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A NEW CONSTRUCTION OF RECURSION OPERATORS FOR SYSTEMS OF THE HYDRODYNAMIC TYPE

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We consider a certain class of two-dimensional systems of the hydrodynamic type that contains all examples known to us and can be associated with a class of linear wave equations possessing an algebra of ladder operators. We use this to give a simple construction of recursion operators for these systems, not always coinciding with those of Sheftel and Teshukov.

1. Introduction

We consider systems of the hydrodynamic type in the sense of Dubrovin and Novikov (see [1, 2]). In particular, we consider the recursion operators introduced and discussed in [3, 4]. Sheftel showed [5] that the general recursion operator associated with a general two-dimensional diagonal hydrodynamic system contains two arbitrary functions of a single variable that satisfy a differential constraint. However, when the hydrodynamic system belongs to the class considered in this paper, these arbitrary functions take the specific form of one of three monomials, and there are therefore only three independent recursion operators for a given system. The class we consider is fairly general and contains all examples known to us.

The systems we consider are associated with a generalized Euler–Poisson–Darboux (EPD) *linear wave equation* in the sense that for each such wave equation, we construct a family of commuting hydrodynamic systems and the corresponding triple of recursion operators. Specific systems correspond to particular solutions of this wave equation. It is easy to construct ladder operators of these wave equations with which it is possible to generate hierarchies of connected solutions. The corresponding hydrodynamic systems are then connected through Teshukov-type recursion operators. These can then be used to generate "higher symmetries" by acting with them upon some simple symmetries that are not of the hydrodynamic type.

Most of this paper is concerned with two-dimensional systems for simplicity of exposition, but our formulas are easily extended to higher dimensions (see Sec. 5).

2. A class of two-dimensional systems

In two dimensions, every system of the hydrodynamic type can be diagonalized. Therefore, without loss of generality, we start with the diagonal system

$$q^i_t = v^i(\mathbf{q})\, q^i_x, \quad i = 1, 2. \tag{1}$$

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We can use v^i to define a diagonal metric with covariant components g_{ij} in the usual way [2]:

$$\frac{\partial_j v^i}{v^j - v^i} = \frac{1}{2}\partial_j \log g_{ii}, \quad i \neq j, \tag{2}$$

where $\partial_j \equiv \partial/\partial q^j$.

All the examples known to us satisfy the condition

$$\partial_2 (v^1 \varphi^1) = \partial_1 (v^2 \varphi^2), \tag{3}$$

where $\varphi^i = \varphi^i(q^i)$ are each functions of a single variable. This is equivalent to the condition

$$\partial_2 (\varphi^1 \log g_{11}) + \partial_1 (\varphi^2 \log g_{22}) = 0. \tag{4}$$

These relations imply the existence of functions $V(q^1, q^2)$ and $G(q^1, q^2)$ satisfying

$$v^1 = \frac{\partial_1 V}{\varphi^1}, \qquad v^2 = \frac{\partial_2 V}{\varphi^2}, \tag{5}$$

$$\log g_{11} = \frac{\partial_1 G}{\varphi^1}, \qquad \log g_{22} = -\frac{\partial_2 G}{\varphi^2}. \tag{6}$$

These functions are related by

$$K(\mathbf{q})\, \partial_1\partial_2 V + \frac{1}{\varphi^1}\,\partial_1 V - \frac{1}{\varphi^2}\,\partial_2 V = 0, \tag{7}$$

where $K(\mathbf{q}) = 2/\partial_1\partial_2 G$. (We note that the degenerate case where $\partial_1\partial_2 G = 0$ is excluded because this leads to $\partial_2 v^1 = \partial_1 v^2 = 0$ and Eqs. (1) then decouple.) We thus have *linear wave equation* (7) for V with coefficients depending on the three functions $\varphi^i(q^i)$ and $G(q^1, q^2)$. For specific coefficients, this equation arises in many places in mathematics, such as in the theory of separable Hamiltonian systems [8].

To proceed further, we restrict these functions as follows:

1. We choose φ^i in the form

$$\varphi^i(q^i) = \frac{1}{\alpha_i (q^i)^n}, \quad \alpha_i \in \mathbb{R}. \tag{8}$$

2. We require the form of wave equation (7) to be invariant under the transformation

$$\overline{V}(\bar{q}^1, \bar{q}^2) = V(q^1, q^2)\,(q^1)^{\alpha_2}(q^2)^{\alpha_1}, \quad \bar{q}^i = f_i(q^i). \tag{9}$$

We immediately find the exact form of $K(\mathbf{q})$ to be

$$K_n(q^1, q^2) = q^1 q^2 \left((q^1)^{n-1} - (q^2)^{n-1}\right), \tag{10}$$

where the suffix $n \neq 1$ refers to the power of q^i in (8). The remaining invariance conditions give

$$f_i(q^i) \equiv \bar{q}^i = \frac{1}{q^i}, \quad i = 1, 2, \tag{11}$$

for $n \neq 1$.

These two further restrictions give us the linear wave equation

$$L_n V \equiv q^1 q^2 \big((q^1)^{n-1} - (q^2)^{n-1}\big)\, \partial_1 \partial_2 V + \alpha_1 (q^1)^n\, \partial_1 V - \alpha_2 (q^2)^n\, \partial_2 V = 0, \tag{12}$$

which can be viewed as a generalisation of the EPD linear wave equation [7] (corresponding to $n = 0$ and $\alpha_1 = \alpha_2$). Some properties of the EPD wave equation are given in the appendix.

Remark. Because $V = 1$ is a trivial solution of (12), we can immediately generate a nontrivial solution

$$V = (q^1)^{-\alpha_2}\, (q^2)^{-\alpha_1} \tag{13}$$

from which we can obtain some v_i using Eqs. (5).

Formula (2) allows us to obtain g_{ij} for the *whole* hierarchy by substituting any particular v^i (such as (13)) or by substituting (5) into (2) with (8) and using (12) to obtain

$$\partial_i \log g_{jj} = \frac{2\alpha_j (q^j)^{n-1}}{q^i\big((q^j)^{n-1} - (q^i)^{n-1}\big)},$$

which gives

$$g_{11} = \frac{(q^2)^{2\alpha_1 n/(n-1)} \gamma^1(q^1)}{((q^1)^n q^2 - (q^2)^n q^1)^{2\alpha_1/(n-1)}}, \qquad g_{22} = \frac{(q^1)^{2\alpha_2 n/(n-1)} \gamma^2(q^2)}{((q^2)^n q^1 - (q^1)^n q^2)^{2\alpha_2/(n-1)}}. \tag{14}$$

3. Ladder and recursion operators

We now consider operators that act on a solution of a linear partial differential equation to create a new (or possibly the same) solution. This is just an operator analogue of a symmetry but is called a *ladder operator* in quantum mechanics and special function theory and a *recursion operator* in the theory of integrable equations [9]. In the latter case, we are interested in finding "commuting flows" ("generalized symmetries") of a *nonlinear* equation. Symmetries satisfy the "linearized equation," which yields the appropriate linear operator for this case. In this paper, we use the term "ladder operator" when referring to the EPD equation and "recursion operator" for systems of the hydrodynamic type. We use relation (5) between (12) and (1) to construct recursion operators for the latter from ladder operators for the former.

Let L be a linear partial differential operator and R an operator (generally integral-differential) that is our ladder or recursion operator, in which case it must satisfy the condition

$$[R, L] = k(q^1, q^2) L, \tag{15}$$

where the brackets denote the commutator and $k(q^1, q^2)$ is a function of q^i determined by this relation. This condition guarantees that if $Lf = 0$, then $L(Rf) = 0$.

3.1. The EPD ladder operator. We now consider generalized EPD wave equation (12) and construct the corresponding ladder operators of the form

$$r = \xi^1\, \partial_1 + \xi^2\, \partial_2 + \xi^0, \tag{16}$$

where $\xi^i = \xi^i(q^1, q^2)$, $i = 0, 1, 2$, satisfying (15) with L_n.

Lemma 1. *There are just three nontrivial operators of form* (16) *satisfying* (15)*; these operators are given by*

$$r_1 = (q^1)^{2-n}\,\partial_1 + (q^2)^{2-n}\,\partial_2 + \alpha_2 (q^1)^{1-n} + \alpha_1 (q^2)^{1-n},$$
$$r_2 = q^1\,\partial_1 + q^2\,\partial_2,$$
$$r_3 = (q^1)^n\,\partial_1 + (q^2)^n\,\partial_2$$

up to either multiplying by or adding a constant.

Using this flexibility of multiplication and addition, we can gauge the commutation relations of r_i to those of a standard basis in the algebra $sl(2,\mathbb{C})$:

$$r_+ = \frac{1}{n-1} r_1, \qquad r_0 = \frac{1}{n-1} r_2 + \frac{\alpha_1 + \alpha_2}{2(n-1)}, \qquad r_- = \frac{1}{n-1} r_3, \tag{17}$$

with $n \neq 1$. The operators r_+, r_0, and r_- have the commutation relations

$$[r_+, r_0] = r_+, \qquad [r_0, r_-] = r_-, \qquad [r_+, r_-] = 2r_0.$$

This algebra has the Casimir operator

$$C_n = r_+ r_- + r_- r_+ - 2r_0^2,$$

which is explicitly given by (we recall that $n \neq 1$)

$$C_n = \frac{2}{(n-1)^2}\left((q^2)^{1-n} - (q^1)^{1-n}\right) L_n - \left(n - 1 + \frac{\alpha_1 + \alpha_2}{2}\right)\frac{\alpha_1 + \alpha_2}{(n-1)^2}.$$

3.2. The hydrodynamic recursion operator. We now use relation (5) to construct a recursion operator for (1) corresponding to each of the ladder operators in (17). Our calculation is purely algebraic, and it is thus very simple to write these operators. One of the recursion operators is nonlocal but is calculated algebraically. Furthermore, ladder operator (16) is scalar, whereas the corresponding hydrodynamic recursion operator is matrix. It is obviously much easier to build a matrix object from a *known* scalar object than to calculate the matrix object from basic definition (15).

We first recall that a general ladder operator for wave equation (12) has form (16). Using relations (5), we can draw the diagram

$$\begin{array}{ccc} V & \xrightarrow{\frac{q^j_x}{\varphi^j}\partial_j} & v_j q^j_x \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{R}^i_j} \\ rV & \xrightarrow{\frac{q^i_x}{\varphi^i}\partial_i} & \mathcal{R}^i_j(v_j q^j_x) \end{array}, \tag{18}$$

which commutes only if $\mathcal{R}$ is a recursion operator. To find the components $\mathcal{R}^i_j$, we need only solve the algebraic system of equations

$$\sum_{j=1}^{2} \mathcal{R}^i_j\left(\frac{q^j_x}{\varphi^j}\partial_j V\right) = \frac{q^i_x}{\varphi^i}\partial_i(rV), \quad i = 1, 2. \tag{19}$$

When solving Eqs. (19), we use (12) to eliminate the mixed derivatives of V. We find that the recursion operators must be first-order differential and that they necessarily include a nonlocal term if $\partial_i\xi^0 \neq 0$.

For general r and φ^i, we can solve Eqs. (19) to find the following class of recursion operators:

$$\begin{aligned}
\mathcal{R}^1_1 &= \xi^1 D_x \frac{1}{q^1_x} + \xi^1 \frac{\partial_1\varphi^1}{\varphi^1} + \frac{\xi^1 q^2_x - \xi^2 q^1_x}{q^1_x K_n \varphi^1} + \partial_1\xi^1 + \xi^0 + \partial_1\xi^0 \frac{q^1_x}{\varphi^1} D_x^{-1}\varphi^1, \\
\mathcal{R}^1_2 &= \frac{\xi^2 q^1_x - \xi^1 q^2_x}{q^2_x K_n \varphi^1} + \partial_1\xi^0 \frac{q^1_x}{\varphi^1} D_x^{-1}\varphi^2, \\
\mathcal{R}^2_1 &= \frac{\xi^2 q^1_x - \xi^1 q^2_x}{q^1_x K_n \varphi^2} + \partial_2\xi^0 \frac{q^2_x}{\varphi^2} D_x^{-1}\varphi^1, \\
\mathcal{R}^2_2 &= \xi^2 D_x \frac{1}{q^2_x} + \xi^2 \frac{\partial_2\varphi^2}{\varphi^2} + \frac{\xi^1 q^2_x - \xi^2 q^1_x}{q^2_x K_n \varphi^2} + \partial_2\xi^2 + \xi^0 + \partial_2\xi^0 \frac{q^2_x}{\varphi^2} D_x^{-1}\varphi^2.
\end{aligned}$$

As shown above (see Sec. 3.1), generalized EPD wave equation (12) admits only three ladder operators, r_-, r_0, and r_+, giving rise to three independent functions ξ^i (for a given $n \neq 1$), which in turn yield a three-parameter family of recursion operators $\{\mathcal{R}_-, \mathcal{R}_0, \mathcal{R}_+\}$. These operators satisfy (15) with L being the matrix linear differential operator defined by the right-hand side of

$$\eta^i_t = v^i \eta^i_x + \sum_j q^i_x \frac{\partial v^i}{\partial q^j} \eta^j.$$

In this formula, η^i represent the components of the symmetry,

$$q^i_\tau = \eta^i(\mathbf{q}, \mathbf{q}_x, \dots),$$

which is not necessarily of the hydrodynamic type. The operators $\mathcal{R}_-$ and $\mathcal{R}_0$ are purely differential, but $\mathcal{R}_+$ has a nonzero nonlocal term. Fixing n (and the parameters α_i) means fixing a hydrodynamic *hierarchy*, not a single system. We also recall that the nontrivial solution

$$V = (q^1)^{-\alpha_2}(q^2)^{-\alpha_1},$$

generated from the trivial solution $V = 1$, is independent of n.

3.3. The Sheftel–Teshukov recursion operators. Recursion operators for diagonal systems of the hydrodynamic type were previously studied by Sheftel [3, 5] and Teshukov [4], who considered recursion operators of the form $A_1 D_x + A_0$ for functions A_i of $\mathbf{q}, \mathbf{q}_x, \dots$. These recursion operators are all calculated by solving the recursion-operator equation directly. Teshukov showed that an n-dimensional semi-Hamiltonian system of the hydrodynamic type admits a recursion operator of the form

$$R^i_j = \left(\delta^i_j C^i D_x + \Gamma^i_{ij}(q^i_x C^j - q^j_x C^i) + \delta^i_j C^j \sum_k q^k_x \Gamma^i_{ik}\right) \frac{1}{q^j_x}, \tag{21}$$

where $\Gamma^i_{ij} = (1/2)(\partial_j \log g_{ii})$ for $i \neq j$ are the usual Christoffel connection coefficients of some metric and each $C^i = C^i(q^i)$ is a function of single variable. The constraint to be imposed on R^i_j can be derived by demanding that (21) satisfy the recursion-operator equation. This is equivalent to requiring that (21) map (1) onto a *commuting* hydrodynamic flow. Acting on (1) with operator (21) gives another *diagonal* hydrodynamic system $q^i_t = w^i q^i_x$ with

$$w^i = \sum_j C^j(\partial_j v^i + \Gamma^i_{ij} v^i).$$

The diagonality follows from the expression of Γ^i_{ij} in terms of v^i. Two hydrodynamic systems $q^i_{t_1} = v^i q^i_x$ and $q^i_{t_2} = w^i q^i_x$ for $i = 1, 2, \ldots, k$ are symmetries of each other if and only if

$$\frac{\partial_j v^i}{v^j - v^i} = \frac{\partial_j w^i}{w^j - w^i}, \quad i \neq j, \quad i, j = 1, 2, \ldots, k. \tag{22}$$

For metric (14), this condition (for $k = 2$) leads to the conditions on $C^i(q^i)$ and γ^i

$$\begin{aligned}(\dot{C}^1 + \dot{C}^2) q^1 q^2 \big((q^1)^{n-1} - (q^2)^{n-1}\big) + n(C^2 q^1 - C^1 q^2)\big((q^1)^{n-1} + (q^2)^{n-1}\big) + \\ + 2C^1 (q^2)^n - 2C^2 (q^1)^n = 0,\end{aligned} \tag{23}$$

$$\begin{aligned}(n-1)\left(nC^1 q^2 - nC^2 q^1 + q^1 q^2 \left(\dot{C}^2 - \dot{C}^1 + C^1 \frac{\dot{\gamma}^1}{\gamma^1} - C^2 \frac{\dot{\gamma}^2}{\gamma^2}\right)\right)\big((q^1)^{n-1} - (q^2)^{n-1}\big) + \\ + 2\alpha_2 C^2 q^1 \big((q^1)^{n-1} - n(q^2)^{n-1}\big) + 2\alpha_1 C^1 q^2 \big((q^2)^{n-1} - n(q^1)^{n-1}\big) + \\ + 2\alpha_1 (n-1) C^2 (q^1)^n + 2\alpha_2 (n-1) C^1 (q^2)^n = 0,\end{aligned} \tag{24}$$

where $\dot{C}^i$ and $\dot{\gamma}^i$ are derivatives with respect to the argument of the function. Equation (23) is easy to solve. Differentiating (23) $n+1$ times with respect to either q^1 or q^2 leads to equations that are for only one of the two functions $C^i(q^i)$. We find that C^1 and C^2 are the same function, but of their respective arguments:

$$C^i = S(q^i), \qquad S(x) = \beta_0 x^n + \beta_1 x + \beta_2 x^{2-n}, \quad n \neq 1. \tag{25}$$

Substituting these forms for C^i in (24) leads to equations for the metric functions γ^i, which have the following solutions:

1. if $C^i = (q^i)^n$, then $\gamma^i = \beta^i (q^i)^{2\alpha_i n/(n-1)}$ for $i = 1, 2$,
2. if $C^i = (q^i)^{2-n}$, then $\gamma^i = \beta^i (q^i)^{2(\alpha_i + (n-1)\alpha_j - (n-1)^2)/(n-1)}$ for $j \neq i = 1, 2$, and
3. if $C^i = q^i$, then $\gamma^i = \beta^i (q^i)^{(2\alpha_i + \lambda_i)/(n-1)}$ for $i = 1, 2$,

where β_i and λ are arbitrary constants. This result proves that for the class under consideration, there are exactly three Sheftel–Teshukov-type recursion operators. Two of these operators are the same (up to an additive constant) as constructed via the ladder operators of the EPD equation (with $C^i = \xi^i$ and for these choices of C^i and γ^i). The Sheftel–Teshukov operators contain γ^i in the term Γ^i_{ii}, whereas ours do not. Recursion operators (20) are valid for these same three cases of ξ^i, but for any γ^i. However, this has no consequence for the hydrodynamic systems, because these are independent of γ^i. The nonlocal recursion operator that we found does not belong to the Sheftel–Teshukov class.

3.4. Hierarchies of symmetries. By construction, recursion operators (20) produce diagonal hydrodynamic symmetries when acting on diagonal systems of the hydrodynamic type. The action of $\mathcal{R}$ on a general system of form (1) is given by

$$\sum_j \mathcal{R}^i_j v^j q^j_x = \left[\sum_j \xi^j \partial_j v^i + \left(\partial_i \xi^i + \xi^0 + \xi^i \frac{\partial_i \varphi^i}{\varphi^i}\right) v^i + \frac{\partial_i \xi^0}{\varphi^i} V\right] q^i_x, \tag{26}$$

where V is the solution of (12) corresponding to v^i. This just corresponds to going around three sides of our commutative diagram.

It is also possible to generate *higher* symmetries by starting with a symmetry of a nonhydrodynamic type. For instance, if we choose v^i to be a homogeneous function of q^1 and q^2, system (1) admits a scaling symmetry, which can be written in the evolutionary form as

$$q^i_{t_s} = xq^i_x + atq^i_t - bq^i = xq^i_x + atv^i q^i_x - bq^i, \tag{27}$$

where the constants a and b are the respective scale weights of t and q^i. If the function v^i is of the homogeneous degree m, its scale weight is mb. For the weights of the two sides of Eq. (1) to balance, we must have $a = 1 - mb$. Acting on (27) with any of our recursion operators yields a second-order symmetry of (1). Because the q^i_t term is of the hydrodynamic type, it is just mapped onto one of the hydrodynamic symmetries, which appears additively in the eventual formula. The nonlocal term in $\mathcal{R}_+$ could yield nonlocal symmetries, but this can be avoided by choosing $b = 1/(n-1)$. The resulting higher symmetry generally depends on x *explicitly*. When φ^i has form (8), applying recursion operator (20) to (27) with $b = 1/(n-1)$ gives

$$\begin{aligned} q^i_\tau = \sum_j \mathcal{R}^i_j q^j_{t_s} &= \frac{\xi^i q^i q^i_{xx}}{(n-1)(q^i_x)^2} + 2\xi^i + (\partial_i \xi^i + \xi^0)\frac{q^i}{1-n} + \\ &+ \frac{\alpha_i (q^i)^n}{(1-n)K_n}\left(\frac{\xi^i q^i q^j_x}{q^i_x} + \frac{\xi^j q^j q^i_x}{q^j_x} - (\xi^i q^j + \xi^j q^i)\right) + atw^i q^i_x + \\ &+ \left(\frac{-n\xi^i}{q^i} + \partial_i \xi^i + \xi^0 + \frac{\partial_i \xi^0 q^i}{1-n} + \frac{\alpha_i \partial_i \xi^0}{\alpha_j (1-n)}(q^i)^n (q^j)^{1-n}\right) x q^i_x, \end{aligned} \tag{28}$$

where w^i is the expression in the square brackets in (26).

Neither the Sheftel–Teshukov nor our recursion operators have the *hereditary* property, and the algebra of symmetries is therefore generally non-Abelian. While the hydrodynamic symmetries mutually commute, the higher-order ones generally do not.

4. Examples in two dimensions

We consider some examples of hydrodynamic systems in our class. The simplest cases have $n = 0$ and $\alpha_1 = \alpha_2 = \alpha$. There are many well-known examples even in this subclass. We do not explicitly construct the recursion operators, because these look complicated but are easily obtained by substituting in (20).

4.1. The case $n = 0$. With $\alpha_1 = \alpha_2 = \alpha$, Eq. (12) is the usual EPD linear wave equation,

$$(q^2 - q^1)\partial_1 \partial_2 V + \alpha(\partial_1 V - \partial_2 V) = 0. \tag{29}$$

It is known that Eq. (29) has infinitely many homogeneous polynomial solutions [1] (also see the appendix). For a fixed α, each solution corresponds to a particular member of the hydrodynamic hierarchy, with each being a symmetry of the others (satisfying condition (22)).

This observation guarantees that for constructing the recursion operators, it suffices to find the solutions that generate the simplest hydrodynamic systems. We let $V_N(\alpha) = V_N(q^1, q^2; \alpha)$ denote the Nth-order homogeneous polynomial solution(s) of (29). Some of these are

$$V_1(\alpha) = q^1 + q^2, \tag{30}$$

$$V_2(\alpha) = \begin{cases} (q^1)^2 + \dfrac{2\alpha}{\alpha+1} q^1 q^2 + (q^2)^2, & \alpha \neq -1, \\ q^1 q^2, & \alpha = -1, \end{cases} \tag{31}$$

$$V_3(\alpha) = \begin{cases} (q^1)^3 + \dfrac{3\alpha}{\alpha+2}((q^1)^2 q^2 + q^1 (q^2)^2) + (q^2)^3, & \alpha \neq -2, \\ (q^1)^2 q^2 + q^1 (q^2)^2, & \alpha = -2. \end{cases} \tag{32}$$

Solution $V_1(\alpha)$ does not generate a nontrivial hydrodynamic system, because the corresponding v^i are constants and system (1) then decouples. In contrast, $V_2(\alpha)$ generates an infinite family containing some well-known examples. Using relations (5), we can find v^i for $V_2(\alpha)$ (removing an inessential constant factor):

$$\begin{aligned} &v^1(\alpha) = \partial_1 V_2(\alpha) = q^1 + \frac{\alpha}{\alpha+1} q^2, \quad v^2(\alpha) = \partial_2 V_2(\alpha) = q^2 + \frac{\alpha}{\alpha+1} q^1, \quad \alpha \neq -1, \\ &v^1 = q^2, \qquad v^2 = q^1, \qquad \alpha = -1. \end{aligned} \tag{33}$$

The hydrodynamic system corresponding to (33) is in the class of the so-called linearly degenerate systems and is very well studied [9, 10]. To demonstrate the symmetry properties, we consider $V_3(-1)$ and write the corresponding w^i (using (5)),

$$w^1 = (q^1)^2 - 2q^1q^2 - (q^2)^2, \quad w^2 = (q^2)^2 - 2q^1q^2 - (q^1)^2.$$

Because v^i and w^i satisfy identity (22), the corresponding hydrodynamic systems commute. Other known examples in this class consist of the cases where $\alpha = \pm 1/2$. If $\alpha = 1/2$, we obtain the system

$$q^1_t = (3q^1 + q^2)q^1_x, \quad q^2_t = (3q^2 + q^1)q^2_x,$$

which describes shallow water waves (a special case of the quasi-classical limit of coupled Korteweg–de Vries equations studied in [11]). If $\alpha = -1/2$, we obtain the system

$$q^1_t = (q^1 - q^2)q^1_x, \qquad q^2_t = (q^2 - q^1)q^2_x,$$

which is simply related to the dispersionless Toda hierarchy discussed in [12].

4.2. The case $n = 2$. This case is a bit more complicated than that where $n = 0$, partly because we do not have a general polynomial solution form. But it is still possible to generate examples of hydrodynamic systems. For instance, setting $\alpha_1 = \alpha_2 = -1$, we obtain the generalized EPD wave equation

$$q^1q^2(q^1 - q^2)\partial_1\partial_2 V - (q^1)^2\partial_1 V + (q^2)^2\partial_2 V = 0. \tag{34}$$

A nontrivial solution of this equation produced from the trivial solution $V = 1$ is $V = q^1q^2$; this then leads to

$$v^1 = (q^1)^2\partial_1 V = (q^1)^2q^2, \qquad v^2 = (q^2)^2\partial_2 V = q^1(q^2)^2,$$

which belong to the Temple class [9]. Acting with r_- (or with $\mathcal{R}_-$) gives

$$w^1 = (q^1)^2q^2(2q^1 + q^2), \qquad w^2 = q^1(q^2)^2(q^1 + 2q^2).$$

5. N-component systems

It is easy to extend the results obtained to a hydrodynamic system in N dimensions:

$$q^i_t = v^i(\mathbf{q})q^i_x, \quad i = 1, 2, \ldots, N.$$

In N dimensions, known examples still satisfy equations analogous to (4),

$$\partial_j(v^i\varphi^i) = \partial_i(v^j\varphi^j), \quad i \neq j, \tag{35}$$

which gives the metric constraint

$$\partial_j(\varphi^i \log g_{ii}) + \partial_i(\varphi^j \log g_{jj}) = 0.$$

Relation (35) entails the existence of a function $V(\mathbf{q})$, which gives

$$v^i = \frac{\partial_i V}{\varphi^i} \quad \forall i = 1, 2, \ldots, N.$$

Using (2), we find the multicomponent generalized EPD equations for V,

$$K^{ij}\partial_i\partial_j V + \frac{\partial_i V}{\varphi^i} - \frac{\partial_j V}{\varphi^j} = 0 \quad \forall i \neq j,$$

where $K^{ij}(\mathbf{q}) = 2/(\varphi^i \partial_j \log g_{ii})$ and

$$g_{ii} = \gamma^i(q^i) \prod_{j\neq i} \frac{(q^j)^{2\alpha_i\alpha_{ij}n/(n-1)}}{((q^i)^n q^j - (q^j)^n q^i)^{2\alpha_i\alpha_{ij}/(n-1)}}.$$

We note that there exist $(1/2)N(N-1)$ such equations. Definition (8) remains the same, but invariance condition (9) becomes

$$\bar{V}(\mathbf{q}) = V(\mathbf{q}) \prod_{k=1}^{N} (q^k)^{\alpha_k}, \quad \bar{q}^i = f^i(q^i),$$

where $\alpha_k = \prod_{j\neq k} \alpha_j$. Under this invariance condition, we obtain the $(1/2)N(N-1)$ equations,

$$L_n^{ij} V \equiv \frac{1}{\alpha_{ij}} q^i q^j ((q^i)^{n-1} - (q^j)^{n-1})\partial_i\partial_j V + \alpha_i (q^i)^n \partial_i V - \alpha_j (q^j)^n \partial_j V = 0,$$

where $\alpha_{ij} = \prod_{k\neq i,j} \alpha_k$. For each equation in this system, we can find the ladder operators

$$\begin{aligned}
r_1^{ij} &= (q^i)^{2-n}\partial_i + (q^j)^{2-n}\partial_j + \alpha_j (q^i)^{1-n} + \alpha_i (q^j)^{1-n}, \\
r_2^{ij} &= q^i\partial_i + q^j\partial_j, \\
r_3^{ij} &= (q^i)^n\partial_i + (q^j)^n\partial_j
\end{aligned}$$

up to multiplying by or adding a constant.

As before, we can construct a commutative diagram and solve the algebraic equations for the components of the recursion operator $\mathcal{R}$, and it is simple to extend our previous examples to the N-dimensional case.

6. Conclusions

We have shown how recursion operators for systems of the hydrodynamic type can be related to the ladder operators of a generalized EPD equation, which is used to generate the functions v^i in (1) by formula (5). This reflects the existence of a rich family of symmetries in the context of semi-Hamiltonian systems of the hydrodynamic type. This is to be contrasted with the usual "soliton equations," such as the Korteweg–de Vries equation, where the hierarchy of symmetries is usually discrete.

Recently, there has been much interest in *nonhomogeneous* systems of the hydrodynamic type, such as the Gibbons–Tsarev equation [13]

$$q_t^1 = q^2 q_x^1 + \frac{1}{q^1 - q^2}, \qquad q_t^2 = q^1 q_x^2 + \frac{1}{q^2 - q^1}.$$

This equation appears to have only a *finite* number of symmetries and is therefore not expected to have a recursion operator. Because the Tsarev theorem [2] on conservation laws, symmetries, and the generalized hodograph transformation does not hold in this case, such equations are indeed very interesting for future investigations.

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Appendix: The Euler–Poisson–Darboux equation

We give some useful properties of the EPD linear wave equation,

$$L_0 u \equiv (x-y)u_{xy} + \alpha u_y - \beta u_x = 0. \tag{36}$$

For the simpler case $\alpha = \beta$, this equation can be derived from

$$u_{rr} + \frac{m}{r}u_r - u_{tt} = 0$$

by the simple coordinate transformation $x = t + r$, $y = t - r$ with $m = 2\alpha$ [7].

The general formula for the Nth-order homogeneous polynomial solution of (36) is given by $u(x,y) = \mathcal{F}_N(x,y)$ with

$$\mathcal{F}_N(x,y) = \sum_{i+j=N} \frac{\alpha_i \beta_j}{i!j!}\, x^i y^j, \tag{37}$$

where $(\alpha)_i = \alpha(\alpha+1)\cdots(\alpha+i-1)$ for $i \geq 1$ and $(\alpha)_0 = 1$ with $i,j \in \mathbb{Z}_+ \cup \{0\}$. This formula is given in [7] for the $\alpha = \beta$ reduction.

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