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Is the largest Lyapunov exponent preserved in embedded dynamics?

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Abstract

The method of reconstruction for an n-dimensional system from observations is to form vectors of m consecutive observations, which for $m > 2n$, is generically an embedding. This is Takens' result. Our analytical examples show that it is possible to obtain *spurious* Lyapunov exponents that are even larger than the largest Lyapunov exponent of the original system. Therefore, we present examples where the largest Lyapunov exponent may not be preserved under Takens' embedding theorem.

1. Introduction

Lyapunov exponents measure the rate of divergence or convergence of two nearby initial points of a dynamical system. A positive Lyapunov exponent measures the average exponential divergence of two nearby trajectories whereas a negative Lyapunov exponent measures exponential convergence of two nearby trajectories. If a discrete nonlinear system is dissipative, a positive Lyapunov exponent quantifies a measure of chaos.

The introduction of Lyapunov exponents to economics was in [1]. Brock and Sayers [2] note that the Wolf [3] algorithm is sensitive to the number of observations as well as to the degree of measurement or system noise in the observations. This observation started a search for new algorithmic designs with improved finite sample properties. The search for an algorithm to calculate Lyapunov exponents with desirable finite sample properties has gained momentum in the last few years. Abarbanel et al. [4–6], Ellner et al. [7], McCaffrey et al. [8], Gençay and Dechert [9] and Dechert and Gençay [10] came up with improved algorithms for the calculation of the Lyapunov exponents from observed data. Gençay [11] worked on the calculation of the Lyapunov exponents with noisy data when feedforward networks were used as the estimation technique.

The main algorithmic design in all papers above is to embed the observations in an m-dimensional space, then by theorems of Mañé [12] and Takens [13] the observations are used to reconstruct the dynamics on the attractor. The Jacobian of the reconstructed dynamics as demonstrated in [14,15] is then used to calculate the

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Lyapunov exponents of the unknown dynamics. The method of reconstruction for a n-dimensional system from observations is to form vectors of m consecutive observations, which for $m > 2n$ is generically an embedding. The Jacobian methods for Lyapunov exponents utilize a function of m variables to model the data and the Jacobian matrix is constructed at each point in the orbit of the data. When embedding occurs at dimension $m = n$, then the Lyapunov exponents of the reconstructed dynamics are the Lyapunov exponents of the original dynamics. However, if embedding only occurs for an $m > n$, then the Jacobian method yields m Lyapunov exponents, only n of which are the Lyapunov exponents of the original system. The problem is that as currently used, the Jacobian method is applied to the full m-dimensional space of the reconstruction, and not just to the n-dimensional manifold that is the image of the embedding map. Our examples show that it is possible to get *spurious* Lyapunov exponents that are even larger than the largest Lyapunov exponent of the original system.

2. The Jacobian algorithm

The Lyapunov exponents for a dynamical system, $f: R^n \to R^n$, with the trajectory, $x_{t+1} = f(x_t)$, $t = 0, 1, 2, \ldots,$ are measures of the average rate of divergence or convergence of a typical trajectory.[1] For an n-dimensional system as above, there are n exponents which are customarily ranked from largest to smallest $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$.

It is a consequence of Oseledec's [16] theorem that the Lyapunov exponents exist for a broad class of functions.[2] The additional properties of Lyapunov exponents and a formal definition are given in [20].

In practice one rarely has the advantage of observing the state of the system, x_t, let alone knowing the actual functional form f which generates the dynamics. The model which is widely used is that associated with the dynamical system there is an observer function $h: R^n \to R$ which generates the observations, $y_t = h(x_t)$. It is assumed that all that is available to the researcher is the sequence $\{y_t\}$. For notational purposes, let

$$y_t^m = (y_t, y_{t+1}, \ldots, y_{t+m-1}). \tag{1}$$

If the set $\overline{U}$ is compact manifold then for $m \geqslant 2n+1$

$$J^m(x) = \big(h(x), h\big(f(x)\big), \ldots, h\big(f^{m-1}(x)\big)\big) \tag{2}$$

generically is an embedding.[3] For $m \geqslant 2n+1$ there exists a function $g: R^m \to R^m$ such that $y_{t+1}^m = g(y_t^m)$ where $y_{t+1}^m = (y_{t+1}, y_{t+2}, \ldots, y_{t+m})$. But notice that

$$y_{t+1}^m = J^m(x_{t+1}) = J^m\big(f(x_t)\big). \tag{3}$$

Hence from Eqs. (1) and (3) $J^m(f(x_t)) = g(J^m(x_t))$.

The function g is topologically conjugate to f. This implies that g inherits the dynamical properties of f. Dechert and Gençay [10] prove the following theorem to show that n of the Lyapunov exponents of g are the Lyapunov exponents of f.

Theorem 2.1 (Dechert and Gençay [10]). *Assume that M is a smooth manifold dimension n, $f: M \to M$ and $h: M \to R$ are (at least) $\mathbb{C}^2$. Define $J^m: M \to R^m$ by $J^m(x) = (h(x), h(f(x)), \ldots, h(f^{m-1}(x)))$. Let $\mu_1(x) \geqslant \mu_2(x) \geqslant \cdots \geqslant \mu_n(x)$ be the eigenvalues of the symmetric matrix $(DJ^m)'_x(DJ^m)_x$, and suppose that $\inf_{x\in M} \mu_n(x) > 0$, $\sup_{x\in M} \mu_1(x) < \infty$. Let $\lambda_1^f \geqslant \lambda_2^f \geqslant \cdots \geqslant \lambda_n^f$ be the Lyapunov exponents of f and $\lambda_1^g \geqslant \lambda_2^g \geqslant \cdots \geqslant \lambda_m^g$ be the Lyapunov exponents of g, where $g: J^m(M) \to J^m(M)$ and $J^m(f(x)) = g(J^m(x))$ on M. Then generically $\{\lambda_i^f\} \subset \{\lambda_i^g\}$.*

By Theorem 2.1, n of the Lyapunov exponents of g are the Lyapunov exponents of f. The approach of Gençay and Dechert [9] is to estimate the function g based on the data sequence $\{J^m(x_t)\}$, and to calculate the Lyapunov exponents of g.

[1] The trajectory is also written in terms of the iterates of f. With the convention that f^0 is the identity map, and $f^{t+1} = f \circ f^t$, then we also write, $x_t = f^t(x_0)$. A trajectory is also called an orbit in the dynamical system literature.

[2] Also see [17–19] for precise conditions and proofs of the theorem.

[3] By *generic* is meant that in every neighborhood of f and h there are functions $\tilde{f}$ and $\tilde{h}$ so that the function J^m corresponding to these functions is an embedding of the attractor of $\tilde{f}$ and the image of the image of the attractor under J^m. Here $2n+1$ is the worst-case upper limit.

From Eq. (1) the mapping g which is to be estimated may be taken [4] to be

$$g:\begin{bmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+m-1} \end{bmatrix} \to \begin{bmatrix} y_{t+1} \\ y_{t+2} \\ \vdots \\ v(y_t, y_{t+1}, \ldots, y_{t+m-1}) \end{bmatrix} \tag{4}$$

and this reduces to estimating $y_{t+m} = v(y_t, y_{t+1}, \ldots, y_{t+m-1})$. Here v is an unknown map. Linearization of the map g yields $\Delta y^m_{t+1} = (Dg)_{y^m_t} \Delta y^m_t$. The solution can be written as $\Delta y^m_t = (Dg^t)_{y^m_0} \Delta y^m_0$.

The Lyapunov exponents can be calculated from the eigenvalues of the matrix $(Dg^t)_{y^m_0}$ using QR decomposition. This method is discussed in [14,15,21] and a modified version is presented in [6].

3. An example

If x is a fixed point, then the subspaces $V^j_t = V^j$ do not depend upon t. Let us consider the mapping $f(x)$ at the fixed point x. Choose $V^1 = R^2$, $V^2 = \operatorname{span}\{(0,1)\}$ and $V^3 = \{0\}$. For $|\mu_1| > |\mu_2|$ consider [5]

$$Df(x) = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}. \tag{5}$$

This will satisfy parts (1) and (2) of Definition in [20] and we will have

$$\lambda_1 = \lim_{t\to\infty} t^{-1} \ln\left(\left|\mu_1^t v_1 + \mu_2^t v_2\right|\right) = \ln|\mu_1|$$
$$\text{for } v \in V^1 \setminus V^2,$$
$$\lambda_2 = \lim_{t\to\infty} t^{-1} \ln\left(\left|\mu_1^t v_1 + \mu_2^t v_2\right|\right) = \ln|\mu_2|$$
$$\text{for } v \in V^2 \setminus V^3.$$

This definition mainly generalizes the idea of eigenvalues to give average linearized contraction and expansion rates on a trajectory. An attractor is a set of points towards which the trajectories of f converge. More precisely, Λ is an attractor if there is an open set $U \subset R^n$ with $\Lambda \subset U$, $f(\overline{U}) \subset U$ and $\Lambda = \bigcap_{t\geqslant 0} f^t(U)$ where $\overline{U}$ is the closure of U. The attractor Λ is said to be indecomposable if there is no proper subset of Λ which is also an attractor. An attractor can be chaotic or ordinary (or nonchaotic). There is more than one definition of a chaotic attractor in the literature. In practice the presence of a positive Lyapunov exponent is taken as a signal that the attractor is chaotic.

Now, suppose that the observations come from the following:

$$y = h(x) = x_1 + x_2, \tag{6}$$

where $h: R^2 \to R$. Let us consider a 3-embedding history generated from $h(x)$ so that,

$$J^3(x) = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \\ \mu_1^2 & \mu_2^2 \end{bmatrix} x \quad \text{and} \tag{7}$$

$$J^3 \circ f(x) = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_1^2 & \mu_2^2 \\ \mu_1^3 & \mu_2^3 \end{bmatrix} x. \tag{8}$$

Let

$$g(y) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\mu_1\mu_2 & \mu_1+\mu_2 \end{bmatrix} y$$

for $y \in R^3$. Then

$$g \circ J^3(x) = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_1^2 & \mu_2^2 \\ \mu_1^3 & \mu_2^3 \end{bmatrix} x = J^3 \circ f(x).$$

Therefore, the condition for conjugacy is satisfied. Also,

$$(Dg)_y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\mu_1\mu_2 & \mu_1+\mu_2 \end{bmatrix}. \tag{9}$$

Let $W^1 = R^3$, $W^2 = \operatorname{span}\{(1,0,0), (1,\mu_2,\mu_2^2)\}$, $W^3 = \operatorname{span}\{(1,0,0)\}$ and $W^4 = \{0\}$. Then

$$(Dg)_y\left(W^1\right) = \operatorname{span}\left\{\left(1,\mu_1,\mu_1^2\right), \left(1,\mu_2,\mu_2^2\right)\right\} \subset W^1,$$
$$(Dg)_y\left(W^2\right) = \operatorname{span}\left\{\left(1,\mu_2,\mu_2^2\right)\right\} \subset W^2 \quad \text{and}$$
$$(Dg)_y\left(W^3\right) = \{0\} \subset W^3.$$

(Notice that the sets $(Dg)_y W^j$ can be proper subsets of W^j. In this example, this comes about since the dynamics of g are not of full dimension, which is immediately apparent from Eq. (9).) If $v \in V^1 \setminus V^2$

[4] Here, the time step is assumed to be equal to the delay time.

[5] This example is from Guckenheimer and Holmes [20].

then

$$v=\alpha\begin{bmatrix}1\\0\end{bmatrix}+\beta\begin{bmatrix}0\\1\end{bmatrix},\quad \alpha\neq 0,\quad \text{and}$$

$$\left(DJ^3\right)v=\alpha\begin{bmatrix}1\\\mu_1\\\mu_1^2\end{bmatrix}+\beta\begin{bmatrix}1\\\mu_2\\\mu_2^2\end{bmatrix}.$$

Here, $\alpha\neq 0$ implies that $(DJ^3)v\in W^1\setminus W^2$. If $v\in V^2\setminus V^3$ then

$$v=\beta\begin{bmatrix}0\\1\end{bmatrix},\quad \beta\neq 0,\quad \text{and}$$

$$\left(DJ^3\right)v=\beta\begin{bmatrix}1\\\mu_2\\\mu_2^2\end{bmatrix}.$$

Also $\beta\neq 0$ implies that $(DJ^3)v\in W^2\setminus W^3$. If $w\in W^1\setminus W^2$ then

$$w=\alpha\begin{bmatrix}1\\\mu_1\\\mu_1^2\end{bmatrix}+\beta\begin{bmatrix}1\\\mu_2\\\mu_2^2\end{bmatrix}+\gamma\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \alpha\neq 0,$$

and

$$\left|(Dg)_y^t w\right|=\left|\alpha\mu_1^t\begin{bmatrix}1\\\mu_1\\\mu_1^2\end{bmatrix}+\beta\mu_2^t\begin{bmatrix}1\\\mu_2\\\mu_2^2\end{bmatrix}\right|.$$

Hence $\lim_{t\to\infty}t^{-1}\ln|(Dg^t)_y w|=\ln|\mu_1|$.

If $w\in W^2\setminus W^3$ then

$$w=\beta\begin{bmatrix}1\\\mu_2\\\mu_2^2\end{bmatrix}+\gamma\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \beta\neq 0,\quad \text{and}$$

$$\left|\left(Dg^t\right)_y w\right|=\left|\beta\mu_2^t\begin{bmatrix}1\\\mu_2\\\mu_2^2\end{bmatrix}\right|.$$

Hence $\lim_{t\to\infty}t^{-1}\ln|(Dg^t)_y w|=\ln|\mu_2|$.

If $w\in W^3\setminus W^4$ then

$$w=\gamma\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \gamma\neq 0$$

and $|(Dg^t)_y w|=0$. Therefore

$$\lim_{t\to\infty}t^{-1}\ln\left|(Dg)_y^t w\right|=-\infty.$$

This example shows Theorem2.1 at work. The two largest Lyapunov exponents of g are the Lyapunov exponents of f, and in this example the 'spurious' third exponent of g is $-\infty$.

4. Spurious Lyapunov exponents

In [9,22] the numerical studies demonstrated that the n Lyapunov exponents of f turned out to be the largest n Lyapunov exponents of g. These results were obtained by using an observation function of the form

$$h(x_1,x_2,\ldots,x_n)=x_1 \tag{10}$$

which has been widely used in simulation studies of nonlinear dynamical systems.

Consider the following variation to the example in the previous section. The dynamics are the same linear dynamics of Eq. (5) and the observation function is the same as Eq. (6). From this we obtain the same embedding equations as (7) and (8). Now however, consider the following function g: for any $a\in R$, let

$$g(y)=\begin{bmatrix}a & 1-a\left(\mu_1^{-1}+\mu_2^{-1}\right) & a\mu_1^{-1}\mu_2^{-1}\\ 0 & 0 & 1\\ 0 & -\mu_1\mu_2 & \mu_1+\mu_2\end{bmatrix}y \tag{11}$$

for $y\in R^3$. Notice that this is not in the form of Eq. (4), however it does satisfy

$$g\circ J^3(x)=\begin{bmatrix}\mu_1 & \mu_2\\ \mu_1^2 & \mu_2^2\\ \mu_1^3 & \mu_2^3\end{bmatrix}x=J^3\circ f(x)$$

and therefore the condition for conjugacy is satisfied. [6] Also,

$$(Dg)_y=\begin{bmatrix}a & 1-a\left(\mu_1^{-1}+\mu_2^{-1}\right) & a\mu_1^{-1}\mu_2^{-1}\\ 0 & 0 & 1\\ 0 & -\mu_1\mu_2 & \mu_1+\mu_2\end{bmatrix}.$$

If $|\mu_2|>|a|$, let $W^1=R^3$, $W^2=\text{span}\{(1,0,0),(1,\mu_2,\mu_2^2)\}$, $W^3=\text{span}\{(1,0,0)\}$ and $W^4=\{0\}$. Then if $a=0$,

$$(Dg)_y\left(W^1\right)=\text{span}\left\{\left(1,\mu_1,\mu_1^2\right),\left(1,\mu_2,\mu_2^2\right)\right\}\subset W^1,$$
$$(Dg)_y\left(W^2\right)=\text{span}\left\{\left(1,\mu_2,\mu_2^2\right)\right\}\subset W^2,\quad \text{and}$$
$$(Dg)_y\left(W^3\right)=\{0\}\subset W^3.$$

[6] This shows that there can be many functions which can generate the same dynamics. In our case we are interested in the impact that the observer function has on this multiplicity of representations, g.

If $a \neq 0$ then

$(Dg)_y(W^1) = W^1, \qquad (Dg)_y(W^2) = W^2$ and $(Dg)_y(W^3) = W^3$.

If $v \in V^1 \setminus V^2$ then

$$v = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \alpha \neq 0, \quad \text{and}$$

$$(DJ^3)v = \alpha \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix}.$$

Here, $\alpha \neq 0$ implies that $(DJ^3)v \in W^1 \setminus W^2$. If $v \in V^2 \setminus V^3$ then

$$v = \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \beta \neq 0, \quad \text{and} \quad (DJ^3)v = \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix}.$$

Also $\beta \neq 0$ implies that $(DJ^3)v \in W^2 \setminus W^3$. If $w \in W^1 \setminus W^2$ then

$$w = \alpha \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \alpha \neq 0$$

and

$$\left|(Dg^t)_y w\right| = \left|\alpha \mu_1^t \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \mu_2^t \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma a^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right|.$$

Hence $\lim_{t\to\infty} t^{-1} \ln|(Dg^t)_y w| = \ln|\mu_1|$.

If $w \in W^2 \setminus W^3$ then

$$w = \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \beta \neq 0, \quad \text{and}$$

$$\left|(Dg^t)_y w\right| = \left|\beta \mu_2^t \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma a^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right|.$$

Hence $\lim_{t\to\infty} t^{-1} \ln|(Dg^t)_y w| = \ln|\mu_2|$.

If $w \in W^3 \setminus W^4$ then

$$w = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \gamma \neq 0,$$

and $|(Dg^t)_y w| = |\gamma||a|^t$. Therefore $\lim_{t\to\infty} t^{-1} \ln|(Dg)_y^t w| = \ln|a|$. Note that if $a = 0$ then this third 'spurious' Lyapunov exponent is $-\infty$.

If $|\mu_1| > |a| > |\mu_2|$ then the subspace W^3 above needs to be changed so that $W^3 = \text{span}\{(1, \mu_2, \mu_2^2)\}$. Then $(Dg)_y(W^1) = W^1$, $(Dg)_y(W^2) = W^2$ and $(Dg)_y(W^3) = W^3$. The three Lyapunov exponents are: $\ln|\mu_1|, \ln|a|, \ln|\mu_2|$. If $|a| > |\mu_1|$ then change the subspaces so that $W^2 = \text{span}\{(1, \mu_1, \mu_1^2), (1, \mu_2, \mu_2^2)\}$, $W^3 = \text{span}\{(1, \mu_2, \mu_2^2)\}$ and again $(Dg)_y(W^1) = W^1$, $(Dg)_y(W^2) = W^2$ and $(Dg)_y(W^3) = W^3$ will hold. The three Lyapunov exponents are: $\ln|a|, \ln|\mu_1|, \ln|\mu_2|$.

Notice that in all cases the two Lyapunov exponents of f are two of the Lyapunov exponents of g. The third Lyapunov exponent of g can be of any magnitude. The problem comes from the fact that the partial derivatives of g do not necessarily lie in the tangent space of the image of the attractor under the Takens embedding (2). It raises the question of how to identify the n *true* Lyapunov exponents of f from the $m - n$ *spurious* Lyapunov exponents that make up the Lyapunov exponents of g.

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