equation $\dot{x}_1 = u_1$ and the expression of the control $u_1(x, t)$ (see [10], for example). ■

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Fixed Zeros of Decentralized Control Systems

Konur A. Ünyelioğlu, Ümit Özgüner, and A. Bülent Özgüler

***Abstract*—This paper considers the notion of *decentralized fixed zeros* for linear, time-invariant, finite-dimensional systems. For an N-channel plant that is free of unstable decentralized fixed modes, an unstable decentralized fixed zero of Channel i $(1 \leq i \leq N)$ is defined as an element of the closed right half-plane, which remains as a blocking zero of that channel under the application of every set of $N-1$ controllers around the other channels, which make the resulting single-channel system stabilizable and detectable. This paper gives a complete characterization of unstable decentralized fixed zeros in terms of system-invariant zeros.**

***Index Terms*—Decentralized control, fixed zeros, linear systems, stabilization.**

I. INTRODUCTION

The main objective of this paper is to give a definition and a characterization of unstable decentralized fixed zeros of a linear, time-invariant, finite-dimensional plant.

Consider the N-channel decentralized plant Z in Fig. 1, which is assumed to be free of unstable decentralized fixed modes [13]. Let $i \in \{1, \cdots, N\}$ be fixed. Assume, without loss of generality, $i = 1$. Let the closed-loop transfer matrix between u_1 and y_1 be denoted by $\hat{Z}_{11}$, where the dependence of $\hat{Z}_{11}$ on the controllers $Z_{c2}, \cdots, Z_{cN}$ is suppressed for simplicity.

An unstable decentralized fixed zero of Channel 1 is defined as an element of the closed right half-plane, which remains as a blocking zero [2], [3] of $\hat{Z}_{11}$ for the application of every collection of $N-1$ local controllers $Z_{c2}, \cdots, Z_{cN}$, which yield that the partially closed-loop system is stabilizable and detectable around Channel 1.

Decentralized fixed zeros deserve attention because of the performance limitations they impose on various sensitivity minimization problems, which can be explained by referring to Figs. 2 and 3, where $Z_{c1}, \cdots, Z_{cN}$ are local controllers to achieve two objectives: 1) closed-loop stability and 2) minimization of the H_∞ norm of the transfer matrix between w and z in Fig. 2.

In Fig. 2, the signal w is a noise affecting the first channel observation. In Fig. 3, the signal r is a reference signal to be tracked by the first channel output y_1. The transfer matrix between r and the error signal e is identical to the one between w and z in Fig. 2. It is easy to compute the transfer matrix between w and z (or the *sensitivity function around Channel 1*) equals $S := (I + \hat{Z}_{11} Z_{c1})^{-1}$. Let $Z_{c1}, Z_{c2}, \cdots, Z_{cN}$ be any collection of local controllers satisfying the closed-loop stability. From [8, Remark and Theorem 3.2] (see also Lemma 2 in the next section), the controllers $Z_{c2}, \cdots, Z_{cN}$ yield that the closed-loop system is stabilizable and detectable around Channel 1 in the partially closed-loop configuration of Fig. 1. Then, observe, at each unstable decentralized fixed zero s_0 of Channel 1, $\|S(s_0)\| = 1$, regardless of the controllers chosen. In other words, 1) the sensitivity of the closed-loop

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K. A. Ünyelioğlu was with the Department of Electrical Engineering, The Ohio State University, Columbus, OH 43210-1272 USA.

Ü. Özgüner is with the Department of Electrical Engineering, The Ohio State University, Columbus, OH 43210-1272 USA.

A. B. Özgüler is with the Department of Electrical and Electronics Engineering, Bilkent University, Ankara, 06533 Turkey (e-mail: ozguler@ee.bilkent.edu.tr).

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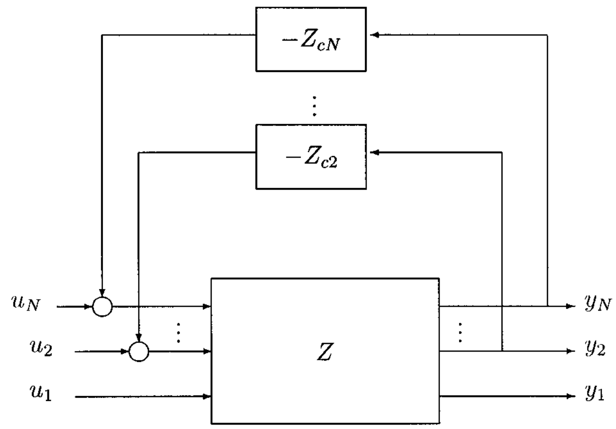

Fig. 1. Partially closed-loop system.

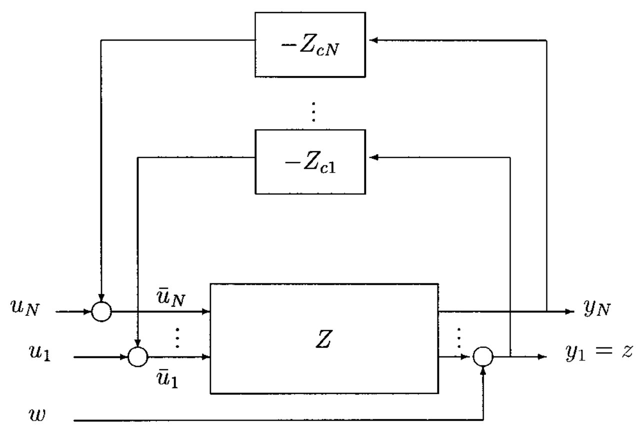

Fig. 2. Disturbance attenuation.

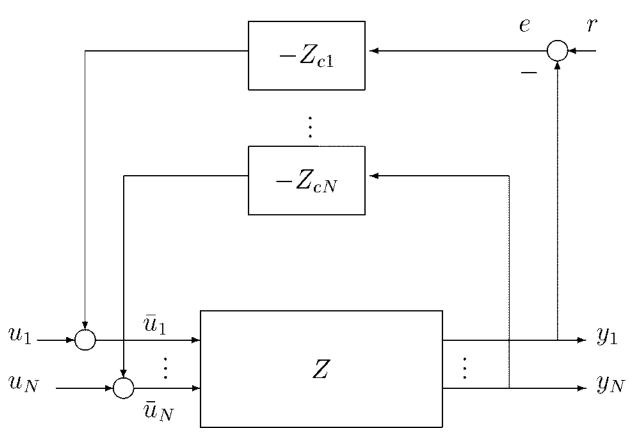

Fig. 3. Reference tracking.

system against the disturbance signals affecting the first channel measurement, and 2) the tracking error with respect to the reference signals to be followed by the first channel output cannot be minimized at those frequencies matching the decentralized fixed zeros of Channel 1.

The rest of the paper is organized as follows. Section II includes the notation, terminology, and the definitions of certain mathematical concepts. Section III gives a precise definition of the concept of decentralized fixed zeros and provides their characterization in terms of the invariant zeros of certain subsystems. Section IV is devoted to some concluding remarks. The Appendix contains the proof of the main result.

II. Notation and Preliminaries

Let $\mathcal{C}$ denote the field of complex numbers. We let $\mathrm{Re}(s)$ denote the real part of $s \in \mathcal{C}$ and define $\mathcal{C}_+ = \{s \in \mathcal{C} | \mathrm{Re}(s) \geq 0\}$, and $\mathcal{C}_{+e} = \mathcal{C}_+ \cup \{\infty\}$. The set of proper real rational functions in the indeterminate s is denoted by $\boldsymbol{P}$ and the set of stable proper real rational functions of s by $\boldsymbol{S}$. The set $\boldsymbol{P}_s$ denotes the set of real rational functions whose denominator polynomials have no roots in $\mathcal{C}_+$. In other words, $\boldsymbol{P}_s$ is the set of stable (but not necessarily proper) rational functions. By I_r, we denote the identity matrix of size r and, by $0_{r\times t}$, the zero matrix with r rows and t columns. The subscript is dropped if the size is clear from the context. The transpose of a matrix B is denoted by B'. Let A be a matrix over ring $\mathcal{C}$ or ring $\boldsymbol{P}$. Then, the notation $A = 0$ is equivalent to saying A is identically zero; i.e., every entry of A is the zero element of the associated ring. If A is over $\boldsymbol{P}$, $\mathrm{rank}\, A$ is the rank of A over $\boldsymbol{P}$ and $\mathrm{rank}\, A(s)$ is the rank of $A(s)$ over $\mathcal{C}$, where $s \in \mathcal{C}_+$ is such that it is not a pole of A.

Let $y = Zu$ and $y_c = Z_c u_c$ be the transfer matrix representations of a plant and a compensator, respectively, where $Z \in \boldsymbol{P}^{p\times r}$ and $Z_c \in \boldsymbol{P}^{r\times p}$. The plant and the compensator are interconnected according to the rules $u = v_e - y_c$, $u_c = v_{ce} + y$, where v_e and v_{ce} denote some external inputs to the closed-loop system. The closed-loop system is *well defined* if $(I + ZZ_c)$ is nonsingular and $(I + ZZ_c)^{-1}$ is over $\boldsymbol{P}$, in which case the transfer matrix description for the closed-loop system is $[y' \quad y_c']' = G[v_e' \quad v_{ce}']'$, where

$$G := \begin{bmatrix} Z - ZZ_c(I+ZZ_c)^{-1}Z & -ZZ_c(I+ZZ_c)^{-1} \\ Z_c(I+ZZ_c)^{-1}Z & Z_c(I+ZZ_c)^{-1} \end{bmatrix}.$$

We say (Z, Z_c) is a *stable pair* if the closed-loop system is well defined and G is a matrix over $\boldsymbol{S}$ [12]. The following statements are equivalent by definition: (Z, Z_c) is a stable pair; Z_c *stabilizes* Z; Z_c *is a stabilizing controller for* Z; and *the closed-loop system associated with the pair* (Z, Z_c) *is stable*. The set of stabilizing controllers of Z will be denoted by $\Sigma[Z]$.

Let a bicoprime fractional representation of Z over $\boldsymbol{S}$ be given by

$$Z = PQ^{-1}R. \tag{1}$$

An element s_0 of $\mathcal{C}_e$ is called a *blocking zero* of $Z \in \boldsymbol{P}^{p\times r}$ if $Z(s_0) = 0$ [2], [3]. An unstable blocking zero can also be characterized via the proper stable Rosenbrock system matrix

$$\Pi := \begin{bmatrix} Q & R \\ -P & 0 \end{bmatrix}$$

associated with a bicoprime fractional representation (1). A number $s_0 \in \mathcal{C}_{+e}$ is an unstable blocking zero of Z if and only if $\mathrm{rank}\, \Pi(s_0) = \mathrm{size}(Q)$. Given a (not necessarily bicoprime) fractional representation (1), a number $s_0 \in \mathcal{C}_{+e}$ is called an *unstable invariant zero* associated with the lth invariant factor of Π (or of the system (P, Q, R)) if $\mathrm{rank}\, \Pi(s_0) \leq l - 1$. Now, let $Z_c = P_c Q_c^{-1}$ be a right coprime fractional representation of Z_c over $\boldsymbol{S}$. Then, (Z, Z_c) is a stable pair if and only if the matrix

$$\begin{bmatrix} Q & RP_c \\ -P & Q_c \end{bmatrix} \tag{2}$$

is unimodular over $\boldsymbol{S}$ [1] or, equivalently, invertible over $\boldsymbol{S}$.

We denote by $\mathcal{N}$ the ordered set of integers $\{1, 2, \cdots, N\}$. Let $Z = [Z_{ij}]$, $Z_{ij} \in \boldsymbol{P}^{p_i \times r_j}$, $i, j \in \mathcal{N}$, be an N-channel plant. *Decentralized stabilization problem* (DSP) is defined as determining a controller $Z_c = \mathrm{diag}\{Z_{c1}, \cdots, Z_{cN}\}$, where $Z_{ci} \in \boldsymbol{P}^{r_i \times p_i}$, $i \in \mathcal{N}$, such that (Z, Z_c) is stable. If such a Z_c exists, we say Z_c solves DSP for Z.

By definition, this is equivalent to saying Z_c is a decentralized stabilizing controller for Z. Let the matrices P and R in (1) be partitioned as $P = [P_1' \quad \cdots \quad P_N']'$ and $R = [R_1 \quad \cdots \quad R_N]$, where $P_i Q^{-1} R_j = Z_{ij}$. DSP for Z is solvable if and only if Z has no *unstable decentralized fixed modes* [13]. An equivalent solvability condition can be given in terms of the fractional representation above as follows. For a proper subset $\mathcal{L}$ of $\mathcal{N}$, define $\mathcal{N} - \mathcal{L}$ to be the complement of $\mathcal{L}$ in $\mathcal{N}$. For a set $\mathcal{K}$ of positive indexes, $R_\mathcal{K}$ denotes the submatrix of R consisting of R_i's with indexes in $\mathcal{K}$. $P_\mathcal{K}$ is defined similarly.

Lemma 1: DSP is solvable if and only if for every proper subset $\mathcal{L}$ of $\mathcal{N}$, [8], [5, Ch. 4], it holds that

$$\operatorname{rank}\begin{bmatrix} Q & R_\mathcal{L} \\ -P_{\mathcal{N}-\mathcal{L}} & 0 \end{bmatrix}(s) \geq \operatorname{size}(Q), \qquad \forall s \in \mathcal{C}_+. \tag{3}$$

For all other undefined terminology and notation pertaining to the algebraic and topological structure of the ring $\boldsymbol{S}$ and for matrices over $\boldsymbol{S}$, we refer the reader to [7], [11], and [12].

III. Decentralized Fixed Zeros

Let Z be the transfer matrix of an N-channel system $(N > 1)$, so it is in the partitioned form $Z = [Z_{ij}]$, where $Z_{ij} \in \boldsymbol{P}^{p_i \times r_j}$, $i, j \in \mathcal{N}$ such that $\sum_{i=1}^{N} p_i = p$ and $\sum_{i=1}^{N} r_i = r$. Let a bicoprime fractional representation of Z over $\boldsymbol{S}$ be given by

$$Z = [\, P_1' \quad \cdots \quad P_N' \,]' \, Q^{-1} \, [\, R_1 \quad \cdots \quad R_N \,] \tag{4}$$

for some $P_i \in \boldsymbol{S}^{p_i \times q}$, $R_i \in \boldsymbol{S}^{q \times r_i}$, $i = 1, \cdots, N$, and $Q \in \boldsymbol{S}^{q \times q}$, so $Z_{ij} = P_i Q^{-1} R_j$, $i, j = 1, \cdots, N$. For each $i \in \boldsymbol{N}$, define the matrix shown at the bottom of the page, where $P_{cj} Q_{cj}^{-1} = Z_{cj}$, $j = 1, \cdots, N$, $j \neq i$, are coprime representations over $\boldsymbol{S}$. If the controllers Z_{cj}, $j = 1, \cdots, N$, $j \neq i$, are such that the representation above is bicoprime. Then, it is said the transfer matrix $\Phi_i(Z_{c1}, \cdots, Z_{c(i-1)}, Z_{c(i+1)}, \cdots, Z_{cN})$ is stabilizable and detectable around Channel i [7, Ch. 7]. In other words, $\mathcal{Z}_{ci}$ is the set of all controllers, which, when applied around the Channels $1, \cdots, i-1, i+1, \cdots, N$, make the resulting single-channel system around Channel i stabilizable and detectable. A relation between $\mathcal{Z}_{ci}$ and the set of decentralized stabilizing controllers of Z is constructed by the following lemma, a proof of that can be obtained via [8, Remark and Theorem 3.2].

Lemma 2: For any $\operatorname{diag}\{Z_{c1}, \cdots, Z_{cN}\}$ solving DSP for Z, $(Z_{c1}, \cdots, Z_{c(i-1)}, \; Z_{c(i+1)}, \cdots, Z_{cN}) \in \mathcal{Z}_{ci}$, for all $i \in \mathcal{N}$. Conversely, for a fixed $i \in \mathcal{N}$, consider any $(Z_{c1}, \cdots, Z_{c(i-1)}, \; Z_{c(i+1)}, \cdots, Z_{cN}) \in \mathcal{Z}_{ci}$. Then, Z_{ci} exists such that $\operatorname{diag}\{Z_{c1}, \cdots, Z_{c(i-1)}, Z_{ci}, Z_{c(i+1)}, \cdots, Z_{cN}\}$ solves DSP for Z.

Let $i \in \mathcal{N}$ be fixed. A number $s_0 \in \mathcal{C}_{+e}$ is called an unstable *decentralized fixed zero of Channel* i of the N-channel system Z if s_0 is a blocking zero of $\Phi_i(Z_{c1}, \cdots, Z_{c(i-1)}, Z_{c(i+1)}, \cdots, Z_{cN})$ for every element $(Z_{c1}, \cdots, Z_{c(i-1)}, \; Z_{c(i+1)}, \cdots, Z_{cN})$ of $\mathcal{Z}_{ci}$. That is, s_0 is called an unstable decentralized fixed zero of Channel i of Z, if s_0 appears as a blocking zero of Channel i in the partially closed-loop system resulting from the application of every $N-1$ local controllers around the other channels, which yield that the single-channel system around Channel i is stabilizable and detectable. For some local controllers in $\mathcal{Z}_{ci}$, an element s_0 of $\mathcal{C}_{+e}$ can appear as a blocking zero at Channel i in the partially closed-loop system, regardless of whether s_0 is a decentralized fixed zero. If s_0, however, is not a decentralized fixed zero, it can always be removed by the application of some other local controllers in $\mathcal{Z}_{ci}$.

The following theorem is the main result of this paper and gives an explicit characterization of unstable decentralized fixed zeros. Using the Fuhrmann equivalence over $\boldsymbol{P}_s$ of any two bicoprime fractional representations of Z [6], the characterization below does not depend on a particular bicoprime representation of Z.

Theorem 1: Let an N-channel transfer matrix $Z = [Z_{ij}]$ have no $\mathcal{C}_+$ decentralized fixed modes and have the bicoprime fractional representation (4). Define $\mathcal{L} = \mathcal{N} - \{i\}$. Let $i \in \mathcal{N}$ be fixed. A number $s_0 \in \mathcal{C}_{+e}$ is an unstable decentralized fixed zero of Channel i of the N-channel system Z if and only if for some subset $\mathcal{K}$ of $\mathcal{L}$ the following holds:

$$\operatorname{rank}\begin{bmatrix} Q & R_i & R_\mathcal{K} \\ -P_i & 0 & 0 \\ -P_{\mathcal{L}-\mathcal{K}} & 0 & 0 \end{bmatrix}(s_0) = q(= \operatorname{size}(Q)). \tag{5}$$

$$\mathcal{Z}_{ci} = \left\{ \begin{array}{l} \left(Z_{c1}, \cdots, Z_{c(i-1)}, Z_{c(i+1)}, \cdots, Z_{cN}\right) \in \boldsymbol{P}^{r_1 \times p_1} \times \cdots \times \boldsymbol{P}^{r_{i-1} \times p_{i-1}} \times \boldsymbol{P}^{r_{i+1} \times p_{i+1}} \times \cdots \times \boldsymbol{P}^{r_N \times p_N} \,| \\ \Phi_i\left(Z_{c1}, \cdots, Z_{c(i-1)}, Z_{c(i+1)}, \cdots, Z_{cN}\right) := \\ [P_i \; 0 \cdots 0 \; 0 \cdots 0] \begin{bmatrix} Q & R_1 P_{c1} & \cdots & R_{i-1} P_{c(i-1)} & R_{i+1} P_{c(i+1)} & \cdots & R_N P_{cN} \\ -P_1 & Q_{c1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -P_{i-1} & 0 & \cdots & Q_{c(i-1)} & 0 & \cdots & 0 \\ -P_{i+1} & 0 & \cdots & 0 & Q_{c(i+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -P_N & 0 & \cdots & 0 & 0 & \cdots & Q_{cN} \end{bmatrix}^{-1} \begin{bmatrix} R_i \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ is bicoprime.} \end{array} \right\}$$

Remark 1: Whenever $\mathcal{K}$ or $\mathcal{L}-\mathcal{K}$ is empty, the corresponding block in (5) does not appear. For instance, when $N = 2$, the set of unstable decentralized fixed zeros of Channel 1 is

$$\left\{ s_0 \in \mathcal{C}_{+e} \,\middle|\, \operatorname{rank}\begin{bmatrix} Q & R_1 & R_2 \\ -P_1 & 0 & 0 \end{bmatrix}(s_0) = q \text{ or } \operatorname{rank}\begin{bmatrix} Q & R_1 \\ -P_1 & 0 \\ -P_2 & 0 \end{bmatrix}(s_0) = q \right\}.$$

Similarly, when $N = 3$, the set of unstable decentralized fixed zeros of Channel 1 is given by

$$\left\{ s_0 \in \mathcal{C}_{+e} \,\middle|\, \operatorname{rank}\begin{bmatrix} Q & R_1 & R_2 & R_3 \\ -P_1 & 0 & 0 & 0 \end{bmatrix}(s_0) = q \text{ or } \operatorname{rank}\begin{bmatrix} Q & R_1 \\ -P_1 & 0 \\ -P_2 & 0 \\ -P_3 & 0 \end{bmatrix}(s_0) = q \text{ or } \operatorname{rank}\begin{bmatrix} Q & R_1 & R_2 \\ -P_1 & 0 & 0 \\ -P_3 & 0 & 0 \end{bmatrix}(s_0) = q \text{ or } \operatorname{rank}\begin{bmatrix} Q & R_1 & R_3 \\ -P_1 & 0 & 0 \\ -P_2 & 0 & 0 \end{bmatrix}(s_0) = q \right\}.$$

Remark 2: The result of the theorem can be equivalently stated as follows. Let Z in (4) be free of unstable decentralized fixed modes. A number $s_0 \in \mathcal{C}_{+e}$ is a decentralized fixed zero of Channel i if and only if it is an invariant zero associated with the $q + 1$st invariant factor of one of the subsystems

$$\left(\begin{bmatrix} P_i \\ -P_{\mathcal{L}-\mathcal{K}} \end{bmatrix}, Q, [R_i \quad R_{\mathcal{K}}]\right).$$

Remark 3: The characterization in the theorem has been given, starting with a particular fractional representation as in (1) or (4) of Z. This is only for notational convenience. The result of the theorem extends to the more general bicoprime representation

$$Z = \begin{bmatrix} P_1 \\ \vdots \\ P_N \end{bmatrix} Q^{-1} [R_1 \quad \cdots \quad R_N] + \begin{bmatrix} W_{11} & \cdots & W_{1N} \\ \vdots & & \vdots \\ W_{N1} & \cdots & W_{NN} \end{bmatrix} \tag{6}$$

as follows. A number $s_0 \in \mathcal{C}_{+e}$ is an unstable decentralized fixed zero of Z of Channel i; i.e., it is a blocking zero of any partially closed-loop system obtained by applying local controllers around the channels $1, \cdots, i-1, i+1, \cdots, N$ such that the closed-loop system is stabilizable and detectable and free of unstable decentralized fixed modes, if and only if for some subset $\mathcal{K}$ of $\mathcal{L}$ the following holds:

$$\operatorname{rank}\begin{bmatrix} Q & R_i & R_{\mathcal{K}} \\ -P_i & W_{ii} & W_{\{i\}\mathcal{K}} \\ -P_{\mathcal{L}-\mathcal{K}} & W_{(\mathcal{L}-\mathcal{K})\{i\}} & W_{(\mathcal{L}-\mathcal{K})\mathcal{K}} \end{bmatrix}(s_0) = q (= \operatorname{size}(Q))$$

where $W_{\mathcal{MN}}$ denotes the submatrix of $[W_{ij}]$ in (6) consisting of W_{mn}'s with $m \in \mathcal{M}, n \in \mathcal{N}$. Given a state-space representation $Z = H(sI - F)^{-1}G + J$, a fractional representation of the type (6) can be readily obtained by letting $(P, Q, R, W) := (H/(s+\sigma), (sI - F)/(s+\sigma), G, J)$, where σ is an arbitrary positive real number.

Remark 4: By hypothesis of the theorem, Z has no unstable decentralized fixed modes, which implies

$$\operatorname{rank}\begin{bmatrix} Q & R_i & R_{\mathcal{K}} \\ -P_i & 0 & 0 \\ -P_{\mathcal{L}-\mathcal{K}} & 0 & 0 \end{bmatrix}(s_0) \geq q$$

for any $s_0 \in \mathcal{C}_{+e}$, as by Lemma 1, each matrix above has a submatrix of rank more than q. We can then use "=" and "$\leq$" interchangably in (5).

Remark 5: In [8], a *hierarchically stable design procedure* for decentralized stabilizing controllers has been proposed, where at each step the local compensator can be chosen as a stabilizing compensator of the respective channel in the closed-loop system. Let $s_0 \in \mathcal{C}_{+e}$ not be an unstable decentralized fixed zero of Channel 1, and consider any permutation $\{i_2, \cdots, i_{N-1}, i_N\}$ of $\{2, \cdots, N-1, N\}$. Lemma 4(ii) (Appendix), the proof of [Only If] part of the Theorem (Appendix), and [8, Thm. 4.2] show, in a hierarchically stable design procedure following the order i_N, $i_{N-1}, \cdots, i_2$, 1 (i.e., a local controller is first applied to Channel i_N, then Channel i_{N-1}, etc.) for almost all[1] local compensators stabilizing the respective channel in the partially closed-loop system, s_0 is not a blocking zero of Channel 1. This result is needed in the synthesis of decentralized stabilizing controllers achieving a tracking objective (see Example 4 below).

Examples 1: Consider a 2 × 2 plant

$$Z = \begin{bmatrix} \frac{s-3}{s+1} & \frac{s-2}{s+1} \\ \frac{2(s-3)}{s-1} & \frac{s-2}{s-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{s-1}{s+1} \end{bmatrix}^{-1} \begin{bmatrix} \frac{s-3}{s+1} & \frac{s-2}{s+1} \\ \frac{2(s-3)}{s+1} & \frac{s-2}{s+1} \end{bmatrix}.$$

By the theorem, the only unstable decentralized fixed zero of Channel 1 is 3 and the only unstable decentralized fixed zero of Channel 2 is 2.

Example 2: In this example, we show an unstable decentralized fixed zero can also be a pole of the plant. Consider the following 2 × 2 plant

$$Z = \begin{bmatrix} 0 & 1 \\ 1 & \frac{s+1}{s-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{s-1}{s+1} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ \frac{s-1}{s+1} & 1 \end{bmatrix}.$$

The only unstable decentralized fixed zero of Channel 1 is one, which is also a pole.

Example 3: Consider the stable transfer matrix

$$Z := \begin{bmatrix} \frac{s}{(s+1)^2} & \frac{0.1s}{(s+1)^2} & \frac{0.1s}{(s+1)^2} \\ \frac{1}{(s+1)} & \frac{1}{(s+1)} & \frac{0.1s}{(s+1)^2} \\ \frac{0.1s}{(s+1)^2} & \frac{0.1s}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}.$$

It represents the following input/output relation:

$$[y_1 \quad y_2 \quad y_3]' = Z[\overline{u}_1 \quad \overline{u}_2 \quad \overline{u}_3]'.$$

Assume the objective is to design a decentralized controller consisting of three scalar local controllers Z_{c1}, Z_{c2}, Z_{c3} to guarantee that the output y_1 tracks the step inputs at steady state while maintaining the stability of the system [consider Fig. 3, where $N = 3, \overline{u}_i = -Z_{ci}y_i$, $i = 2, 3, \overline{u}_1 = Z_{c1}(r - y_1), u_i = 0, i = 1, 2, 3$]. Obtain the bicoprime

[1]The term "almost all" is defined with respect to the subspace topology induced by graph topology [7, Ch. 1], [12].

fractional representation (4) of Z over $\boldsymbol{S}$ such that $Q = I_3$, $P_1 = [1\ \ 0\ \ 0]$, $P_2 = [0\ \ 1\ \ 0]$, $P_3 = [0\ \ 0\ \ 1]$, and R_i equals the ith column of Z, $i = 1, 2, 3$. Observe

$$\operatorname{rank}\begin{bmatrix} Q & R_1 & R_2 \\ -P_1 & 0 & 0 \\ -P_3 & 0 & 0 \end{bmatrix}(0) = 3$$

implying that zero is an unstable decentralized fixed zero associated with Channel 1. In other words, no decentralized stabilizing feedback is available to achieve that y_1 tracks the step inputs at steady state.

Example 4: To illustrate the synthesis of a decentralized stabilizing compensator as in Fig. 3, which guarantees the output y_1 tracks the step inputs at steady state while maintaining the stability of the system, suppose in the previous example $Z(1,2)$ is changed to $Z(1,2) = (0.1(s+0.5)/(s+1)^2)$. In this new system, Channel 1 becomes free of unstable decentralized fixed zeros. In this case, a decentralized controller can be designed to achieve the tracking objective as follows. Let Z_{c3} be any controller stabilizing $Z(3,3)$ and apply Z_{c3} to the third control channel of Z. The controller Z_{c3} should satisfy that 1) the resulting two-channel partially closed-loop system, denoted by $\tilde{Z}$, is stabilizable, detectable, and free of unstable decentralized fixed modes, and 2) Channel 1 of $\tilde{Z}$ is devoid of decentralized fixed zeros at the origin. [Even if Z_{c3} does not satisfy both 1) and 2), from Remark 5, any neighborhood of Z_{c3} contains a controller satisfying both 1) and 2). So no loss of generality occurs by assuming Z_{c3} satisfies both 1) and 2).] Now, let Z_{c2} be any controller stabilizing the second channel transfer function of $\tilde{Z}$, and apply Z_{c2} to the second control channel of $\tilde{Z}$. Via Remark 5, we can assume, possibly by slightly perturbing Z_{c2}, the resulting single-channel partially closed-loop system is stabilizable, detectable, and free of blocking zeros at the origin. It is now well known how to design a controller for that single-channel system that achieves stability and the desired tracking objective (see, for example, [4, Ch. 9]).

IV. Conclusions

This paper gives a characterization of unstable decentralized fixed zeros in terms of the plant-invariant zeros. The motivation for studying the decentralized fixed zeros originates from the performance limitations imposed by decentralized feedback structures, especially in the tracking and regulation problems. Because an unstable decentralized fixed zero associated with a particular channel appears as a blocking zero of that channel under any decentralized stabilizing controller, it prescribes a bound beyond which the norm of the sensitivity function cannot be minimized by a stabilizing decentralized controller.

In [11], decentralized blocking zeros that determine the solvability conditions for the decentralized strong stabilization problem have been described in terms of decentralized fixed zeros. For 2 × 2 decentralized systems, the notion of decentralized fixed zeros and its implications on H_∞ sensitivity minimization problem have earlier been studied in [10].

Appendix

The following easy technical result is Lemma A.1 in [9].

Lemma 3: Let $\tilde{D} \in \boldsymbol{S}^{\tilde{p}\times\tilde{r}}$, $\tilde{E} \in \boldsymbol{S}^{\tilde{p}\times\tilde{n}}$, $\tilde{F} \in \boldsymbol{S}^{\tilde{m}\times\tilde{r}}$ and $X_0 \in \boldsymbol{S}^{\tilde{n}\times\tilde{m}}$, where $\tilde{p} \geq 2$, $\tilde{r} \geq 2$. Let q_0 be an integer satisfying $0 < q_0 < \min(\tilde{p}, \tilde{r})$ such that $\operatorname{rank}(\tilde{D} + \tilde{E}X_0\tilde{F}) \geq q$, for all $z \in \mathcal{C}_{+e}$. Then, given $z_0 \in \mathcal{C}_{+e}$, any ball about $X_0 \in \boldsymbol{S}^{\tilde{n}\times\tilde{m}}$ contains a $\overline{X}_0$ for which $\operatorname{rank}(\tilde{D} + \tilde{E}\overline{X}_0\tilde{F})(z_0) > q_0$ if and only if

$$\operatorname{rank}[\tilde{D}\quad \tilde{E}](z_0) > q_0 \quad \text{and} \quad \operatorname{rank}[\tilde{D}'\quad \tilde{F}']'(z_0) > q_0.$$

We need Lemma 4 below in the proof of the Theorem. Lemma 4(i) can be proven using [11, Lemma 6]. The proof of Lemma 4(ii) is based on Lemma 3 and is straightforward.

Lemma 4: Consider $T_i \in \boldsymbol{S}^{t_i\times q}$, $S_i \in \boldsymbol{S}^{q\times s_i}$, $i = 1, 2$, and a biproper $Q_{11} \in \boldsymbol{S}^{q\times q}$ such that $(Q_{11}, [S_1\quad S_2])$ and $(Q_{11}, [T_1'\quad T_2']')$ are left and right coprime, respectively, and the two-channel plant $[T_1'\quad T_2']'Q_{11}^{-1}[S_1\quad S_2]$ has no unstable decentralized fixed modes. Define $Z_{11} = T_1 Q_{11}^{-1} S_1$. Let

$$\mathcal{Z}_c := \left\{ Z_c = P_c Q_c^{-1} \in \boldsymbol{P}^{s_2\times t_2} \text{ for right coprime}(Q_c, P_c) \,\middle|\, [T_2\quad 0]\begin{bmatrix} Q_{11} & S_1P_c \\ -T_1 & Q_c \end{bmatrix}^{-1}\begin{bmatrix} S_2 \\ 0 \end{bmatrix} \text{ is bicoprime} \right\}.$$

i) For any $s_0 \in \mathcal{C}_{+e}$ satisfying

$$\operatorname{rank}\begin{bmatrix} Q_{11} & S_2 & S_1 \\ -T_2 & 0 & 0 \end{bmatrix}(s_0) \leq q \text{ or}$$
$$\operatorname{rank}\begin{bmatrix} Q_{11} & S_2 \\ -T_2 & 0 \\ -T_1 & 0 \end{bmatrix}(s_0) \leq q, \tag{7}$$

it holds that s_0 is a blocking zero of

$$G_{Z_c} := [T_2\quad 0]\begin{bmatrix} Q_{11} & S_1P_c \\ -T_1 & Q_c \end{bmatrix}^{-1}\begin{bmatrix} S_2 \\ 0 \end{bmatrix} \tag{8}$$

for all $Z_c = P_c Q_c^{-1} \in \mathcal{Z}_c$, where the fractional representation of Z_c is coprime.

ii) Let (7) fail for some $s_0 \in \mathcal{C}_{+e}$. Then, for almost all $Z_c \in \Sigma[Z_{11}]$, s_0 is not a blocking zero of G_{Z_c}, where the term "almost all" is defined with respect to the subspace topology induced by graph topology.

Proof of the Theorem: We prove the theorem for the case $N = 3$. The case $N > 3$ can be handled via induction in a straightforward way. **[If]** Assume, for notational simplicity, $i = 1$. Let two coprime fractions $P_{c2}Q_{c2}^{-1}$, $P_{c3}Q_{c3}^{-1}$ over $\boldsymbol{S}$ be such that

$$[P_1\quad 0\quad 0]\begin{bmatrix} Q & R_2P_{c2} & R_3P_{c3} \\ -P_2 & Q_{c2} & 0 \\ -P_3 & 0 & Q_{c3} \end{bmatrix}^{-1}\begin{bmatrix} R_1 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

is bicoprime. It holds that [8, Thm. 3.2]

$$\begin{bmatrix} P_1 & 0 \\ P_2 & 0 \end{bmatrix}\begin{bmatrix} Q & R_3P_{c3} \\ -P_3 & Q_{c3} \end{bmatrix}^{-1}\begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \tag{10}$$

is also bicoprime, and the two-channel system (10) has no $\mathcal{C}_+$ decentralized fixed modes. Let $s_0 \in \mathcal{C}_{+e}$ be such that

$$\operatorname{rank}\begin{bmatrix} Q & R_1 & R_2 & R_3 \\ -P_1 & 0 & 0 & 0 \end{bmatrix}(s_0) \leq q \text{ or}$$
$$\operatorname{rank}\begin{bmatrix} Q & R_1 & R_2 \\ -P_1 & 0 & 0 \\ -P_3 & 0 & 0 \end{bmatrix}(s_0) \leq q. \tag{11}$$

Equation (11) implies

$$\operatorname{rank}\left[\begin{array}{cc|cc} Q & R_3P_{c3} & R_1 & R_2 \\ -P_3 & Q_{c3} & 0 & 0 \\ \hline -P_1 & 0 & 0 & 0 \end{array}\right](s_0) \leq q + p_3. \tag{12}$$

Similarly, if $s_0 \in \mathcal{C}_{+e}$ satisfies

$$\operatorname{rank}\begin{bmatrix} Q & R_1 \\ -P_1 & 0 \\ -P_2 & 0 \\ -P_3 & 0 \end{bmatrix}(s_0) \le q \text{ or}$$

$$\operatorname{rank}\begin{bmatrix} Q & R_1 & R_3 \\ -P_1 & 0 & 0 \\ -P_2 & 0 & 0 \end{bmatrix}(s_0) \le q, \tag{13}$$

then,

$$\operatorname{rank}\left[\begin{array}{cc|c} Q & R_3P_{c3} & R_1 \\ -P_3 & Q_{c3} & 0 \\ \hline -P_2 & 0 & 0 \\ -P_1 & 0 & 0 \end{array}\right](s_0) \le q + p_3. \tag{14}$$

Because the statement holds true for $N = 2$, any $s_0 \in \mathcal{C}_{+e}$ for which (12) or (14) holds is a decentralized fixed zero of Channel 1 of the two-channel system (10). Now, by Lemma 3(i), $s_0 \in \mathcal{C}_{+e}$ is a blocking zero of (9). Because $P_{c2}Q_{c2}^{-1}$, $P_{c3}Q_{c3}^{-1}$ are arbitrary, $s_0 \in \mathcal{C}_{+e}$ is an unstable decentralized fixed zero of Channel 1 of Z. This completes the proof.

[**Only If**] For $N = 2$, the proof follows from Lemma 4(ii). For $N = 3$, let $Z_{c3} = P_{c3}Q_{c3}^{-1} \in \Sigma[P_3Q^{-1}R_3]$ for a right coprime pair of matrices (P_{c3}, Q_{c3}) be such that the fraction in (10) is bicoprime and the two-channel transfer matrix in (10) has no $\mathcal{C}_+$ decentralized fixed modes. Such a Z_{c3} exists via [8, Thm 3.2] and the fact that Z has no $\mathcal{C}_+$ decentralized fixed modes. Let $s_0 \in \mathcal{C}_{+e}$ be such that (11) and (13) both fail. Using Lemma 3, we can perturb P_{c3} and Q_{c3} slightly to $\overline{P}_{c3} = P_{c3} + \Delta_P$ and $\overline{Q}_{c3} = Q_{c3} + \Delta_Q$ to ensure $\overline{P}_{c3}\overline{Q}_{c3}^{-1} = (P_{c3} + \Delta_P)(Q_{c3} + \Delta_Q)^{-1}$ is still a right coprime fraction, $\overline{P}_{c3}\overline{Q}_{c3}^{-1} \in \Sigma[P_3Q^{-1}R_3]$

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} Q & R_3\overline{P}_{c3} & R_1 & R_2 \\ -P_3 & \overline{Q}_{c3} & 0 & 0 \\ -P_1 & 0 & 0 & 0 \end{bmatrix}(s_0) \\
&\quad = \operatorname{rank}\left(\begin{bmatrix} Q & 0 & R_1 & R_2 \\ -P_3 & 0 & 0 & 0 \\ -P_1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} R_3 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \overline{P}_{c3} & 0 & 0 \\ 0 & \overline{Q}_{c3} & 0 & 0 \end{bmatrix}\right) \\
&\quad > q + p_3, \\
&\operatorname{rank}\begin{bmatrix} Q & R_3\overline{P}_{c3} & R_1 \\ -P_3 & \overline{Q}_{c3} & 0 \\ -P_1 & 0 & 0 \\ -P_2 & 0 & 0 \end{bmatrix}(s_0) \\
&\quad = \operatorname{rank}\left(\begin{bmatrix} Q & 0 & R_1 \\ -P_3 & 0 & 0 \\ -P_1 & 0 & 0 \\ -P_2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} R_3 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \overline{P}_{c3} \\ \overline{Q}_{c3} \end{bmatrix}\begin{bmatrix} 0 & I & 0 \end{bmatrix}\right) \\
&\quad > q + p_3,
\end{aligned}$$

and the fractional representation of the two-channel plant

$$\overline{Z} := \begin{bmatrix} P_1 & 0 \\ P_2 & 0 \end{bmatrix}\begin{bmatrix} Q & R_3(P_{c3} + \Delta_P) \\ -P_3 & (Q_{c3} + \Delta_Q) \end{bmatrix}^{-1}\begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}$$

is bicoprime and devoid of unstable decentralized fixed modes. Applying the result for $N = 2$ to $\overline{Z}$, s_0 is not an unstable decentralized fixed zero of Channel 1 of $\overline{Z}$. Consequently, s_0 is not an unstable decentralized fixed zero of Channel 1 of Z. This completes the proof.

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