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A fast algorithm for constructing Arf closure and a conjecture

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ABSTRACT

In this article, we give a fast and an easily implementable algorithm for computing the Arf closure of an irreducible algebroid curve (or a branch). Moreover, we study the relation between the branches having the same Arf closure and their regularity indices. We give some results and a conjecture, which are steps towards the interpretation of Arf closure as a specific way of taming the singularity.

1. Introduction

Canonical closure of a local ring constructed by Cahit Arf solves the problem of determining the characters of a space curve singularity [1]. The characters of a plane curve singularity, introduced first by Du Val in 1942, are special integers, which determine the multiplicity sequence of the plane curve singularity, [15]. Contrary to the well-known plane case, in which the characteristic exponents, the multiplicity sequence, the semigroup of the singularity and the characters determine each other, it was not known how

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to obtain the characters in the space case, until Cahit Arf developed his theory [1]. In 1946, Arf showed that the characters of a space branch could be obtained from the completion of the local ring corresponding to the branch by constructing its canonical closure, later known as Arf closure [1]. Since then, many algebraic geometers and algebraists have worked on Arf rings and Arf closure, [3,8,9,13,19]. Moreover, Arf semigroups and their applications in coding theory have been a recent area of interest [5,6,10,14,20]. For a good survey and a quick introduction to Arf theory, see [22].

In spite of all this interest in Arf rings and Arf closure, there is not a fast implementable algorithm for the computation of Arf closure in the literature. The construction method given by Arf cannot be implemented as an algorithm, without finding a bound, that determines up to which degree a division series should be expanded, and finding an efficient bound is not easy at all. The construction of Arf closure by using Hamburger–Noether matrices presented by Castellanos does not give an answer to this problem either [12]. The only implemented algorithm is given by Arslan [2]. The algorithm uses Arf's construction method and starts with determining the semigroup of values of the branch and its conductor, but determining the semigroup of values and the conductor of a branch is a difficult problem, which has been studied by many mathematicians and different algorithms have been given [11,18]. Noting that the special case of this problem is the famous Fobenius problem (or coin problem) makes it clear, why this problem is difficult, and it is unnecessary to mention that there is a vast literature on the Frobenius problem.

Our main objects of interest in this article are space curve singularities. Following Castellanos and Castellanos [12] and using their notation, we consider a space curve singularity as an algebroid curve $C = \operatorname{Spec}(R)$, where $(R, \mathfrak{m}, k)$ is a local ring, complete for the $\mathfrak{m}$-adic topology, with Krull dimension 1, and having k as a coefficient field. We work with irreducible algebroid curves (or branches), in other words R will always be a domain. In this case, it can be shown that $R \cong k[\![\varphi_1(t), ..., \varphi_n(t)]\!] \subset k[\![t]\!]$, where $\varphi_1(t), ..., \varphi_n(t)$ are power series in t, see [12] or [18]. Hence, we will be working with subrings of $k[\![t]\!]$. Here, the set $\{\varphi_1(t), ..., \varphi_n(t)\}$ is a parametrization of the curve C and the minimum possible n is called its embedding dimension. It is denoted by $\operatorname{embdim}(C)$ and is also equal to $\dim_k \mathfrak{m}/\mathfrak{m}^2$. The minimum order of the series of any parametrization of the curve C is called its multiplicity, which is also equal to the multiplicity of the local ring R.

By using the notation in [22], we denote the semigroup of orders of the local ring $R = k[\![\varphi_1(t), ..., \varphi_n(t)]\!]$ by $W(R)$. For $n \in \mathbb{N}$, $I_n = \{r \in R \mid \operatorname{ord}(r) \geq n\}$ and $I_n/S_n = \{r \cdot S_n^{-1} \mid \operatorname{ord}(r) \geq n\}$, where $S_n \in R$ has order n. In general, I_n/S_n is not a ring. The ring generated by I_n/S_n is denoted by $[I_n]$. A ring is called an Arf ring, if $I_n/S_n = [I_n]$ for any n in its semigroup of orders. For a local ring $R \subset k[\![t]\!]$, the smallest Arf ring containing R is called the Arf closure of R. In [1], Arf not only defines the Arf closure, but also gives a method for its construction: The Arf closure of $R = k[\![\varphi_1(t), ..., \varphi_n(t)]\!]$ can be presented as $R^* = k + kF_0 + kF_0F_1 + ... + kF_0...F_{l-2} + k[\![t]\!]F_0...F_{l-1}$, where $R_0 = R$, F_i is a smallest ordered element of R_i with $a_i = \operatorname{ord}(F_i)$ and $R_i = [I_{a_{i-1}}]$ for $i = 1, ..., l$ and $R_l = k[\![t]\!]$. Hence, that $W(R^*) = \{0, a_0, a_0 + a_1, ..., a_0 + ... + a_{l-2}, a_0 + ... + a_{l-1} + \mathbb{N}\}$ and it is shown in [1] that the multiplicity sequence of C is $(a_0, a_1, ..., a_{l-1}, 1, 1, ...)$. In [16], an instructive and detailed example is given with a discussion on the geometric aspects of the problem.

Recalling that determining the semigroup of values of a branch is a difficult problem, the construction of Arf closure by computing R_i's in each step is not efficient at all. In the next section, we give an easily implementable algorithm for constructing the Arf closure by avoiding these difficult and time consuming computations.

2. An algorithm for the computation of Arf closure

Let C be the branch with the corresponding local ring $R = k[\![\varphi_1(t), ..., \varphi_n(t)]\!]$ and $W(R) = \{i_0, i_1, ..., i_{h-1}, i_h + \mathbb{N}\}$, where $i_0 = 0$ and $i_1 = \mathrm{ord}(\varphi_1(t))$. The ring R can be presented as

$$R = k + kS_{i_1} + kS_{i_2} + ... + kS_{i_{h-1}} + k[\![T]\!]S_{i_h}$$

where S_{i_j}'s are elements of R of order i_j chosen such that $\varphi_1(t), ..., \varphi_n(t)$ are among them. Hence, $S_{i_1} = \varphi_1(t)$. By using this notation, we have the following obvious lemma:

Lemma 2.1. $[I_{i_1}] = k[\![\varphi_1, \frac{\varphi_2}{\varphi_1}, ..., \frac{\varphi_n}{\varphi_1}]\!]$.

Proof.

$$[I_{i_1}] = \sum k\left(\frac{S_{i_2}}{\varphi_1}\right)^{\alpha_2}\left(\frac{S_{i_3}}{\varphi_1}\right)^{\alpha_3}...\left(\frac{S_{i_{h-1}}}{\varphi_1}\right)^{\alpha_{h-1}} + k[\![T]\!]\frac{S_{i_h}}{\varphi_1}$$

where the sum is over all $\alpha_2, \alpha_3, ..., \alpha_{h-1}$ satisfying

$$\alpha_2(i_2 - i_1) + \alpha_3(i_3 - i_1) + ... + \alpha_{h-1}(i_{h-1} - i_1) < (i_h - i_1)$$

and S_{i_r} is an element of R of order i_r. It is obvious that $k[\![\varphi_1, \frac{\varphi_2}{\varphi_1}, ..., \frac{\varphi_n}{\varphi_1}]\!] \subset [I_{i_1}]$. To prove that $[I_{i_1}] \subset k[\![\varphi_1, \frac{\varphi_2}{\varphi_1}, ..., \frac{\varphi_n}{\varphi_1}]\!]$, it is enough to show $\frac{S_{i_j}}{\varphi_1}$ is contained in $k[\![\varphi_1, \frac{\varphi_2}{\varphi_1}, ..., \frac{\varphi_n}{\varphi_1}]\!]$ for any $i_j \in W(R)$. Since S_{i_j} is an element of R, it can written as $S_{i_j} = \sum a_{\alpha_1\alpha_2...\alpha_n}(\varphi_1^{\alpha_1}...\varphi_n^{\alpha_n})$ and

$$\frac{S_{i_j}}{\varphi_1} = \sum a_{\alpha_1\alpha_2...\alpha_n}(\varphi_1)^{\alpha_1+\alpha_2+...+\alpha_n-1}\left(\frac{\varphi_2}{\varphi_1}\right)^{\alpha_2}...\left(\frac{\varphi_n}{\varphi_1}\right)^{\alpha_n}$$

Since each summand in the expression of $\frac{S_{i_j}}{\varphi_1}$ is an element of the ring $k[\![\varphi_1, \frac{\varphi_2}{\varphi_1}, ..., \frac{\varphi_n}{\varphi_1}]\!]$, $\frac{S_{i_j}}{\varphi_1}$ is also an element of the same ring for any $i_j \in W(R)$, showing that $[I_{i_1}] \subset k[\![\varphi_1, \frac{\varphi_2}{\varphi_1}, ..., \frac{\varphi_n}{\varphi_1}]\!]$, $\square$

Remark 2.2. Note that $[I_{i_1}]$ is the local ring corresponding to the blow up of the branch of the curve C at the origin.

As a consequence, the parametrization corresponding to R_i can be obtained from the parametrization corresponding to R_{i-1} by doing power series divisions, and these parameterizations can be used to determine F_i's to construct the Arf closure. The problem that should be solved is up to which degree the division series must be expanded so that no information is lost. To solve this problem, we first recall the following important theorem, showing that every power series parametrization of a branch can be interchanged with a polynomial parametrization.

Theorem 2.3. *(See [11].) Let C be a branch with the parametrization $\{\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)\}$. Let c be the conductor of the semigroup of values of C. Then any parametrization $\{\phi_1(t), \phi_2(t), ..., \phi_n(t)\}$ with $\phi_i \equiv \varphi_i \pmod{t^c}$ for $1 \leq i \leq n$ gives the branch C.*

By using this theorem, and observing that the conductor of R_i is smaller than the conductor of R_{i-1} where $R_0 = R$, we can immediately propose a bound for expanding the division series: the conductor c of $W(R)$. Unfortunately, as we have mentioned above, determining c from the parametrization is a problem on its own. We aim to propose a bound without determining the conductor c of $W(R)$. To do this, we first construct a blow up schema, which not only summarizes the construction of the Arf closure of the ring $R = k[\![\varphi_1, ..., \varphi_n]\!]$, but is also essential in the proof of our main theorem.

$$\begin{array}{llll} \textit{Column 1} & \textit{Column 2} & \textit{Column } n & \textit{smallest ordered element} \\ \varphi_1^{(0)}(t), & \varphi_2^{(0)}(t), & \ldots, \varphi_n^{(0)}(t), & \longrightarrow F_0 = t^{a_0} + \text{higher degree terms} \\ \varphi_1^{(1)}(t), & \varphi_2^{(1)}(t), & \ldots, \varphi_n^{(1)}(t), & \longrightarrow F_1 = t^{a_1} + \text{higher degree terms} \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_1^{(l)}(t), & \varphi_2^{(l)}(t), & \ldots, \varphi_n^{(l)}(t), & \longrightarrow F_l = t + \text{higher degree terms} \end{array} \tag{2.1}$$

Here $\varphi_i^{(0)} = \varphi_i$, and

$$\varphi_i^{(j)}(t) = \begin{cases} \varphi_i^{(j-1)}(t), & \text{if } F_{j-1} = \varphi_i^{(j-1)}(t) \\ \frac{\varphi_i^{(j-1)}(t)}{F_{j-1}} - c_{ij}, & \text{if } F_{j-1} \neq \varphi_i^{(j-1)}(t) \end{cases} \tag{2.2}$$

$c_{ij} \in k$ and $c_{ij} \neq 0$ if and only if $\operatorname{ord}(\varphi_i^{(j-1)}(t)) = \operatorname{ord}(F_{j-1})$. Note also that $\operatorname{ord}(F_{l-1}) \geq 2$ with $a_0 \geq a_1 \geq ... \geq a_{l-1} \geq 2$. Then $R_j = k[\![\varphi_1^{(j)}(t), \varphi_2^{(j)}(t), ..., \varphi_s^{(j)}(t)]\!]$ and the Arf closure of R is:

$$R^* = k + kF_0 + kF_0F_1 + ... + kF_0F_1...F_{l-2} + F_0...F_{l-1}k[\![t]\!].$$

Let c^* denote the conductor of the semigroup $W(R^*)$ of the Arf closure. As $W(R^*) = \{0, a_0, a_0 + a_1, ..., a_0 + ... + a_{l-2}, a_0 + ... + a_{l-1} + \mathbb{N}\}$, $c^* = a_0 + ... + a_{l-1}$, and thus $R^* = k + k\overline{F_0} + k\overline{F_0}\,\overline{F_1} + ... + k\overline{F_0}\,\overline{F_1}...\overline{F_{l-2}} + t^{c^*}k[\![t]\!]$, where $\overline{F_j} = F_j \pmod{t^{c^*}}$. Hence, it is enough to find $\overline{F_j}$'s instead of the exact F_j's for constructing the Arf closure R^*. We can now state our main theorem.

Theorem 2.4. *The rings $k[\![\varphi_1, ..., \varphi_n]\!]$ and $k[\![\phi_1, ..., \phi_n]\!]$ have the same Arf closure, where $\phi_j \equiv \varphi_j \pmod{t^{c^*+1}}$ for $1 \le j \le n$.*

Proof. We have to show that we do not lose any significant data by expanding the division series up to degree $c^* + 1$, while determining the series $\varphi_i^{(j)}(t)$. We prove our claim in two parts by considering the constants 'c_{ij}'s and by using the blow up schema (2.1) and the notation there. Note that, if a column in the blow up schema (2.1) does not enter the algorithm, this means that the algorithm is the same without that column. Therefore, losing monomials in that column has no effect on the computation of the Arf closure. Hence, without loss of generalization, we can assume that all of the columns enter the algorithm at least once. Using the schema (2.1) again, this is equivalent to saying that for all i, $\varphi_i^{(j)} = F_j$ for at least one j.

We first consider the case with zero constants. In other words, recalling Eq. (2.2), $c_{ij} = 0$ for all i, j, so that all the important monomials are the smallest ordered terms of $\varphi_i^{(0)}$'s.

In this situation, for all i, $\text{ord}(\varphi_i^{(0)}) \ge \text{ord}(\varphi_i^{(1)}) \ge ... \ge \text{ord}(\varphi_i^{(k)})$ and,

$$\varphi_i^{(j)}(t) = \varphi_i^{(k)}(t) \prod_{l \in I_{ij}} F_l \quad \text{where } I_{ij} = \left\{l : j \le l < k \text{ and } F_l \ne \varphi_i^{(l)}(t)\right\}$$

Then, $\text{ord}(\varphi_i^{(0)}) = \sum_{l \in I_{i0}} a_l + \text{ord}(\varphi_i^{(k)}(t))$. We have observed that $\varphi_i^{(j)}(t) = F_j$ for some j. Therefore, $\sum_{l \in I_{i0}} a_l + \underbrace{\underbrace{\text{ord}(F_j)}_{a_j}}_{\text{ord}(\varphi_i^{(j)})} \le c^*$. Also, since $\text{ord}(\varphi_i^{(k)}) \le \text{ord}(\varphi_i^{(j)})$ for all $j < k$, we have $\text{ord}(\varphi_i^{(0)}) = \sum_{l \in I_{i0}} a_l + \text{ord}(\varphi_i^{(k)}) \le c^*$ for all i. So, expanding the division series up to the degree $(c^* + 1)$ is enough to construct the Arf closure.

If $c_{ij} \ne 0$ for some i and j, we do an induction on the number of j's for which $c_{ij} \ne 0$ for some i.

- $n = 1$ ($c_{ij} \ne 0$ for only one i and for some j)

$$\begin{array}{llll}
\varphi_1^{(0)}(t), \ \dots \ \varphi_i^{(0)}(t), & & \dots \ \varphi_s^{(0)}(t), & \longrightarrow F_0 \\
\vdots \qquad \vdots & & \vdots & \vdots \\
\varphi_1^{(j)}(t), \ \dots \ \varphi_i^{(j)}(t) = \frac{\varphi_i^{(j-1)}(t)}{F_{j-1}} - c_{ij}, & & \dots \ \varphi_s^{(j)}(t), & \longrightarrow F_j \\
\vdots \qquad \vdots & & \vdots & \vdots \\
\varphi_1^{(k)}(t), \ \dots \ \varphi_j^{(k)}(t), & & \dots \ \varphi_s^{(k)}(t), & \longrightarrow F_k = t + \dots
\end{array}$$

Let

$$\varphi_i^{(j)}(t) = b_1 t^{\alpha_1} + b_2 t^{\alpha_2} + \dots = \frac{\varphi_i^{(j-1)}(t)}{F_{j-1}} - c_{ij} \quad (\alpha_1 < \alpha_2). \tag{2.3}$$

It's enough to show that, we don't lose the term t^{α_1} in the steps $0, 1, ..., j-1$ by expanding the division series up to the degree (c^*+1). The reason is that after the j-th step, there are no nonzero c_{ij}'s and from the previous part, we know that $\alpha_1 < a_j + a_{j+1} + \dots + a_k < c^*$. (Note that, without loss of generalization, we can assume that the ith column enters the algorithm at least once after the j-th step.)

Hence, by Eq. (2.3),

$$\varphi_i^{(j-1)}(t) = \left(c_{ij} + b_1 t^{\alpha_1} + b_2 t^{\alpha_2} + \dots\right) F_{j-1}$$

and

$$\varphi_i^{(0)}(t) = \left(c_{ij} + b_1 t^{\alpha_1} + b_2 t^{\alpha_2} + \dots\right) \prod_{l \in \Lambda_{i,0}} F_l$$

where $\Lambda_{i,0} = \{l : 0 \le l \le j-1 \text{ and } F_l \ne \varphi_i^{(l)}(t)\}$. As $\alpha_1 \le a_j + \dots + a_{k-1}$ and $c^* = a_0 + ... + a_{j-1} + a_j + ... + a_{k-1}$, we can say that $\varphi_i^{(0)}(t) \bmod t^{c^*+1}$ contains the term, which gives t^{α_1} in the j-th step. This shows that expanding the division series up to the degree $c^* + 1$ in steps $0, 1, ..., j-1$ guarantees that the term t^{α_1} is obtained in the j-th step.

- Assume the claim is true for a branch having $c_{ij} \ne 0$ for $n-1$ different j's. We take any branch having $c_{ij} \ne 0$ for n different j's.

Let the first constant appears at the i_0-th column, j_0-th step. Then,

$$\begin{array}{llll}
\varphi_1^{(0)}(t), \ \dots \ \varphi_i^{(0)}(t), & & \dots \ \varphi_s^{(0)}(t), & \longrightarrow F_0 \\
\vdots \qquad \vdots & & \vdots & \vdots \\
\varphi_1^{(j_0)}(t), \ \dots \ \varphi_{i_0}^{(j_0)}(t) = \frac{\varphi_{i_0}^{(j_0-1)}(t)}{F_{j_0-1}} - c_{i_0 j_0}, & & \dots \ \varphi_s^{(j_0)}(t), & \longrightarrow F_{j_0} \\
\vdots \qquad \vdots & & \vdots & \vdots \\
\varphi_1^{(k)}(t), \ \dots \ \varphi_j^{(k)}(t), & & \dots \ \varphi_s^{(k)}(t), & \longrightarrow F_k = t + \dots
\end{array}$$

From the induction assumption, for the steps starting with j_0-th one, it is sufficient to expand the division series up to degree $a_{j_0}+a_{j_0+1}+\ldots+a_{k-1}$, since that is the conductor of the Arf closure of the ring $k[\![\varphi_1^{(j_0)},\varphi_2^{(j_0)},\ldots,\varphi_s^{(j_0)}]\!]$. In other words, all the significant monomials that determine the Arf closure have orders less than or equal to $a_{j_0}+\ldots+a_{k-1}$ at j_0-th step. Then, as in the first part of the induction hypothesis, since

$$\varphi_{i_0}^{(j_0)}(t)=b_1t^{\alpha_1}+b_2t^{\alpha_2}+\ldots=\frac{\varphi_{i_0}^{(j_0-1)}(t)}{F_{j_0-1}}-c_{i_0j_0}\quad(\alpha_1<\alpha_2),\tag{2.4}$$

we can write $\varphi_{i_0}^{(0)}$ as:

$$\varphi_{i_0}^{(0)}(t)=\left(c_{ij}+b_1t^{\alpha_1}+b_2t^{\alpha_2}+\ldots\right)\prod_{l\in\Lambda_{i_0,0}}F_l,$$

where $\Lambda_{i_0,0}=\{l:0\le l\le j_0-1\text{ and }F_l\ne\varphi_{i_0}^{(l)}(t)\}$. Then all the important monomials in the first steps have order less than or equal to $\operatorname{ord}(\prod_{l\in\Lambda_{i_0,0}}F_l)=a_0+a_1+\ldots+a_{j_0-1}$ plus $a_{j_0}+a_{j_0+1}+\ldots+a_{k-1}$, which is equal to c^*. Hence, by truncating the division series in mod t^{c^*+1}, all the significant terms to construct the Arf closure are preserved. □

Remark 2.5. We should note that Theorem 2.4 does not say that the parameterizations $\{\varphi_1^{(0)}(t),\varphi_2^{(0)}(t),\ldots,\varphi_s^{(0)}(t)\}$ and $\{\phi_1^{(0)}(t),\phi_2^{(0)}(t),\ldots,\phi_s^{(0)}(t)\}$ (where $\phi_j^{(0)}$ is the truncation of the series $\varphi_j^{(0)}$ in mod t^{c^*+1}) provide the same curve. Expanding the division series up to degree c^*+1 in each blow up does not guarantee to obtain the blow up ring $R_j=k[\![\varphi_1^{(j)}(t),\varphi_2^{(j)}(t),\ldots,\varphi_s^{(j)}(t)]\!]$ exactly, but it guarantees to construct the Arf closure correctly.

Example 2.6. Let us compute the Arf closure of the ring $k[\![t^4,t^6+t^9,t^{14}]\!]$ with $c^*=10$

$$\begin{array}{llll}
\mathbf{R}_0: t^4 & t^6+t^9\ \ t^{14} & \longrightarrow & F_0=t^4\\
\mathbf{R}_1: t^4 & t^2+t^5\ \ t^{10} & \longrightarrow & F_1=t^2+t^5\\
\mathbf{R}_2: t^2-t^5+t^8-t^{11} & t^2+t^5\ \ t^8-t^{11} & \longrightarrow & F_2=t^2+t^5\\
\mathbf{R}_3: -2t^3+3t^6-4t^9 & t^2+t^5\ \ t^6-2t^9 & \longrightarrow & F_3=t^2+t^5\\
\mathbf{R}_4: -2t+5t^4-9t^7+14t^{10} & t^2+t^5\ \ t^4+\ldots & \longrightarrow & F_4=-2t+5t^4-9t^7+14t^{10}
\end{array}$$

The Arf Closure is, $R=k+kt^4+k(t^6+t^9)+kt^8+k[\![t]\!]t^{10}$, and the multiplicity sequence of the corresponding branch is: $(4,2,2,2,1,1)$

We now have a bound for expanding the division series to determine the Arf closure. It is obvious that c^*+1 is a much better bound than c, since c^*+1 is much smaller than c. (Recall that $R\subset R^*$ and $W(R)\subset W(R^*)$. Thus, $c^*\le c$.) But, it looks like as if we are in a vicious circle: We want to determine the Arf closure and the multiplicity sequence, but we need the conductor of the Arf closure for this. Hence, we ask the following question:

'Is there a way to find the conductor of the Arf closure or to give a bound for it without knowing the Arf closure?' To answer that question, we first focus on plane branches, but before that we give the following general remark for all branches.

Remark 2.7. Let C be a branch given with the parametrization $\{\varphi_1(t), ..., \varphi_n(t)\}$ and the multiplicity sequence $a_0, a_1, ..., a_{k-1}, 1, 1....$ Then the conductor c^* of the Arf closure of the ring $R = k[\![\varphi_1(t), ..., \varphi_n(t)]\!]$ is equal to the sum $a_0 + a_1 + ... + a_{k-1}$.

3. A bound for c^*

Let C be a plane algebroid curve with primitive parametrization $\{x(t), y(t)\}$ with $x(t), y(t) \in k[\![t]\!]$, where k is algebraically closed of characteristic 0. With a coordinate change and interchanging x and y if necessary, we can assume that

$$x(t) = t^n \quad \text{and} \quad y(t) = \sum a_i t^i$$

with $\operatorname{ord}(y(t)) > n$ and $a_i \in k$. If $\beta_0 := n$; $\beta_1 :=$ smallest power appearing in $y(t)$, that is not divisible by n; $e_1 := \gcd(\beta_0, \beta_1)$; continuing inductively, $\beta_i :=$ smallest power for which $\gcd(\beta_0, \beta_1, ..., \beta_i) < \gcd(\beta_0, \beta_1, ..., \beta_{i-1})$; $e_i = \gcd(\beta_0, \beta_1, ..., \beta_i)$ and $e_q = 1$, the set $\{\beta_0, \beta_1, ..., \beta_q\}$ is called the **characteristic exponents** of C [4]. $M(n, m)$ denoting the sequence of divisors in the Euclidean algorithm of n and m, the multiplicity sequence of C is

$$M(\beta_0, \beta_1), M(e_1, \beta_2 - \beta_1), M(e_2, \beta_3 - \beta_2), \ldots, M(e_{q-1}, \beta_q - \beta_{q-1}), 1, 1, ... \qquad (3.1)$$

We first give the conductor of the Arf closure of a plane branch in terms of its characteristic exponents.

Theorem 3.1. *Let C be a plane algebroid curve with characteristic exponents $\{\beta_0, \beta_1, ..., \beta_q\}$. Then*

$$c^* = \beta_0 + \beta_q - 1.$$

To prove this theorem, we need two lemmas.

Lemma 3.2. *Let $a_1, a_2, \ldots, a_k$ be natural numbers s.t.* $\gcd(a_1, \ldots, a_k) = 1$. *Define* $b_1 = a_1$, $b_i = \gcd(b_{i-1}, a_i) = \gcd(a_1, \ldots, a_i)$ $(1 < i \le k)$ *inductively. Then,*

$$\gcd(b_{i-1}, a_i - a_{i-1}) = b_i$$

Proof. Since $b_i = \gcd(b_{i-1}, a_i)$ and $b_{i-1} = \gcd(b_{i-2}, a_{i-1})$, we have $b_{i-1} = b_i h$, $a_i = b_i r$, $a_{i-1} = b_{i-1} p$ and $b_{i-2} = b_{i-1} q$, where $\gcd(h, r) = 1$ and $\gcd(p, q) = 1$. Then $\gcd(b_{i-1}, a_i - a_{i-1}) = \gcd(b_i h, b_i(r - hp)) = b_i$ as $\gcd(h, r) = 1$. □

The second lemma is the following, and an equivalent is given in [7].

Lemma 3.3. *(See [7].) Let n and m be two natural numbers, $e = \gcd(n, m)$ and also let d_i be the divisors obtained by applying the Euclidean algorithm to n and m. In this case $\sum_i d_i = n + m - e$.*

Proof of Theorem 3.1. Recalling Remark 2.7 and Eq. (3.1), it is enough to show that the sum $M(\beta_0, \beta_1) + M(e_1, \beta_2 - \beta_1) + M(e_2, \beta_3 - \beta_2) + \ldots + M(e_{q-1}, \beta_q - \beta_{q-1})$ is equal to $\beta_0 + \beta_1 - 1$. By Lemma 3.3, the sum of the multiplicities is $\beta_0 + \beta_1 - \gcd(\beta_0, \beta_1) + \gcd(\beta_0, \beta_1) + \beta_2 - \beta_1 + \gcd(\gcd(\beta_0, \beta_1), \beta_2 - \beta_1) + ... + \gcd(\beta_1, ..., \beta_{q-1}) + \beta_q - \beta_{q-1} - 1$ and by Lemma 3.2, this is equal to $\beta_0 + \beta_q - 1$, which completes the proof. □

Having determined c^* in the plane case, we can give a bound for c^* in the space case by using Theorem 3.1.

Let C be an algebroid curve with $\operatorname{embdim}(C) > 2$ and with local ring $R = k[\![\varphi_1(t), \varphi_2(t), ..., \varphi_s(t)]\!]$, where

$$\begin{aligned}
&\varphi_1(t) = t^{m_{11}} \\
&\varphi_2(t) = a_{21}t^{m_{21}} + a_{22}t^{m_{22}} + ... + a_{2r_2}t^{m_{2r_2}} \\
&\vdots \\
&\varphi_s(t) = a_{s1}t^{m_{s1}} + a_{s2}t^{m_{s2}} + ... + a_{sr_s}t^{m_{sr_s}}.
\end{aligned}$$

$m_{11} \le m_{21} \le .... \le m_{s1}$ and $\gcd(m_{11}, m_{21}, m_{22}, ..., m_{2r_2}, ..., m_{s1}, m_{s2}, ..., m_{sr_s}) = 1$ Then, we can always determine constants $b_2, ..., b_s$ such that in the sum $\varphi(t) = b_2\varphi_2(t) + b_3\varphi_3(t) + ... + b_s\varphi_s(t)$, none of the m_{ij}'s vanish and the greatest common divisor of the powers of the terms of $\varphi(t)$ and $\varphi_1(t)$ is equal to $\gcd(m_{11}, ..., m_{s1}, m_{s2}, ..., m_{sr_s}) = 1$. If we consider the plane curve branch $\tilde{C}$ with the local ring $\tilde{R} = k[\![\varphi_1(t), \varphi(t)]\!]$,

$$\tilde{R} \subset R \Rightarrow \tilde{R}^* \subset R^* \Rightarrow W(\tilde{R}^*) \subset W(R^*) \Rightarrow \tilde{c}^* \ge c^*.$$

That is, c^* is always greater than or equal to the smallest characteristic exponent of $\tilde{C}$ plus greatest characteristic exponent of $\tilde{C}$ minus 1. As the smallest characteristic exponent is equal to m_{11} and the greatest characteristic exponent of $\tilde{C}$ is less than or equal to m_{sr_s}, $c^* \le m_{11} + m_{sr_s} - 1$, and we can state the following theorem, which gives the bound to determine the Arf closure correctly.

Theorem 3.4. *Let $R = k[\![\varphi_1(t), \varphi_2(t), ..., \varphi_s(t)]\!]$ be a branch, where*

$$\begin{aligned}
&\varphi_1(t) = t^{m_{11}} \\
&\varphi_2(t) = a_{21}t^{m_{21}} + a_{22}t^{m_{22}} + ... + a_{2r_2}t^{m_{2r_2}} \\
&\vdots \\
&\varphi_s(t) = a_{s1}t^{m_{s1}} + a_{s2}t^{m_{s2}} + ... + a_{sr_s}t^{m_{sr_s}}
\end{aligned}$$

$m_{11} \leq m_{21} \leq .... \leq m_{s1} \leq ... \leq m_{sr_s}$ and $\gcd(m_{11}, m_{21}, ..., m_{2r_2}, ..., m_{s1}, ..., m_{sr_s}) = 1$. Using the bound $m_{11} + m_{sr_s}$ for the truncation of the division series in each blow up is sufficient to construct the Arf Closure correctly.

Example 3.5. Recall Example 2.6. To compute the Arf closure of the ring $k[\![t^4, t^6+t^9, t^{14}]\!]$, we can now use the bound $4 + 14 = 18$ while expanding the division series and we get:

$\mathbf{R}_0 : t^4$	$t^6 + t^9 \ t^{14}$
$\mathbf{R}_1 : t^4$	$t^2 + t^5 \ t^{10}$
$\mathbf{R}_2 : t^2 - t^5 + t^8 - t^{11} + t^{14} - t^{17}$	$t^2 + t^5 \ t^8 - t^{11} + t^{14} - t^{17}$
$\mathbf{R}_3 : -2t^3 + 3t^6 - 4t^9 + 5t^{12} - 6t^{15} + 6t^{18}$	$t^2 + t^5 \ t^6 - 2t^9 + 8t^{12} - 4t^{15} + 4t^{18}$
$\mathbf{R}_4 : -2t + 5t^4 - 9t^7 + 14t^{10} - 20t^{13} + 26t^{16}$	$t^2 + t^5 \ t^4 + ...$

As $F_0 = t^4$, $F_1 = F_2 = F_3 = t^2 + t^5$ and $F_4 = -2t + 5t^4 - 9t^7 + 14t^{10} - 20t^{13} + 26t^{16}$, the Arf Closure is $R = k + kt^4 + k(t^6 + t^9) + kt^8 + k[\![t]\!]t^{10}$, and the multiplicity sequence of the corresponding branch is: $(4, 2, 2, 2, 1, 1)$. Hence, the results are the same with what we have found in Example 2.6.

4. Hilbert functions of local rings having the same Arf closure

In this section, we present a conjecture of Arslan and Sertöz and give examples supporting this conjecture obtained by using the algorithm given above. First, we characterize the Hilbert function of an Arf ring.

Recall that the Hilbert function $H_R(n)$ of the local ring R with the maximal ideal $\mathfrak{m}$ is defined to be the Hilbert function of the associated graded ring $gr_{\mathfrak{m}}(R) = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. In other words,

$$H_R(n) = H_{gr_{\mathfrak{m}}(R)}(n) = \dim_{R/\mathfrak{m}}\left(\mathfrak{m}^n/\mathfrak{m}^{n+1}\right), \quad n \geq 0.$$

The Hilbert series of R is defined to be

$$HS_R(t) = \sum_{n \in \mathbb{N}} H_R(n)t^n.$$

It has been proved by Hilbert and Serre that, $HS_R(t) = \frac{h(t)}{(1-t)^d}$, where $h(t)$ is a polynomial with coefficients from $\mathbb{Z}$, $h(1)$ is the multiplicity of R and d is the Krull dimension of R. It is also known that there is a polynomial $P_R(n) \in \mathbb{Q}[n]$ called the Hilbert polynomial of R such that $H_R(n) = P_R(n)$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. The smallest n_0 satisfying this condition is the regularity index of the Hilbert function of R. We first show that the regularity index of the Hilbert function of an Arf ring is 1.

Theorem 4.1. *(See [21, Theorems 1 and 2].) Let R be a local Cohen–Macaulay ring of dimension d and multiplicity e. Then*

$$\operatorname{embdim}(R) \leq e - d + 1,$$

and if there is equality (R has maximal embedding dimension), then the associated graded ring of R is Cohen–Macaulay.

Theorem 4.2. *(See [19, Theorem 2.2].) An Arf ring has maximal embedding dimension.*

Corollary 4.3. *The associated graded rings of Arf rings are Cohen Macaulay.*

Proof. This is a direct consequence of Theorems 4.1 and 4.2, as $d = 1$ for Arf rings. □

We have $\dim(R) = \dim gr(R)$ and $\operatorname{embdim}(R) = \operatorname{embdim}(gr(R))$. For a graded ring G and a nonzero divisor $x \in G$ of degree 1, we have $\operatorname{embdim}(G/x) = \operatorname{embdim}(G) - 1$, $e(G/x) = e(G)$, and $\dim G/x = \dim G - 1$. Hence, G has maximal embedding dimension if and only if G/x has maximal embedding dimension. Furthermore, $HS_G(t) = \frac{HS_{G/x}(t)}{1-t}$. (Also, note that to guarantee the existence of a nonzero divisor of degree 1, the field has to be infinite, and there is standard trick for extending the field.) A 0-dimensional graded ring of maximal embedding dimension and multiplicity e has Hilbert series $1+(e-1)t$, so a 1-dimensional graded ring of maximal embedding dimension has Hilbert series $\frac{1+(e-1)t}{(1-t)}$.

As a consequence, we can state the next theorem.

Theorem 4.4. *Let R be a local ring and R^* its Arf Closure. Then the Hilbert series of R^* is:*

$$P_{R^*}(t) = \frac{1 + (e-1)t}{1-t}.$$

We have shown that the regularity index of the Hilbert function of an Arf ring is 1. This shows that, although an Arf ring is not generally regular, it is very close to being regular, so we have the following question: Can we interpret the Arf closure as a specific way of taming the singularity? With this question in mind and recalling that the Arf closure of a ring is obtained by enlarging the ring with the addition of new elements in a certain manner, we can try to understand the effect of adding an element on the regularity index of the Hilbert function. The following conjecture due to Arslan and Sertöz says that, while constructing the Arf closure, the addition of a new element results with a ring having a Hilbert function with a smaller or an equal regularity index:

Conjecture 4.5. *If R_1 and R_2 are two local rings having the same Arf closure with $R_1 \subset R_2$ and $P_{R_1}(t) = \frac{h_1(t)}{1-t}$, $P_{R_2}(t) = \frac{h_2(t)}{1-t}$, then we have*

$$\operatorname{degree}(h_1) \geq \operatorname{degree}(h_2).$$

Note that the regularity indices of R_1 and R_2 are $\operatorname{degree}(h_1)$ and $\operatorname{degree}(h_2)$. Moreover, the claim of the conjecture is not true for two arbitrary local rings, one of which contains the other:

Example 4.6. Consider the rings $R_1 = k[\![t^{10}, t^{15}, t^{17}, t^{18}]\!]$ and $R_2 = k[\![t^{10}, t^{11}, t^{15}, t^{17}, t^{18}]\!]$, which do not have the same Arf closure. Although $R_1 \subset R_2$, we have $P_{R_1}(t) = \frac{1+3t+4t^2+2t^3}{1-t}$ and $P_{R_2}(t) = \frac{1+4t+4t^2+t^4}{1-t}$.

Lastly, we give some examples supporting the conjecture. The next table presents rings having the Arf closure

$$k[\![t^{12}, t^{18}, t^{25}, t^{26}, t^{27}, t^{28}, t^{29}, t^{31}, t^{32}, t^{33}, t^{34}, t^{35}]\!]$$

and their Hilbert series. Observe that while getting closer to the Arf closure, the degrees of the $h(t)$'s, and so the regularity indices never increase.

Rings	Hilbert Series
$k[\![t^{12}, t^{18}, t^{25}, t^{26}]\!]$	$1 + 3t + 4t^2 + 3t^3 + t^4$
$k[\![t^{12}, t^{18}, t^{25}, t^{26}, t^{27}]\!]$	$1 + 4t + 5t^2 + 2t^3$
$k[\![t^{12}, t^{18}, t^{25}, t^{26}, t^{27}, t^{28}]\!]$	$1 + 5t + 5t^2 + t^3$
$k[\![t^{12}, t^{18}, t^{25}, t^{26}, t^{27}, t^{28}, t^{29}]\!]$	$1 + 6t + 5t^2$
$k[\![t^{12}, t^{18}, t^{25}, t^{26}, t^{27}, t^{28}, t^{29}, t^{31}]\!]$	$1 + 7t + 4t^2$
$k[\![t^{12}, t^{18}, t^{25}, t^{26}, t^{27}, t^{28}, t^{29}, t^{31}, t^{32}]\!]$	$1 + 8t + 3t^2$
$k[\![t^{12}, t^{18}, t^{25}, t^{26}, t^{27}, t^{28}, t^{29}, t^{31}, t^{32}, t^{33}]\!]$	$1 + 9t + 2t^2$
$k[\![t^{12}, t^{18}, t^{25}, t^{26}, t^{27}, t^{28}, t^{29}, t^{31}, t^{32}, t^{33}, t^{34}]\!]$	$1 + 10t + t^2$
$k[\![t^{12}, t^{18}, t^{25}, t^{26}, t^{27}, t^{28}, t^{29}, t^{31}, t^{32}, t^{33}, t^{34}, t^{35}]\!]$	$1 + 11t$

The next table presents rings having the Arf closure

$$k[\![t^{12}, t^{16} + t^{30}, t^{20}, t^{31}, t^{33}, t^{34}, t^{35}, t^{37}, t^{38}, t^{39}, t^{41}, t^{42}]\!].$$

Rings	Hilbert Series
$k[\![t^{12}, t^{16} + t^{30}, t^{31}]\!]$	$1 + 2t + 2t^2 + 2t^3 + 2t^4 + 2t^5 + t^6$
$k[\![t^{12}, t^{16} + t^{30}, t^{20}, t^{31}]\!]$	$1 + 3t + 5t^2 + 3t^3$
$k[\![t^{12}, t^{16} + t^{30}, t^{20}, t^{31}, t^{33}]\!]$	$1 + 4t + 7t^2$
$k[\![t^{12}, t^{16} + t^{30}, t^{20}, t^{31}, t^{33}, t^{34}, t^{35}]\!]$	$1 + 6t + 5t^2$
$k[\![t^{12}, t^{16} + t^{30}, t^{20}, t^{31}, t^{33}, t^{34}, t^{35}, t^{37}, t^{38}, t^{39}, t^{41}, t^{42}]\!]$	$1 + 11t$

(Here, the Arf closure computations are done by using the SINGULAR [17] library "ArfClosure.lib", which you can find in [23]. The library uses the Arf construction algorithm given above.)

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