^{1/\gamma}$ (m is the mass of bosons and h is Planck's constant), and

$$F(\gamma) = \int_0^1 \frac{x^{1/\gamma - 1} dx}{\sqrt{1-x}}, \tag{4}$$

and

$$G(\gamma) = \int_0^\infty \frac{x^{1/\gamma - 1/2} dx}{e^x - 1} = \Gamma(1/\gamma + 1/2) \zeta(1/\gamma + 1/2), \tag{5}$$

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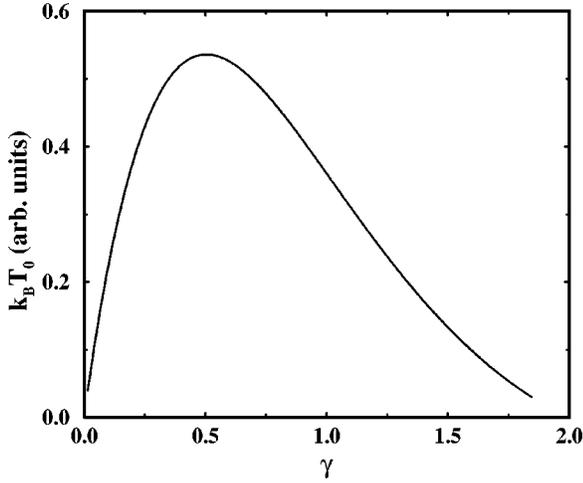
FIG. 1. The variation of the critical temperature T_0 with the external potential exponent γ.

in which $\Gamma(x)$ and $\zeta(x)$ are the gamma and the Riemann zeta functions, respectively. The total energy of the noninteracting system for $T < T_0$ ($\mu = 0$) is given by

$$\frac{\langle E \rangle}{N k_B T_0} = \frac{\Gamma(1/\gamma + 3/2)\zeta(1/\gamma + 3/2)}{\Gamma(1/\gamma + 1/2)\zeta(1/\gamma + 1/2)}\left(\frac{T}{T_0}\right)^{1/\gamma + 3/2}. \quad (6)$$

Figure 1 shows the variation of the critical temperature T_0 as a function of the exponent γ in the trapping potential. It should be noted that T_0 vanishes for harmonic potential due to the divergence of the function $G(\gamma = 2)$. It appears that the maximum T_0 is attained for $\gamma \approx 0.5$, and for a constant trap potential [i.e., $V_{\text{ext}}(x) = V_0$] the BEC disappears, consistent with the Hohenberg theorem.

We are interested in how the short-range interaction effects modify the picture presented above. To this end, we employ the mean-field formalism and describe the collective dynamics of a Bose condensate by its macroscopic time-dependent wave function $\Upsilon(x,t) = \Psi(x)\exp(-i\mu t)$, where μ is the chemical potential. The condensate wave function $\Psi(x)$ satisfies the Gross-Pitaevskii (GP) equation [24,25]

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_{\text{ext}}(x) + 2g n_1(x) + g\Psi^2(x)\right]\Psi(x) = \mu\Psi(x), \quad (7)$$

where g is the repulsive, short-range interaction strength, and $n_1(x)$ is the average noncondensed particle distribution function. We treat the interaction strength g as a phenomenological parameter without going into the details of actually relating it to any microscopic description [26]. In the semiclassical two-fluid model [27,28] the noncondensed particles can be treated as bosons in an effective potential [21,29]

$$V_{\text{eff}}(x) = V_{\text{ext}}(x) + 2g n_1(x) + 2g\Psi^2(x). \quad (8)$$

The density distribution function is given by

$$n_1(x) = \int \frac{dp}{2\pi\hbar}\frac{1}{\exp\{[p^2/2m + V_{\text{eff}}(x) - \mu]/k_B T\} - 1}, \quad (9)$$

and the total number of particles N fixes the chemical potential through the relation

$$N = N_0 + \int \frac{\rho(E)dE}{\exp[(E - \mu)/k_B T] - 1}, \quad (10)$$

where $N_0 = \int \Psi^2(x)dx$ is the number of condensed particles, and the semiclassical density of states is determined by

$$\rho(E) = \frac{\sqrt{2m}}{h}\int_{V_{\text{eff}}(x) < E}\frac{dx}{\sqrt{E - V_{\text{eff}}(x)}}. \quad (11)$$

The GP equation yields a simple solution when the kinetic energy term is neglected (the Thomas-Fermi approximation),

$$\Psi^2(x) = \frac{\mu - V_{\text{ext}}(x) - 2g n_1(x)}{g}\theta[\mu - V_{\text{ext}}(x) - 2g n_1(x)], \quad (12)$$

where $\theta[x]$ is the unit step function. More precisely, the Thomas-Fermi approximation [7,20,30] would be valid when the interaction energy $\sim g N_0/\Lambda$ far exceeds the kinetic energy $\hbar^2/2m\Lambda^2$, where Λ is the spatial extent of the condensate cloud. For a linear trap potential (i.e., $\gamma = 1$), a variational estimate for Λ is given by $\Lambda = [\hbar^2/2m(\pi/2)^{1/2}2a/V_0]^{1/3}$. We note that the Thomas-Fermi approximation would break down for temperatures close to T_0 where N_0 is expected to become very small.

The above set of equations [Eqs. (9)–(12)] must be solved self-consistently to obtain the various physical quantities such as the chemical potential $\mu(N,T)$, the condensate fraction N_0/N, and the effective potential V_{eff}. In a 3D system, Minguzzi et al. [21] solved a similar system of equations numerically and also introduced an approximate semianalytical solution by treating the interaction effects perturbatively. Motivated by the success [21,22] of the perturbative approach we consider a weakly interacting system in 1D. To zero order in $g n_1(r)$, the effective potential becomes

$$V_{\text{eff}}(x) = \begin{cases} V_{\text{ext}}(x) & \text{if } \mu < V_{\text{ext}}(x) \\ 2\mu - V_{\text{ext}}(x) & \text{if } \mu > V_{\text{ext}}(x). \end{cases} \quad (13)$$

Figure 2 displays the typical form of the effective potential within our semianalytic approximation scheme. The most noteworthy aspect is that the effective potential that affects the bosons acquires a double-well shape because of the interactions. We can explain this result by a simple argument. Let the number of particles in the left and right wells be N_L and N_R, respectively, so that $N = N_L + N_R$. The nonlinear or interaction term in the GP equation may be approximately regarded as $V = N_L^2 + N_R^2$. Therefore the problem reduces to the minimization of the interaction potential V, which is achieved for $N_L = N_R$.

The number of condensed atoms is calculated to be

$$N_0 = \frac{2\gamma a}{(1 + \gamma)g V_0^{1/\gamma}}\mu^{1/\gamma + 1}. \quad (14)$$

The density of states is given by
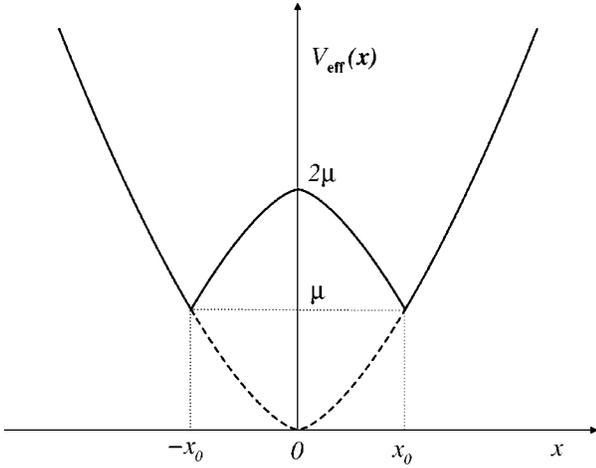
FIG. 2. Effective potential $V_{\text{eff}}(x)$ in the presence of interaction $[x_0 = (\mu/V_0)^{1/\gamma}a]$. Thick dotted line represents the lower part of the external potential $V_{\text{ext}}(x)$.

$$\rho(E) = \kappa \begin{cases} H(\gamma,\mu,E)(2\mu-E)^{1/\gamma-1/2} & \text{if } \mu < E < 2\mu \\ F(\gamma)E^{1/\gamma-1/2} & \text{if } E > 2\mu, \end{cases}$$

$$(15)$$

where

$$H(\gamma,\mu,E) = \int_1^{E/(2\mu-E)} \frac{x^{1/\gamma-1}dx}{\sqrt{x-1}}.$$

Using the above density of states, conservation of total number of particles gives us a transcendental equation for the chemical potential

$$N = N_0 + \kappa(k_BT)^{1/\gamma+1/2}I(\gamma,\mu,T), \qquad (16)$$

where

$$I(\gamma,\mu,T) = F(\gamma)\int_{2\mu/k_BT}^{\infty} \frac{x^{1/\gamma-1/2}dx}{ze^x-1}$$

$$+ \int_{\mu/k_BT}^{2\mu/k_BT} H(\gamma,\mu,xk_BT)$$

$$\times \frac{(2\mu/k_BT-x)^{1/\gamma-1/2}dx}{ze^x-1},$$

in which $z = e^{-\mu/k_BT}$. The chemical potential $\mu(N,T)$ is determined from the solution of Eq. (16). Finally, the total energy of the interacting system can be written as

$$\langle E \rangle = [\langle E \rangle_{\text{nc}}(N-N_0)/2 + \langle E \rangle_c]/N, \qquad (17)$$

where $\langle E \rangle_{\text{nc}}$ is the energy of the noncondensed particles

$$\langle E \rangle_{\text{nc}} = \int \frac{E\rho(E)dE}{\exp[(E-\mu)/k_BT]-1}$$

$$= \kappa(k_BT)^{1/\gamma+1/2}J(\gamma,\mu,T), \qquad (18)$$

where
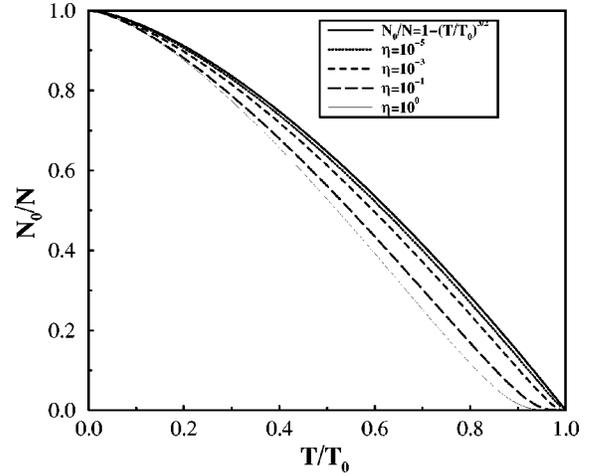
FIG. 3. The condensate fraction N_0/N versus temperature T/T_0 for $N = 10^5$ and for various interaction strengths η.

$$J(\gamma,\mu,T) = \int_{2\mu/k_BT}^{\infty} \frac{x^{1/\gamma+1/2}dx}{ze^x-1}$$

$$+ \int_{\mu/k_BT}^{2\mu/k_BT} H(\gamma,\mu,x) \frac{(2\mu/k_BT-x)^{1/\gamma+1/2}dx}{ze^x-1}$$

and $\langle E \rangle_c$ is the energy of the particles in the condensate,

$$\langle E \rangle_c = \frac{g}{2} \int \Psi^4(x)dx = \frac{2a\gamma^2\mu^{2+1/\gamma}}{(1+\gamma)(2\gamma+1)gV_0^{1/\gamma}}. \qquad (19)$$

The kinetic energy of the condensed particles is neglected within our Thomas-Fermi approximation to the GP equation.

III. RESULTS AND DISCUSSION

Up to now we have based our formulation on arbitrary γ, but in the rest of this work we shall present our results for $\gamma = 1$. Our calculations show that the results for other values of γ are qualitatively similar. In Figs. 3 and 4 we calculate the condensate fraction as a function of temperature for various values of the interaction strength $\eta = g/V_0a$ (at constant
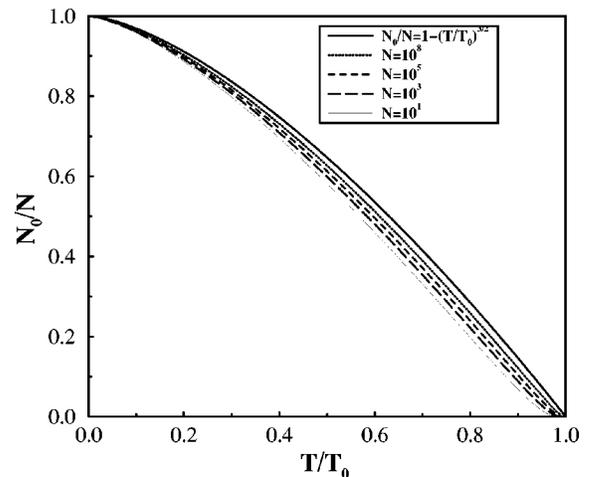
FIG. 4. The condensed fraction N_0/N versus temperature T/T_0 for $\eta = 0.001$ and for different numbers of particles N.
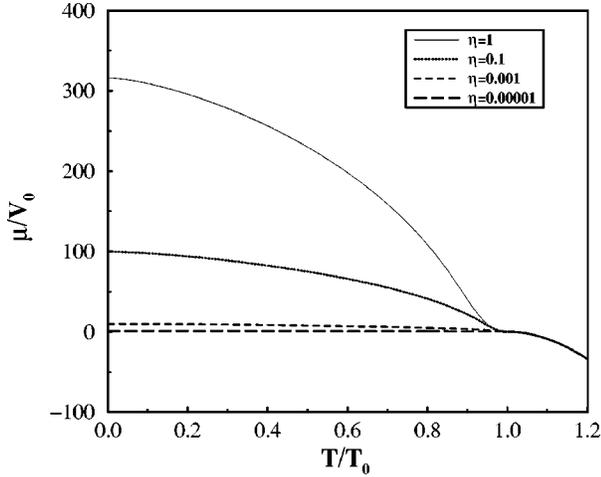
FIG. 5. The temperature dependence of the chemical potential $\mu(N,T)$ for various interaction strengths and for $N=10^5$ particles.

FIG. 7. The temperature dependence of the total energy of 1D Bose gas for various interaction strengths η and the Maxwell-Boltzmann distribution for $N=10^5$ particles.

$N=10^5$) and different numbers of particles (at constant $\eta=0.001$), respectively. We observe that as the interaction strength η is increased, the depletion of the condensate becomes more appreciable (Fig. 3). As shown in the corresponding figures, a significant fraction of the particles occupies the ground state of the system for $T<T_0$.

The temperature dependence of the chemical potential is plotted in Figs. 5 and 6 for various interaction strengths (constant $N=10^5$) and different numbers of particles (constant $\eta=0.001$), respectively. Effects of interactions on $\mu(N,T)$ are seen as large deviations from the noninteracting behavior for $T<T_0$.

In Fig. 7 we show the ground state energy of an interacting 1D system of bosons as a function of temperature for different interaction strengths. For small η, and $T<T_0$, $\langle E\rangle$ is similar to that in a noninteracting system. As η increases, some differences start to become noticeable, and for $\eta\approx 1$ we observe a small bump developing in $\langle E\rangle$. This may indicate the breakdown of our approximate scheme for large enough interaction strengths, as we can find no fundamental reason for such behavior. It is also possible that the Thomas-Fermi approximation employed is violated as the transition

to a condensed state is approached. Although it is conceivable to imagine the full solution of the mean-field equations [Eqs. (9)–(12)] may remedy the situation for larger values of η, the PIMC simulations [19] also seem to indicate that the condensation is inhibited for strongly interacting systems. The results for the specific heat calculated from the total energy curves, i.e., $C_V=d\langle E\rangle/dT$, are depicted in Fig. 8. The sharp peak at $T=T_0$ tends to be smoothed out with increasing interaction strength. It is known that the effects of a finite number of particles are also responsible for such behavior [20]. In our treatment these two effects are not disentangled. It was pointed out by Ingold and Lambrecht [14] that the identification of the BEC should also be based on the behavior of C_V around $T\approx T_0$. Our calculations indicate that the peak structure of C_V remains even in the presence of weak interactions, thus we are led to conclude that a true transition to a Bose-Einstein condensed state is predicted within the present approach.

IV. CONCLUDING REMARKS

In this work we have applied the mean-field, semiclassical two-fluid model to interacting bosons in 1D power-law trap

FIG. 6. The temperature dependence of the chemical potential $\mu(N,T)$ for different numbers of particles N and for $\eta=0.001$.
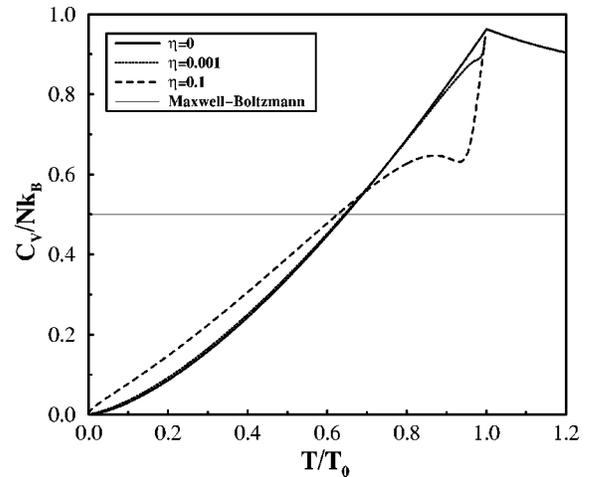
FIG. 8. The temperature dependence of the specific heat C_V for various interaction strengths η and the Maxwell-Boltzmann distribution for $N=10^5$ particles.

potentials. We have found that for a range of interaction strengths the behavior of the thermodynamic quantities resembles that of noninteracting bosons. Thus BEC in the sense of macroscopic occupation of the ground state occurs when the short-range interparticle interactions are not too strong. Our results are in qualitative agreement with the recent PIMC simulations [19] of similar systems. Both 2D and 1D simulation results [19,23] indicate a phase transition for a finite number system, in contrast to the situation in the thermodynamic limit. Since systems of much larger size can be studied within the present approach, our work complements the PIMC calculations.

The possibility of studying the tunneling phenomenon of condensed bosons in spatially different regions separated by a barrier has recently attracted some attention [31–34]. In particular, Dalfovo et al. [32] have shown that a Josephson-type tunneling current may exist for bosons under the influence of a double-well trap potential. Zapata et al. [34] have estimated the Josephson coupling energy in terms of the condensate density. It is interesting to speculate on such a pos-sibility in the present case, since the effective potential in our description is of the form of a double-well potential (cf. Fig. 2). In our treatment, the interaction effects modify the single-well trap potential into one which exhibits two minima. Thus if we think of this effective potential as the one affecting the condensed bosons and according to the general arguments [31–34] based on two weakly connected systems we should have an oscillating flux of particles when the chemical potential in the two wells is different. Any configuration with $N_L \neq N_R$, which is always the case for an odd number of bosons, will result in an oscillatory motion. It would be interesting to explore these ideas in future work.

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